Assigment 1

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Q1:

1. Consider the statemetn: $[f(n) = O(g(n))$ and $f(n) \neq o(g(n)] \Longrightarrow [f(n) = \Theta(g(n))]$. State whether this statement is true or false. If your answer is "true" , then prove it; if it is "false', then provide a counter example.

> *The answer is True.*
> *Since we know that $f(n) = O(g(n))$ and $f(n) \neq o(g(n))$:*
>
> $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty \text{ also } \neq 0$$
>
> *Since the algorithms' cost would never decrease with the working load goes up, we have :*
>
> $$\lim_{n\to\infty} \frac{f(n)}{g(n)} \in (0, \infty)$$
>
> *From above, we know that the limitation ratio must lays within 0 and ∞, which holding both tight upper bound and tight lower bound.*

Q2:

3.2-4 ★
Is the function $\lceil \lg n \rceil!$ polynomially bounded? Is the function $\lceil \lg \lg n \rceil!$ polynomially bounded?

> - *If a function $f(n)$ is polynomially bounded, then there must be constant values c, k, n_0, such that for all $n \geq n_0$, $f(n) \leq cn^k$. Hence, $\lg(f(n)) \leq k \lg cn \leq kc \lg n$, therefore, we know that $\lg(f(n)) = O(\lg n)$.*

- *To the other side: if we know that $\lg(f(n)) = O(\lg n)$, we could conclude that $f(n)$ is polynomially bounded.*

From above, we know that proving a function $f(n)$ is polynomially bounded is equivalent to prove that $\lg(f(n)) = O(\lg(n))$.

- *Function $\lceil \lg n \rceil!$*

  - *First we concern about $\lg(n!)$'s complexity:*
    - *According to Stirling's approximation:*

$$\begin{aligned} \lg(n!) &= \lg(\sqrt{2\pi n}(\frac{n}{e})^n (e^{\alpha_n})) \\ &= \lg(\sqrt{2\pi n}(\frac{n}{e})^n (1 + \Theta(\frac{1}{n}))) \\ &= \Theta(\sqrt{n}) + \Theta(n \lg n) + \Theta(\frac{1}{n}) \\ &= \Theta(n \lg n) \end{aligned}$$

  - *Second, we want to show $\lceil \lg n \rceil = \Theta(\lg n)$:*

    - $\lceil \lg n \rceil \leq \lg n$
    - $\lceil \lg n \rceil < \lg n + 1 \leq 2 \lg n$ *for* $\forall n \geq 2$

  - *Third, to $\lg(\lceil \lg n \rceil !)$, we could take $\lceil \lg n \rceil$ as n in previous prove :*

$$\begin{aligned} \lg(\lceil \lg n \rceil !) &= \Theta(\lceil \lg n \rceil \lg \lceil \lg n \rceil) \\ &= \Theta(\lg n (\lg \lg n)) \\ &= \omega(\lg n) \\ &\neq O(\lg n) \end{aligned}$$

  So, $\lceil \lg n \rceil!$ is not polynomially bounded.

- *Function $\lceil \lg \lg n \rceil!$, the steps are all the same:*

  - *First, we have to know that any polylogarithmic function grows slowly than any positive polynomial function*
    - *Suppose for constants $a, b > 0$, $\lg^b n = o(n^a)$.*
    - *Substitute n with $\lg n$, b with 2 and a with 1:* $\lg^2(\lg n) = o(\lg n)$

$$
\begin{aligned}
\lg(\lceil \lg\lg n \rceil !) &= \Theta(\lceil \lg\lg n \rceil \lg \lceil \lg\lg n \rceil) \\
&= \Theta((\lg\lg n)(\lg\lg\lg n)) \\
&= o((\lg\lg n)^2) \\
&= o(\lg^2(\lg n)) \\
&= o(\lg n)
\end{aligned}
$$

So, $\lceil \lg\lg n \rceil !$ is polynomially bounded.

