

## Assignment 2

Jiadao Zou --- jxz172230

Q1:

4.3-6

Show that the solution to  $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$  is  $O(n \lg n)$ .

Choose  $n_1$  such that  $n \geq n_1$  implies  $n/2 + 17 \leq 3n/4$ . We'll find  $c, d$  such that  $T(n) \leq cn \lg n - d$ .

$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + 17 + n \\ &\leq 2(c(n/2 + 17) \lg(n/2 + 17) - d) + n \\ &\leq cn \lg(n/2 + 17) + 34c \lg(n/2 + 17) - 2d + n \\ &\leq cn \lg(3n/4) + 34c \lg(3n/4) - 2d + n \\ &= cn \lg n - d + cn \lg(3/4) + 34c \lg(3n/4) - d + n \end{aligned}$$

Take  $c = -2/\lg(3/4)$  and  $d = 34$ . Then we have  $T(n) \leq cn \lg n - d + k \lg(n) - n$ , where  $k \in \text{Integer}$ . Since  $\lg(n) = o(n)$ , there exists  $n_2$  such that  $n \geq n_2$  implies,  $n \geq k \lg(n)$ . Letting  $n_0 = \max\{n_1, n_2\}$ , we have that  $n \geq n_0$  implies  $T(n) \leq cn \lg n - d$ . Therefore, get proved.

4.3-9

Solve the recurrence  $T(n) = 3T(\sqrt{n}) + \lg n$  by making a change of variables. Your solution should be asymptotically tight. Do not worry about whether values are integral.

First, we could use  $2^k$  to replace  $n$ :

$$\begin{aligned} T(n) &= 3T(\sqrt{k/2}) + \lg n \\ T(2^k) &= 3T(2^{k/2}) + k \end{aligned}$$

Now, use  $S(k)$  as  $T(2^k)$ , Integer  $k > 0$ :

$$S(k) = 3S(k/2) + \lg k$$

If we guess that,  $S(k) \leq c \cdot k^{\lg 3} + dk$ :

$$\begin{aligned}
S(k) &= 3S(k/2) + \lg k \\
&\leq 3[c(k/2)^{\lg 3} + d(k/2)] + k \\
&= \frac{3c}{2^{\lg 3}} k^{\lg 3} + \left(\frac{3}{2}d + 1\right)k \\
&= C \cdot k^{\lg 3} + \left(\frac{3}{2}d + 1\right)k
\end{aligned}$$

If  $d \leq -2$  :

$$S(k) \leq C \cdot k^{\lg 3} + dk$$

If we guess that,  $S(k) \geq c \cdot k^{\lg 3} + dk$ :

$$\begin{aligned}
S(k) &= 3S(k/2) + \lg k \\
&\geq 3[c(k/2)^{\lg 3} + d(k/2)] + k \\
&= \frac{3c}{2^{\lg 3}} k^{\lg 3} + \left(\frac{3}{2}d + 1\right)k \\
&= C \cdot k^{\lg 3} + \left(\frac{3}{2}d + 1\right)k
\end{aligned}$$

If  $d \geq -2$  :

$$S(k) \geq C \cdot k^{\lg 3} + dk$$

Thus,  $S(k) = \Theta(k^{\lg 3})$ , according to our transformation:  $T(n) \rightarrow T(2^k) = S(k)$ . Therefore we have  $T(n) = \Theta((\lg n)^{\lg 3})$

## Q3

### 4.4-2

Use a recursion tree to determine a good asymptotic upper bound on the recurrence  $T(n) = T(n/2) + n^2$ . Use the substitution method to verify your answer.

