EE 600 SOLUTION SET # 1

Professor Edward J. Delp **April 21, 1998**

1.1

$$
\begin{aligned}
A \cup B &= \{x : x \in A \text{ or } x \in B\} = \{2 \leq x \leq 6\} \\
A \cap B &= \{x : x \in A \text{ and } x \in B\} = \{3 \leq x \leq 5\} \\
A^c &= \{x : x \in \mathbf{S} \text{ but } x \notin A\} = \{-\infty < x < 2\} \cup \{5 < x < \infty\} \\
A \setminus B &= \{x : x \in A \text{ but } x \notin B\} = \{2 \leq x < 3\}
\end{aligned}
$$

1.2 Suppose $s \in A$. Then, since $A \subset B$, $s \in B$. Now, since $B \subset C$, $s \in C$. $\therefore s \in A \Rightarrow s \in C$. Hence $A \subset C$.

1.3 Do this in two parts:

$$\text{Show } A \subset B \Rightarrow A \cup B = B \quad (A \subset B \text{ only if } A \cup B = B) \tag{1}$$
$$\text{and } A \subset B \Leftarrow A \cup B = B \quad (A \subset B \text{ if } A \cup B = B). \tag{2}$$

Part 1: ($\Rightarrow$) Suppose $A \subset B$. Then $s \in A \Rightarrow s \in B$. Hence, $A \cup B = \{x : x \in A \text{ or } x \in B\} = \{x : x \in B\} = B$.

Part 2: ($\Leftarrow$) Suppose $A \cup B = B$ and $s \in A$. Then $s \in A \cup B = B$. Hence $s \in A \Rightarrow s \in B \quad (A \subset B)$.

1.4

$$
\begin{aligned}
& B \setminus A = \{x : x \in B \text{ and } x \notin A\} \\
& x \notin A \Leftrightarrow x \in A^c \\
& \therefore B \setminus A = \{x : x \in B \text{ and } x \in A^c\} = B \cap A^c
\end{aligned}
$$

1.5 Use mathematical induction.

Math induction:

1. Prove statement is true for $n = n_0$.
2. Assume statement is true for $n = k$ and use that fact to prove that the statement is true for $n = k + 1$.

Proof of 1. Let $n_0 = 1$. The set $\mathbf{S}$ consists of 1 element. Call it s_1. In this case $\mathbf{S} = \{s_1\}$ and $\mathbf{S}$, $\emptyset$ are the only subsets of $\mathbf{S}$.

$$\therefore \text{There are } 2^1 \text{ subsets.}$$

Proof of 2. Now suppose that if the set $\mathbf{S}_k$ is composed of $n = k$ elements, then there are 2^k subsets of $\mathbf{S}_k$.

Let the set $\mathbf{S}_k = \{s_1, s_2, \ldots, s_k\}$ and let the set $\mathbf{S}_{k+1}$ be the universal set composed of all the elements of the set $\mathbf{S}_k$ and a new element, s_{k+1}. Then

$$\mathbf{S}_{k+1} = \{s_1, s_2, \ldots, s_{k+1}\}.$$

Now all the subsets of $\mathbf{S}_k$ are also subsets of $\mathbf{S}_{k+1}$. In addition, we may form a new set A' from each subset A of $\mathbf{S}_k$ by adding s_{k+1} to A.

$$A' = A \cup \{s_{k+1}\}.$$

There are no subsets of $\mathbf{S}_{k+1}$ which cannot be formed in one of the two ways above. Hence, since there are 2^k subsets in $\mathbf{S}_k$, there are

$$2^k + 2^k = 2^{k+1}$$

subsets of $\mathbf{S}_{k+1}$.

1.6 Suppose $s \in \left(\bigcap_j A_j\right)^c$. Then $s \notin \bigcap_j A_j$. Therefore, there exists at least one j such that $s \notin A_j$. Call it j_0.

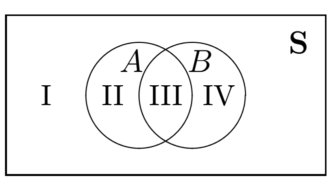
Then $s \notin A_{j_0} \Rightarrow s \in A_{j_0}^c$.
Therefore $s \in \bigcup_j A_j^c$ (since $A_{j_0}^c \subset \bigcup_j A_j^c$).

$$\therefore \left(\bigcap_j A_j\right)^c \subset \bigcup_j A_j^c$$

Now suppose that $s \in \bigcup_j A_j^c$. Then, there exists j_0 such that $s \in A_{j_0}^c$. $\therefore s \notin A_{j_0}$. $\therefore$ $s \notin \bigcap_j A_j$. Hence $s \in \left(\bigcap_j A_j\right)^c$.

$$\therefore \bigcup_j A_j^c \subset \left(\bigcap_j A_j\right)^c$$

1.7 Consider four possibilities given by the Venn diagram[1] below



[1]**N.B. Do NOT use Venn diagrams on the examinations. No credit will be given for any solution which uses a Venn diagram.**

Case I: s not in A and not in B.
Then $s \notin A \setminus B$, $s \notin B \setminus A$.
$\therefore s \notin (A \setminus B) \cup (B \setminus A)$.

Case II: s in A but not in B.
$s \in A, s \notin B \Rightarrow s \notin A \setminus B$
$\therefore s \in (A \setminus B) \cup (B \setminus A)$

Case III: s in B but not in A.
$s \in B, s \notin A \Rightarrow s \in B \setminus A$
$\therefore s \in (A \setminus B) \cup (B \setminus A)$

Case IV: $s \in A, s \in B$.
$s \in A \Rightarrow s \notin B \setminus A$. $s \in B \Rightarrow s \notin A \setminus B$
$\therefore s \notin (A \setminus B) \cup (B \setminus A)$.

1.8 Define $G \setminus F = \{s : s \in G \text{ and } s \notin F\}$
To show $F \triangle G = G \setminus F$ we must show:

(1) $F \triangle G \subseteq G \setminus F$

(2) $G \setminus F \subseteq F \triangle G$

Proof:

(1) Let $s \in F \triangle G$ then we have two cases:
Case 1: Let $s \in F$ and $s \notin G$. By assumption $F \subset G$ or equivalently $s \in F \Rightarrow s \in G$, so we get $s \in G$. This is a contradiction so this case cannot happen and we need not consider it.
Case 2: $s \notin F$ and $s \in G$ and so by definition $s \in G \setminus F$ This proves (1).

(2) Let $s \in G \setminus F$ then $s \in G$ and $s \notin F$ and so by definition $s \in F \triangle G$
This proves (2).
(1) and (2) $\Rightarrow F \triangle G = G \setminus F$. □

1.9 Let $\mathbf{S}$ have N elements with $1 \leq N < \infty$. Given any collection $\mathbf{C}$ of subsets of $\mathbf{S}$, the field containing $\mathbf{C}$, i.e., $\mathcal{F}(\mathbf{C})$, has at most 2^N elements (since the power set of $\mathbf{S}$ has 2^N elements). Any infinite sequence of elements of $\mathcal{F}(\mathbf{C})$ has therefore at most 2^N distinct entries.

So, let $A_n \in \mathcal{F}(\mathbf{C})$, $n = 1, 2, \ldots$. Then, there exists a K such that for any $n > K$, there exists m, $1 \leq m \leq K$ such that $A_n = A_m$. So, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{K} A_i$. But since $\bigcup_{i=1}^{K} A_i$ is a finite union of elements of $\mathcal{F}(\mathbf{C})$, it is in $\mathcal{F}(\mathbf{C})$ and hence

$$\bigcup_{i=1}^{\infty} A_i \left(= \bigcup_{i=1}^{K} A_i \right) \in \mathcal{F}(\mathbf{C}).$$

Thus $\mathcal{F}(\mathbf{C})$ is a σ-algebra.

1.10 The finest partition of $\mathbf{S}$ given by A, B, and C contains the 3 elements:

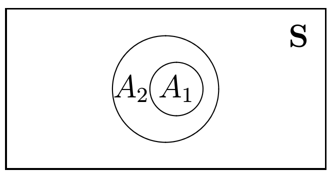
$$\{s_1\}, \{s_3, s_5\}, \{s_2, s_4, s_6\}$$

So the smallest σ-algebra is given by

$$\begin{aligned}\mathcal{F}(A,B,C) = &\{\emptyset, \mathbf{S}, \{s_1\}, \{s_3, s_5\}, \{s_2, s_4, s_6\}, \{s_1, s_3, s_5\}, \{s_1, s_2, s_4, s_6\},\\ &\{s_2, s_3, s_4, s_5, s_6\}\}.\end{aligned}$$

$\mathcal{F}(A,B,C)$ is simply the collection of all possible unions of the elements in the partition.

1.11 From the Venn diagram[2], we see that



the finest partition of $\mathbf{S}$ consists of the 3 sets A_1 , $A_2 \setminus A_1$, A_2^c. Hence, $\mathcal{F}(\beta) = \{A_1, A_2 \setminus A_1, A_2^c, A_2, A_1 \cup A_2^C, A_1^C, \emptyset, \mathbf{S}\}$.

1.12 Recall that $A \setminus B = A \cap B^c$. Let $\mathcal{F}$ be the σ-algebra on $\mathbf{S}$. $B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$
$A, B^c \in \mathcal{F} \Rightarrow A \cap B^c \in \mathcal{F}$
$\therefore A \setminus B \in \mathcal{F}$

1.13 Finest Partition of $\mathbf{S}$ that contains $\beta : \{[0, \frac{1}{2}), \{\frac{1}{2}\}, (\frac{1}{2}, 1]\}$

$$\therefore \mathcal{F}(\beta) = \left\{\left[0, \tfrac{1}{2}\right), \left\{\tfrac{1}{2}\right\}, \left(\tfrac{1}{2}, 1\right], \left[0, \tfrac{1}{2}\right], \left[0, \tfrac{1}{2}\right) \cup \left(\tfrac{1}{2}, 1\right], \left[\tfrac{1}{2}, 1\right], [0,1], \emptyset\right\}$$

1.14 Generate the following matrix of rationals, which includes them all, some even more than once.

$$\begin{array}{ccccccccc}
1 & \frac{1}{1} & \boxed{\frac{2}{1}} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \cdots \\
2 & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \boxed{\frac{4}{2}} & \frac{5}{2} & \frac{6}{2} & \cdots \\
3 & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \boxed{\frac{6}{3}} & \cdots \\
4 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\
 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \cdots \\
 & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \frac{5}{6} & \frac{6}{6} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

[2]**This is the last Venn diagram you will ever see in EE 600. See Problem 1.7.**

Note, for example, that all the circled entries are 1 and all the boxed entries are 2.

Now, order the elements of this matrix by starting in the upper left corner and taking succeeding diagonal "rows" from top to bottom. The first 10 terms in this ordering are shown here.

N	1	2	3	4	5	6	7	8	9	10	$\cdots$
	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{2}$	$\frac{3}{1}$	$\frac{2}{2}$	$\frac{1}{3}$	$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{4}$	$\cdots$
Q	$\downarrow$	$\downarrow$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\downarrow$	$\cdots$
	1	2	$\frac{1}{2}$	3	$\cancel{1}$	$\frac{1}{3}$	4	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{4}$	$\cdots$

In this way we establish a unique one-to-one correspondence between the rationals and the integers. Note that because of the redundancy in the way we arranged the rationals ("1" appears many times in the matrix, for example), not all integers have a rational assigned to them. Thus, we are not even using all the integers: ($n = 5$ is unused, since it corresponds to 1, which was already assigned to $n = 1$).

Hence, the rationals are countably infinite.

EE 600 SOLUTION SET # 2

Professor Edward J. Delp **April 21, 1998**

2.1 **a)** $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

Now $A \cup B \supseteq A$.

$\therefore P(A \cup B) \geq P(A) = 1$

$\therefore P(A \cup B) = 1$

$\therefore P(A \cap B) = 1 + 1 - 1 = 1$

b) If $A = B$ with probability 1, then the probability of the set containing members of A or B but not of both is zero. That is $P[(A \setminus B) \cup (B \setminus A)] = 0$.

Since $A \setminus B$ and $B \setminus A$ are disjoint, we have $P[(A \setminus B) \cup (B \setminus A)] = P(A \setminus B) + P(B \setminus A) = 0$.

$\therefore P(A \setminus B) = P(B \setminus A) = 0$.

Now $P(A) = P[(A \cap B) \cup P(A \setminus B)] = P(A \cap B) + P(A \setminus B) = P(A \cap B)$.

Similarly, we get $P(B) = P(A \cap B)$.

$\therefore P(A) = P(B) = P(A \cup B)$

2.2 Since $A \cap A^c = \emptyset$, $P(A) + P(A^c) = P(A \cup A^c) = P(\mathbf{S}) = 1$.

$\therefore P(A^c) = 1 - P(A)$.

2.3 $\mathbf{S} \cap \emptyset = \emptyset$

$\mathbf{S} \cup \emptyset = \mathbf{S}$

$\therefore P(\mathbf{S} \cup \emptyset) = P(\mathbf{S}) + P(\emptyset) = 1 + P(\emptyset)$.

But $\mathbf{S} \cup \emptyset = \mathbf{S}$, $\therefore P(\mathbf{S} \cup \emptyset) = P(\mathbf{S}) = 1$.

$\therefore 1 + P(\emptyset) = 1 \Rightarrow P(\emptyset) = 0$

2.4 **a)** $B = B \cap \mathbf{S} = B \cap (A \cup A^c) = (B \cap A) \cup (B \cap A^c) = (B \cap A) \cup (B \setminus A)$

Since $B \cap A$ and $B \setminus A$ are disjoint, we have $P(B) = P(B \cap A) + P(B \setminus A)$.

$\therefore P(B \setminus A) = P(B) - P(A \cap B)$

b) If $A \subset B$, then $A \cap B = A$ and $P(A \cap B) = P(A)$.

$\therefore P(B \setminus A) = P(B) - P(A)$

2.5 **a)** $A = (A \setminus B) \cup (A \cap B) \Longrightarrow P(A) = P(A \setminus B) + P(A \cap B)$

$B = (B \setminus A) \cup (B \cap A) \Longrightarrow P(B) = P(B \setminus A) + P(B \cap A)$

hence $P(A) + P(B) = P(A \setminus B) + P(B \setminus A) + 2P(A \cap B)$.

$$\therefore P(A) + P(B) - P(A \cap B) = P(A \setminus B) + P(B \setminus A) + P(A \cap B) \qquad \text{(I)}$$

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$$

$$\therefore P(A \cup B) = P(A \setminus B) + P(B \setminus A) + P(A \cap B) \qquad \text{(II)}$$

Since the right sides of (I) and (II) are equal, so are the left sides

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

b) $\therefore$ Since $P(A \cap B) \geq 0$, $P(A \cup B) \leq P(A) + P(B)$.

c)

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P\left[\left(\bigcup_{i=1}^{n-1} A_i\right) \cup A_n\right] \leq P\left(\bigcup_{i=1}^{n-1} A_i\right) + P(A_n)$$

$$\begin{aligned}\therefore P\left(\bigcup_{i=1}^{n} A_i\right) &\leq P\left[\left(\bigcup_{i=1}^{n-2} A_i\right) \bigcup A_{n-1}\right] + P(A_n) \\ &\leq P\left(\bigcup_{i=1}^{n-2} A_i\right) + P(A_{n-1}) + P(A_n)\end{aligned}$$

Continuing in this fashion, we obtain the result.

2.6 **a)** $A \cup (A^c \cap B) = (A \cup A^c) \cap (A \cup B) = \mathbf{S} \cap (A \cup B) = A \cup B$

$\therefore P\big[A \cup (A^c \cap B)\big] = P(A \cup B)$

b) Using DeMorgan's Law, $A^c \cup B^c = (A \cap B)^c$

$$\begin{aligned}\therefore P(A^C \cup B^C) &= P\big[(A \cap B)^c\big] = 1 - P(A \cap B) \\ &= 1 - \big[P(A) + P(B) - P(A \cup B)\big] \\ &= 1 + P(A \cup B) - P(A) - P(B)\end{aligned}$$

2.7 $A \cap B = \emptyset \Longrightarrow A \subseteq B^c$

$\therefore P(A) \leq P(B^c)$

2.8

$$\begin{aligned}P(A \cup B \cup C) =& P\big[A \cup (B \cup C)\big] = P(A) + P(B \cup C) - P\big[A \cap (B \cup C)\big] \\ =& P(A) + P(B) + P(C) - P(B \cap C) - P\big[A \cap (B \cup C)\big] \\ & - \Big[P\big[(A \cap B) \cup (A \cap C)\big]\Big] \\ =& P(A) + P(B) + P(C) - P(B \cap C) \\ & - \Big[P(A \cap B) + P(A \cap C) - P\big[(A \cap C) \cap (B \cap C)\big]\Big] \\ =& P(A) + P(B) + P(C) - P(B \cap C) \\ & - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)\end{aligned}$$

2.9

$$\begin{aligned}\mathbf{S} =& \{X+Y : X = \text{ \# of dots on 1 die, } Y = \text{\# of dots on the other die }\}\\ =&\{2,3,\ldots,11,12\}\end{aligned}$$

Each (X,Y) pair has equal probability of happening and there are 36 unique (X,Y) pairs

$$\therefore P(X=x_i, Y=y_i) = \frac{1}{36} \qquad x_i = 1,\ldots,6, \quad y_i = 1,\ldots,6$$

a)

$$\begin{aligned}A =&\{x_i + y_i = 5\}\\ =&\big\{(x_i=1, y_i=4) \text{ or } (x_i=2, y_i=3) \text{ or } (x_i=3, y_i=2)\\ &\text{ or } (x_i=4, y_i=1)\big\}\end{aligned}$$

$$\therefore P(A) = P(1,4) + P(2,3) + P(3,2) + P(4,1) = 4\left(\frac{1}{36}\right) = \frac{1}{9}$$

b)

$$B = \{X+Y \text{ divisible by } 3\} = \{X+Y : X+Y = 3 \text{ or } 6 \text{ or } 9 \text{ or } 12\}$$

$$\begin{aligned}\therefore P(B) =&P(X+Y=3) + P(X+Y=6) + P(X+Y=9)\\ &+ P(X+Y=12)\\ =&\frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{1}{3}\end{aligned}$$

2.10 A and B independent $\implies P(A \cap B) = P(A)P(B)$

$P(A) = P\big[A \cap (B \cup B^c)\big] = P\big[(A \cap B) \cup (A \cap B^c)\big] = P(A \cap B) + P(A \cap B^c)$

$\therefore P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)\big[1 - P(B)\big] = P(A)P(B^c)$

$\therefore A$ and B^c are independent.

2.11 If A and B are statistically independent $\implies P(A \cap B) = P(A)P(B)$.

If A and B are mutually exclusive $\implies P(A \cap B) = 0$.

