EE 600 PROBLEM SET # 1

Professor Edward J. Delp **April 21, 1998**

1.1 $\mathbf{S}$ is the set of all real numbers x. With

$$A = \{2 \leq x \leq 5\}$$
$$B = \{3 \leq x \leq 6\}$$

find $A \cup B$, $A \cap B$, A^c, and $A \setminus B$.

1.2 Show that the inclusion relation in transitive; that is, show that: if $A \subset B$ and $B \subset C$, then it follows that $A \subset C$.

1.3 Let A and B be arbitrary sets. Show that $A \subset B$ if and only if $A \cup B = B$

1.4 Let A and B be arbitrary sets. Show that $B \setminus A = B \cap A^c$.

1.5 Show that if $\mathbf{S}$ contains n elements, then there are 2^n possible subsets of $\mathbf{S}$.

1.6 Prove DeMorgan's Law:

$$\left(\bigcap_j A_j\right)^c = \bigcup_j A_j^c$$

1.7 Show that the set D of all elements that belong either to A or to B but not to both is given by

$$D = (A \setminus B) \cup (B \setminus A)$$

1.8 Given two sets F and G, their *symmetric difference*, denoted by $F \triangle G$, is defined as:

$$F \triangle G = \{s : s \in F \text{ or } s \in G \text{ but not both}\}$$

Show that if $F \subset G$, then $F \triangle G = G \setminus F$.

1.9 Show that if the space $\mathbf{S}$ is finite, then any algebra in $\mathbf{S}$ is a σ-algebra.

1.10 The space $\mathbf{S}$ has six elements, i.e.,

$$\mathbf{S} = \{s_1, s_2, s_3, s_4, s_5, s_6\}.$$

Find the smallest σ-algebra containing the sets

$$A = \{s_1\}$$
$$B = \{s_1, s_3, s_5\}$$
$$C = \{s_2, s_4, s_6\}.$$

1.11 Let $\beta = \{A_1, A_2\}$ where $A_1 \subset A_2 \subset \mathbf{S}$. List all the sets in the σ-algebra generated by β.

1.12 Show that if A and B belong to a σ-algebra in $\mathbf{S}$, then $A \setminus B$ also belongs to the σ-algebra.

1.13 Let $\mathbf{S} = [0, 1]$ and $\beta = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right]\right\}$. Find $\mathcal{F}(\beta)$, the σ-algebra generated by β.

1.14 Prove that the set of rationals

$$Q = \left\{\frac{m}{n} : m = 0, \pm 1, \pm 2, \ldots\ ;\ n = \pm 1, \pm 2, \ldots\right\}$$

is a countably infinite set.

EE 600 PROBLEM SET # 2

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2.1 **a)** Show that if $P(A) = P(B) = 1$, then $P(A \cap B) = 1$.

b) If $A = B$ with probability 1, then $P(A) = P(B) = P(A \cap B)$.

2.2 Given any event A in a probability space with probability $P(A)$, show that

$$P(A^c) = 1 - P(A).$$

2.3 Show that if $\emptyset$ is the null event, then

$$P(\emptyset) = 0.$$

2.4 **a)** Let A and B be arbitrary events in a probability space. Show that

$$P(B \setminus A) = P(B) - P(A \cap B).$$

b) Show that if $A \subset B$, then

$$P(B \setminus A) = P(B) - P(A).$$

2.5 Let A and B be two arbitrary events in a probability space.

a) Show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

b) Show that, therefore $P(A \cup B) \leq P(A) + P(B)$.

c) Let $A_1, \ldots, A_n$ be arbitrary events in a probability space. Show that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

(Generalization of b).

2.6 Let A and B be events for which only $P(A)$, $P(B)$, and $P(A \cup B)$ are known. In terms of these probabilities, find

a) $P\left(A \cup (A^c \cap B)\right)$,

b) $P\left(A^c \cup B^c\right)$.

2.7 Show that if $A \cap B = \emptyset$, then $P(A) \leq P(B^c)$.

2.8 Show that for any events A, B, and C,

$$\begin{aligned} P(A \cup B \cup C) =& P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C). \end{aligned}$$

2.9 If two fair dice are rolled once, what is the probability that the total number of dots showing is

a) equal to 5?

b) divisible by 3?

2.10 If the events A and B are statistically independent, show that the events A and B^c are independent.

2.11 Is it possible that two events are independent and mutually exclusive?

EE 600 PROBLEM SET # 3

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3.1 Suppose that events $A_1, A_2, \ldots, A_m$ partition a sample space $\mathbf{S}$, and that $P(A_i) > 0$, $\forall i = 1, \ldots, m$. Show that, if B is an arbitrary event in $\mathbf{S}$, then

$$P(B) = \sum_{i=1}^{m} P(B \mid A_i) P(A_i)$$

3.2 Let A, B and C be events in a probability space $\mathbf{S}$. Show that, if A, B and C are pairwise independent, and if $P(A \cap B \cap C) = P(A)P(B)P(C)$, then

$$\begin{aligned} P(A \mid B \cap C) &= P(A) \\ P(B \mid A \cap C) &= P(B) \\ P(C \mid A \cap B) &= P(C) \end{aligned}$$

3.3 A box has 10 balls, 6 of which are black and 4 of which are white. Three balls are removed from the box, their color unnoted. Find the probability that a fourth ball removed from the box is white. Assume that the 10 balls are equally likely to be drawn from the box.