The tree has  $\lg n + 1$  levels, and the subproblem size of a node at depth  $i$  is  $n/2^i$ . The total cost of the tree at depth is  $(\frac{n}{2^i})^2 \cdot 1^i$ :

$$\begin{aligned}
T(n) &= \sum_{i=0}^{\lg n} \left(\frac{n}{2^i}\right)^2 \cdot 1^{\lg i} + \Theta(1) \\
&< \sum_{i=0}^{\infty} (1/4)^i n^2 + \Theta(1) \\
&= \frac{1 - 0}{1 - 1/4} n^2 + \Theta(1) \\
&= \Theta(n^2)
\end{aligned}$$

In the following, use substitution method to verify that  $T(n) \leq cn^2$

$$\begin{aligned} T(n) &\leq c(n/2)^2 + n^2 \\ &= c\frac{n^2}{4} + n^2 \\ \text{if } \left(\frac{c}{4} + 1\right) &\geq c \implies c \geq \frac{3}{4} : \\ &\leq cn^2 \end{aligned}$$

#### 4.4-6

Argue that the solution to the recurrence  $T(n) = T(n/3) + T(2n/3) + cn$ , where  $c$  is a constant, is  $\Omega(n \lg n)$  by appealing to the recursion tree.

Examining the tree, we observe that the cost at each level of the tree is exactly  $cn$ . To find a lower bound on the cost of the algorithm, we need a lower bound on the height of the tree. The shortest simple path from root to leaf is found by following the left child at each node. Since we divide by 3 at each step, we see that this path has length  $\log_3 n$ , so the cost of the algorithm is  $cn(\log_3 n + 1) \geq cn \log_3 n = \frac{c}{\lg 3} n \lg n = \Omega(n \lg n)$ .

#### Problem 4-3

Give asymptotic upper and lower bounds for  $T(n)$  in each of the following recurrences. Assume that  $T(n)$  is constant for sufficiently small  $n$ . Make your bounds as tight as possible, and justify your answers.

**b:**  $T(n) = 3T(n/3) + n/\lg n$

According to M.T.  $a=3$ ,  $b=3$ ,  $\log_b a = 1$ , and  $f(n) = n/\lg n$

At the first look,  $f(n) = n/\lg n = 2^{\lg n}/\lg n \implies 0$ , when  $n \rightarrow \infty$ . So,

$f(n) = o(n^{\log_a b}) = O(n^{1-\epsilon})$ ,  $\exists \epsilon : 1 \gg \epsilon > 0$ . So we guess case 1 apply to this equation.

In the following, I would try to prove this guess by deduction:

First, show that  $T(n) \leq n \lg n$ :

$$\begin{aligned} T(n) &= 3T(n/3) + n/\lg n \\ &\leq cn \lg n - cn \lg(3) + n/\lg n \\ &= cn \lg n + \left(\frac{1}{\lg n} - c \lg 3\right)n \\ &\leq cn \lg n \end{aligned}$$

Now, we want to prove  $T(n) \geq cn^{1-\epsilon}$  for every  $\epsilon > 0$ :

$$\begin{aligned}
T(n) &= 3T(n/3) + n/\lg n \\
&\geq 3c/3^{1-\epsilon} n^{1-\epsilon} + n/\lg n \\
&= 3^\epsilon c n^{1-\epsilon} + n/\lg n
\end{aligned}$$

Also, we want to show  $T(n) \leq cn^{1-\epsilon}$  could be like that:

$$\begin{aligned}
3^\epsilon c n^{1-\epsilon} + n/\lg n &\geq c n^{1-\epsilon} \\
3^\epsilon + n^\epsilon / (c \lg n) &\geq 1
\end{aligned}$$

Since  $3^\epsilon > 1$  and  $\lg n = o(n^\epsilon)$ , the above equation holds.  
The function is soft Theta of  $n$ .