Hence the answer is *no* unless either $P(A)$ or $P(B)$ is zero or both are zero.

EE 600 SOLUTION SET # 3

Professor Edward J. Delp **April 21, 1998**

3.1

$$\bigcup_{i=1}^{m} A_i = \mathbf{S}, A_i \cap A_j = \emptyset, i \neq j$$

$$B = B \cap \mathbf{S} = B \cap \left(\bigcup_{i=1}^{m} A_i \right) = \bigcup_{i=1}^{m} (B \cap A_i) \qquad \text{(union of disjoint sets)}$$

$$\therefore P(B) = P\left(\bigcup_{i=1}^{m} B \cap A_i \right) = \sum_{i=1}^{m} P(B \cap A_i) = \sum_{i=1}^{m} P(B \mid A_i) P(A_i)$$

3.2

$$\begin{aligned} P(A \mid B \cap C) &= \frac{P\big(A \cap (B \cap C)\big)}{P(B \cap C)} \quad \text{if } P(B \cap C) > 0 \\ &= \frac{P(A)P(B)P(C)}{P(B)P(C)} \\ &= P(A) \end{aligned}$$

The others are identical.

3.3 If nothing is known about which three balls were removed, then the event {fourth ball is white} must have the same probability as {first ball is white}, which is easily found.

$$P\{\text{fourth ball is white}\} = \frac{4}{10}.$$

This *assumes* that each ball has an equal chance of being chosen. If the above justification doesn't satisfy you, then rewrite $P\{\text{fourth ball is white}\}$ as

$$\sum_{i=0}^{3} P\{\text{fourth ball is white} \mid i \text{ white balls chosen in first 3}\} \\ P\{i \text{ white balls chosen in first 3}\}.$$

3.4 Define the following events:

$B_1 = \{\text{1st ball is black}\}$

$B_2 = \{\text{2nd ball is black}\}$

$B_3 = \{\text{3rd ball is black}\}$

The probability we are asked to compute is $P(B_1 \cap B_2 \cap B_3 \mid B_1 \cup B_2 \cup B_3)$

$$P(B_1 \cap B_2 \cap B_3 | B_1 \cup B_2 \cup B_3) = \frac{P\big[(B_1 \cap B_2 \cap B_3) \cap (B_1 \cup B_2 \cup B_3)\big]}{P(B_1 \cup B_2 \cup B_3)}$$

But $(B_1 \cap B_2 \cap B_3) \cap (B_1 \cup B_2 \cup B_3) = B_1 \cap B_2 \cap B_3$.

$$\begin{aligned}
\therefore P(B_1 \cap B_2 \cap B_3 | B_1 \cup B_2 \cup B_3) &= \frac{P(B_1 \cap B_2 \cap B_3)}{P(B_1 \cup B_2 \cup B_3)} \\
&= \frac{P(B_3 \mid B_1 \cap B_2) P(B_2 \mid B_1) P(B_1)}{1 - P(B_1^c \cap B_2^c \cap B_3^c)} \\
&= \frac{5}{29} \text{ (after some work).}
\end{aligned}$$

3.5 Let L_j denote the event that the j^{th} person draws a long straw and S_j the event he/she draws the short one. If you are the j^{th} person, then the event that gets you killed is $L_1 \cap L_2 \cap \cdots \cap L_{j-1} \cap S_j$. All other events result in your survival. Thus

$$\begin{aligned}
P(\text{death}) &= P(L_1 \cap L_2 \cap \cdots \cap L_{j-1} \cap S_j) \\
&= P(L_1) P(L_2 | L_1) P(L_3 | L_1, L_2) \cdots P(L_{j-1} | L_1, \ldots, L_{j-2}) P(S_j | L_1, \ldots, L_{j-1}) \\
&= \left(\frac{r-1}{r}\right)\left(\frac{r-2}{r-1}\right)\left(\frac{r-3}{r-2}\right) \cdots \left(\frac{r-j+1}{r-j+2}\right)\left(\frac{1}{r-j+1}\right) \\
&= \frac{1}{r} \qquad \text{(independent of } j\text{)}
\end{aligned}$$

$\therefore$ it does not matter where you stand.

3.6 If $P(AC) = P(A)P(C)$, then A and C are independent.

$P(AC) = 1/3$

$P(A) = P(AC \cup AD) = P(AC) + P(AD) = 1/3 + 1/6 = 1/2$

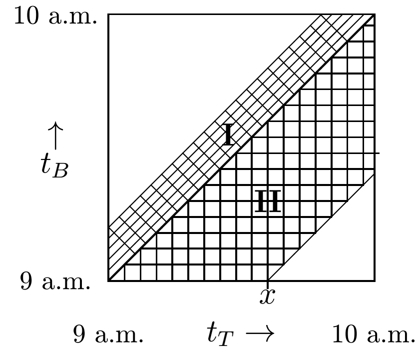
$P(C) = P(AC \cup BC) = P(AC) + P(BC) = 1/3 + 1/6 = 1/2$

$\therefore P(A)P(C) = 1/4$

$\therefore$ *not* independent.

3.7 Let t_T and t_B denote the arrival times of the train and the bus, respectively. The space of all possible outcomes can be represented as a rectangle of the form

$$\{0 \leq t_T \leq 60, 0 \leq t_B \leq 60\}$$



The event that they arrive at the same time is represented by the heavy diagonal line ($t_T = t_B$). Suppose the train arrives first. Then, they will meet if the bus arrives within 10 minutes ($t_T \leq t_B \leq t_T + 10$). Theis event is representated by region I. Similarly, the event that the bus arrives first and they meet is given by region II. The probability that they *do not meet* is given by

$$\frac{\frac{1}{2}(60-x)^2 + \frac{1}{2}50^2}{3600} = 0.5 \Longrightarrow x = 26.83 \text{ minutes.}$$

3.8 $P(x = i) = \frac{1}{200}$, $i = 1, 2, \ldots, 200$.

$A = \{7, 14, 21, \ldots, 187, 196\}$. There are 28 integers divisible by 7 in $[1, 200]$.
$\therefore P(A) = \frac{28}{200} = 0.14$.

$B = \{13, 16, 19, \ldots, 193, 196, 199\}$ (63 of them) $\therefore P(B) = \frac{63}{200} = 0.315$.

$C = \{1, 2, 3, \ldots, 19\} \quad \therefore P(C) = \frac{19}{200} = 0.095$.

3.9 We must test probabilities pairwise, threewise, etc. That is,

$$P(A_{\alpha_1} \cap \cdots \cap A_{\alpha_k}) = P(A_{\alpha_1}) \cdots P(A_{\alpha_k})$$

for all possible subsets $\{\alpha_1, \ldots, \alpha_k\}$ of the set $\{1, \ldots, n\}$. For $k = 2$, we have pairwise; $k = 3$, we have threewise, etc. There are $\binom{n}{k} = \frac{n!}{k!(n-k!)}$ subsets with k elements out of n. Therefore, we have to test $\sum_{k=2}^{n} \binom{n}{k}$ relations.

$$\sum_{k=2}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} - \sum_{k=0}^{1} \binom{n}{k} = 2^n - (1 + n)$$

3.10 Let $\mathbf{S} = \{1, 2, \ldots, 12\}$

$P(\text{uppermost face} = i) = 1/12$.

a) $P(7) = 1/12$.

b) $P(> 4) = P(5 \text{ or } 6 \text{ or } \cdots \text{ or } 12) = P(5) + P(6) + \cdots + P(12) = 8/12 = 2/3$.

c) $P(\text{odd \# other than } 5) = P(1 \text{ or } 3 \text{ or } 7 \text{ or } 9 \text{ or } 11) = 5/12$.

d) $P(>6,<10) = P(7 \text{ or } 8 \text{ or } 9) = 3/12 = 1/4.$

3.11 We are interested in the event that the player wins. Call this event W. If we let x_1 be the outcome of the first throw, then we may partition W as follows:

$$W = \bigcup_{k=2}^{12} \big(W \cap \{x_1 = k\}\big)$$

$$\therefore P(W) = \sum_{k=2}^{12} P\big(W \cap \{x_1 = k\}\big) \qquad \text{(I)}$$

Now if $x_1 = 2$, 3 or 12, the player (shall we call him Edward?) loses. Hence,

$$W \cap \{x_1 = k\} = \emptyset,\ k = 2, 3, 12.$$

If $x_1 = 7$ or 11, however, Edward wins on the first roll.

$$W \cap \{x_1 = k\} = \{x_1 = k\},\ k = 7, 11.$$

Hence, we may rewrite Equation (I) as

$$\begin{aligned} P(W) &= P(x_1 = 7) + P(x_1 = 11) + \sum_{c \in \{4,5,6,8,9,10\}} P\big(W \cap \{x_1 = c\}\big) \\ &= \frac{2}{9} + \sum_{c \in \{4,5,6,8,9,10\}} P\big(W \cap \{x_1 = c\}\big) \end{aligned}$$

Now, for a fixed c in $\{4, 5, 6, 8, 9, 10\}$,

$$W \cap \{x_1 = c\} = \bigcup_{n=2}^{\infty} \big(W_n \cap \{x_1 = c\}\big) \qquad \text{(II)}$$

where W_n is the event that Edward wins at the n^{th} step. For $n \geq 2$ and fixed c in $\{4, 5, 6, 8.9, 10\}$,

$$W_n \cap \{x_1 = c\} = \{x_1 = c\} \cap \{x_2 \notin 7, c\} \cap \cdots \cap \{x_{n-1} \notin 7, c\} \cap \{x_n = c\}.$$

$$\therefore P\big(W_n \cap \{x_1 = c\}\big) = \big[P(x = c)\big]^2 \big[P(x \notin \{7, c\})\big]^{n-2}.$$

Therefore, from Equation (II),

$$\begin{aligned} P\big(W \cap \{x_1 = c\}\big) &= \big[P(x = c)\big]^2 \sum_{n=2}^{\infty} \Big[P\big(n \notin \{7, c\}\big)\Big]^{n-2} \\ &= \frac{\big[P(x = c)\big]^2}{P(x = c) + P(x = 7)} \end{aligned}$$

Hence, we get

$$
\begin{aligned}
P(W) &= \frac{2}{9} + \sum_{c\in\{4,5,6,8,9,10\}} \frac{\left[P(x=c)\right]^2}{P(x=c)+P(x=7)} \\
&= \frac{2}{9} + \left(\frac{1}{36} + \frac{2}{45} + \frac{25}{11\times 36} + \frac{25}{11\times 36} + \frac{2}{45} + \frac{1}{36}\right) \\
&= 0.49\overline{29}
\end{aligned}
$$

3.12 We partition the population into 3 subsets: $A = \{\text{those who always tell the truth}\}$, $B = \{\text{those who always lie}\}$, $C = \{\text{those who lie half the time}\}$. If you select an individual at random, the probablity that he/she is in one of the above classes is:

$$
P(A) = \frac{1}{2} \qquad P(B) = \frac{1}{10} \qquad P(C) = \frac{2}{5}
$$

a) Let $T = \{\text{the answer is true}\}$.

$$
\begin{aligned}
P(T) &= P(T \mid A)P(A) + P(T \mid B)P(B) + P(T \mid C)P(C) \\
&= (1)(1/2) + (0)(1/10) + (1/2)(2/5) \\
&= 0.7
\end{aligned}
$$

b) Let $N = \{\text{the answer is “No”}\}$.

$$
\begin{aligned}
P(C \mid N) &= \frac{P(C\cap N)}{P(N)} \\
&= \frac{P(N \mid C)P(C)}{P(N)} \\
&= \frac{P(N \mid C)P(C)}{P(N \mid A)P(A) + P(N \mid B)P(B) + P(N \mid C)P(C)} \\
&= \frac{(1/2)(2/5)}{(0)(1/2) + (0)(1/10) + (1/2)(2/5)} \\
&= 1.
\end{aligned}
$$

EE 600 SOLUTION SET # 4

Professor Edward J. Delp **April 21, 1998**

4.1 For every $\alpha \in (-\infty, \infty)$, we must have the set $\{s \in \mathbf{S} : X(s) \le \alpha\}$ be an element of $\mathcal{F}$.

$$\begin{aligned} \alpha \le 1 :&\ \{s \in \mathbf{S} : I_A(s) \le \alpha\} = \mathbf{S} \in \mathcal{F} \\ 0 \le \alpha < 1 :&\ \{s \in \mathbf{S} : I_A(s) \le \alpha\} = A^c \in \mathcal{F} \\ \alpha < 0 :&\ \{s \in \mathbf{S} : I_A(s) \le \alpha\} = \emptyset \in \mathcal{F} \end{aligned}$$

Hence, the indicator function on any set in $\mathcal{F}$ is a valid random variable.

4.2 **a)**

$$\begin{aligned} F_x(+\infty) &= P(x \le +\infty) = P(\mathbf{S}) = 1 \\ F_x(-\infty) &= P(x \le -\infty) = P(\emptyset) = 0 \end{aligned}$$

b) Since $\{X : X \le a\} \subset \{X : X \le b\}$, we know $P\{X : X \le a\} \le P\{X : X \le b\}$. $\therefore F_X(a) \le F_X(b)$.

c)

$$\begin{aligned} P\big(X \in (a, b]\big) &= P(a < X \le b) = P\big(\{X \le b\} \setminus \{X \le a\}\big) \\ &= P(X \le b) - P(X \le a) = F_X(b) - F_X(a) \end{aligned}$$

$$P\big(X \in (a, b)\big) = F_X(b) - F_X(a) - P(X = b)$$

$$P\big(X \in [a, b]\big) = F_X(b) - F_X(a) + P(X = a)$$

$$P\big(X \in [a, b)\big) = F_X(b) - F_X(a) - P(X = b) + P(X = a)$$

4.3

$$f_X(x) = \lim_{x_2 \to x} \frac{F_X(x_2) - F_X(x)}{x_2 - x}, \quad x_2 > x.$$

Since $F_X(x_2) - F_X(x) \ge 0$ and $x_2 > x$, $f_X(x)$ is the limit of the ratio of two positive quantities, we deduce

$$f_X(x) \ge 0, \quad x \in (-\infty, \infty).$$

4.4 **a)** yes

b) no; not continuous from the right

c) no; not monotone increasing

d) no; $\lim_{x\to-\infty} F(x) \neq 0$

e) yes

4.5

$$F_X(x \mid A) = P(X \le x \mid A) = \frac{P\big(\{X \le x\} \cap A\big)}{P(A)} = \frac{P(A \mid X \le x)P(X \le x)}{P(A)}$$
$$= \frac{P(A \mid X \le x)F_X(x)}{P(A)} \qquad \text{(assuming } P(A) \neq 0\text{).}$$

4.6

$$\{X \le x\} = \{X \le x\} \cap \mathbf{S} = \{X \le x\} \cap \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{X \le x\} \cap A_i$$
$$\therefore F_X(x) = P(X \le x) = P\Big(\bigcup_{i=1}^{n} \{X \le x\} \cap A_i\Big) = \sum_{i=1}^{n} P\Big(\{X \le x\} \cap A_i\Big)$$
$$= \sum_{i=1}^{n} P(X \le x \mid A_i)P(A_i) = \sum_{i=1}^{n} F_X(x \mid A_i)P(A_i)$$

4.7 Since $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$, we get $\int_0^{\infty} be^{-x}\,dx = 1 \Longrightarrow b = 1$.

$$P(X \le x_m) = \int_0^{x_m} e^{-x}\,dx = 0.5 \Longrightarrow \left.\frac{e^{-s}}{-1}\right|_0^{x_m} = 0.5 \Longrightarrow x_m = \ln 2.$$

4.8 $(x-10)^2 < 4 \Longrightarrow 8 < x < 12$.
$\therefore f_X\left(x \mid (X-10)^2 < 4\right) = f_X(x \mid 8 < X < 12)$.

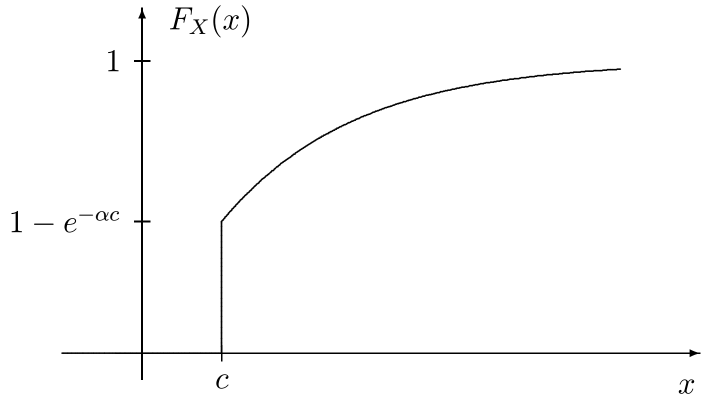
$$F_X(x \mid 8 < X < 12) = \begin{cases} 1, & x \ge 12 \\ \dfrac{F_X(x) - F_X(8)}{F_X(12) - F_X(8)}, & 8 \le x < 12 \\ 0, & x < 8 \end{cases}$$
$$\therefore f_X(x \mid 8 < X < 12) = \begin{cases} \dfrac{f_X(x)}{F_X(12) - F_X(8)}, & 8 \le x < 12 \\ 0, & \text{otherwise.} \end{cases}$$

From a table of standard normal values, $F_X(12) - F_X(8) = 0.955$.

4.9 Let $c = 0$. Then $F_X(x) = \left(1 - e^{-\alpha x}\right)U(x)$.

$$\therefore f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0, & x < 0. \end{cases}$$

Let $c > 0$. Then $F_X(x)$ looks like this:



$$\therefore f_X(x) = \begin{cases} \alpha e^{-\alpha x}, & x > c \\ 0, & x < c \end{cases}$$

Notice that in both these cases, the derivative is undefined at $x = c$. In particular, when $x > c$, $F_X(x)$ has a step at $x = c$. Hence, $f_X(x)$ will contain an impulse at $x = c$ which has height $F_X(c) = 1 - e^{-\alpha c}$. To see that this is so, use the following property of $f_X(x)$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = \int_{-\infty}^{c^-} f_X(x)\,dx + \int_{c^-}^{c^+} f_X(x)\,dx + \int_{c^+}^{\infty} f_X(x)\,dx$$

$$1 = \int_{c^-}^{c^+} f_X(x)\;dx + \int_{c^+}^{\infty} \alpha e^{-\alpha x}\;dx = \int_{c^-}^{c^+} f_X(x)\;dx + \left.\frac{\alpha e^{-\alpha x}}{-\alpha}\right|_{c^+}^{\infty}$$

$$1 = \int_{c^-}^{c^+} f_X(x)\;dx + e^{-\alpha c} \Longrightarrow \int_{c^-}^{c^+} f_X(x)\,dx = 1 - e^{-\alpha c}.$$

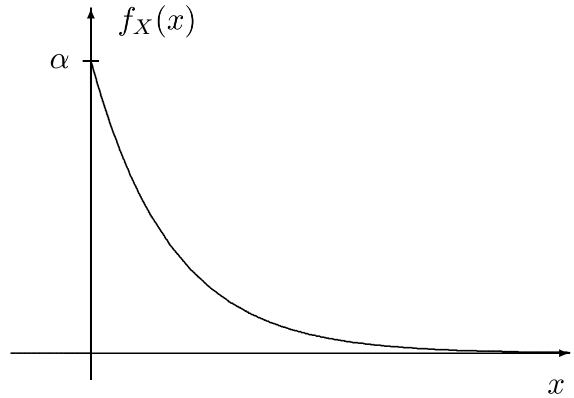
The only function which yields $1 - e^{-\alpha c}$ when integrated over (c^-, c^+) is an impulse of weight $1 - e^{-\alpha c}$.

