Assigment 2

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Q1:

4.3-6

> Show that the solution to $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$ is $O(n \lg n)$.

Choose n_1 such that $n \geq n_1$ implies $n/2 + 17 \leq 3n/4$. We'll find c, d such that $T(n) \leq cn \lg n - d$.

$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + 17) + n \\ &\leq 2(c(n/2+17)\lg(n/2+17) - d) + n \\ &\leq cn\lg(n/2+17) + 34c\lg(n/2+17) - 2d + n \\ &\leq cn\lg(3n/4) + 34c\lg(3n/4) - 2d + n \\ &= cn\lg n - d + cn\lg(3/4) + 34c\lg(3n/4) - d + n \end{aligned}$$

Take $c = -2/\lg(3/4)$ and $d = 34$. Then we have $T(n) \leq cn\lg n - d + k\lg(n) - n$, where$k \in Integer$. Since $\lg(n) = o(n)$, there exists n_2 such that $n \geq n_2$ implies, $n \geq k\lg(n)$. Letting $n_0 = \max\{n_1, n_2\}$, we have that $n \geq n_0$ implies $T(n) \leq cn\lg n - d$. Therefore, get proved.

4.3-9

> Solve the recurrence $T(n) = 3T(\sqrt{n}) + \log n$ by making a change of variables. Your solution should be asymptotically tight. Do not worry about whether values are integral.

First, we could use 2^k to replace n:

$$\begin{aligned} T(n) &= 3T(\sqrt{k/2}) + \lg n \\ T(2^k) &= 3T(2^{k/2}) + k \end{aligned}$$

Now, use $S(k)$ as $T(2^k)$,$Integer\ k > 0$:

$$S(k) = 3S(k/2) + \lg k$$

If we guess that, $S(k) \leq c \cdot k^{\lg 3} + dk$:

$$
\begin{aligned}
S(k) &= 3S(k/2) + \lg k \\
&\le 3[c(k/2)^{\lg 3} + d(k/2)] + k \\
&= \frac{3c}{2^{\lg 3}} k^{lg3} + (\frac{3}{2}d + 1)k \\
&= C \cdot k^{lg3} + (\frac{3}{2}d + 1)k
\end{aligned}
$$

$$
\begin{aligned}
\text{If } d \le -2: \\
S(k) &\le C \cdot k^{\lg 3} + dk
\end{aligned}
$$

If we guess that, $S(k) \ge c \cdot k^{\lg 3} + dk$:

$$
\begin{aligned}
S(k) &= 3S(k/2) + \lg k \\
&\ge 3[c(k/2)^{\lg 3} + d(k/2)] + k \\
&= \frac{3c}{2^{\lg 3}} k^{lg3} + (\frac{3}{2}d + 1)k \\
&= C \cdot k^{lg3} + (\frac{3}{2}d + 1)k
\end{aligned}
$$

$$
\begin{aligned}
\text{If } d \ge -2: \\
S(k) &\ge C \cdot k^{\lg 3} + dk
\end{aligned}
$$

Thus, $S(k) = \Theta(k^{\lg 3})$, according to our transformation: $T(n) \to T(2^k) = S(k)$. Therefore we have $T(n) = \Theta((\lg k = n)^{\lg 3})$

Q3

4.4-2

> Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = T(n/2) + n^2$. Use the substitution method to verify your answer.

The tree has $\lg n + 1$ levels, and the subproblem size of a node at depth i is $n/2^i$. The total cost of the tree at depth is $(\frac{n}{2^i})^2 \cdot 1^i$:

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\lg n} (\frac{n}{2^i})^2 \cdot 1^{\lg i} + \Theta(1) \\
&< \sum_{i=0}^{\infty} (1/4)^i n^2 + \Theta(1) \\
&= \frac{1-0}{1-1/4} n^2 + \Theta(1) \\
&= \Theta(n^2)
\end{aligned}
$$

In the following, use substuition method to verify that $T(n) \le cn^2$

$$\begin{aligned} T(n) &\le c(n/2)^2 + n^2 \\ &= c\frac{n^2}{4} + n^2 \\ &\text{if } (\frac{c}{4}+1) \ge c \Longrightarrow c \ge \frac{3}{4}: \\ &\le cn^2 \end{aligned}$$

4.4-6

> Argue that the solution to the recurrence $T(n) = T(n/3) + T(2n/3) + cn$, where c is a constant, is $\Omega(n \lg n)$ by appealing to the recursion tree.

Examining the tree, we observe that the cost at each level of the tree is exactly cn. To find a lower bound on the cost of the algorithm, we need a lower bound on the height of the tree. The shortest simple path from root to leaf is found by following the left child at each node. Since we divide by 3 at each step, we see that this path has length $\log_3 n$, so the cost of the algorithm is $cn(\log_3 n + 1) \ge cn \log_3 n = \frac{c}{\lg 3} n \lg n = \Omega(n \lg n)$.