**c:**  $T(n) = 4T(n/2) + n^2\sqrt{n}$

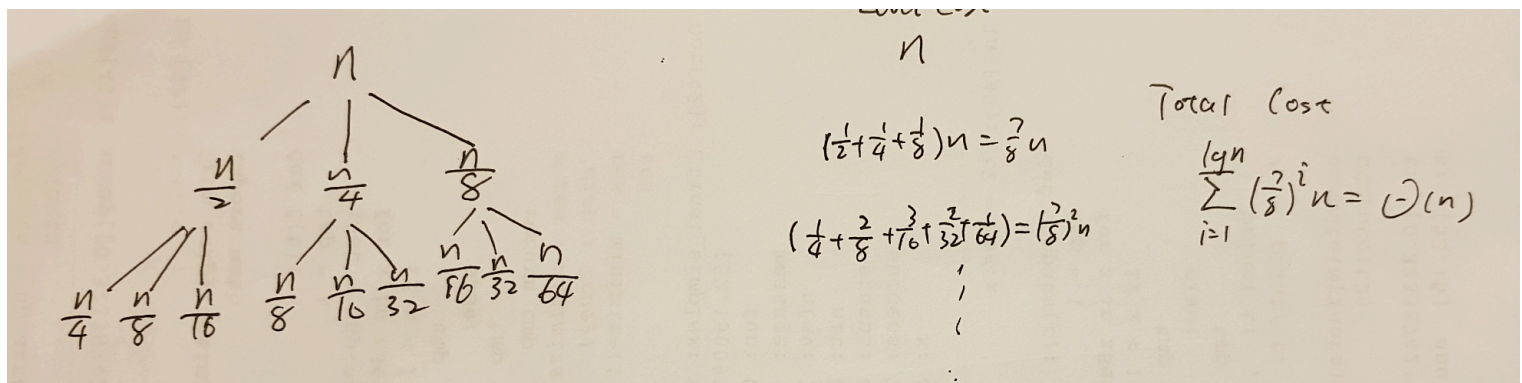
Look at the equation:  $T(n) = 4T(n/2) + n^{5/2}$ , according to M.T., we have  $a=4$ ,  $b=2$ ,  $\log_b a = 2$ , and  $f(n) = n^{5/2} = \Theta(n^{2+\epsilon})$  for  $\epsilon = 1/2$ .

So we look at case 3 regulation part 2:

$$af(n/b) = 4(n/2)^2 \sqrt{n/2} = n^{5/2}/\sqrt{2} \leq cn^{5/2}, \forall c \in [\frac{1}{\sqrt{2}}, 1)$$

Since part 1 and part 2 are satisfied, case 3 is applied.

**f:**  $T(n) = T(n/2) + T(n/4) + T(n/8) + n$



So, we want to prove the guess of recursion tree model.

First, to show that  $T(n) = O(n)$  by substitution  $T(n) \leq cn$ :

$$\begin{aligned}
T(n) &= T(n/2) + T(n/4) + T(n/8) + n \\
&\leq c\frac{n}{2} + c\frac{n}{4} + c\frac{n}{8} + n \\
&= (\frac{7c}{8} + 1)n \\
&\leq cn, \text{ when } c \geq 8
\end{aligned}$$

Also, since  $T(n) = T(n/2) + T(n/4) + T(n/8) + n \geq n$ , thus  $T(n) = \Omega(n)$ .  
 So, we have  $T(n) = \Theta(n)$

## Q4

### 4-4 Fibonacci numbers

This problem develops properties of the Fibonacci numbers, which are defined by recurrence (3.22). We shall use the technique of generating functions to solve the Fibonacci recurrence. Define the **generating function** (or **formal power series**)  $\mathcal{F}$  as

$$\begin{aligned}\mathcal{F}(z) &= \sum_{i=0}^{\infty} F_i z^i \\ &= 0 + z + z^2 + 2z^3 + 3z^4 + 5z^5 + 8z^6 + 13z^7 + 21z^8 + \cdots ,\end{aligned}$$

where  $F_i$  is the  $i$ th Fibonacci number.