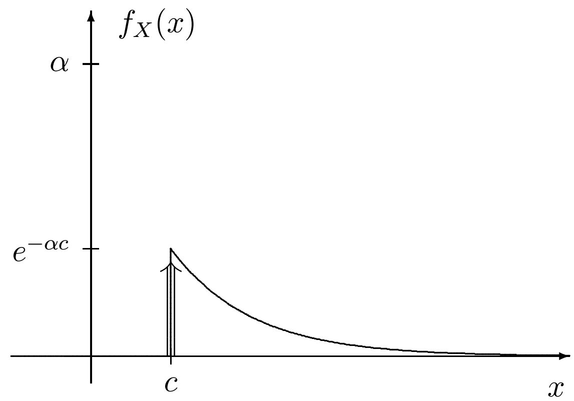
$\therefore$ For $c > 0$,

$$f_X(x) = \begin{cases} 0, & x < c \\ \alpha e^{-\alpha x} + (1 - e^{-\alpha c})\delta(x - c), & x \geq c. \end{cases}$$

or

$$f_X(x) = \alpha e^{-\alpha x} U(x - c) + \left(1 - e^{-\alpha c}\right)\delta(x - c)$$





We don't need to consider $c < 0$ because $F_X(x)$ is not a valid distribution function in that case.

4.10 $F(x) = P(X \leq x)$. If $x > b$, then $P(X \leq x) = 1$ since $a \leq X \leq b$. Similarly $P(X \leq x) = 0$ if $x \leq a$.

$\therefore F(x) = 1,\ x > b$; $F(x) = 0,\ x < a$.

4.11 If $X(s) \leq Y(s)$, $\forall s \in \mathbf{S}$, then $\{s : X(s) \leq \omega\} \supseteq \{s : Y(s) \leq \omega\}$.

$\therefore$ Since $A \supseteq B \Longrightarrow P(A) \geq P(B)$, we have $P\{s : X(s) \leq \omega\} \geq P\{s : Y(s) \leq \omega\}$ or $F_X(\omega) \geq F_Y(\omega)$.

EE 600 SOLUTION SET # 5

Professor Edward J. Delp **April 21, 1998**

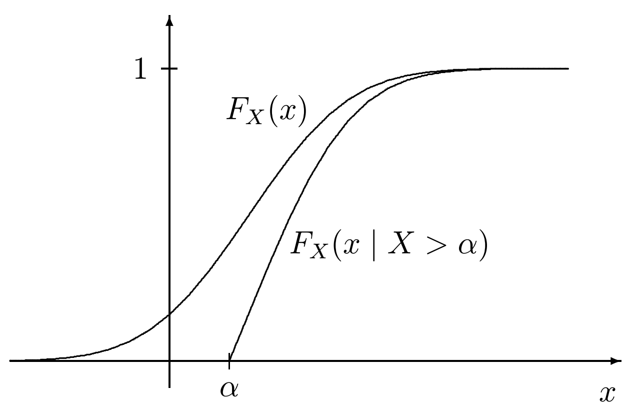
5.1 In both cases, we assume $P(M) \neq 0$.

$$F_X(x \mid M) = P(X \leq\mid M) = \frac{P\Big(\{X \leq x\} \cap M\Big)}{P(M)}$$
$$f_X(x \mid M) = \frac{\partial}{\partial x} F_X(x \mid M)$$

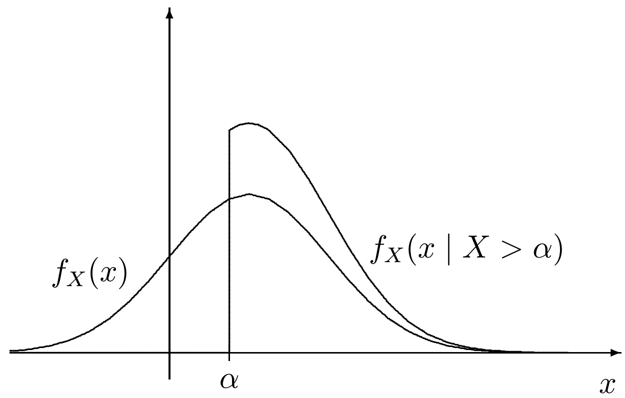
a)

$$F_X(x \mid X > \alpha) = \frac{P\Big(\{X \leq x\} \cap \{X > \alpha\}\Big)}{P(X > \alpha)} = \begin{cases} \dfrac{P(\alpha < X \leq x)}{1 - F_X(\alpha)}, & x > \alpha \\ \dfrac{P(\emptyset)}{1 - F_X(\alpha)}, & x \leq \alpha \end{cases}.$$

$$\therefore F_X(x \mid X > \alpha) = \begin{cases} \dfrac{F_X(x) - F_X(\alpha)}{1 - F_X(\alpha)}, & x > \alpha \\ 0, & x \leq \alpha \end{cases}.$$



$$\therefore f_X(x \mid X > \alpha) = \begin{cases} \dfrac{f_X(x)}{1 - F_X(\alpha)}, & x > \alpha \\ 0, & x \leq \alpha \end{cases}.$$

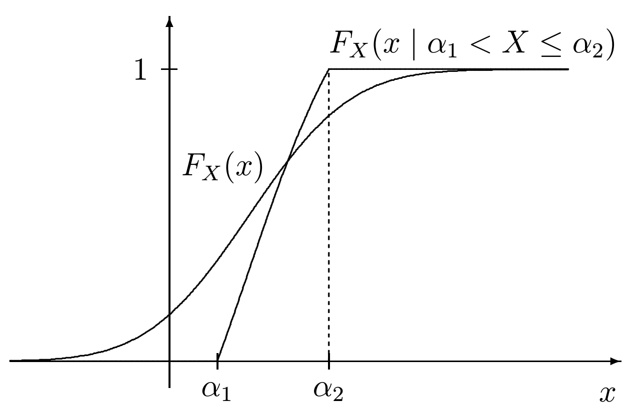


b)

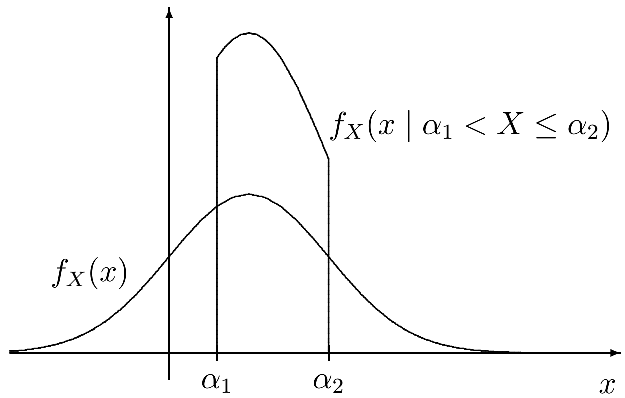
$$F_X(x \mid \alpha_1 < X \le \alpha_2) = \frac{P\Big(\{X \le x\} \cap \{\alpha_1 < X \le \alpha_2\}\Big)}{P(\alpha_1 < X \le \alpha_2)}$$

$$= \begin{cases} \dfrac{P(\alpha_1 < X \le \alpha_2)}{P(\alpha_1 < X \le \alpha_2)} = 1, & x > \alpha_2 \\ \dfrac{P(\alpha_1 < X \le x)}{P(\alpha_1 < X \le \alpha_2)}, & \alpha_1 \le x \le \alpha_2 \\ \dfrac{P(\emptyset)}{P(\alpha_1 < X \le \alpha_2)} = 0, & x < \alpha_1. \end{cases}$$

$$\therefore F_X(x \mid \alpha_1 < X \le \alpha_2) = \begin{cases} 1, & x > \alpha_2 \\ \dfrac{F_X(x) - F_X(\alpha_1)}{F_X(\alpha_2) - F_X(\alpha_1)}, & \alpha_1 \le x \le \alpha_2 \\ 0, & x < \alpha_1 \end{cases}$$



$$\therefore f_X(x \mid \alpha_1 < X \le \alpha_2) = \begin{cases} \dfrac{f_X(x)}{F_X(\alpha_2) - F_X(\alpha_1)}, & \alpha_1 \le x \le \alpha_2 \\ 0, & \text{elsewhere.} \end{cases}$$



5.2

$$F_X(x \mid \{X > 1\}) = \frac{P(\{X \le x, X > 1\})}{P(\{X > 1\})}$$

$$\text{for } x \le 1 \quad F_X(x \mid \{X > 1\}) = \frac{P(\emptyset)}{P(\{X > 1\})} = 0$$

$$\text{for } x \ge 1 \quad F_X(x \mid \{X > 1\}) = \frac{P(\{1 < X \le x\})}{P(\{X > 1\})} = \frac{F_X(x) - F_X(1)}{1 - F_X(x)}$$

$F_X(x) = 1 - e^{-x}$, $x \ge 0$ so

$$\frac{1 - e^{-x} - 1 + e^{-1}}{1 - 1 + e^{-1}} = \frac{e^{-1} - e^{-x}}{e^{-1}} = \frac{1}{2}$$

solve for x:

5.3

$$e^a = 1 + \frac{a}{1!} + \frac{a^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{a^k}{k!}$$

Multiply both sides by e^{-a}:

$$1 = e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{k!} = \sum_{k=0}^{\infty} e^{-a} \frac{a^k}{k!}$$

5.4

$$
\begin{aligned}
P(A \mid x \le X \le x+\Delta x) &= \frac{P\Big(A \cap \{x \le X \le x+\Delta x\}\Big)}{P(x \le X \le x+\Delta x)} \\
&= \frac{P(x \le X \le x+\Delta x \mid A)}{P(x \le X \le x+\Delta x)} P(A) \\
&= \frac{\Big[P(x \le X \le x+\Delta x \mid A)/\Delta x\Big]}{\Big[P(x \le X \le x+\Delta x)/\Delta x\Big]} P(A)
\end{aligned}
$$

$$
\begin{aligned}
\therefore P(A \mid X = x) &= \lim_{\Delta x \to 0} P(A \mid x \le X \le x+\Delta x) \qquad \text{(by Continuity Theorem)} \\
&= \lim_{\Delta x \to 0} \frac{\Big[P(x \le X \le x+\Delta x \mid A)/\Delta x\Big]}{\Big[P(x \le X \le x+\Delta x)/\Delta x\Big]} P(A) \\
&= \frac{f_X(x \mid A)}{f_X(x)} P(A)
\end{aligned}
$$

In the last line, we used the fact that the limit of the ratio of two functions exists and is the ratio of the two limits if the two limits exist.

$$
\therefore P(A \mid X = x) f_X(x) = f_X(x \mid A) P(A)
$$

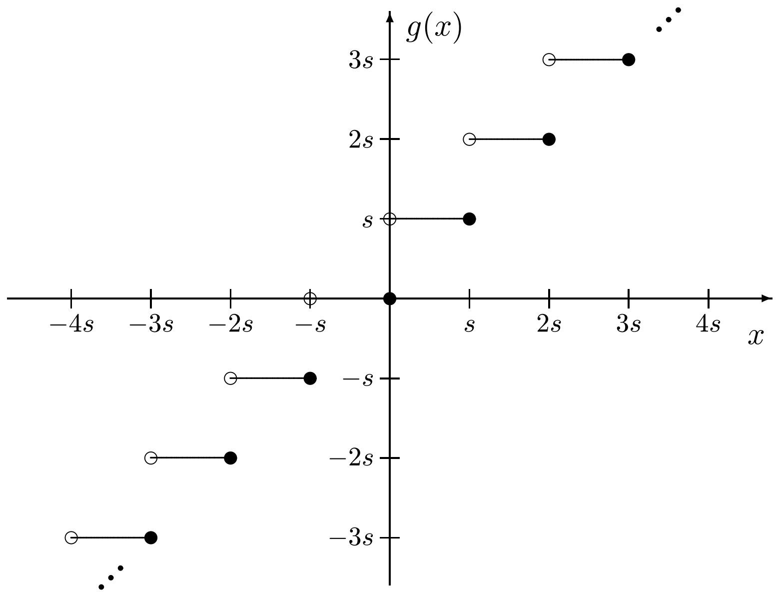
$$
\therefore \int_{-\infty}^{+\infty} P(A \mid X = x) f_X(x)\, dx = \int f_X(x \mid A) P(A)\, dx = P(A)
$$

5.5

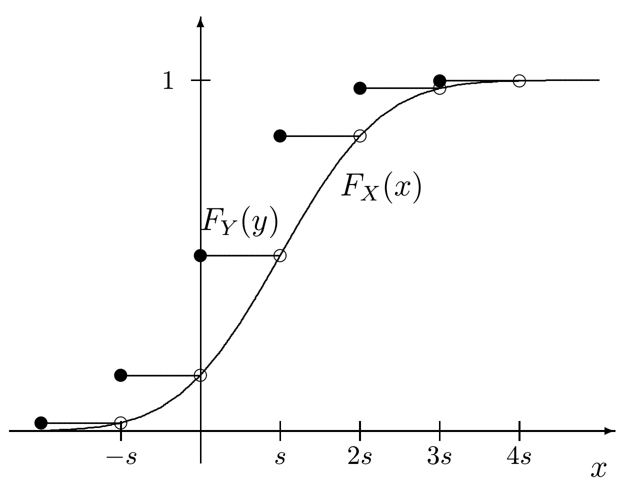
$$
A = A \cap \mathbf{S} = A \cap \Big\{\{X \le x\} \cup \{X > x\}\Big\} = \Big(A \cap \{X \le x\}\Big) \cup \Big(A \cap \{X > x\}\Big)
$$

$$
\begin{aligned}
\therefore P(A) &= P\Big(A \cap \{X \le x\}\Big) + P\Big(A \cap \{X > x\}\Big) \\
&= P(A \mid X \le x) P(X \le x) + P(A \mid X > x) P(X > x) \\
&= P(A \mid X \le x) F_X(x) + P(A \mid X > x)\Big[1 - F_X(x)\Big]
\end{aligned}
$$

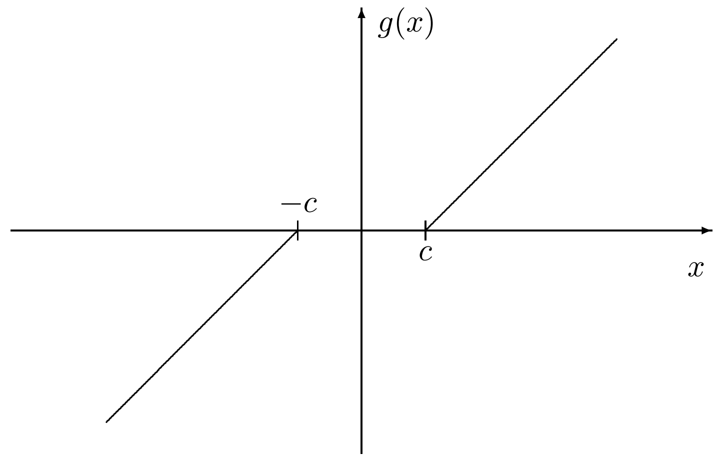
5.6



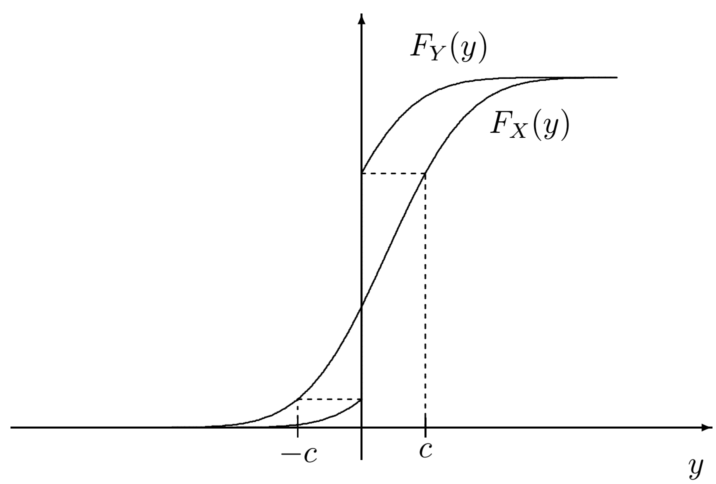
$$\{Y \leq y\} = \{X \leq (n+1)s\} \quad ns \leq y < (n+1)s$$
$$\therefore P(Y \leq y) = P(X \leq (n+1)s\} \quad ns \leq y < (n+1)s$$
$$\therefore F_Y(y) = F_X((n+1)s) \quad ns \leq y < (n+1)s$$



5.7



$$\{Y \leq y\} = \begin{cases} \{X \leq y + c\}, & y \leq 0 \\ \{X \leq y - c\}, & y < 0 \end{cases}$$



5.8 Note that since $y(x)$ has the form of a distribution function, it is monotone increasing. Hence, the inverse is also monotone increasing.

$$\{Y \leq y\} = \begin{cases} \{F_X(x) \leq y\} = \mathbf{S}, & y \leq 1 \\ \{X \leq F_X^{-1}(y)\}, & 0 \leq y < 1 \\ \emptyset, & y < 0 \end{cases}$$

$$\therefore F_Y(y) = \begin{cases} 1, & y \geq 1 \\ P\Big(X \leq F_X^{-1}(y)\Big) = F_X\Big(F_X^{-1}(y)\Big) = y, & 0 \leq y < 1 \\ 0, & y < 0 \end{cases}$$

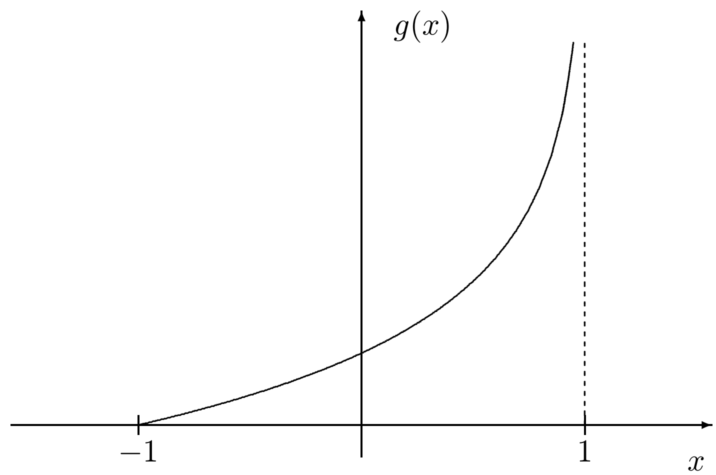
$\therefore Y$ is Uni$(0, 1)$.