3.4 With the same box composition as in the above problem, find the probability that all three balls will be black if it is known that at least one of the removed balls is black.

3.5 You are one of r people where $r > 1$. A suicidal mission is to be assigned to the person who draws the short straw from a set of r straws. What place in line should you take to maximize your chance of survival? (other than $r + 1$!)

3.6 Consider a game which consists of two successive trials. The first trial has outcomes A or B and the second has outcomes C and D. The probabilities for the four possible outcomes of the game are as follows:

Outcome:	AC	AD	BC	BD
Probability:	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Are A and C statistically independent? Prove your answer.

3.7 A train and a bus arrive at the station at random between 9 a.m. and 10 a.m. The train stops for 10 minutes and the bus for x minutes. Find x so that the probability that the bus and the train will meet is 0.5.

3.8 Let $\mathbf{S} = \{x \text{ integer} : x \in [1, 200]\}$. Define events A, B and C as

$$\begin{aligned} A &= \{x \in \mathbf{S} : x \text{ is divisible by } 7\} \\ B &= \{x \in \mathbf{S} : x = 3n + 10 \text{ for some positive } n\} \\ C &= \left\{x \in \mathbf{S} : x^2 + 1 \leq 375\right\} \end{aligned}$$

Compute $P(A)$, $P(B)$ and $P(C)$ where $P(\cdot)$ is the equally likely probability measure on the events of $\mathbf{S}$.

3.9 Show that to test n events for independence one must test $2^n - n - 1$ relations.

3.10 A dodecahedron (a geometrical solid with 12 equal faces) has its faces marked with the integers from 1 to 12. When it is rolled, the outcome is the number on its uppermost face. Find the probability of each of the following events:

- **a)** obtaining the number 7.
- **b)** obtaining a number higher than 4.
- **c)** obtaining an odd number other than 5.
- **d)** obtaining a number more than 6 and less than 10.

3.11 The game of craps is played with two dice as follows: In a particular game, one person throws the dice. He wins on the first throw if he gets 7 or 11 points; he loses on the first throw if he gets 2, 3 or 12 points. If he gets 4, 5, 6, 8, 9 or 10 points on the first throw, he continues to throw the dice repeatedly until he produces either a 7 or the number first thrown; in the former case he loses, in the latter he wins. What is the probability of winning?

3.12 In a certain country, one-half of the people always tell the truth, one-tenth of the people always lie, and the remaining people tell the truth one-half of the time.

- **a)** A stranger stops an individual at random and asks him a question about the time of day. What is the probability that the answer is true?
- **b)** The stranger asks the question, "Do you always tell the truth?" If the answer is "No," what is the probability that the individual asked is one who tells the truth only one-half the time?

EE 600 PROBLEM SET # 4

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4.1 Consider the indicator function I_A, of the event $A \in \mathbb{F}$, where $\mathbb{F}$ is a σ-field. I_A is defined by

$$I_A = \begin{cases} 1, & s \in A \\ 0, & s \in A^c \end{cases}$$

where s is an elementary outcome in $\mathbf{S}$. Hence, if the outcome of an experiment is in A, then $I_A = 1$; if the outcome is not in A, then $I_A = 0$. Is the indicator function a valid random variable?

4.2 Let X be an arbitrary random variable defined on a probability space. Show that the following properties of the probability distribution function are true:

a) $\lim_{x \to +\infty} F_X(x) = 1$; $\lim_{x \to -\infty} F_X(x) = 0$

b) If $b \geq a$, then $F_X(b) \geq F_X(a)$.

c) $P\big(X \in (a, b]\big) = P\big(a < X \leq b\big) = F_X(b) - F_X(a)$.

Also think about $P\big(X \in (a, b)\big)$, $P\big(X \in [a, b]\big)$, and $P\big(X \in [a, b)\big)$.

4.3 Let X be a random variable with absolutely continuous probability distribution function. Show that $f_X(x) \geq 0$, for all $x \in (-\infty, +\infty)$, where $f_X(\cdot)$ is the probability density function for X. Show that $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$.

4.4 Which of the following are valid probability distribution functions:

a)

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

b)

$$F(x) = \begin{cases} 0, & x \leq c \\ 1, & x > c \end{cases}$$

c)

$$F(x) = \begin{cases} 0, & x < 2 \\ x - 1, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

d)

$$F(x) = 1 - e^{-x}$$

e)

$$F(x) = \big(1 - e^{-x}\big)U(x)$$

In b), let c be any constant, and in e), $U(x)$ is the right continuous unit step function.

4.5 Given a random variable X with probability distribution function $F_X(x)$ and an event A such that $P(A) \neq 0$, show that

$$F_X(x \mid A) = \frac{P(A \mid X \leq x)F_X(x)}{P(A)}.$$

4.6 Let $A_1, A_2, \ldots, A_n$ be a partition of the space $\mathbf{S}$. Show that

$$F_X(x) = \sum_{i=1}^{n} F_X(x \mid A_i)P(A_i).$$

4.7 The *median* of the random variable X is defined as the real number, x_m, such that $P(X \leq x_m) = 0.5$. Find the median of the random variable which has the probability density function $f_X(x) = be^{-x}$, $x \in [0, \infty)$. (Hint: The value of b is fixed by the properties of $f_X(\cdot)$. Find b before finding x_m.)