**a.** Show that  $\mathcal{F}(z) = z + z\mathcal{F}(z) + z^2\mathcal{F}(z)$ .

b. Show that

$$\begin{aligned}\mathcal{F}(z) &= \frac{z}{1 - z - z^2} \\ &= \frac{z}{(1 - \phi z)(1 - \hat{\phi} z)} \\ &= \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \phi z} - \frac{1}{1 - \hat{\phi} z} \right),\end{aligned}$$

where

$$\phi = \frac{1 + \sqrt{5}}{2} = 1.61803 \dots$$

and

$$\hat{\phi} = \frac{1 - \sqrt{5}}{2} = -0.61803 \dots$$

c. Show that

$$\mathcal{F}(z) = \sum_{i=0}^{\infty} \frac{1}{\sqrt{5}} (\phi^i - \hat{\phi}^i) z^i.$$

d. Use part (c) to prove that  $F_i = \phi^i / \sqrt{5}$  for  $i > 0$ , rounded to the nearest integer. (Hint: Observe that  $|\hat{\phi}| < 1$ .)

$$\begin{aligned}a. \quad \mathcal{F}(z) &= \sum_{i=0}^{\infty} \bar{F}_i \cdot z^i && (\bar{F}_0 = 0, \bar{F}_1 = 1, \bar{F}_2 = 1, \dots) \\ &= \bar{F}_0 + \bar{F}_1 z + \sum_{i=2}^{\infty} (\bar{F}_{i-1} + \bar{F}_{i-2}) z^i \\ &= 0 + z + z \sum_{i=2}^{\infty} \bar{F}_{i-1} z^{i-1} + z^2 \sum_{i=2}^{\infty} \bar{F}_{i-2} z^{i-2} \\ &= z + z \sum_{i=1}^{\infty} \bar{F}_i z^i + z^2 \sum_{i=0}^{\infty} \bar{F}_i z^i \\ &= z + z \mathcal{F}(z) + z^2 \mathcal{F}(z)\end{aligned}$$



b. from above question:

$$f(x) - x f(x) - x^2 f(x) = x$$

$$f(x) = \frac{x}{1-x-x^2}$$

$$\therefore \phi - \hat{\phi} = \sqrt{5}, \phi + \hat{\phi} = 1 \text{ and } \phi \hat{\phi} = -1$$

$$f(x) = \frac{x}{1 - (\phi + \hat{\phi})x + \phi \hat{\phi} x^2}$$

$$= \frac{x}{(1 - \phi x)(1 - \hat{\phi} x)}$$

$$= \frac{(\phi - \hat{\phi})x + 1 - 1}{\sqrt{5}(1 - \phi x)(1 - \hat{\phi} x)}$$

$$= \frac{1}{\sqrt{5}} \left( \frac{(1 - \hat{\phi}x) - (1 - \phi x)}{(1 - \phi x)(1 - \hat{\phi} x)} \right) = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right)$$

c. from part b: and  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad (|x| < 1)$

$$f(x) = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right)$$

$$= \frac{1}{\sqrt{5}} \left( \sum_{i=0}^{\infty} (\phi x)^i - \sum_{i=0}^{\infty} (\hat{\phi} x)^i \right)$$

$$= \sum_{i=0}^{\infty} \frac{1}{\sqrt{5}} (\phi^i - \hat{\phi}^i) x^i$$

$$d. f(x) = \sum_{i=0}^{\infty} F_i x^i \quad \left( F_i = \frac{1}{\sqrt{5}} (\phi^i - \hat{\phi}^i) = \frac{\phi^i}{\sqrt{5}} - \frac{\hat{\phi}^i}{\sqrt{5}} \right)$$

Since  $|\hat{\phi}| < 1$ , we have  $\left| \frac{\hat{\phi}^i}{\sqrt{5}} \right| < \left| \frac{\phi^i}{\sqrt{5}} \right| < \frac{1}{\sqrt{5}} < \frac{1}{2}$

So,  $F_i$  must be an integer <sup>nearest</sup> to  $\frac{\phi^i}{\sqrt{5}}$  for Fibonacci numbers are integers.