5.9

$$F_Y(y) = \left[1 - e^{-2y}\right] U(y) \qquad F_X(x) = \frac{1}{2}(x+1), \quad -1 \le x \le 1$$

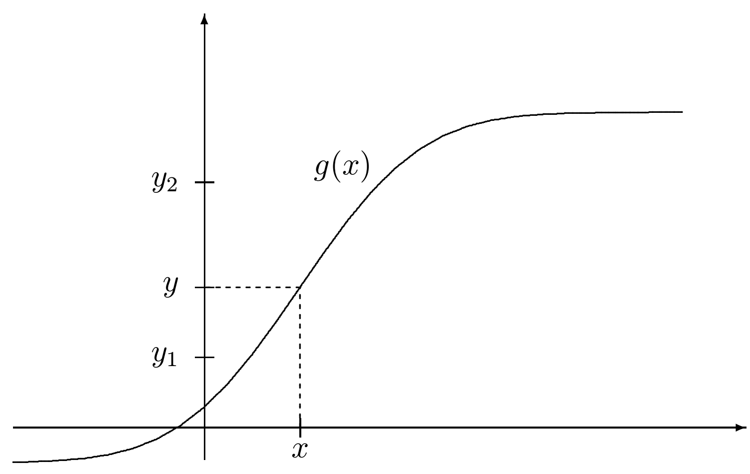
$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= P(Y \le g(x)) \quad \text{(if } g(x) \text{ is monotone increasing)} \\ &= F_Y(g(x)). \end{aligned}$$

$$\therefore \frac{1}{2}(x+1) = 1 - e^{-2g(x)} \Longrightarrow g(x) = \frac{1}{2}\ln\left(\frac{2}{1-x}\right) \qquad -1 \le x < 1$$



5.10 **a)** Monotone increasing:

Given $F_X(x)$ and $F_y(y)$, $y = g(x)$. Note: y is constrained by $g(x)$.



If $y_1 < y = g(x)$ then

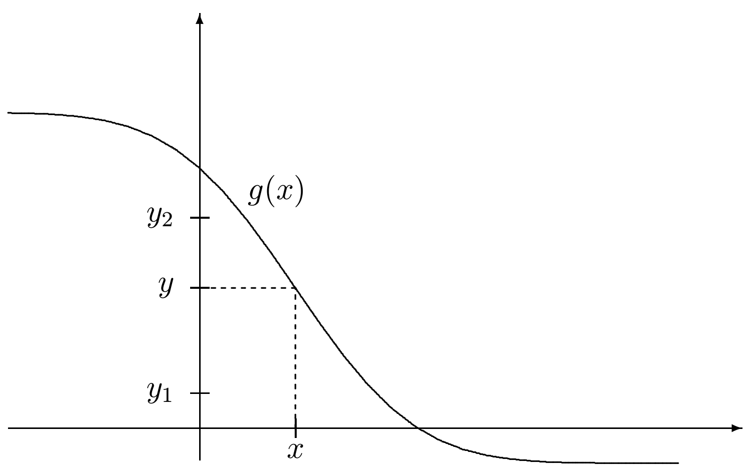
$$
\begin{aligned}
F_{XY}(x,y) &= P(X \le x, Y \le y_1) \\
&= P(\{X \le x\} \cap \{Y \le y_1\}) \\
&= P(\{Y \le y_1\}) \\
&= F_Y(y_1)
\end{aligned}
$$

If $y_2 > y = g(x)$ then

$$
\begin{aligned}
F_{XY}(x,y) &= P(X \le x_1, Y \le y_2) \\
&= P(\{X \le x\} \cap \{Y \le y_2\}) \\
&= P(\{X \le x\}) \\
&= F_x(x)
\end{aligned}
$$

$$
\therefore F_{XY}(x,y) = \begin{cases} F_X(x), & \text{if } y > g(x) \\ F_Y(y), & \text{if } y < g(x) \end{cases}
$$

b) Monotone decreasing:



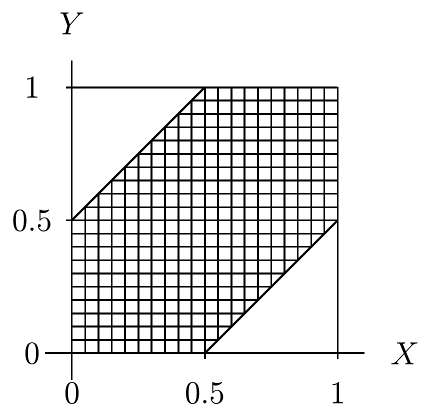
If $y_1 < y = g(x)$ then

$$
\begin{aligned}
F_{XY}(x,y) &= P(X \le x, Y \le y) \\
&= 0
\end{aligned}
$$

If $y_2 > y = g(x)$ then

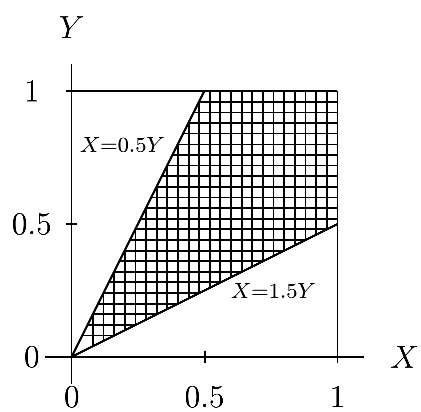
$$
\begin{aligned}
F_{XY}(x,y) &= P(X \le x, Y \le y_2) \\
&= P(X \le x) - P(Y > y_2) \\
&= F_X(x) - \left[1 - F_Y(y_2)\right]
\end{aligned}
$$

5.11 **a)** Since $f_{XY}(x,y) = \begin{cases} 1 & 0 \le x,\ y \le 1 \\ 0 & \text{elsewhere,} \end{cases}$ the probability that (X,Y) lies within a region in $\{[0,1]\times[0,1]\}$ is equal to the area of that region, while the probability that (X,Y) lies outside $\{[0,1]\times[0,1]\}$ is zero. Hence, $P(|X-Y| \le 0.5)$ is the area of the shaded region corresponding to $\{|X-Y| \le 0.5\}$.



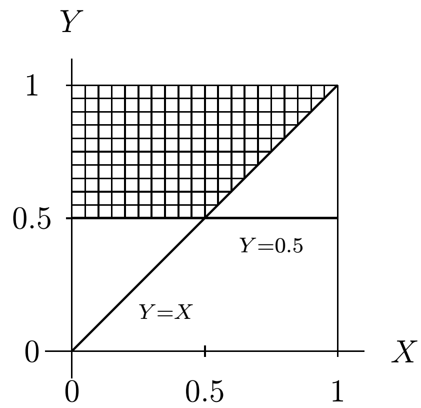
$$P(|X-Y| \le 0.5) = 1 - \tfrac{1}{8} - \tfrac{1}{8} = \tfrac{3}{4}$$

b) $\left|\frac{X}{Y} - 1\right| \le .5 \Longrightarrow -0.5 \le \frac{X}{Y} - 1 \le 0.5 \Longrightarrow 0.5Y \le X \le 1.5Y$



$$\begin{aligned} P\left(\left|\frac{X}{Y} - 1\right| \le 0.5\right) &= 1 - \left(\frac{1}{2}\right)(1)\left(\frac{1}{2}\right) \\ &\quad - \left(\frac{1}{2}\right)(1)\left(\frac{2}{3}\right) \\ &= 1 - \frac{1}{4} - \frac{1}{3} \\ &= \frac{5}{12} \end{aligned}$$

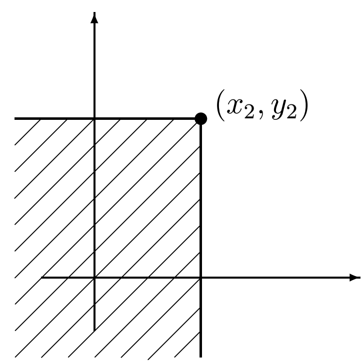
c)



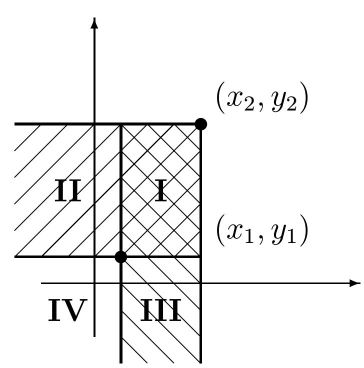
$$
\begin{aligned}
P(Y \geq X \mid Y \geq .5) &= \frac{P(Y \geq X, Y \geq .5)}{P(Y \geq .5} \\
&= \frac{3/8}{1/2} \\
&= 3/4
\end{aligned}
$$

5.12

Consider the event $\{X \leq x_2, Y \leq y_2\}$, given by the shaded region.



This event is composed of four smaller disjoint events as shown here:



$$
\begin{aligned}
\therefore P(X \leq x_2, Y \leq y_2) &= P(\mathbf{I}) + P(\mathbf{II}) + P(\mathbf{III}) = P(\mathbf{IV}) \\
&= P(\mathbf{I}) + \Big(P(\mathbf{II}) + P(\mathbf{IV})\Big) + \Big(P(\mathbf{III}) + P(\mathbf{IV})\Big) - P(\mathbf{IV}) \\
&= P(x_1 < X \leq x_2, y_1 < Y \leq y_2) + P(X \leq x_1, Y \leq y_2) \\
&\quad + P(X \leq x_2, Y \leq y_1) - P(X \leq x_1, Y \leq y_1) \\
\therefore F_{XY}(x_2, y_2) &= P(x_1 < X \leq x_2, y_1 < Y \leq y_2) \\
&\quad + F_{XY}(x_1, y_2) + F_{XY}(x_2, y_1) - F_{XY}(x_1, y_1)
\end{aligned}
$$

Rearranging gives the desired result.

5.13 **a)** No.

By definition,

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}.$$

A valid bivariate distribution of X and Y implies $f_{XY}(x,y) \geq 0$, $\forall x, y$.

For $x > 0$, $y > 0$,

$$\frac{\partial F_{XY}(x,y)}{\partial x} = -e^{-(x+y)}(-1) = e^{-(x+y)}$$
$$\frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} = e^{-(x+y)}(-1) = -e^{-(x+y)}$$

Since $f_{XY}(x,y) \not\geq 0 \forall x > 0, y > 0$, F_{XY} is not a valid bivariate distribution function.

b) No.

F_{XY} not monotone increasing on y.

EE 600 SOLUTION SET # 6

Professor Edward J. Delp **April 21, 1998**

6.1 **a)** To determine the constant b, use $\iint_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1$.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = \int_0^b\int_0^1 3xy\,dx\,dy = 3\left(\int_0^b y\,dy\right)\left(\int_0^1 x\,dx\right)$$
$$= 3\left(\frac{b^2}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}b^2 = 1$$

$$\therefore b = \frac{2}{\sqrt{3}}$$

b) As in a), use $\iint_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1$.

$$\iint_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = \int_0^1\int_0^{0.5} bx(1-y)\,dx\,dy$$
$$= b\left(\int_0^{0.5} x\,dx\right)\left(\int_0^1 (1-y)\,dy\right)$$
$$= b\left(\frac{1}{8}\right)\left(1-\frac{1}{2}\right) = \frac{b}{16} = 1$$

$$\therefore b = 16$$

c)

$$\int_0^2\int_{-1}^1 b(x^2+4y^2)\,dx\,dy = b\left(\left[\int_{-1}^1 x^2\,dx\right]\left[\int_0^2 dy\right] + \left[4\int_{-1}^1 dx\right]\left[\int_0^2 y^2\,dy\right]\right)$$
$$= b\left(\left[23\right]\left[2\right] + 4\left[2\right]\left[\frac{8}{3}\right]\right)$$
$$= b\left(\frac{4}{3}+\frac{64}{3}\right) = b\left(\frac{68}{3}\right) = 1$$

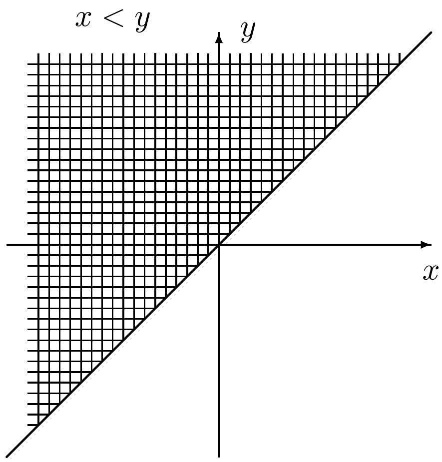
$$\therefore b = \frac{3}{68}$$

6.2 $f_{XY}(x,y) = [e^{-x}U(x)][e^{-y}U(y)] = f_X(x)f_Y(y) \qquad \therefore X, Y$ independent.

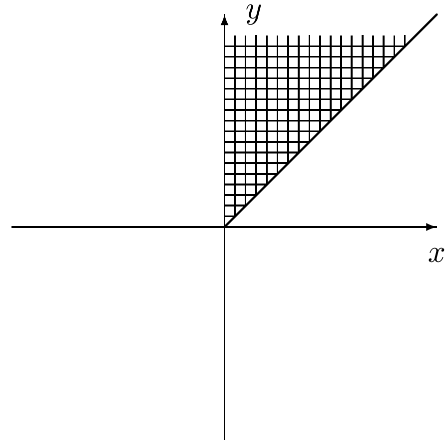
a) $P(X \le x) = F_X(x) = \int_{-\infty}^{x} f_X(s)\,ds = [1-e^{-x}]\,U(x)$

b) $P(Y \le y) = [1-e^{-y}]\,U(y)$

c) $P(X < Y) = P(\text{region depicted in Graph A}) = P(\text{region depicted in Graph B})$



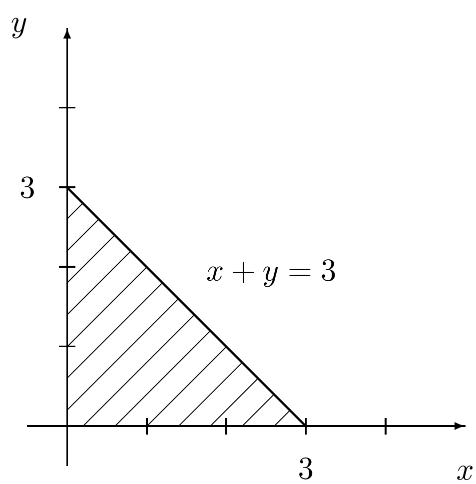
Graph A



Graph B

$$\therefore P(X < Y) = \int_0^\infty \int_0^y f_{XY}(x,y)\,dx\,dy = \int_0^\infty f_Y(y)\left(\int_0^y f_X(x)\,dx\right)dy$$
$$= \int_0^\infty f_Y(y)F_X(y)\,dy$$
$$\therefore P(X < Y) = \int_0^\infty e^{-y}\left[1 - e^{-y}\right]dy = \int_0^\infty e^{-y}dy - \int_0^\infty e^{-2y}dy = 1 - \frac{1}{2} = \frac{1}{2}$$

d) $P(X + Y \leq 3) = P(\text{region in Graph C})$



Graph C

$$\begin{aligned}
\therefore Pr(X+Y\le 3) &= \int_0^3\int_0^{3-y} f_X(x)f_Y(y)\,dx\,dy = \int_0^3 e^{-y}\left(\int_0^{3-y} e^{-x}dx\right)dy\\
&= \int_0^3 e^{-y}\left(\frac{e^{-x}}{-1}\Big|_0^{3-y}\right)dy = \int_0^3 e^{-y}\left(1-e^{-3+y}\right)dy\\
&= \int_0^3 e^{-y}dy - e^{-3}\int_0^3 dy = \left(1-e^{-3}\right) - 3e^{-3}\\
&= 1-4e^{-3}
\end{aligned}$$

6.3 To show that $F_{XY}(x,y)$ is nondecreasing in x, let $x_1 < x_2$. Then, the event $\{X \le x_1, Y \le y\}$ is contained in the event $\{X \le x_2, Y \le y\}$ for all y.

$$\therefore P(X\le x_1, Y\le y) \le P(X\le x_2, Y\le y)$$
$$\therefore F_{XY}(x_1,y) \le F_{XY}(x_2,y), \forall y.$$

Let $y_1 < y_2$. Using an identical argument on Y, we see that $F_{XY}(x,y_1) \le F_{XY}(x,y_2)$. Similarly, if $x_1 < x_2$ and $y_1 < y_2$, then $F_{XY}(x_1,y_1) \le F_{XY}(x_2,y_2)$.

6.4

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \int_{-\infty}^{+\infty} g(x)h(y)\,dy\\
&= g(x)\int_{-\infty}^{+\infty} h(y)\,dy\\
f_Y(y) &= h(y)\int_{-\infty}^{+\infty} g(x)\,dx
\end{aligned}$$

$$\text{Since} \iint_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1, \text{we get} \int_{-\infty}^{+\infty} g(x)\,dx \int_{-\infty}^{+\infty} h(y)\,dy = 1.$$

$$\therefore f_X(x) = \frac{g(x)}{\int_{-\infty}^{+\infty} g(x)\,dx} \qquad f_Y(y) = \frac{h(y)}{\int_{-\infty}^{+\infty} h(y)\,dy}$$

6.5 X and Y are independent since the joint density factors into the product of marginal densities. X is Rayleigh distributed with parameter $\alpha = 1$ and Y is exponentially distributed with parameter $\beta = 1$.

6.6 To get the first equality, use the reasoning in Problem 6.2 c) with the roles of X and Y interchanged and the region of integration the whole $X-Y$ plane. To get the second equality, we use the relation $Pr(Y \le X) = 1 - Pr(X < Y)$.