Q3

3-2 Relative asymptotic growths

Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants. Your answer should be in the form of the table with "yes" or "no" written in each box.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	—	—	✓	✓	—
b.	n^k	c^n	—	—	✓	✓	—
c.	$\sqrt{n}$	$n^{\sin n}$	—	—	—	—	—
d.	2^n	$2^{n/2}$	✓	✓	—	—	—
e.	$n^{\lg c}$	$c^{\lg n}$	✓	—	✓	—	✓
f.	$\lg(n!)$	$\lg(n^n)$	✓	—	✓	—	✓

a. $\lg^k = o(n^\epsilon) \Longrightarrow \omega(\lg^k n) = n^\epsilon, \omega \to \Omega, \omega \to \bar{o} \wedge \bar{O} \Longrightarrow \bar{\Theta}$.
b. $n^k = o(c^n) \Longrightarrow \omega(n^k) = c^n, \omega \to \Omega, \omega \to \bar{o} \wedge \bar{O} \Longrightarrow \bar{\Theta}$.
c. sinfunction changes periodically, so the relationship between those two function is undefined.
d. $\lim\limits_{n\to\infty} \frac{2^n}{2^{n/2}} = \lim\limits_{n\to\infty} (\sqrt{2})^n = \infty$, *which indicates*
$o(2^n) = 2^{n/2}, o \to O, o \to \bar{\omega} \wedge \bar{\Omega} \Longrightarrow \bar{\Theta}$
e. when $c = 1$, $n^{\lg c} = n^0 = 1 = c^{\lg n} = 1^{\lg n}$, *so* $O, \Omega \Longrightarrow \Theta = \bar{o} \wedge \bar{\omega}$
f. $\lg(n!) = \Theta(n \lg n) \Longrightarrow \Theta(\lg n!) = n \lg n = \lg n^n$, *so* $O, \Omega \Longrightarrow \Theta = \bar{o} \wedge \bar{\omega}$

Q4

3-3 *Ordering by asymptotic growth rates*

a. Rank the following functions by order of growth; that is, find an arrangement $g_1, g_2, \ldots, g_{30}$ of the functions satisfying $g_1 = \Omega(g_2)$, $g_2 = \Omega(g_3)$, ..., $g_{29} = \Omega(g_{30})$. Partition your list into equivalence classes such that functions $f(n)$ and $g(n)$ are in the same class if and only if $f(n) = \Theta(g(n))$.

$$2^{2^{n+1}} > 2^{2^n} > (n+1)! > n! \Leftarrow (n \neq \Theta(n^{n+1}e^{-n})) > e^n \Leftarrow (e^n = 2^n(e/2)^n = \omega(n2^n)) >$$
$$n2^n > 2^n > (\frac{3}{2})^n > (\lg n)^{\lg n} = n^{\lg\lg n} \Leftarrow (a^{\log_b c} = a^{\log_b a \cdot \log_a c} = c^{\log_b a}) > (\lg n)! > n^3 >$$
$$n^2 = 4^{\lg n} > n\lg n = lg(n!) > 2^{\lg n} = n > \sqrt{n} = (\sqrt{2})^{\lg n} \Leftarrow \text{take log} >$$
$$2^{\sqrt{2\lg n}} \Leftarrow \text{take log} > \lg^2 n > \ln n > \sqrt{\lg n} >$$
$$\ln\ln n \Leftarrow (\lg 2^{\lg^* n} = \lg^* n \cdot \lg\ln\ln n = \omega(\lg^* n)) >$$
$$2^{\lg^* n} > \lg^* n = \lg^*(\lg n) \Leftarrow (\lg^*(\lg n) = (\lg^* n) - 1) > lg(lg^*)n > n^{1/\lg n} = 2 = 1$$

> - *Most of the functions follow that:*
>   - *Exponential functions grow faster than polynomial functions, and the later grow faster than polylogarithmic functions*
>   - *The base of a logarithmic doesn't matter asymptotically, but the base of an exponential and the degree of a polynomial do matter.*

Q5

5. Show that (i)$\sum_{i=1}^{i=n} i^2 = \Theta(n^3)$

> *Prove:*
>
> *suppose we have:* $H(n) = \frac{\sum_{i=1}^{i=n} i^2}{n^3}$*, also we know that* $H(1) = 1, H(2) = \frac{5}{8}, H(3) = \frac{12}{27}$*, we could make a guess that* $for \forall n, H(n) \in (0, \infty)$*:*
> *Compare the following two expression, for* $\exists integer\ n \in (0, \infty)$*:*
>
> - $H(n) = \frac{\sum_{i=1}^{i=n} i^2}{n^3}$
> - $H(n+1) = \frac{\sum_{i=1}^{i=n} i^2 + (n^2+2n+1)}{n^3 + (3n^2+3n+1)}$

> ***Obviously, the increase in numerator is less than the increase in denominator. So the $H(n)$ is monotonically decreasing.***
>
> - *$H(n)$ won't be 0 because the numerator is always larger than 0.*
> - *As well, $H(n)$ won't be ∞ because the denominator is always larger than 0.*