4.8 The random variable X is $N(10; 1)$, where $N(m_X; \sigma_X)$ denotes a normal (Gaussian) random variable with mean m_X and standard deviation σ_X. Find $f_X(x \mid (x-10)^2 < 4)$.

4.9 Find $f_X(x)$ if $F_X(x) = \big(1 - e^{-\alpha x}\big)U(x - c)$. (Hint: Consider two cases: $c = 0$ and $c > 0$. Why do we not have to consider $c < 0$?)

4.10 Show that, if for the random variable X, $a \leq X(\cdot) \leq b$ for all events A, then $F(x) = 1$ for $x > b$ and $F(x) = 0$ for $x < a$.

4.11 Let X and Y be two random variables defined on the same probability space. Show that if $X(\cdot) \leq Y(\cdot)$ for all events A, then $F_X(\omega) \geq F_Y(\omega)$ for every $\omega \in (-\infty, \infty)$.

EE 600 PROBLEM SET # 5

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5.1 Let X be a random variable with probability distribution function $F_X(x)$ and probability density function $f_X(x)$. Find the conditional probability distribution and probability density functions of X conditioned on the event M where:

a) $M = \{X > \alpha\}$

b) $M = \{\alpha_1 < X \leq \alpha_2\},\ \alpha_1 < \alpha_2$.

Sketch in terms of $F_X(x)$ and $f_X(x)$.

5.2 Let X be a random variable with probability distribution and density functions $F_X(x)$ and $f_X(x)$, respectively. The *median value* of X is that value of x such that $F_X(x) = 0.5$. Let

$$f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

Find the *conditional median* of X conditioned on the event $\{X > 1\}$.

5.3 If a random variable X is of discrete type, taking values at the points $0, 1, \ldots, n, \ldots,$ with

$$P\{X = k\} = e^{-a}\frac{a^k}{k!} \qquad k = 0, 1, \ldots; \qquad a > 0$$

then we say it has a *Poisson distribution* with parameter a. Show that

$$\sum_{k=0}^{\infty} P\{X = k\} = 1.$$

(Hint: expand e^a in a Taylor series.)

5.4 Let the random variable X and the event A be defined on the same probability space. Show that

$$P(A) = \int_{-\infty}^{+\infty} P(A \mid X = x) f_X(x)\, dx$$

where $f_X(x)$ is the probability density function of X.

5.5 Let X and A be defined as above. Show that

$$P(A) = P(A \mid X \leq x)F_X(x) + P(A \mid X > x)\big[1 - F_X(x)\big].$$

5.6 Let X be a continuous random variable and $Y = g(X)$ where $g(x)$ is defined by

$$g(x) = ns \qquad ns < x \le (n+1)s$$

with s a positive constant. Find $F_Y(y)$ and sketch it in terms of $F_X(x)$.

5.7 Let X be as above and $Y = g(X)$ where

$$g(x) = \begin{cases} x + c, & x < -c \\ 0, & -c \le x \le c \\ x - c, & x > c \end{cases}$$

Find $F_Y(y)$ and sketch in terms of $F_X(x)$.

5.8 Let X be a continuous random variable with probability distribution function $F_X(x)$. Let $Y = g(X)$ where $g(x) = F_X(x)$. Show that Y is uniformly distributed on the interval $[0, 1]$.

5.9 Let X be uniform on the interval $[-1, 1]$. Find $g(x)$ such that if $Y = g(X)$, then

$$f_Y(y) = 2e^{-2y}U(y).$$

5.10 Let X and Y be random variables with probability distribution functions of $F_X(x)$ and $F_Y(y)$, respectively. Assume that $Y = g(X)$ where $g(\cdot)$ is a measurable monotonic increasing function. Find the bivariate distribution function of X and Y. Rework the problem assuming $g(\cdot)$ is monotonic decreasing.

5.11 Let X and Y be independent random variables, each uniformly distributed on the interval $[0, 1]$. Find

a) $P\big(|X - Y| \le 0.5\big)$

b) $P\big(\left|\frac{X}{Y} - 1\right| \le 0.5\big)$

c) $P\big(Y \ge X \mid Y \ge 0.5\big)$

5.12 Let the bivariate probability distribution function of X and Y be $F_{XY}(x, y)$. Show that

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$

5.13 Which of the following are valid bivariate probability distribution functions?

a)

$$F_{XY}(x, y) = \begin{cases} 1 - e^{-(x+y)}, & x \ge 0, y \ge 0 \\ 0, & \text{elsewhere.} \end{cases}$$

b)

$$F_{XY}(x,y) = \begin{cases} 0, & x < y \\ 1, & x \geq y \end{cases}$$

EE 600 PROBLEM SET # 6

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6.1 Determine the constant b such that each of the following are valid probability density functions:

a)

$$f_{XY}(x,y) = \begin{cases} 3xy, & 0 \leq x \leq 1;\ 0 \leq y \leq b \\ 0, & \text{elsewhere.} \end{cases}$$

b)

$$f_{XY}(x,y) = \begin{cases} bx(1-y), & 0 \leq x \leq 0.5;\ 0 \leq y \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

c)

$$f_{XY}(x,y) = \begin{cases} b(x^2+4y^2), & 0 \leq |x| < 1;\ 0 \leq y < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

6.2 Let the random variables X and Y be jointly distributed with probability density function:

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)}, & x \geq 0;\ y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find

a) $P(X \leq x)$

b) $P(Y \leq y)$

c) $P(X < Y)$

d) $P(X + Y \leq 3)$

6.3 Show that $F_{XY}(x,y)$ is a nondecreasing function in x or y or both.

6.4 If $f_{XY}(x,y) = g(x)h(y)$, find the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

6.5 Suppose

$$f_{XY}(x,y) = \begin{cases} xe^{-x^2/2}e^{-y}, & x \geq 0;\ y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Are X and Y independent?