6.7

$$F_{XY}(x,y) = \left[(1-e^{-ax})U(x)\right]\left[(1-e^{-ay})U(y)\right] = F_X(x)F_Y(y)$$

a) Since X and Y are independent

$$f_X(x \mid Y=y) = f_X(x) \qquad f_Y(y \mid X=x) = f_Y(y).$$

b) $\therefore X, Y$ independent

6.8 Since f_{XY} is uniform, we can determine the height by using $\iint_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1$.

$$\int_0^1\int_{-1}^0 a dx dy + \int_{-1}^0\int_0^1 a\,dx\,dy = 1 \Longrightarrow a = 1/2.$$

$$\therefore f_{XY} = \begin{cases} 1/2 & \{0 \le x \le 1, 1 \le y \le 0\}\text{or}\{-1 \le x \le 0, 0 \le y \le 1\} \\ 0 & \text{elsewhere.} \end{cases}$$

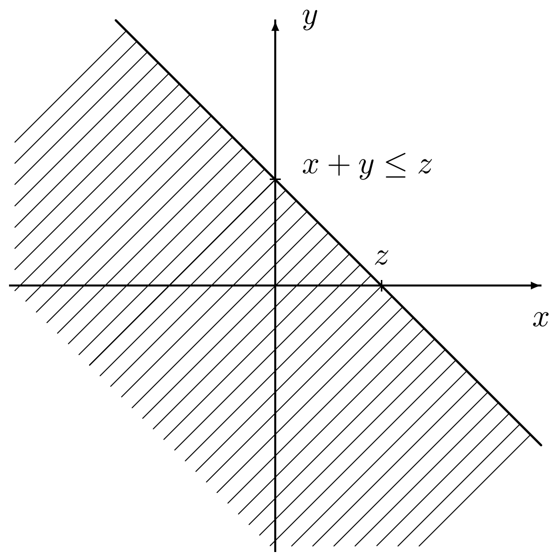
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \begin{cases} 0, & |x| > 1 \\ 1/2, & |x| \le 1 \end{cases} \qquad \therefore X \sim \text{Uniform on } [-1,1]$$

$$f_X(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \begin{cases} 0, & |y| > 1 \\ 1/2, & |y| \le 1 \end{cases} \qquad \therefore Y \sim \text{Uniform on } [-1,1]$$

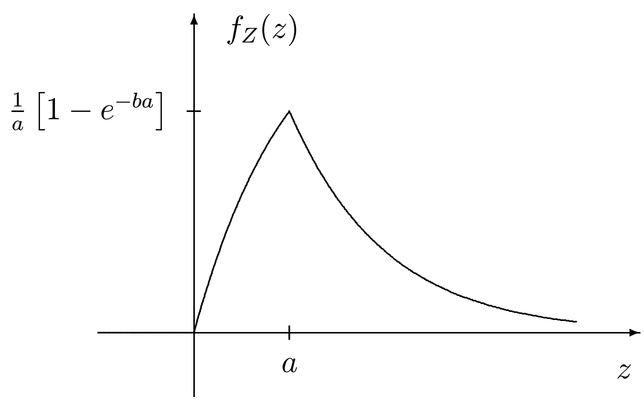
$$\therefore f_{XY}(x,y) \ne f_X(x) f_Y(y) \qquad \therefore X, Y \text{not independent}$$

6.9

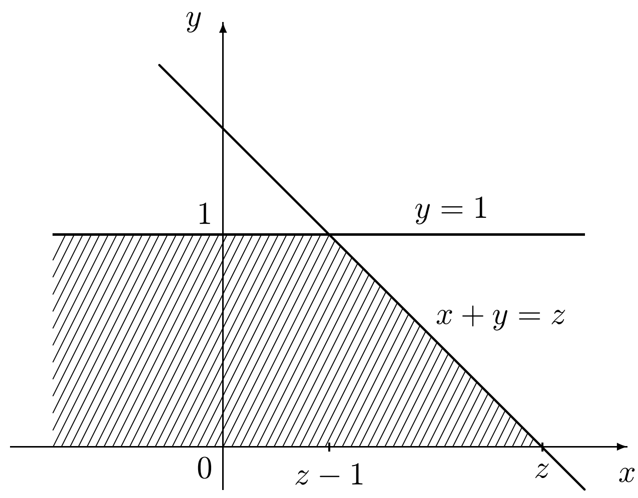
$$F_Z(z) = Pr(Z \le z) = Pr(X+Y \le z) = P(\text{region})$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x,y)\,dx\,dy$$



$$\therefore f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{d}{dz}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x,y)\,dx\,dy\right]$$
$$= \int_{-\infty}^{\infty}\left(\frac{d}{dz}\int_{-\infty}^{z-y} f_{XY}(x,y)\,dx\right)dy = \int_{-\infty}^{\infty} f_{XY}(z-y,y)\,dy$$
$$= \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)\,dy$$
$$= \begin{cases} 0, & z \le 0, \\ \int_0^z \frac{b}{a}e^{-by}dy, & 0 \le z \le a \\ \int_{z-a}^z \frac{b}{a}e^{-by}dy & z \ge a \end{cases} = \begin{cases} 0, & z \le 0 \\ \frac{1}{a}[1-e^{-bz}], & 0 \le z \le a \\ \frac{1}{a}[e^{ba}-1]e^{-bz} & z \ge a \end{cases}$$



6.10



$$P(Z \le z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{z-y} f_{XY}(x,y)\,dx\,dy$$
$$= \int_0^1 \int_{-\infty}^{z-y} f_X(x)\,dx\,dy$$

$$F_Z(z) = \int_0^1 F_X(z-y)\,dy$$
$$f_Z(z) = \int_0^1 f_X(z-y)\,dy$$

Letting $u = z - y$, $du = -dy$,

$$f_Z(z) = \int_z^{z-1} f_X(u)(-du) = \int_{z-1}^{z} f_X(u)\,du$$
$$= F_X(z) - F_X(z-1)$$

EE 600 SOLUTION SET # 7

Professor Edward J. Delp **April 21, 1998**

7.1 Letting $\gamma\begin{pmatrix} x \\ y \end{pmatrix}$ be the mapping from (x, y) to (z, w), we see that

$$\begin{pmatrix} z \\ w \end{pmatrix} = \gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi + y\sin\phi \\ -x\sin\phi + y\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\text{Hence, } \left| J\begin{pmatrix} x \\ y \end{pmatrix} \right| = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{vmatrix} = 1 \text{ and}$$

$$f_{ZW}(z, w) = \frac{f_{XY}(x, y)}{\left| J\begin{pmatrix} x \\ y \end{pmatrix} \right|} = f_{XY}(x, y).$$

$$\text{But } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & cos\phi \end{pmatrix}^{-1}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}$$

$$\therefore x = z\cos\phi - w\sin\phi \text{ and } y = z\sin\phi + w\cos\phi.$$

$$\therefore f_{ZW}(z, w) = f_{XY}(z\cos\phi - w\sin\phi, z\sin\phi + w\cos\phi).$$

7.2 We use $f_{XY}(x, y) = \dfrac{f_{R\Theta}(r, \theta)}{\left| J\begin{pmatrix} r \\ \theta \end{pmatrix} \right|}$.

$$\begin{matrix} X = R\cos\Theta \\ Y = R\sin\Theta \end{matrix} \qquad \therefore \left| J\begin{pmatrix} R \\ \Theta \end{pmatrix} \right| = \begin{vmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{vmatrix} = |R|$$

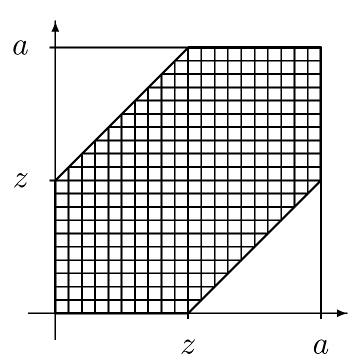
$$\therefore f_{XY}(x, y) = \frac{\frac{r}{2\pi\sigma^2}e^{-r^2/2\sigma^2}U(r)}{|r|} = \frac{1}{2\pi\sigma^2}e^{-(x^2+y^2)/2\sigma^2}U(\sqrt{x^2+y^2})$$

We can ignore the $U(\sqrt{x^2+y^2})$, since it is always 1.

$$\therefore f_{XY}(x, y) = \frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}\frac{1}{\sqrt{2\pi}\sigma}e^{-y^2 2\sigma^2} = f_X(x)f_Y(y)$$

7.3

$$\begin{aligned} \text{First find } P(Z \le z) &= F_Z(z) \\ &= P(|X - Y| \le z) \\ &= P(-z \le X - Y \le z) \\ &= P(\text{region in figure}). \end{aligned}$$



$$f_{XY}(x,y) = \begin{cases} 1/a^2 & 0 \le x, y \le a \\ 0 & \text{elsewhere} \end{cases}$$

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ 1 - \frac{1}{a^2}(a-z)^2 = 2\frac{z}{a}(1 - \frac{z}{2a}), & 0 \le z \le a \\ 1, & z > a \end{cases}$$

$$\therefore f_Z(z) = \frac{2}{a}(1 - \frac{z}{a}), \qquad 0 \le z \le a$$

7.4

$$\begin{aligned} \mu_Y = E(Y) &= E(aX+b) \\ &= \int_{-\infty}^{\infty} (az+b) f_X(x)\,dx \\ &= a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\int_{-\infty}^{\infty} f_X(x)\,dx \\ &= a\mu_X + b \end{aligned}$$

$$\begin{aligned} \sigma_Y^2 = E\big((Y-\mu_Y)^2\big) &= E\big((aX+b-(a\mu_X+b))^2\big) \\ &= E\big(a^2(x-\mu_X)^2\big) \\ &= a^2\sigma_X^2 \end{aligned}$$

7.5

Recall that $I_A = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{if } A^c \text{ occurs.} \end{cases}$

$$E\{I_A\} = \sum_{i=0}^{1} iP(I_A = i) = (0)P(A^C) + (1)P(A) = P(A).$$

7.6

$$P(X=k) = e^a \frac{a^k}{k!}$$

$$\begin{aligned} \therefore E(x) &= \sum_{k=0}^{\infty} k e^a \frac{a^k}{k!} = \sum_{k=1}^{\infty} e^a \frac{a^k}{(k-1)!} = a\sum_{k=1}^{\infty} e^a \frac{a^{k-1}}{(k-1)!} = a\sum_{k=0}^{\infty} e^a \frac{a^k}{k!} \\ &= a(1) = a \end{aligned}$$

$$\begin{aligned} \therefore \sigma_X^2 &= E\big\{(X-m_X)^2\big\} = E\big\{(X-a)^2\big\} = E(X^2) - a^2 \\ &= \sum_{k=0}^{\infty} k^2 e^a \frac{a^k}{k!} - a^2 = a\sum_{k=1}^{\infty} k e^a \frac{a^{k-1}}{(k-1)!} - a^2 = a\sum_{k=0}^{\infty} (k+1) e^a \frac{a^k}{k!} - a^2 \\ &= a(a+1) - a^2 = a \end{aligned}$$

7.7

$$E\{|X|^r\} = \int_{-\infty}^{\infty} |x|^r f_X(x)\,dx = \int_{|x|<\lambda} |x|^r f_X(x)\,dx + \int_{|x|\geq\lambda} |x|^r f_X(x)\,dx$$

Both terms on the right are positive. Therefore, we have the following inequality:

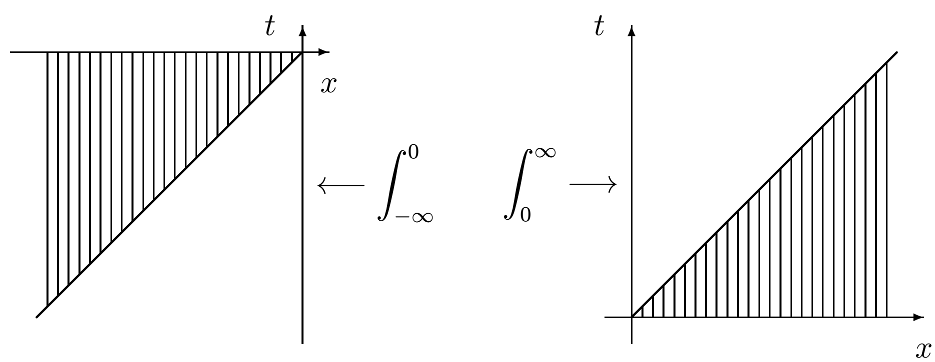
$$E\{|X|^r\} \geq \int_{|x|\geq\lambda} |x|^r f_X(x)\,dx \geq |\lambda|^r \int_{|x|\geq\lambda} f_X(x)\,dx \geq |\lambda|^r P\big(|X| \geq \lambda\big)$$

$$\therefore P\big(|X| \geq \lambda\big) \leq \frac{E\{|X|^r\}}{|\lambda|^r}$$

7.8

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{0} x f_X(x)\,dx + \int_{0}^{\infty} x f_X(x)\,dx \\ &= \int_{-\infty}^{0} \left(-\int_{x}^{0} dt\right) f_X(x)\,dx + \int_{0}^{\infty} \left(\int_{0}^{x} dt\right) f_X(x)\,dx \end{aligned}$$

The regions of integration in the $t - x$ plane are shown here.



Change the order of integration in each.

$$\begin{aligned} E(X) &= \int_{-\infty}^{0} \left(-\int_{-\infty}^{t} f_X(x)\,dx\right) dt + \int_{0}^{\infty} \left(\int_{t}^{\infty} f_X(x)\,dx\right) dt \\ &= -\int_{-\infty}^{0} F_X(t)\,dt + \int_{0}^{\infty} \big[1 - F_X(t)\big]\,dt \end{aligned}$$

7.9 Use the inversion formula $f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega)e^{-j\omega x}\,d\omega$.

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha|\omega|}e^{-j\omega x}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{0} e^{\alpha\omega}e^{-j\omega x}\,d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-\alpha\omega}e^{-j\omega x}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{(\alpha-jx)\omega)}\,d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-(\alpha+jx)\omega}\,d\omega \\
&= \frac{1}{2\pi}\left.\frac{e^{(\alpha-jx)\omega}}{(\alpha-jx)}\right|_{-\infty}^{0} + \frac{1}{2\pi}\left.\frac{e^{-(\alpha+jx)\omega}}{-(\alpha+jx)}\right|_{0}^{\infty} \\
&= \frac{1}{2\pi}\frac{1}{\alpha-jx} + \frac{1}{2\pi}\frac{1}{\alpha+jx} = \frac{1}{2\pi}\frac{2\alpha}{\alpha^2+x^2} \\
&= \frac{\alpha/\pi}{\alpha^2+x^2}
\end{aligned}
$$

This is the *Cauchy* density.

7.10 Since $\phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx$, we know $\phi_X(0) = 1$. We now use

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} g(x)\,dx\right| &\le \int_{-\infty}^{\infty} \left|g(x)\right|dx \\
\left|\phi_X(\omega)\right| &\le \int_{-\infty}^{\infty} \left|f_X(x)e^{j\omega x}\right|dx \\
\left|\phi_X(\omega)\right| &\le \int_{-\infty}^{\infty} f_X(x)\,dx \\
\left|\phi_X(\omega)\right| &\le 1
\end{aligned}
$$

and since $\phi_X(0) = 1$, this implies $\phi_X(\omega)$ takes on its maximum at $\omega = 0$.

7.11 Consider $E\{(aX-Y)^2\}$. Since $(aX-Y)^2 \ge 0, \forall a$, $E\{(aX-Y)^2 =\} \ge 0, \forall a$. Expanding, we have $a^2E(X^2) - 2aE(XY) + E(Y^2) \ge 0, \forall a$. Since this polynomial in a is greater than or equal to 0 for all a, it must be greater than or equal to zero at its minimum. Let's find the value a for which it is a minimum:

$$
\frac{\partial}{\partial a}E\{(aX-Y)^2\} = 2aE(X^2) - 2E(XY) \ge 0 \qquad \therefore a = \frac{E(XY)}{E(X^2)}
$$

Substituting this value back into the original expression, we obtain

$$
\left(\frac{E(XY)}{E(X^2)}\right)^2 E(X^2) - 2\left(\frac{E(XY)}{E(X^2)}\right)E(XY) + E(Y^2) \ge 0
$$

Solving, we obtain $E^2(XY) \le E(X^2)E(Y^2)$.

7.12 Letting $X' = X - m_X$ and $Y' = Y - m_Y$ we see $\rho = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{E(X'Y')}{\sqrt{E(X'^2)E(Y'^2)}}$. Since $E^2(X'Y') \le E(X'^2)E(Y'^2)$, then $\rho \le 1$.

7.13 X, Y uncorrelated $\Longrightarrow C_{XY} = 0$.

$$\begin{aligned}\therefore \sigma^2_{X+Y} &= E\{(X+Y-m_{X+Y})^2\} = E\Big\{\big[(X-m_X)+(Y-m_Y)\big]^2\Big\}\\ &= \sigma^2_X + \sigma^2_Y + 2C_{XY} = \sigma^2_X + \sigma^2_Y\end{aligned}$$

7.14

$$\begin{aligned}E\{g(X)h(Y)\} &= \iint g(x)h(y)f_{XY}(x,y)\,dx\,dy = \iint g(x)h(y)f_X(x)f_Y(y)\,dx\,dy\\ &= \int g(x)f_X(x)\,dx \int h(y)f_Y(y)\,dy = E\{g(X)\}E\{h(Y)\}\end{aligned}$$

7.15 Since $E\Big\{\big[Y-(aX+b)\big]^2\Big\} \geq 0$, $\forall a, b$, the values of a and b which satisfy $E\Big\{\big[Y-(aX+b)\big]^2\Big\} = 0$ must minimize $E\Big\{\big[Y-(aX+b)\big]^2\Big\}$. Let us solve for these values.

$$\left.\begin{aligned}\frac{\partial}{\partial a}E\Big\{\big[Y-(aX+b)\big]^2\Big\} = 0\\ \frac{\partial}{\partial b}E\Big\{\big[Y-(aX+b)\big]^2\Big\} = 0\end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} a &= \frac{C_{XY}}{\sigma^2_X}\\ b &= m_Y - \frac{C_{XY}}{\sigma^2_X}m_X\end{aligned}\right. \quad \text{(after some work)}$$

Expanding $E\Big\{\big[Y-(aX+b)\big]^2\Big\}$, substituting for a and b, then using $\rho_{XY} = 1$, i.e., $C_{XY} = \sigma_X\sigma_Y$, we obtain that

$$E\Big\{\big[Y-(aX+b)\big]^2\Big\}\Bigg|_{\substack{a=\frac{C_{XY}}{\sigma^2_X}\\ b=m_Y-\frac{C_{XY}}{\sigma^2_X}m_X}} = 0$$

and, hence, that $Y = \dfrac{C_{XY}}{\sigma^2_X}X + m_Y - \dfrac{C_{XY}}{\sigma^2_X}$ in the mean square sense.