Problem 4-3

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for sufficiently small n. Make your bounds as tight as possible, and justify your answers.

b: $T(n) = 3T(n/3) + n/\lg n$

According to M.T. $\text{a=3, b=3, } \log_b a = 1, \text{ and } f(n) = n/\lg n$
At the first look, $f(n) = n/\lg n = 2^{\lg n}/\lg n \Longrightarrow 0, \ when\ n \to \infty$. So,
$f(n) = o(n^{log_a b}) = O(n^{1-\epsilon}), \ \exists \epsilon : 1 >> \epsilon > 0$. So we guess case 1 apply to this equation.
In the following, I would try to prove this guess by deduction:
First, show that $T(n) \le n \lg n$:

$$\begin{aligned} T(n) &= 3T(n/3) + n/\lg n \\ &\le cn \lg n - cn \lg(3) + n/\lg n \\ &= cn \lg n + (\frac{1}{\lg n} - c \lg 3)n \\ &\le cn \lg n \end{aligned}$$

Now, we want to prove $T(n) \ge cn^{1-\epsilon}$ for every $\epsilon > 0$:

$$
\begin{aligned}
T(n) &= 3T(n/3) + n/\lg n \\
&\geq 3c/3^{1-\epsilon} n^{1-\epsilon} + n/\lg n \\
&= 3^{\epsilon} c n^{1-\epsilon} + n/\lg n
\end{aligned}
$$

Also, we want to show $T(n) \leq cn^{1-\epsilon}$ could be like that:

$$
\begin{aligned}
3^{\epsilon} c n^{1-\epsilon} + n/\lg n &\geq cn^{1-\epsilon} \\
3^{\epsilon} + n^{\epsilon}/(c \lg n) &\geq 1
\end{aligned}
$$

Since $3^{\epsilon} > 1$ and $\lg n = o(n^{\epsilon})$. the above equation holds.
The function is soft Theta of n.

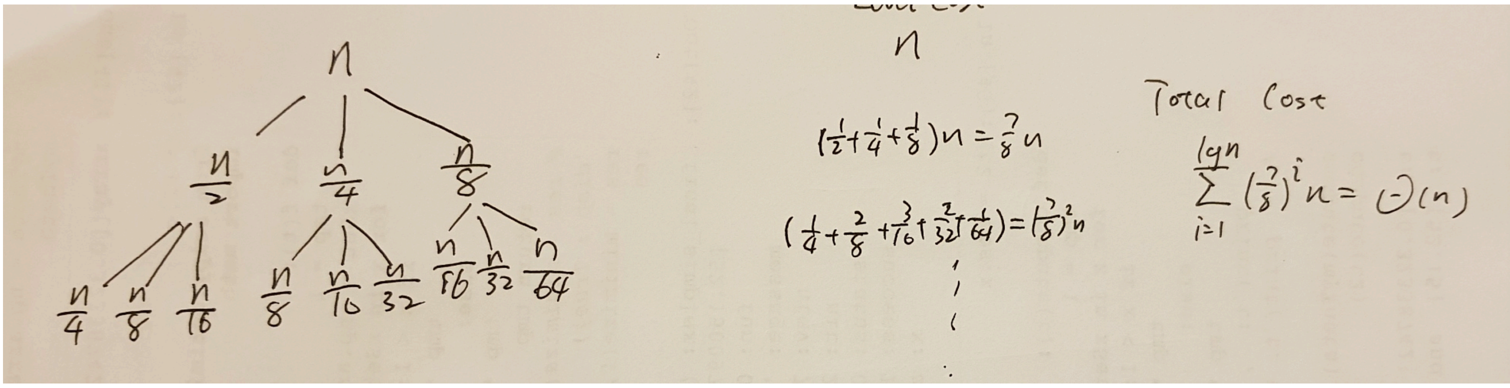
c: $T(n) = 4T(n/2) + n^2\sqrt{n}$

Look at the equation: $T(n) = 4T(n/2) + n^{5/2}$, according to M.T.
$\text{a=4, b=2, } \log_b a = 2, \text{ and } f(n) = n^{5/2} = \Theta(n^{2+\epsilon}) \text{ for } \epsilon = 1/2$
So we look at case 3 regulation part 2:

$$
af(n/b) = 4(n/2)^2\sqrt{n/2} = n^{5/2}/\sqrt{2} \leq cn^{5/2},\ \forall c \in [\frac{1}{\sqrt{2}}, 1)
$$

Since part 1 and part 2 are satisfied, case 3 is applied.