6.6 For two independent, continuous random variables X and Y, show that

$$P(Y \leq X) = \int_{-\infty}^{+\infty} F_Y(x) f_X(x)\, dx = 1 - \int_{-\infty}^{+\infty} F_X(x) f_Y(x)\, dx$$

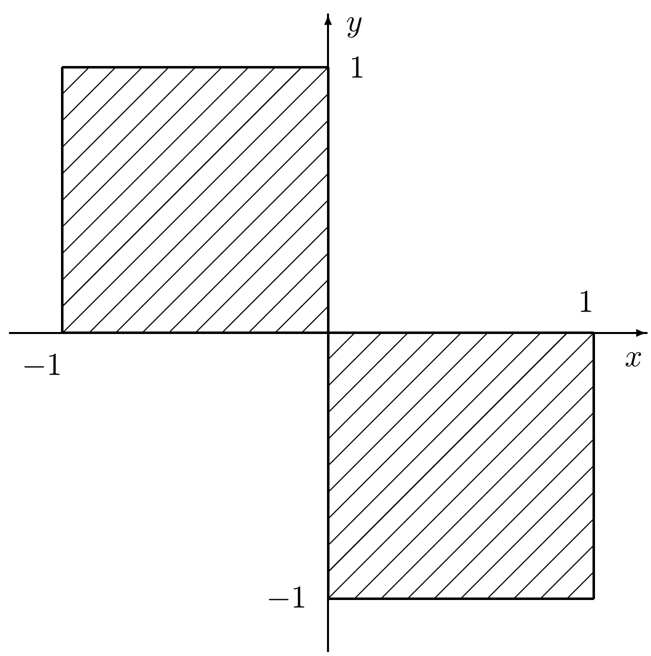
6.7 Suppose that X and Y have a joint distribution given by

$$F_{XY}(x, y) = \left[1 - e^{-ax} - e^{-ay} + e^{-a(x+y)}\right] U(x) U(y).$$

a) Find $f_X(x \mid Y = y)$ and $f_Y(y \mid X = x)$.

b) Are X and Y independent?

6.8 Suppose $f_{XY}(x, y)$ is uniform and supported on the following region in $\mathbb{R}^2$



Find $f_X(x)$ and $f_Y(y)$. Are X and Y independent?

6.9 Suppose X and Y are independent with

$$f_X(x) = \frac{1}{a}\left[U(x) - U(x-a)\right]$$
$$f_Y(y) = be^{-by} U(y).$$

Find and sketch the density of the random variable $Z = X + Y$.

6.10 Let X and Y be independent random variables with Y uniformly distributed from 0 to 1. Let X have probability distribution and density functions $F_X(x)$ and $f_X(x)$ respectively. Form a new random variable Z by using the equation $Z = X + Y$. Show that:

$$f_Z(z) = F_X(z) - F_X(z-1)$$

EE 600 PROBLEM SET # 7

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7.1 Let X and Y be continuous random variables and

$$Z = X\cos\phi + Y\sin\phi$$
$$W = -X\sin\phi + Y\cos\phi$$

where ϕ is some angle. Find f_{ZW} in terms of f_{XY}.

7.2 Let R and Θ be independent random variables such that R has the Rayleigh density

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} U(r)$$

and Θ is uniformly distributed on $[-\pi, \pi]$. Show that $X = R\cos\Theta$ and $Y = R\sin\Theta$ are independent random variables and that each has a normal density, $N(0, \sigma^2)$.

7.3 Let X and Y be independent and uniform on the interval $[0, a]$. Find the probability density function of $Z = |X - Y|$.

7.4 Let X be a continuous random variable with mean μ_X and variance σ_X^2. If $Y = aX + b$, find μ_Y and σ_Y^2.

7.5 Find $E\{I_A\}$ where I_A is the indicator function.

7.6 Let X be a Poisson random variable with parameter a. Find the mean and variance of X.

7.7 Let X be a continuous random variable such that $E\{|X|^r\} < \infty$. Show that

$$P\{|X| \geq \lambda\} \leq \frac{E\{|X|^r\}}{\lambda^r}, \qquad \lambda > 0.$$

7.8 Show that

$$E\{x\} = -\int_{-\infty}^{0} F_X(x)\,dx + \int_{0}^{\infty} \left[1 - F_X(x)\right] dx.$$

7.9 Show that if X is a Cauchy random variable with parameter α, then

$$\phi_X(\omega) = e^{-\alpha|\omega|}$$

where $\phi_X(\omega) = E\{e^{j\omega x}\}$ is the characteristic function for X.

7.10 Let X be a random variable with characteristic function $\phi_X(\omega)$. Show that $\phi_X(\omega)$ takes on its maximum value at the origin ($\omega = 0$).