EE 600 SOLUTION SET # 8

Professor Edward J. Delp **April 21, 1998**

8.1

$$\left.\begin{aligned} Z &= X + Y \\ W &= \frac{X}{X+Y} \end{aligned}\right\} \Longrightarrow \begin{cases} X = ZW \\ Y = Z(1-W) \end{cases} \qquad \text{This mapping is one-to-one.}$$

$$\left|J(x,y)\right| = \frac{1}{\left|J(z,w)\right|} = \frac{1}{\begin{vmatrix} w & z \\ 1-w & -z \end{vmatrix}} = \frac{1}{|-wz - z + wz|} = \frac{1}{|z|}$$

$$\begin{aligned} \therefore f_{ZW}(z,w) &= |z|e^{-zw}e^{-z+zw}U(zw)U(z-zw) \\ &= |z|e^{-z}U(zw)U(z-zw) \end{aligned}$$

$$\left.\begin{aligned} U(zw) &\Longrightarrow f_{ZW}(\cdot) \neq 0 \text{ if } zw > 0 \\ U(z-zw) &\Longrightarrow f_{ZW}(\cdot) \neq 0 \text{ if } z > zw\,(>0) \end{aligned}\right\} \therefore \begin{aligned} z > 0 \\ 0 < w < 1 \end{aligned}$$

$$\therefore f_{ZW} = \begin{cases} ze^{-z}U(z), & 0 < w < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$\therefore f_Z(z) = ze^{-z}U(z) \qquad f_W(w) = 1,\ 0 < w < 1$$

8.2

$$Z = XY \text{ and let } W = Y. \quad \therefore \left|J(x,y)\right| = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = |y|$$

$$\therefore f_{ZW}(z,w) = \frac{f_{XY}(z/w,w)}{|w|} = \frac{1}{|w|}f_X(z/w)f_Y(w)$$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|w|}f_X(z/w)f_Y(w)\,dw = \int_{|z|}^{\infty} \frac{1}{|w|}\frac{1}{\pi}\frac{1}{\sqrt{1-(z/w)^2}}\frac{w}{\alpha^2}e^{-w^2/2\alpha^2}\,dw$$

Here the integral ranges from $|z|$ to ∞ because $f_X(z/w) \neq 0$ only for $w > |z|$.

$$\therefore f_Z(z) = \int_{|z|}^{\infty} \frac{w^{-w^2/2\alpha^2}}{\pi\alpha^2\sqrt{w^2-z^2}}\,dw.$$

$$\text{Now, let } t = \frac{w^2-z^2}{2\alpha^2} \qquad \therefore dt = \frac{w}{\alpha^2}dw.$$

$$\therefore f_Z(z) = \frac{1}{\pi\alpha\sqrt{2}}\int_0^{\infty} \frac{e^{-z^2/2\alpha^2}e^{-t}}{\sqrt{t}}\,dt = \frac{e^{-z^2/2\alpha^2}}{\pi\alpha\sqrt{2}}(\sqrt{\pi}) = \frac{1}{\sqrt{2\pi\alpha^2}}e^{-z^2/2\alpha^2}$$

This is the form of the Gaussian density function.

8.3

$$Y = \sum_{i=1}^{n} \alpha_i X_i \Longrightarrow E(Y) = E\left(\sum_{i=1}^{n} \alpha_i X_i\right) = \sum_{i=1}^{n} \alpha_i E(X_i)$$

$$\begin{aligned} E(Y^2) &= E\left[\left(\sum_{i=1}^{n} \alpha_i X_i\right)\left(\sum_{j=1}^{n} \alpha_j X_j\right)\right] = E\left[\sum\sum_{i,j=1}^{n} \alpha_i \alpha_j X_i X_j\right] \\ &= \sum_{i=1}^{n} \alpha_i^2 E(X_i^2) + \sum\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \alpha_i \alpha_j E(X_i) E(X_j) \end{aligned}$$

$$\begin{aligned} \therefore \sigma_Y^2 &= E(Y^2) - E^2(Y) \\ &= \sum_{i=1}^{n} \alpha_i^2 E(X_i^2) + \sum\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \alpha_i \alpha_j E(X_i) E(X_j) - \sum\sum_{i,j=1}^{n} \alpha_i \alpha_j E(X_i) E(X_j) \\ &= \sum_{i=1}^{n} \alpha_i^2 E(X_i^2) - \sum_{i=1}^{n} \alpha_i^2 E^2(X_i) \\ &= \sum_{i=1}^{n} \alpha_i^2 \sigma_{X_i}^2 \end{aligned}$$

8.4

$$E\big[|X-\alpha|\big] = \int_{-\infty}^{\alpha} (\alpha - x) f_X(x)\,dx + \int_{\alpha}^{\infty} (x-\alpha) f_X(x)\,dx$$

$$\frac{d}{d\alpha} E\big[|X-\alpha|\big] = 0 = 2F_X(\alpha) - 1$$

$$F_X(\alpha) = \frac{1}{2} \Longrightarrow \alpha \sim \text{ median of } X$$

Note: you *cannot* write:

$$\frac{d\,E\big[|X-\alpha|\big]}{d\alpha} = E\left[\frac{d\,|X-\alpha|}{d\alpha}\right].$$

8.5 Show $E|X-a| = E|X-m| + 2\int_a^m (x-a) f_X(x)\,dx$.

Case 1: $a < m$

$$E|X-a| = \int_{-\infty}^{a}(a-x)f_X(x)\,dx + \int_{a}^{\infty}(x-a)f_X(x)\,dx$$
$$= \int_{-\infty}^{a}(a-x)f_X(x)\,dx + \int_{a}^{m}(x-a)f_X(x)\,dx + \int_{m}^{\infty}(x-a)f_X(x)\,dx. \quad (*)$$

$$\begin{aligned} E|X-m| &= E|X-a+a-m| \\ &= \int_{-\infty}^{m} -(x-a+a-m)f_X(x)\,dx + \int_{m}^{\infty}(x-a+a-m)f_X(x)\,dx \\ &= \int_{-\infty}^{m}(a-x)f_X(x)\,dx + (m-a)F_X(m) \\ &\quad + \int_{m}^{\infty}(x-a)f_X(x)\,dx + (a-m)\big(1-F_X(m)\big) \end{aligned}$$

Since $F_X(m) = 1/2$, the second and fourth terms cancel, so

$$E|X-m| = \int_{-\infty}^{a}(a-x)f_X(x)\,dx + \int_{a}^{m}(a-x)f_X(x)\,dx + \int_{m}^{\infty}(x-a)F_X(x)\,dx \quad (**)$$

Comparing $(*)$ and $(**)$, we have the result.

The proof for Case 2: $a > m$ is similar.

8.6 We will use the independence of X and Y and the circular symmetry f_{XY} to derive a differential equation that both f_X and f_Y must satisfy. The solution of this D.E. is a Gaussian function.

$$f_{XY}(x,y) = f_X(x)f_Y(y) = g(\sqrt{x^2+y^2})$$
$$\therefore \frac{\partial}{\partial x} f_X(x)f_Y(y) = \frac{\partial}{\partial x} g(\sqrt{x^2+y^2})$$

$$\therefore f_X'(x)f_Y(y) = g'(\sqrt{x^2+y^2})\frac{1}{2}(x^2+y^2)^{-1/2}2x,\ \forall x,y \in \mathbb{R}$$
$$\text{Also } f_Y'(y)f_X(x) = g'(\sqrt{x^2+Y^2})\frac{1}{2}(x^2+y^2)^{-1/2}2y,\ \forall x,y \in \mathbb{R}$$

$$\therefore \frac{f_X'(x)f_Y(y)}{f_X(x)f_Y'(y)} = \frac{x}{y} \qquad \therefore \frac{f_X'(x)}{xf_X(x)} = \frac{f_Y'(y)}{yf_Y(y)} = c$$

where c is some constant in $\mathbb{R}$.

$$\therefore f_X'(x) = cxf_X(x)$$

The solution is

$$\begin{aligned} f_X(x) &= e^{b+\frac{c}{2}x^2} \\ &= Ae^{\frac{cx^2}{2}} \end{aligned}$$

$f_Y(y)$ obeys the same D.E. $\Longrightarrow$

$$f_Y(y) = Be^{\frac{cx^2}{2}}.$$

This is the form of a Normal density.

8.7 $\underline{Y} = \underline{A}\,\underline{X} \Longrightarrow \underline{\Lambda}_Y = \underline{A}\,\underline{\Lambda}_x \underline{A}^T$. Rearranging, we obtain

$$\underline{\Lambda}_X \underline{A}^T = \underline{A}^{-1} \underline{\Lambda}_Y$$

If $\underline{A}$ is orthogonal, then $\underline{A}^T \underline{A} = \underline{I}$ and $\underline{A}^{-1} = \underline{A}^T$.

Hence, we have $\underline{\Lambda}_X \underline{A}^T = \underline{A}^T \underline{\Lambda}_Y$. We wish $\underline{\Lambda}_y$ to be diagonal matrix (uncorrelated components), so this matrix equation can be rewritten as

$$\underline{\Lambda}_X \left[\underline{a}_1 \cdots \underline{a}_n\right] = \left[\underline{a}_1 \cdots \underline{a}_n\right] \begin{bmatrix} \sigma_{y_1}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{y_n}^2 \end{bmatrix}$$

where $\underline{a}_i$ is the i^{th} column of $\underline{A}^T$, and finally as n separate equations:

$$\underline{\Lambda}_X \underline{a}_i = \lambda_i - \underline{a}_i \quad i = 1, \ldots, n \text{ where } \lambda_i = \sigma_{y_i}^2$$

The solutions ($\underline{a}_i$'s and λ_i's) are the eigenvalues and eigenvectors of the matrix $\underline{\Lambda}_x$. Solve first for the eigenvalues:

$$\det(\underline{\Lambda}_x - \lambda_i \underline{I}) = 0 \Longrightarrow \begin{vmatrix} 4 - \lambda_i & 1 \\ 1 & 1 - \lambda_i \end{vmatrix} = \lambda_i^2 - 5\lambda_i + 3 = 0$$
$$\therefore \lambda_1 = 4.3028 \qquad \lambda_2 = 0.6972$$

For each eigenvalue, find the associated eigenvector:

$$(\Lambda_X - \lambda_i \underline{I})\underline{a}_1 = \begin{pmatrix} -0.3028 & 1 \\ 1 & -3.3028 \end{pmatrix} \underline{a}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \underline{a}_1 = \begin{pmatrix} 1 \\ 0.3028 \end{pmatrix}$$
$$\text{Similarly, we find } \underline{a}_2 = \begin{pmatrix} -0.3028 \\ 1 \end{pmatrix}.$$

We said above that $\underline{A}$ had to be orthogonal. This forces the eigenvectors to have unit length (i.e., $\|\underline{a}_i\| = \sqrt{a_{i1}^2 + a_{i2}^2} = 1$).

$$\therefore \underline{A}^T = \left[\frac{\underline{a}_1}{\|\underline{a}_1\|} \quad \frac{\underline{a}_2}{\|\underline{a}_2\|}\right] = \begin{pmatrix} 0.9571 & -0.2898 \\ 0.2898 & 0.9571 \end{pmatrix}$$
$$\therefore \underline{A} = \left(\underline{A}^T\right)^{-1} = \begin{pmatrix} 0.9571 & 0.2898 \\ -0.2898 & 0.9571 \end{pmatrix}$$

As a check the $\underline{\Lambda}_y$ is diagonal, we compute

$$
\begin{aligned}
\underline{\Lambda}_y &= \underline{A}\,\underline{\Lambda}_X\underline{A}^T \\
&= \begin{pmatrix} 0.9571 & 0.2898 \\ -0.2898 & 0.9571 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0.9571 & -0.2898 \\ 0.2898 & 0.9571 \end{pmatrix} \\
&= \begin{pmatrix} 0.9571 & 0.2898 \\ -0.2898 & 0.9571 \end{pmatrix} \begin{pmatrix} 4.1182 & -0.2021 \\ 1.2469 & 0.6673 \end{pmatrix} \\
&= \begin{pmatrix} 4.303 & 0 \\ 0 & 0.697 \end{pmatrix}
\end{aligned}
$$

8.8 For $\widehat{\sigma}_X^2$ to be unbiased, we require $E\{\widehat{\sigma}_X^2\} = \sigma_X^2$.

$$E\{\widehat{\sigma}_X^2\} = \frac{1}{N}\sum_{i=1}^{N} E\{X_i^2\} - E\left\{\widehat{\overline{X}}^2\right\} = \overline{X^2} - E\left\{\widehat{\overline{X}}^2\right\}$$

$$\widehat{\overline{X}}^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_iX_j \Longrightarrow E\left\{\widehat{\overline{X}}^2\right\} = \frac{1}{N^2}\sum_i\sum_j E\{X_iX_j\}$$

$$i = j \Longrightarrow E\{X_iX_j\} = \overline{X^2}$$

$$i \neq j \Longrightarrow E\{X_iX_j\} = \overline{X}^2$$

$$\therefore E\left\{\widehat{\overline{X}}^2\right\} = \frac{N}{N^2}\overline{X^2} + \frac{N^2-N}{N^2}\overline{X}^2 = \frac{1}{N}\overline{X^2} + \frac{N-1}{N}\overline{X}^2$$

$$\therefore E\{\widehat{\sigma}_X^2\} = \overline{X^2} - \frac{1}{N}\overline{X^2} - \frac{N-1}{N}\overline{X}^2 = \left(\frac{N-1}{N}\right)\sigma_X^2 \neq \sigma_X^2$$

$$\therefore \widehat{\sigma}_X^2 \text{ is biased.}$$

8.9 Adding η to both sides and dividing by 2, we obtain

$$\frac{E\{|x|\} + \eta}{2} = \frac{\sigma}{\sqrt{2\pi}}e^{-\eta^2/2\sigma^2} + \eta\mathbb{G}\Big(\frac{\eta}{\sigma}\Big)$$

as the relation whose validity we need to prove.

$$
\begin{aligned}
\frac{E\{|x|\} + \eta}{2} &= \frac{-\int_{-\infty}^{0} xf_X(x)\,dx + \int_0^\infty xf_X(x)\,dx + \int_{-\infty}^{\infty} xf_X(x)\,dx}{2} \\
&= \int_0^\infty xf_X(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty xe^{-(x-\eta)^2/2\sigma^2}\,dx \\
&\overset{?}{=} \frac{\sigma}{\sqrt{2\pi}}e^{\eta^2/2\sigma^2} + \eta\mathbb{G}\Big(\frac{\eta}{\sigma}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{E\{|x|\} + \eta}{2} &= \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty (x-\eta+\eta)e^{-(x-\eta)^2/2\sigma^2}\,dx \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty (x-\eta)e^{-(x-\eta)^2/2\sigma^2}\,dx + \frac{\eta}{\sigma\sqrt{2\pi}}\int_0^\infty e^{-(x-\eta)^2/2\sigma^2}\,dx
\end{aligned}
$$

Using $g = \dfrac{(x-n)}{\sigma}$ in both integrals, we obtain

$$\frac{E\{|x|\}+\eta}{2} = \frac{\sigma}{\sqrt{2\pi}}\int_{-\eta/\sigma}^{\infty} ye^{-y^2/2}\,dy + \frac{\eta}{\sqrt{2\pi}}\int_{-\eta/\sigma}^{\infty} e^{-y^2/2}\,dy$$

Now $\displaystyle\int_{-\eta/\sigma}^{\eta/\sigma} ye^{-y^2/2}dy = 0$, and $\displaystyle\int_{-\eta/\sigma}^{\infty} e^{-y^2/2}\,dy = \int_{-\infty}^{\eta/\sigma} e^{-y^2/2}\,dy$. Hence,

$$\begin{aligned}\frac{E\{|x|\}+\eta}{2} &= \frac{\sigma}{\sqrt{2\pi}}\int_{\eta/\sigma}^{\infty} ye^{-y^2/2}dy + \eta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\eta/\sigma} e^{y^2/2}\,dy \\ &= \frac{\sigma}{\sqrt{2\pi}}\int_{\eta/\sigma}^{\infty} ye^{-y^2/2}\,dy + \eta\mathbb{G}\Big(\frac{\eta}{\sigma}\Big)\end{aligned}$$

Let $z = y^2$. Then

$$\frac{\sigma}{\sqrt{2\pi}}\int_{\eta/\sigma}^{\infty} ye^{-y^2/2}\,dy = \frac{\sigma}{\sqrt{2\pi}(2)}\int_{\eta^2/\sigma^2}^{\infty} e^{-z/2}\,dz = \frac{\sigma}{\sqrt{2\pi}}e^{-\eta^2/2\sigma^2}$$

$$\therefore \frac{E\{|x|\}+\eta}{2} = \frac{\sigma}{\sqrt{2\pi}}e^{-\eta^2/2\sigma^2} + \eta\mathbb{G}\Big(\frac{\eta}{\sigma}\Big) \text{ as desired.}$$

8.10

$$\eta = \int_0^{\infty} xf_X(x)\,dx \geq \int_{\sqrt{\eta}}^{\infty} xf_X(x)\,dx \geq \sqrt{\eta}\int_{\sqrt{\eta}}^{\infty} f_X(x)\,dx = \sqrt{\eta}P(X \geq \sqrt{\eta})$$

$$\therefore P(X \geq \sqrt{\eta}) \leq \frac{\eta}{\sqrt{\eta}} = \sqrt{\eta}$$

8.11 Let $\underline{R} = \begin{pmatrix}\sigma_{11}^2 & \cdots & \sigma_{1n}^2 \\ \vdots & \ddots & \vdots \\ \sigma_{n1}^2 & \cdots & \sigma_{nn}^2\end{pmatrix}$. Then, since $\underline{R}\,\underline{R}^{-1} = \underline{I}$, the i^{th} row, i^{th} column entry of $\underline{R}\,\underline{R}^{-1}$ is 1. $\therefore \displaystyle\sum_{j=1}^{n}\sigma_{ij}^2 a_{ji} = 1$ where $\underline{R}^{-1} = \begin{pmatrix}a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$.

$$\begin{aligned}\therefore E\{\underline{X}\,\underline{R}^{-1}\underline{X}^T\} &= E\left\{\begin{pmatrix}X_1 & \cdots & X_n\end{pmatrix}\begin{pmatrix}a_n & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}\begin{pmatrix}x_1 \\ \vdots \\ x_n\end{pmatrix}\right\} \\ &= E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} X_i a_{ij} X_j\right\} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\sigma_{ji}^2 \\ &= \sum_{i=1}^{n}(1) = n\end{aligned}$$

8.12 As $n \to \infty$, $X_1 + \cdots + X_n$ approaches a Gaussian random variable with variance $\sigma_1^2 + \cdots + \sigma_n^2 \to \infty$. Hence, the density of $X_1 + \cdots + X_n$ tends to get flatter as $n \to \infty$ and, in particular, if n is large enough, we may say that it is constant in any interval of length 2π.