f: $T(n) = T(n/2) + T(n/4) + T(n/8) + n$



So, we want to prove the guess of recursion tree model.
First, to show that $T(n) = O(n)$ by substitution $T(n) \leq cn$:

$$
\begin{aligned}
T(n) &= T(n/2) + T(n/4) + T(n/8) + n \\
&\leq c\frac{n}{2} + c\frac{n}{4} + c\frac{n}{8} + n \\
&= (\frac{7c}{8} + 1)n \\
&\leq cn,\ when\ c \geq 8
\end{aligned}
$$

Also, since $T(n) = T(n/2) + T(n/4) + T(n/8) + n \geq n$, thus $T(n) = \Omega(n)$.
So, we have $T(n) = \Theta(n)$

Q4

> ***4-4 Fibonacci numbers***
> This problem develops properties of the Fibonacci numbers, which are defined by recurrence (3.22). We shall use the technique of generating functions to solve the Fibonacci recurrence. Define the ***generating function*** (or ***formal power series***) $\mathcal{F}$ as
>
> $$\begin{aligned} \mathcal{F}(z) &= \sum_{i=0}^{\infty} F_i z^i \\ &= 0 + z + z^2 + 2z^3 + 3z^4 + 5z^5 + 8z^6 + 13z^7 + 21z^8 + \cdots , \end{aligned}$$
>
> where F_i is the ith Fibonacci number.
>
> ***a.*** Show that $\mathcal{F}(z) = z + z\mathcal{F}(z) + z^2\mathcal{F}(z)$.

b. Show that

$$\mathcal{F}(z) = \frac{z}{1-z-z^2} = \frac{z}{(1-\phi z)(1-\hat{\phi} z)} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right),$$

where

$$\phi = \frac{1+\sqrt{5}}{2} = 1.61803\ldots$$

and

$$\hat{\phi} = \frac{1-\sqrt{5}}{2} = -0.61803\ldots .$$

c. Show that

$$\mathcal{F}(z) = \sum_{i=0}^{\infty} \frac{1}{\sqrt{5}}(\phi^i - \hat{\phi}^i) z^i .$$

d. Use part (c) to prove that $F_i = \phi^i/\sqrt{5}$ for $i > 0$, rounded to the nearest integer. (*Hint:* Observe that $|\hat{\phi}| < 1$.)

a. $F(z) = \sum_{i=0}^{\infty} F_i \cdot z^i$ $\quad (F_0 = 0, F_1 = 1, F_2 = 1, \ldots)$

$= F_0 + F_1 z + \sum_{i=2}^{\infty} (F_{i-1} + F_{i-2}) z^i$

$= 0 + z + z\sum_{i=2}^{\infty} F_{i-1} \cdot z^{i-1} + z^2 \sum_{i=2}^{\infty} F_{i-2} \cdot z^{i-2}$

$= z + z\sum_{i=1}^{\infty} F_i \cdot z^i + z^2 \sum_{i=0}^{\infty} F_i \cdot z^i$

$= z + z F(z) + z^2 F(z)$

b. from above question:

$$\mathcal{F}(z) - z\mathcal{F}(z) - z^2\mathcal{F}(z) = z$$

$$\mathcal{F}(z) = \frac{z}{1-z-z^2}$$

$\because \phi - \hat{\phi} = \sqrt{5}$, $\phi + \hat{\phi} = 1$ and $\phi\hat{\phi} = -1$

$$\mathcal{F}(z) = \frac{z}{1-(\phi+\hat{\phi})z + \phi\hat{\phi}z^2}$$

$$= \frac{z}{(1-\phi z)(1-\hat{\phi} z)}$$

$$= \frac{(\phi-\hat{\phi})z + 1 - 1}{\sqrt{5}(1-\phi z)(1-\hat{\phi} z)}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{(1-\hat{\phi} z) - (1-\phi z)}{(1-\phi z)(1-\hat{\phi} z)}\right) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right)$$

c. from part b: and $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ $(|x|<1)$

$$\mathcal{F}(z) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\sum_{i=0}^{\infty}(\phi z)^i - \sum_{i=0}^{\infty}(\hat{\phi} z)^i\right)$$

$$= \sum_{i=0}^{\infty}\frac{1}{\sqrt{5}}(\phi^i - \hat{\phi}^i)z^i$$

d. $\mathcal{F}(z) = \sum_{i=0}^{\infty} F_i z^i$ $\quad\left(F_i = \frac{1}{\sqrt{5}}(\phi^i - \hat{\phi}^i) = \frac{\phi^i}{\sqrt{5}} - \frac{\hat{\phi}^i}{\sqrt{5}}\right)$

since $|\hat{\phi}| < 1$, we have $\left|\frac{\hat{\phi}^i}{\sqrt{5}}\right| < \left|\frac{\hat{\phi}}{\sqrt{5}}\right| < \frac{1}{\sqrt{5}} < \frac{1}{2}$

so, F_i must be an nearest integer to $\frac{\phi^i}{\sqrt{5}}$ for Fibonaci numbers are integers.