7.11 Show that $E^2(XY) \leq E(X^2)E(Y^2)$ for any random variables X and Y.

7.12 Show that if $\rho = \dfrac{C_{XY}}{\sigma_X \sigma_Y}$ is the correlation coefficient, then $|\rho| \leq 1$.

7.13 Show that if X and Y are uncorrelated random variables, then the variance of the sum is the sum of the variances, i.e., $\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$.

7.14 Show that if X and Y are independent random variables, then

$$E\big\{g(X)h(Y)\big\} = E\big\{g(X)\big\}E\big\{h(Y)\big\}$$

for any functions g and h.

7.15 Two random variables are said to be equal in the *mean square sense* if

$$E\big\{(X - Y)^2\big\} = 0.$$

Show that if $\rho_{XY} = 1$ where ρ_{XY} is the correlation coefficient, then there exists real numbers a and b such that $Y = aX + b$ in the mean square sense.

EE 600 PROBLEM SET # 8

Professor Edward J. Delp **April 21, 1998**

8.1 Let the random variables X and Y be independent, identically distributed (iid), with

$$f_X(x) = e^{-x}U(x)$$

Let

$$Z = X + Y$$
$$W = \frac{X}{X+Y}$$

Show that Z and W are independent and find the marginal probability densities of Z and W.

8.2 The random variables X and Y are independent with:

$$f_X(x) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$
$$f_Y(y) = \frac{y}{\alpha^2}e^{-\frac{y^2}{2\alpha^2}}U(y)$$

Show that their product $Z = XY$ is Gaussian.

8.3 Let $X_1, X_2, \ldots, X_n$ be random variables that are independent with finite variances. Form

$$Y = \sum_{i=1}^{n} \alpha_i X_i$$

where α_i is real. Find the mean and variance of Y.

8.4 Show that the constant α which minimizes $E\big[|X - \alpha|\big]$ is the median of X. You may assume $F_X(x)$ is absolutely continuous.

8.5 Let X be a random variable with probability density function of $f(x)$ and whose moments exist. Let m be the median of X. Show that for any a:

$$E\big[|X - a|\big] = E\big[|X - m|\big] + 2\int_a^m (x-a)f(x)\,dx$$

8.6 The random variables X and Y are independent and their joint density $f_{XY}(x,y)$ has circular symmetry, i.e., $f_{XY}(x,y) = f_{XY}(r)$, where $r = \sqrt{x^2+y^2}$.

Show that X and Y are normal.

8.7 Let $\underline{X}$ be a 2-dimensional random vector with covariance matrix: $\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$.

Find a linear transformation matrix $\underline{A}$ such that $\underline{Y} = \underline{A}\,\underline{X}$ has uncorrelated components.

8.8 Let X_i, $i = 1, 2, \ldots, N$ be iid random variables. Show that the variance estimator

$$\widehat{\sigma}_x^2 = \frac{1}{N}\sum_{i=1}^{N} X_i^2 - \left(\widehat{\overline{X}}\right)^2$$

is biased, where

$$\widehat{\overline{X}} = \frac{1}{N}\sum_{j=1}^{N} X_j.$$

The following problems, with slight notation changes, may also be found in A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, Inc.

8.9 Show that if the random variable X is $N(\eta;\sigma)$, then

$$E\big\{|x|\big\} = \sigma\sqrt{\frac{2}{\pi}}e^{-\eta^2/2\sigma^2} + 2\eta\mathbb{G}\Big(\frac{\eta}{\sigma}\Big) - \eta$$

where $\mathbb{G}(\cdot)$ is the Gaussian integral

$$\mathbb{G}(x) \triangleq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy$$

8.10 Show that if $X \geq 0$ and $E\{X\} = \eta$, then $P\{X \geq \sqrt{\eta}\} \leq \sqrt{\eta}$.

8.11 Show that if R is the correlation matrix of the random vector $\underline{X} : [X_1, \ldots, X_n]$ and R^{-1} is its inverse, then $E\{\underline{X}R^{-1}\underline{X}^t\} = n$.

8.12 Show that if the random variables X_i are of continuous type and independent, then, for sufficiently large n, the density of $\sin(X_1 + \cdots + X_n)$ is nearly equal to the density of $\sin(X)$, where X is a random variable uniform in the interval $(-\pi, \pi)$.

8.13 Show that if $a_n \to a$ and $E\big\{|X_n - a_n|^2\big\} \to 0$, then $X_n \to a$ in the mean square sense as $n \to \infty$.

EE 600 PROBLEM SET # 9

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9.1 Let X and Y be jointly Gaussian random variables with means m_X and m_Y, variances σ_X^2 and σ_Y^2 and correlation coefficient $\rho = \mathrm{Cov}(X, Y)/\sigma_X\sigma_Y$. Show that

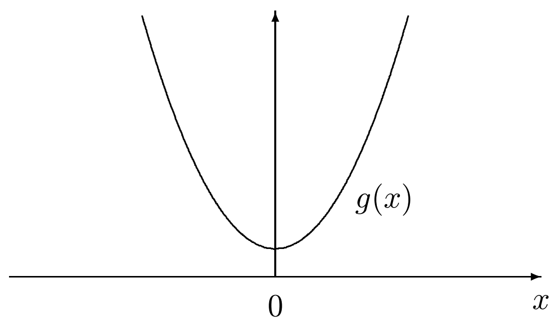
$$\begin{aligned} E\big(Y \mid X = x\big) &= \rho\frac{\sigma_Y}{\sigma_X}(x - m_X) + m_Y \\ &= \frac{\mathrm{Cov}(X, Y)}{\sigma_X^2}(x - m_X) + m_Y \end{aligned}$$