Now, to determine the density of $Y = \sin(X_1 + \cdots + X_n)$, it is enough to know the density of $(X_1 + \cdots + X_n) \bmod 2\pi$ by the periodicity of sin. Therefore, we compute $f_Z(z)$ where $Z = \left(\sum_{i=1}^{n} X_i\right) \bmod 2\pi$.

$$F_Z(z) = P(Z \geq z) = \sum_{i=1}^{n} \int_{i2\pi}^{i2\pi+z} f_{X_1+\cdots+X_n}(x)\,dx$$

$$\therefore f_Z(z) = \sum_{i=1}^{n} f_{X_1+\cdots+X_n}(i2\pi + z)$$

But $f_{X_1+\cdots+X_n}$ is constant in any interval of length 2π. Therefore $f_{X_1+\cdots+X_n}$ does not depend on z. $\therefore f_Z(z)$ does not depend on z. $\therefore f_Z(z) \sim U(0, 2\pi)$.

$\therefore$ The density of $\sin(z)$ is $f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}} & |y| < 1 \\ 0 & \text{elsewhere} \end{cases}$

8.13 As $a_n \to a$, $E\{|X_n - a_n|^2\} \to 0$. We want $E\{|X_n - a|^2\} \to 0$

$$E\{|X_n - a|^2\} \leq E\{|X_n - a_n|^2 + |a_n - a|^2\} = E\{|X_n - a_n|^2\} + |a_n - a|^2$$

But the right side $\to 0$. $\therefore$ The left side $\to 0$.

EE 600 SOLUTION SET # 9

Professor Edward J. Delp **April 21, 1998**

9.1 $E\{Y \mid X = x\} = \int y f_{Y|X}(y \mid X = x)\, dy$

$$
\begin{aligned}
f_{Y|X}(y \mid X = x) &\\
&= \frac{f_{XY}(x,y)}{f_X(x)}\\
&= \frac{\left[2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right]^{-1} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x-m_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right) + \left(\frac{y-m_Y}{\sigma_Y}\right)^2\right)\right]}{\left[\sqrt{2\pi}\sigma_X\right]^{-1} \exp\left[\frac{-1}{2}\left(\frac{x-m_X}{\sigma_X}\right)^2\right]}
\end{aligned}
$$

Note: Here we implicitly use the fact that $X \sim N(m_X, \sigma_X^2)$ if X and Y are jointly Gaussian. This is justified since, by the definition of jointly Gaussian random variables, any linear combination, $aX + bY$, is a Gaussian random variable. Take $a = 1$, $b = 0$.

$$
\begin{aligned}
f_{Y|X}(y \mid X = x) &\\
&= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x-m_X}{\sigma_X}\right)^2 (1-(1-\rho^2)) -2\rho\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right) + \left(\frac{y-m_Y}{\sigma_Y}\right)^2\right)\right]\\
&= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{y-m_Y}{\sigma_Y} - \rho\left(\frac{x-m_X}{\sigma_X}\right)\right)^2\right]\\
&= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)\sigma_Y^2}\left(y - m_Y - \frac{\rho\sigma_Y}{\sigma_X}(x-m_X)\right)^2\right]
\end{aligned}
$$

This is a Gaussian density itself with variance $\sigma_Y^2(1-\rho^2)$ and mean $m_Y + \frac{\rho\sigma_Y}{\sigma_X}(x-m_X)$. Hence, $E\{Y \mid X = x\}$ is the mean of this density.

$$E\{Y \mid X = x\} = m_Y + \frac{\rho\sigma_Y}{\sigma_X}(x - m_X)$$

9.2 The least-mean-square-error linear predictor for Y in terms of X is

$$\widehat{Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X^2}(X - m_X) + m_Y$$

$$m_X = E\{X\} = E\{\cos\Phi\} = \int_0^{2\pi} \cos\phi f_\Phi(\phi)\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} \cos\phi\,d\phi = 0$$
$$m_Y = 0$$

$$\mathrm{Cov}(X,Y) = E\{(X-m_X)(Y-m_Y)\} = E\{XY\} = \int_0^{2\pi} \cos\phi\sin\phi\left(\frac{1}{2\pi}\right)d\phi = 0$$

$\therefore \widehat{Y} = 0$ (Why does this make sense in this example?)

9.3 If we assume the conditional density is symmetric about its mean, then

$$f_{Y|X}(\eta + \epsilon \mid X = x) = f_{Y|X}(\eta - \epsilon \mid X = x)$$

where $\eta = E\{Y|X = x\}$.

That $g(x)$ is even and concave means that $g(x) = g(-x)$ and $|a| > |b| \Longrightarrow g(a) > g(b)$. We wish to minimize $E\left\{g\big(Y - \phi(x)\big)\right\}$ by choosing the proper $\phi(\cdot)$. Rewrite the expected value:

$$\begin{aligned} E\{g\big(Y - \phi(X)\big)\} &= \iint g\big(y - \phi(x)\big) f_{XY}\,dx\,dy \\ &= \int \left[\int g\big(y - \phi(x)\big) f_{Y|X}(y \mid X = x)\,dy\right] f_X(x)\,dx. \end{aligned}$$

Note that, if we assume $g(\cdot)$ is nonnegative, then the integrand of the x integral is nonnegative, $\forall x$. Hence if we minimize the inner integral for each x, then we minimize the entire expression.

To minimize $\int_{-\infty}^{\infty} g\big(y - \phi(x)\big) f_{Y|X}(g \mid X = x)\,dy$, we take the partial derivative with respect to ϕ and set it to 0:

$$\frac{\partial}{\partial\phi}\int_{-\infty}^{\infty} g\big(y - \phi(x)\big) f_{Y|X}(y \mid X = x)dy = -\int_{\infty}^{\infty} g'\big(y - \phi(x)\big) f_{Y|X}(y \mid X = x)\,dy = 0$$

Ignoring the negative sign

$$\begin{aligned} &\int_{\infty}^{\infty} g'\big(y - \phi(x)\big) f_{Y|X}(y \mid X = x)\,dy \\ &\quad = \int_{-\infty}^{\phi(x)} g'(y - \phi(x)) f_{Y|X}(y \mid X = x)\,dy + \int_{\phi(x)}^{\infty} g'(y - \phi(x)) f_{Y|X}(y \mid X = x)\,dy \\ &\quad = 0 \end{aligned}$$

Let $y' = 2\phi(x) - y$ in the second integral (treating $\phi(x)$ like the constant it is). The purpose is to get both integrals to have the same limits.

$$\begin{aligned} y = \phi(x) &\Longrightarrow y' = \phi(x) \\ y = +\infty &\Longrightarrow y' = -\infty \end{aligned} \qquad dy = -dy'$$

$$\begin{aligned}
\int_{\infty}^{\infty} & g'\big(y-\phi(x)\big) f_{Y|X}(y \mid X = x)\, dy \\
&= \int_{-\infty}^{\phi(x)} g'(y-\phi(x)) f_{Y|X}(y \mid Y = x)\, dy \\
&\quad - \int_{\phi(x)}^{-\infty} g'(\phi(x)-y') f_{Y|X}(2\phi(x)-y' \mid X = x)\, dy' \\
&= \int_{-\infty}^{\phi(x)} g'(y-\phi(x)) f_{Y|X}(y \mid X = x)\, dy \\
&\quad + \int_{-\infty}^{\phi(x)} g'(\phi(x)-y) f_{Y|X}(2\phi(x)-y \mid X = x)\, dy \\
&= 0
\end{aligned}$$

Now, since $g(a) = g(-a)$, $g'(a) = -g'(-a)$. Using this fact in the second integral and combining the two:

$$\int_{-\infty}^{\infty} g'\big(y-\phi(x)\big)\Big[f_{Y|X}(y \mid X = x) - f_{Y|X}\big(2\phi(x)-y \mid X = x\big)\Big]\, dy = 0$$

If we can make the quantity in brackets 0, $\forall x$, then we have it. Use the symmetry of $f_{Y|X}$ about the mean.

$$f_{Y|X}\big(y \mid X = x\big) = f_{Y|X}\big(2\phi(x)-y \mid X = x\big) \quad \text{if} \begin{pmatrix} y = \eta + \epsilon \\ 2\phi(x) - y = \eta - \epsilon \end{pmatrix}$$

Solving the equations simultaneously, we obtain

$$\phi(x) = \eta = E\{Y \mid X = x\}$$

Note: We have only shown that $\phi(x) = E\{Y \mid X = x\}$ *extremizes* the mean cost; to finish, use the concavity of $g(\cdot)$ to show that it is actually a minimum, i.e., $\left(\frac{\partial^2}{\partial\phi^2} E\left\{g\big(Y-\phi(x)\big)\right\} > 0\right)$.

9.4 Let $g(x)$ be a general nonlinear estimate and $\phi(x)$ be the *optimum* nonlinear mean-square estimate $\big(\phi(x) = E\{Y \mid X = x\}\big)$. Then

$$E\Big\{\big(Y - g(x)\big)^2\Big\} = E\big\{Y^2\big\} - 2E\big\{Yg(x)\big\} + E\big\{g^2(x)\big\}.$$

Look at the second term.

$$E\{Y(g(x)\} = \iint y g(x) f_{XY}(x,y)\,dx\,dy = \int \left[\int y f_{Y|X}(y \mid X = x)\,dy\right] g(x) f_X(x)\,dx$$
$$= \int E\{Y \mid X = x\} g(x) f_X(x)\,dx = E\{\phi(x) g(x)\}$$

$\therefore$ For a general estimator, $\widehat{Y} = g(x)$

$$E\left\{\left(Y - g(x)\right)^2\right\} = E\{Y^2\} - 2E\{\phi(x)g(x)\} + E\{g^2(x)\}.$$

For the optimal mean-square estimator, $\widehat{Y}_{\text{opt}} = E\{Y \mid X\} = \phi(x)$

$$E\left\{\left(Y - \phi(x)\right)^2\right\} = E\{Y^2\} - 2E\{\phi^2(x)\} + E\{\phi^2(x)\}$$
$$= E\{Y^2\} - E\{\phi^2(x)\}$$

9.5 Assume the observations are independent of each other. Use the estimator

$$\widehat{m} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

where

X = random variable whose mean we are estimating
X_i = i^{th} observation of X
$\widehat{m}$ = estimator

Recall (or derive): $\sigma^2_{\widehat{m}} = \dfrac{\sigma^2_X}{N}$ (variance of estimator)

We want $\sigma^2_{\widehat{m}} \leq 0.5 m_X$. $\quad \therefore \dfrac{1}{N}\sigma^2_X \leq 0.5 m_X \quad \therefore N \geq \dfrac{2\sigma^2_X}{m_X}$

For $X \sim \text{Uni}(-5,7)$, $m_X = 1$, $\sigma^2_X = 12$. $\quad \therefore N \geq 24$

9.6 An estimator is *efficient* if the variance of the estimator is equal to the Cramer-Rao lower bound.

$$\sigma^2_{\widehat{\mu}} \leq \frac{1}{E\left\{\left(\dfrac{\partial \log f_{\underline{X}}(\underline{x})}{\partial \mu}\right)^2\right\}} \quad \text{where } \underline{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix}$$

The X_is are independent, identically distributed $\Longrightarrow$

$$f_{\underline{X}}(\underline{x}) = f_{X_1,\ldots,X_N}(x_1,\ldots,x_N)$$
$$= f_{X_1}(x_1)\cdots f_{X_N}(x_N)$$
$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-1}{2\sigma^2}(x_i - \mu)^2\right]$$

$$\therefore \log f_{\underline{X}}(\underline{x}) = \log \prod_{i=1}^{N} f_{X_i}(x_i) = \sum_{i=1}^{N} \log f_{X_i}(x_i)$$
$$= \sum_{i=1}^{N} \log \left(\sqrt{2\pi}\sigma\right)^{-1} + \sum_{i=1}^{N} \log \left[exp\left(\frac{-1}{2\sigma^2}(x_i-\mu)^2\right)\right]$$
$$= N \log \left(\sqrt{2\pi}\sigma\right)^{-1} - \sum_{i=1}^{N} \frac{(x_i-\mu)^2}{2\sigma^2}$$

$$\therefore \frac{\partial \log f_{\underline{X}}(\underline{x})}{\partial \mu} = -\sum_{i=1}^{N} \frac{1}{2\sigma^2}\frac{\partial}{\partial\mu}(x_i-\mu)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\mu)$$

$$\therefore E\left\{\left(\frac{\partial \log f_{\underline{X}}(\underline{x})}{\partial \mu}\right)^2\right\} = E\left\{\frac{1}{\sigma^4}\left(\sum_{i=1}^{N}(x_i-\mu)\right)^2\right\} = \frac{1}{\sigma^4}N\sigma^2 = \frac{N}{\sigma^2}$$

$$\therefore \frac{1}{E\left\{\left(\frac{\partial \log f_{\underline{X}}(\underline{x})}{\partial \mu}\right)^2\right\}} = \frac{\sigma^2}{N}$$

But $\mathrm{var}(\widehat{\mu}) = \dfrac{\sigma^2}{N}$ also.

$$\therefore \mathrm{var}(\widehat{\mu}) = \frac{1}{E\left\{\left(\frac{\partial \log f_{\underline{X}}(\underline{x})}{\partial \mu}\right)^2\right\}}$$

9.7 Define the sequence of random variables $Y_i = X_i - X$. Since X_i converges in mean square to X, Y_i converges in mean square to zero.

Use the generalized Chebyshev Inequality with $\alpha = 0$, $n = 2$:

$$P\left\{|Y_i|^2 \geq \epsilon\right\} \leq \frac{E\left\{|Y_i|^2\right\}}{\epsilon^2} \qquad (*)$$

Since Y_i converges in mean square to zero, the right-hand side of $(*)$ goes to zero as $i \to \infty$. Thus the left-hand side must also go to zero.

$$\Longrightarrow P\left\{|X_i - X|^2 \geq \epsilon\right\} \to 0 \text{ as } i \to \infty$$
$$\Longrightarrow P\left\{|X_i - X| \geq \sqrt{\epsilon}\right\} \to 0$$

so X_i converges to X in probability.

9.8

$$\eta(t) = E\{X(t)\} = \int e^{at} f_A(a)\,da$$

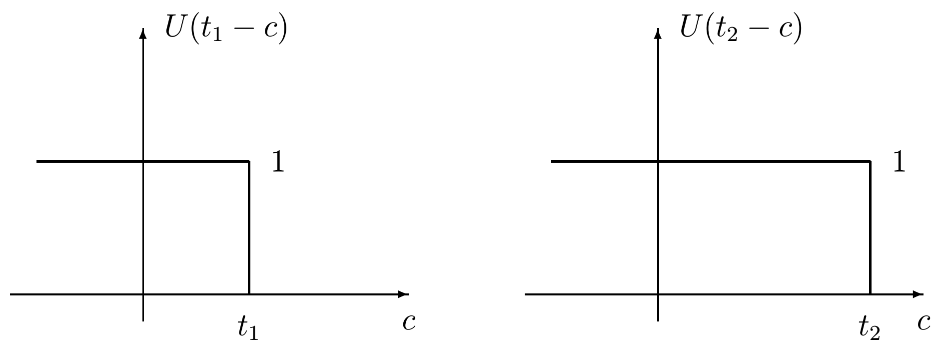
$$R(t_1,t_2) = E\left\{e^{At_1}e^{At_2}\right\} = E\left\{e^{A(t_1+t_2)}\right\} = \int e^{a(t_1+t_2)} f_A(a)\,da$$

$X(t)$ is a function of the random variable A for each value of t. Use the transformation technique with $f(x,t) = \dfrac{f_A(a)}{|g'(a)|}$ where $g(a) = e^{at}$. Hence, $g'(a) = te^{at} = tX(t)$.

$$\therefore f(x,t) = \frac{f_a\left(\frac{\ln X(t)}{t}\right)}{|tX(t)|} = \frac{f_a\left(\frac{\ln X(t)}{t}\right)}{|t|X(t)}$$

9.9

$$\begin{aligned} R_X(t_1,t_2) &= E\big\{X(t_1)X(t_2)\big\} = E\big\{U(t_1-c)U(t_2-c)\big\} \\ &= \int_0^T U(t_1-c)U(t_2-c)f_C(c)\,dc \\ &= \int_0^T U(t_1-c)U(t_2-c)\frac{1}{T}\,dc \end{aligned}$$



Case 1: $t_1 < 0$, $t_2 < 0$

$$\int_0^T U(t_1-c)U(t_2-c)(\frac{1}{T})\,dc = 0$$

Case 2: $0 \le \min(t_1,t_2) < T$

$$\int_0^T U(t_1-c)U(t_2-c)\frac{1}{T}\,dc = \int_0^{\min(t_1,t_2)} \frac{1}{T}\,dc = \frac{\min(t_1,t_2)}{T}$$

Case 3: $\min(t_1,t_2) \ge T$

$$\int_0^T U(t_1-c)U(t_2-c)\frac{1}{T}\,dc = 1$$

$$\therefore R_X(t_1,t_2) = \begin{cases} 0, & \min(t_1,t_2) < 0 \\ \dfrac{\min(t_1,t_2)}{T}, & 0 \le \min(t_1,t_2) \le T \\ 1, & \min(t_1,t_2) \ge T \end{cases}$$

9.10 **a)** continuous parameter

b) $E\{X(t)\} = E\{A \sin \omega_0 t\} = E\{A\} \sin \omega_0 t = 0, \forall t$

c) $X\left(\frac{\pi}{2\omega_0}\right) = A \sin\left(\omega_0 \frac{\pi}{2\omega_0}\right) = A \sin \frac{\pi}{2} = A$

$\therefore X\left(\frac{\pi}{2\omega_0}\right) \sim \text{uni}(-1, 1)$

9.11 **a)**

$$\begin{aligned} \text{var}(X(t)) &= E\left\{\left[X(t) - \eta_X(t)\right]^2\right\} = E\{X^2(t)\} \text{ (zero-mean)} \\ &= R_X(0) = 68 \end{aligned}$$

b)

$$Y(t) = X(t) + AX(t-2)$$

$$\begin{aligned} \therefore E\{Y^2(t)\} &= E\{[X(t) + AX(t-2)]^2\} \\ &= E(X^2(t)) + 2AE(X(t)X(t-2)) + A^2E(X^2(t-2)) \\ &= R_X(0) + 2AR_X(2) + A^2R_X(0) \\ &= (1 + A^2)R_X(0) + 2AR_X(2) = (a + A^2)68 + (2A)68 \\ &= 68(1 + 2A + A^2) = 68(a + A)^2 \geq 0 \end{aligned}$$

$$\therefore A = -1 \Longrightarrow E\{Y^2(t)\} = 0(!)$$

Note: If $R_X(\tau)$ is periodic, it can be shown that almost each sample function of $X(t)$ is periodic with the same period. If that period is T, then

$$X(t) = X(t + kT)$$

in the mean square sense, $\forall$ integers k.