9.2 Suppose that the random variable ϕ is uniformly distributed on $[0, 2\pi]$ and let the random variables X and Y be defined by

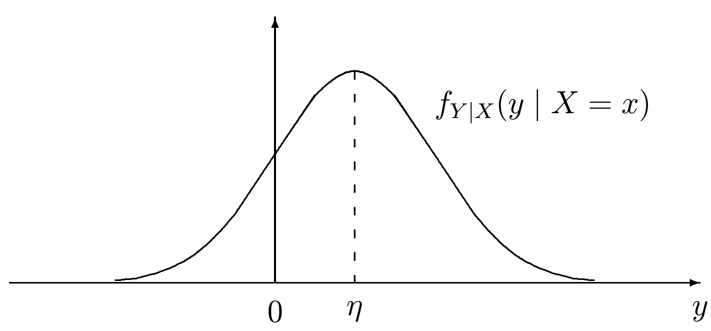
$$X = \cos\phi \qquad Y = \sin\phi.$$

Determine the least-mean-square-error linear predictor for Y, given that $X = x$ and $-1 \leq x \leq 1$.

9.3 In the approximation of the random variable Y by $\phi(X)$, one may use the "mean cost" $E\big\{g\big[Y - \phi(X)\big]\big\}$, where $g(x)$ is a given function. Show that, if $g(x)$ is an even concave function as below, then the "mean cost" is minimum if $\phi(X) = E\big\{Y \mid x\big\}$.



You may assume the conditional density of Y given the event $\{X = x\}$ is symmetric about its mean, as shown below.



The following two problems, with slight notation changes, may also be found in A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, Inc.

9.4 Show that if $\phi(x) = E\{Y \mid x\}$ is the nonlinear mean-square estimate of Y in terms of X, then

$$E\Big\{\big[Y - \phi(X)\big]^2\Big\} = E\{Y^2\} - E\{\phi^2(X)\}$$

9.5 We observe repeated trials of a random variable which is uni$(-5, 7)$. If we wish to estimate its mean value with a variance that is no greater than 50% of the true mean, how many samples must we observe?

9.6 Let X_i, $i = 1, \ldots, N$ be i.i.d. Gaussian random variables with mean μ and variance σ^2. Show that the sample mean is an efficient estimate of the mean.

9.7 Derive the following theorem:

THEOREM: Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables that converge in mean square to the random variable X. Then the sequence converges to X in probability.

Hint: The Generalized Chebyshev Inequality (discusssed below) or a variation of it may be helpful.

Let X be an arbitrary random variable and $\alpha \in \mathbb{R}$, $n \in \mathbb{Z}$ be arbitrary,then the *Generalized Chebyshev Inequality* is:

$$P\big\{|X - \alpha|^n \geq \epsilon\big\} \leq \frac{E\big\{|X - \alpha|^n\big\}}{\epsilon^n}$$

The following two problems, with slight notation changes, may also be found in A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, Inc.

9.8 The continuous parameter random process $X(t) = e^{At}$ is a family of exponentials depending on the random variable A. Express the mean $\eta(t)$, the autocorrelation function $R(t_1, t_2)$, and the first-order probability density function $f(x; t)$ of $X(t)$ in terms of the probability density function $f_A(a)$ of A.

9.9 The random variable C is uniform in the interval $(0, T)$. Find the autocorrelation function of X_t if $X_t = U(t - C)$.

9.10 A random process has sample functions of the form

$$X(t) = A \sin(\omega_0 t)$$

where A is a random variable, uni$(-1, 1)$, and ω_0 is fixed.

a) Is the process continuous-parameter or discrete-parameter?

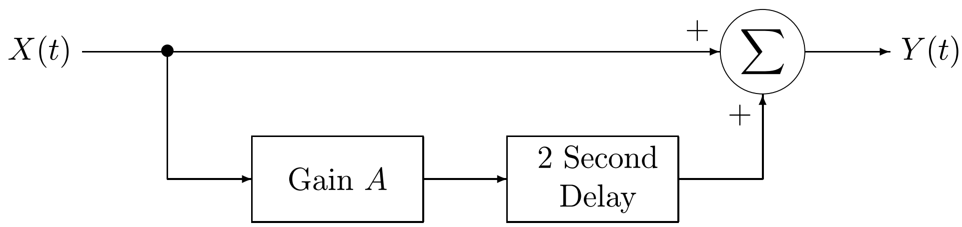
b) Find $E\{X(t)\} = \overline{X}(t)$.

c) Find the probability density function of $X(\pi/2\omega_0)$.

9.11 Consider a zero-mean strict-sense stationary random process with

$$R_X(\tau) = 50\cos(20\pi\tau) + 18cos(30\pi\tau)$$

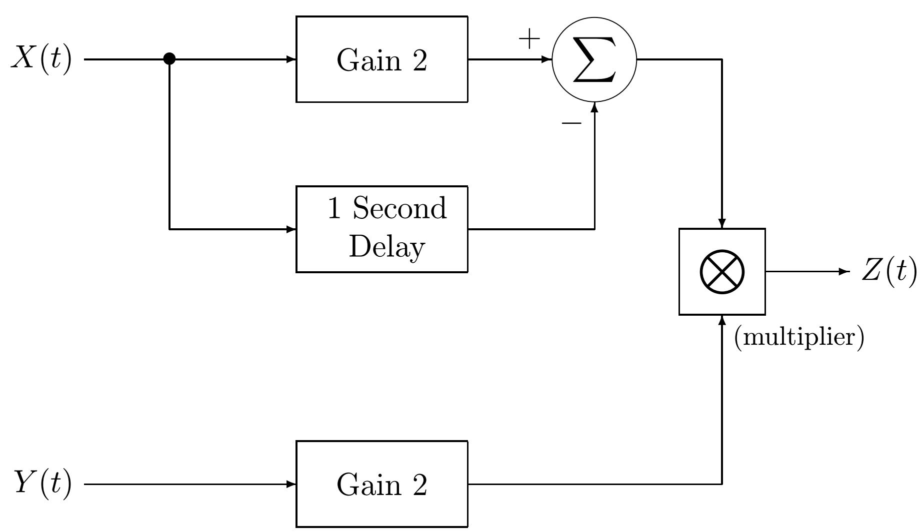
as input to the following system:



a) Find the variance of $X(t)$.

b) What value of A will minimize the mean square value of $Y(t)$?

9.12 Let $X(t)$ and $Y(t)$ be jointly wide-sense stationary random processes.



Given that $R_{XY}(\tau) = 10e^{-2\tau}U(\tau)$, find $E\{Z(t)\}$.

EE 600 PROBLEM SET # 10

Professor Edward J. Delp **April 21, 1998**

The following 6 problems, with slight notation changes, may also be found in A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, Inc.

10.1 Show that if Ψ is a random variable with $\Phi(\lambda) = E\{e^{j\lambda\Psi}\}$ and $\Phi(1) = \Phi(2) = 0$, then the process $X_t = \cos(\omega t + \Psi)$ is wide-sense stationary. Find $E\{X(t)\}$ and $R_X(\tau)$ if Ψ is uniform in the interval $(-\pi, \pi)$.

10.2 Show that if $X(t)$ is a strict-sense stationary process and Ψ is a random variable independent of $X(t)$, then the process $Y_t = X(t - \Psi)$ is strict-sense stationary.

10.3 Show that if $X(t)$ is a stationary process with derivative $X'(t)$, then for a given t the random variables $X(t)$ and $X'(t)$ are orthogonal and uncorrelated.

10.4 Let $X(t)$ and $V(t)$ be continuous parameter random processes. Find the mean and variance of the random variable

$$N_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt, \text{ where } X(t) = 10 + V(t)$$

for $T = 5$ and for $T = 100$. Assume that $E\{V(t)\} = 0$, $R_V(\tau) = 2\delta(\tau)$.

10.5 Show that $|R_{xy}(\tau)| \leq \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$.

10.6 Show that if X(t) is a wide-sense stationary random process and $S = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} X(kT)$ then

$$E\{S^2\} = \frac{1}{2\pi n^2}\int_{-\infty}^{\infty} S_X(\omega)\frac{sin^2 n\omega T/2}{sin^2\omega T/2}\,d\omega$$

10.7 An ergodic random process has $R_X(\tau) = 5e^{-|x|} + 100$. If 100 independent samples of this process are observed and used to estimate the mean value, what percentage of the true mean value will the RMS error of this estimate be?

$$\text{RMS error} \triangleq \sqrt{E\{(\hat{m} - m)^2\}}$$

where m is the mean value and $\hat{m}$ is the estimate of the mean.

10.8 A wide-sense stationary random process has an autocorrelation function of the form

$$R_X(\tau) = \begin{cases} 4 + \frac{2}{3}\left(1 - \frac{|\tau|}{3}\right), & |\tau| \leq 3 \\ 4 & |\tau| > 3. \end{cases}$$

Find the spectral density.

10.9 White noise with spectral density 1 V^2/Hz is input to a linear system with impulse response

$$h(t) = U(t) - U(t-1)$$

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

a) Determine $R_{XY}(t_1, t_2)$.

b) Determine $R_{YY}(t_1, t_2)$.

If they are wide-sense stationary, then write them in terms of $|t_2 - t_1| = \tau$.

EE 600 PROBLEM SET # 11

Professor Edward J. Delp **April 21, 1998**

11.1 Let $X(t)$ be a white noise process bandlimited to 50 Hz with normalized power of 100 V^2. It is desired to generate a discrete parameter random process, Y_n, by taking instantaneous samples of $X(t)$:

$$Y_n = X(nT_s)$$

where $T_s = 0.01$ seconds. Find:

- **a)** the mean of Y_n.
- **b)** the autocorrelation function of Y_n.

11.2 Let $X(t)$ be a wide-sense stationary random process with autocorrelation function $R_X(\tau)$ and spectral density $S_X(\omega)$. Let $\alpha, \beta \in \mathbb{R}$. Form a new process:

$$Y(t) = \alpha X(\beta t)$$

Find:

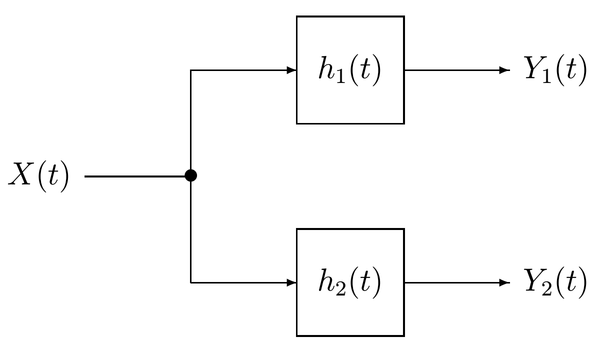
- **a)** the autocorrelation function of $Y(t)$
- **b)** the spectral density of $Y(t)$

11.3 Given a linear time invariant system with transfer function $H(\omega)$. White noise with power of 100 V^2 per 1 kHz band is input into this system. The cross-correlation function of the input X_t and output Y_t is determined to be:

$$R_{XY}(\tau) = \begin{cases} 25e^{-6\tau} & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

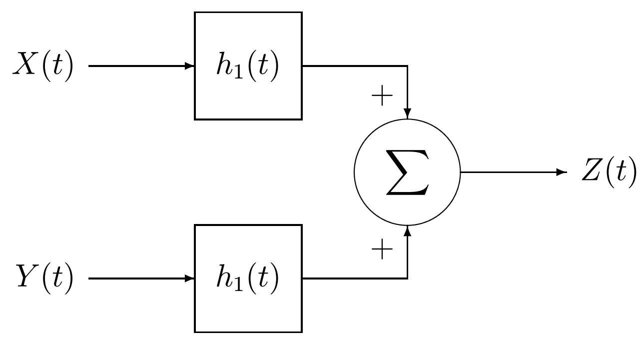
Find the output power spectral density function and output autocorrelation function.

11.4 Let $X(t)$ be a strictly stationary continuous parameter random process. Let $X(t)$ be the input to two linear time-invariant systems with $Y_1(t)$ and $Y_2(t)$ as the respective outputs as shown below:



Derive a sufficient condition on the frequency response of the two systems such that $Y_1(t)$ and $Y_2(t)$ are uncorrelated.

11.5 Given the following system:



where $X(t)$ and $Y(t)$ are zero mean wide-sense stationary random processes with $X(t)$ and $Y(t)$ statistically independent and with:

$$R_X(\tau) = e^{-9|\tau|} \qquad R_Y(\tau) = e^{-16|\tau|}$$

The impulse responses are:

$$h_1(t) = e^{-t}u(t) \qquad h_2(t) = e^{-2t}u(t)$$

Find the autocorrelation function and power spectral density of $Z(t)$.

11.6 A simple processing step is sometimes used in the analysis of trends on the stock market. This simple tool is commonly known as a *two day moving average.* It is described by a difference equation of the form:

$$Y_n = \frac{X_n + X_{n-1}}{2}$$

where X_n is the Dow Jones average for the current day, and X_{n-1} the Dow Jones average for the previous day. Y_n is the output. Let's use a *simple* model of the Dow Jones average X_n. Let X_n be zero mean independent identically distributed with unit variance. We can now think of X_n and Y_n as random processes.

a) Is X_n stationary? If not, is X_n wide-sense stationary? Justify your answer.

b) Find the autocorrelation function of Y_n. Is Y_n wide-sense stationary?

c) Rework part (b) but now use a recursive averaging system given by:

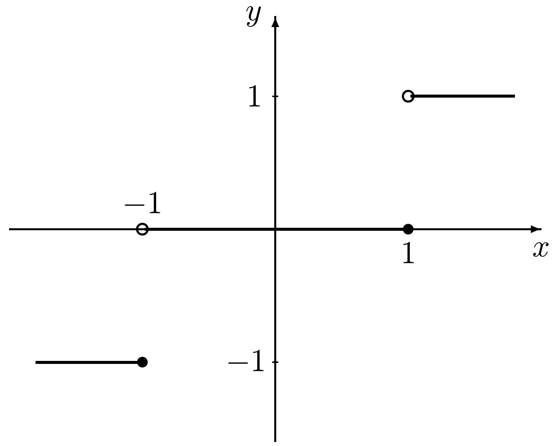
$$Y_n = \alpha Y_{n-1} + X_n$$

where $0 < \alpha < 1$.

11.7 Let the random variable X have the probability density function

$$f(x) = \frac{1}{2}e^{-|x|} \qquad x \in (-\infty, \infty)$$

It is desired to quantize X using the three level quantizer shown below.



Let Y denote the random variable at the output of the quantizer, i.e.,

$$Y = \begin{cases} -1 & X \leq -1 \\ 0 & -1 < X \leq 1 \\ 1 & X > 1 \end{cases}$$

a) Find the conditional probability density function of X conditioned on $\{Y = -1\}$, i.e., $f_X(x \mid Y = -1)$.

b) Let $Q = Y - X$ be defined as the *quantization error* (or *quantization noise*). Find the conditional probability density function of the quantization error conditioned on $\{Y = -1\}$, i.e., $f_Q(q \mid Y = -1)$.

c) Find the probability density function of Q.

11.8 Let $X(t)$ be a wide-sense stationary random process with first order probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

It is also known that $X(t)$ is bandlimited white noise with bandlimit 200 Hz. Let $X(t)$ be the input to a system described by

$$y(t) = \frac{d\,x(t)}{dt}$$

where $y(t)$ is the output.

Find the mean square value of the output.