Hence, since $2 = kT$ for our case, the choice of $A = -1$ gives

$$Y(t) = X(t) - X(t-2) = X(t) - X(t - kT) = 0 \text{ m.s.}$$

This is why $E\{Y^2(t)\} = 0$ for $A = -1$.

9.12

$$\begin{aligned} Z(t) &= \left[2X(t) - X(t-1)\right]\left[2Y(t)\right] \\ &= 4X(t)Y(t) - 2X(t-1)Y(t) \end{aligned}$$

$$\begin{aligned} \therefore E\{Z(t)\} &= E\{YX(t) - 2X(t-1)Y(t)\} \\ &= 4R_{XY}(0) - 2R_{XY}(-1) \\ &= 4(10) - 2(0) = 40 \end{aligned}$$

$$\begin{aligned} \text{Note: } R_{XY}(\tau) &= E\{X(t+\tau)Y(t)\} \\ &\neq E\{X(t)Y(t+\tau)\} \end{aligned}$$

EE 600 SOLUTION SET # 10

Professor Edward J. Delp **April 21, 1998**

10.1 Given $\Phi(1) = \Phi(2) = 0$, we ask what this implies about Ψ.

$$\Phi(1) = E\{e^{i\Psi}\} = E\{\cos\Psi + i\sin\Psi\} = E\{\cos\Psi\} + iE\{\sin\Psi\}$$

If a complex number equals 0, then both its real and imaginary parts do.

$$\therefore E\{\cos\Psi\} = 0 \qquad E\{\sin\Psi\} = 0$$

Similarly,

$$\Phi(2) = 0 \Longrightarrow E\{\cos 2\Psi\} = E\{\sin 2\Psi\} = 0$$

To show $X(t) = \cos(\omega t + \Psi)$ is wide-sense stationary, we must show that $E\{X(t)\}$ is a constant and $E\{X(t_1)X(t_2)\}$ is a function of $|t_1 - t_2|$ only.

$$\begin{aligned} E\{X(t)\} &= E\{\cos(\omega t + \Psi)\} \\ &= E\{\cos\omega t\cos\Psi - \sin\omega t\sin\Psi\} \\ &= \cos\omega t E\{\cos\Psi\} - \sin\omega t E\{\sin\Psi\} = 0 \end{aligned}$$

$$\begin{aligned} R_X(t_1,t_2) &= E\{X(t_1)X(t_2)\} = E\{\cos(\omega t_1 + \Psi)\cos(\omega t_2 + \Psi)\} \\ &= E\left\{\frac{1}{2}\cos\big[\omega(t_1+t_2) + 2\Psi\big] + \frac{1}{2}\cos\big(\omega(t_1 - t_2)\big)\right\} \\ &= \frac{1}{2}E\{\cos\omega(t_1+t_2)\cos 2\Psi\} - \frac{1}{2}E\{\sin\omega(t_1+t_2)\sin 2\Psi\} \\ &\quad + \frac{1}{2}\cos(\omega|t_1 - t_2|) \\ &= \frac{1}{2}\cos(\omega|t_1 - t_2|) \end{aligned}$$

$$\therefore \quad \left.\begin{array}{c} \Phi(1) = \Phi(2) = 0 \\ X(t) = \cos(\omega t + \Psi) \end{array}\right\} \Longrightarrow X(t) \text{ wide-sense stationary}$$

If $\Psi \sim \text{uni}[-\pi,\pi]$, then $\Phi(1) = \Phi(2) = 0$.

$$\therefore E\{X(t)\} = 0 \qquad R_X(\tau) = \frac{1}{2}\cos(\omega\tau).$$

10.2 Recall

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X = x) f_X(x)\, dx. \tag{$*$}$$

Let $A = \{Y(t) \leq y\}$ and $X = \underline{\epsilon}$. The formula becomes

$$P\big(Y(t) \leq y\big) = F_Y(y,t) = \int_{-\infty}^{\infty} F_Y(y,t \mid \underline{\epsilon} = \epsilon) f_{\underline{\epsilon}}(\epsilon)\, d\epsilon = E\big\{F_Y(y,t \mid \underline{\epsilon})\big\}$$

$$\therefore F_Y(y,t) = E\big\{F_Y(y,\epsilon \mid \underline{\epsilon}\big\}$$

On the left, we have $F_Y(y,t)$, which is a *number* depending on y and t. On the right, we have the expectation of $F_Y(y,t \mid \underline{\epsilon})$ which is a function of the random variable $\underline{\epsilon}$, and hence is random itself. Let us rewrite $E\big\{F_Y(y,t \mid \underline{\epsilon})\big\}$.

$$F_Y(y,t \mid \underline{\epsilon} = \epsilon) = P\big(Y(t) \leq y \mid \underline{\epsilon} = \epsilon\big) = P\big(X(t - \underline{\epsilon}) \leq y \mid \underline{\epsilon} = \epsilon\big)$$

Since we are given that $\underline{\epsilon} = \epsilon$, we may replace $\underline{\epsilon}$ by ϵ (ϵ is a number, like 10 or 84).

$$F_Y(y,t \mid \underline{\epsilon} = \epsilon) = P(X(t-\epsilon) \leq y | \underline{\epsilon} = \epsilon) = P(X(t-\epsilon) \leq y) = F_X(y)$$

We may remove the condition $\underline{\epsilon} = \epsilon$ because $X(t-\epsilon)$ is independent of $\underline{\epsilon}$ and we may write $Pr(x(t-\epsilon) \leq y) = F_X(y)$ because $X(t)$ is strict-sense stationary. Plug this into our expression for $F_Y(y,t)$.

$$F_Y(y,t) = \int_{-\infty}^{\infty} F_X(y) f_{\underline{\epsilon}}(\epsilon)\, d\epsilon = F_X(y) \int_{-\infty}^{\infty} f_{\underline{\epsilon}}(\epsilon)\, d\epsilon = F_X(y)$$

Hence, the 1$^{\text{st}}$ order density of the random process $Y(t)$ does not depend on t. To summarize what we've done so far, we used the formula $(*)$ to show that $Y(t)$ is first order sationary; i.e., that $F_y(y,t)$ does not depend on t. The problem asks you to show that $Y(t)$ is strict-sense stationary. This requires showing the the n^{th} order distribution $F_Y(y_1, y_2, \ldots, y_n; t_1, t_2, \ldots, t_n)$ is unchanged by the addition of a constant to all the t_i. You would do this by defining the event A in the formula as $A = \big\{Y(t_1) \leq y_1, \ldots, Y(t_n) \leq y_n\big\}$ and proceeding as we just did.

10.3 Because a differentiator is a linear system whose output is the derivative of the input, it can be shown that

$$\eta_{X'}(t) = \eta'_X(t) \qquad R_{XX'}(\tau) = -R'_{XX}(\tau)$$

$\therefore$ Since $\eta_X(t) = \eta$ (no time-dependence), $\eta_{X'}(t) = \eta'_X(t) = 0$. Now, since $R_{XX'}(\tau) = -R'_{XX}(\tau)$, $R_{XX'}(0) = 0$.

$$\left.\begin{aligned} \therefore E\big\{X(t)X'(t)\big\} &= R_{XX'}(0) = 0 \\ \text{But } E\big\{X'(t)\big\} &= \eta'_X(t) = 0 \end{aligned}\right\} \therefore E\big\{X(t)X'(t)\big\} = E\big\{X(t)\big\}E\big\{X'(t)\big\} = 0$$

$\therefore X(t)$ are uncorrelated and orthogonal.

10.4

$$
\begin{aligned}
E\{N_T\} &= E\left\{\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right\} = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt \\
&= \frac{1}{2T}\int_{-T}^{T}\big(10 + E\{V(t)\}\big)\;dt = \frac{1}{2T}10(2T) = 10
\end{aligned}
$$

$$
\therefore \sigma^2_{N_T} = E\{N_T^2\} - E^2\{N_T\} = E\{N_T^2\} - 100
$$

$$
\begin{aligned}
E\{N_T^2\} &= E\left\{\left(\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right)\left(\frac{1}{2T}\int_{-T}^{T} X(s)\,ds\right)\right\} \\
&= \frac{1}{4T^2}\iint_{-T}^{T} E\{X(t)X(s)\}\,dt\,ds \\
&= \frac{1}{4T^2}\iint_{-T}^{T} E\{(10+V(t))(10+V(s))\}\,dt\,ds \\
&= \frac{1}{4T^2}\iint_{-T}^{T} \big(100 + R_v(t-s)\big)\,dt\,ds \\
&= 100 + \frac{1}{4T^2}\iint_{-T}^{T} 2\delta(t-s)\,dt\,ds \\
&= 100 + \frac{1}{4T^2}\int_{-T}^{T} 2\,ds = 100 + \frac{1}{T}
\end{aligned}
$$

$$
\therefore \sigma^2_{N_T} = 100 + \frac{1}{T} - 100 = \frac{1}{T}.
$$

10.5

$$
\begin{aligned}
&E\left\{\big(X(t+\tau)+Y(t)\big)^2\right\} \ge 0 \\
&\therefore E\{X^2(t+\tau) + 2X(t+\tau)Y(t) + Y^2(t)\} \ge 0 \\
&\therefore \frac{1}{2}\big[R_X(0)+R_Y(0)\big] \ge -R_{XY}(\tau)
\end{aligned}
$$

$$
\frac{1}{2}[R_X(0)+R_Y(0)] \ge \big|R_{XY}(\tau)\big|
$$

$$
\begin{aligned}
&E\{(X(t+\tau)-Y(t))^2\} \ge 0 \\
&\therefore E\{X^2(t+\tau) - 2X(t+\tau)Y(t) + Y^2(t)\} \ge 0 \\
&\therefore \frac{1}{2}[R_X(0)+R_Y(0)] \ge R_{XY}(\tau)
\end{aligned}
$$

10.6 Since $S = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} X(kT)$

$$\begin{aligned}
\therefore E(S^2) &= E\left[\left(\frac{1}{n}\sum_{k=1}^{n} X(kT)\right)\left(\frac{1}{n}\sum_{j=1}^{n} X(jT)\right)\right] \\
&= \frac{1}{n^2} E\left\{\sum\sum_{i,j=1}^{n} X(kT)X(jT)\right\} \\
&= \frac{1}{n^2}\sum\sum_{i,j=1}^{n} R_X\big((k-j)T\big)
\end{aligned}$$

Now, $R_X\big((k-j)T\big) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_X(\omega) e^{i\omega(k-j)T}\, d\omega$

$$\begin{aligned}
\therefore E(S^2) &= \frac{1}{n^2}\sum\sum_{i,j=1}^{n}\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega) e^{i\omega(k-j)T}\, d\omega \\
&= \frac{1}{2\pi n^2}\int_{-\infty}^{\infty} S_X(\omega)\left(\sum\sum_{i,j=1}^{n} e^{i\omega(k-j)T}\right) d\omega
\end{aligned}$$

$$\begin{aligned}
\sum\sum_{i,j=1}^{n} e^{i\omega(k-j)T} &= \sum\sum_{i,j=1}^{n} e^{i\omega kT} e^{-i\omega jT} = \left(\sum_{k=1}^{n} e^{i\omega kT}\right)\left(\sum_{j=1}^{n} e^{-i\omega jT}\right) \\
&= \left(e^{i\omega T}\sum_{k=0}^{n-1} e^{i\omega kT}\right)\left(e^{i\omega T}\sum_{j=0}^{n-1} e^{i\omega jT}\right) \\
&= \left(\frac{1-e^{i\omega nT}}{1-e^{i\omega T}}\right)\left(\frac{1-e^{-i\omega nT}}{1-e^{-i\omega T}}\right)\cancelto{1}{\left(e^{i\omega T} e^{-i\omega T}\right)} \\
&= \frac{e^{i\omega nT/2}}{e^{i\omega T/2}}\frac{\left(e^{-i\omega nT/2}-e^{i\omega nT/2}\right)}{\left(e^{-i\omega T/2}-e^{i\omega T/2}\right)}\frac{e^{-i\omega nT/2}}{e^{-i\omega T/2}}\frac{\left(e^{i\omega nT/2}-e^{-i\omega nT/2}\right)}{\left(e^{i\omega T/2}-e^{-i\omega T/2}\right)} \\
&= \cancelto{1}{\left(\frac{e^{i\omega nT/2}e^{-i\omega nT/2}}{e^{i\omega T/2}e^{-i\omega T/2}}\right)}\frac{(-2i\sin\omega nT/2)}{(-2i\sin\omega T/2)}\frac{(2i\sin\omega nT/2)}{(2i\sin\omega T/2)} \\
&= \frac{\sin^2(\omega nT/2)}{\sin^2(\omega T/2)}
\end{aligned}$$

$$\therefore E(S^2) = \frac{1}{2\pi n^2}\int_{-\infty}^{\infty} S_X(\omega)\frac{sin^2(\omega nT/2)}{sin^2(\omega T/2)}\, d\omega$$

10.7 $X(t)$ ergodic and not periodic. $\quad \therefore E^2\{X(t)\} = 100$

$$\therefore \big|E\{X(t)\}\big| = 10$$

$\widehat{m} = \frac{1}{100} \sum_{i=1}^{100} X(t_i)$; $X(t_i)$ independent of each other.

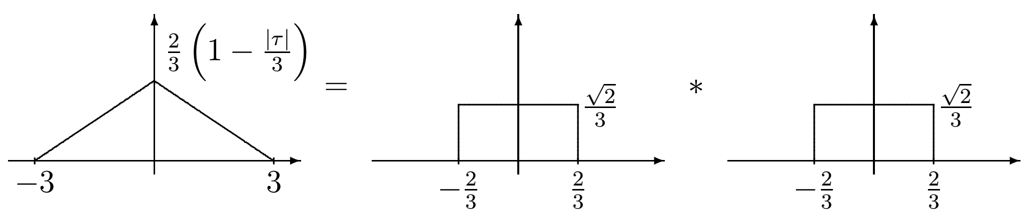
$$\therefore \sigma^2_{\widehat{m}} = E\big\{(\overline{m} - \overline{\widehat{m}})^2\big\} = E\big\{(\widehat{m} - m)^2\big\} = \frac{\sigma^2_X}{100}$$

$$\sigma^2_X = R_X(0) - m^2_X = 5 \qquad \therefore \sigma^2_{\widehat{m}} = \frac{5}{100} = 0.05$$

$$\therefore \frac{\sqrt{\sigma^2_{\widehat{m}}}}{|m|} = \frac{\sqrt{.05}}{10} = 0.0224 \Longrightarrow \%\text{ error } = 2.24\%$$

10.8

$$S_X(\omega) = \mathcal{F}\big\{R_X(\tau)\big\} = 8\pi\delta(\omega) + \mathcal{F}\left\{\frac{2}{3}\left(1 - \frac{|\tau|}{3}\right)\big[U(\tau+3) - U(\tau-3)\big]\right\}$$



$$\therefore \mathcal{F}\left\{\triangle\right\} = \left[\mathcal{F}\left\{\sqcap\right\}\right]^2 = \left(\frac{2\sin\left(\frac{3}{2}\omega\right)}{\omega}\right)^2 = \frac{4}{\omega^2}\sin^2\left(\frac{3}{2}\omega\right)$$

$$\therefore S_X(\omega) = 8\pi\delta(\omega) + \frac{4\sin^2\left(\frac{3}{2}\omega\right)}{\omega^2}$$

10.9 $R_X(\tau) = \delta(\tau)$.

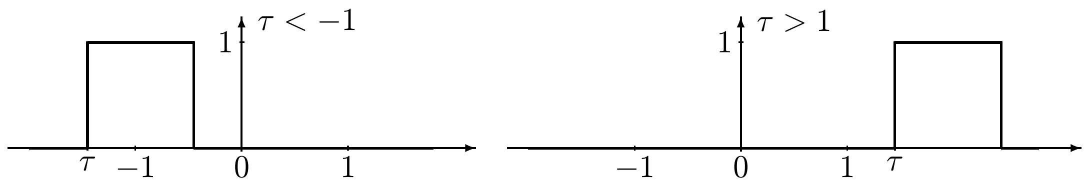
a)

$$\begin{aligned} R_{XY}(t_1, t_2) &= E\big\{X(t_1)Y(t_2)\big\} = E\{X(t_1)\int_0^1 h(s)X(t_2 - s)\,ds\} \\ &= \int_0^1 h(s)R_X(t_1 - t_2 + s)\,ds = \int_0^1 h(s)\delta(t_1 - t_2 + s)\,ds \\ &= h(t_2 - t_1) = h(-\tau) \qquad (\tau = t_1 - t_2) \end{aligned}$$

b)

$$
\begin{aligned}
R_{YY}(t_1,t_2) &= E\big\{Y(t_1)Y(t_2)\big\}\\
&= E\left\{\left(\int_0^1 h(s)X(t_1-s)\,ds\}\right)\left(\int_0^1 h(t)X(t_2-t)\,dt\}\right)\right\}\\
&= \iint_0^1 h(t)h(s)R_X(t_1-s-t_2+t)\,ds\,dt\\
&= \iint_0^1 h(t)h(s)\delta(t_1-s-t_2+t)\,ds\,dt\\
&= \int_0^1 h(t)h(t_1-t_2+t)\,dt = \int_0^1 h(t_1-t_2+t)\,dt = \int_0^1 h(\tau+t)\,dt
\end{aligned}
$$

Suppose $\tau < -1$ or $\tau > 1$. Then $h(\tau + t)$ looks like:

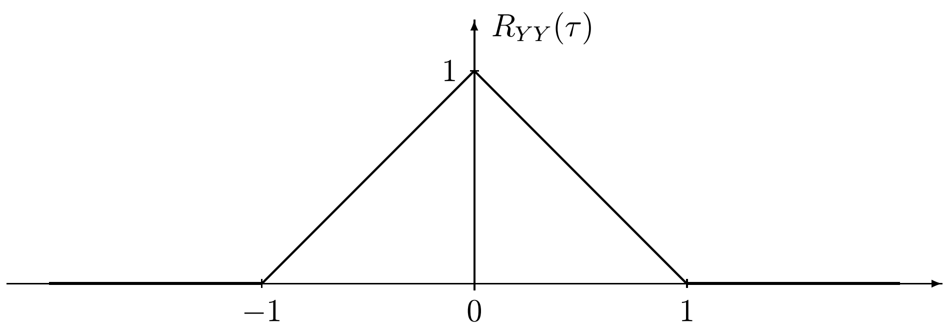


$$\therefore \int_0^1 h(\tau+t)\,dt = 0 \qquad |\tau| > 1$$

$$\text{Suppose } -1 \le \tau \le 0. \text{ Then } \int_0^1 h(\tau+t)\,dt = \int_0^\tau (1)\,dt = \tau + 1$$

$$\text{Suppose } 0 \le \tau \le 1. \text{ Then } \int_0^1 h(\tau+t)\,dt = \int_\tau^1 (1)\,dt = 1 - \tau$$

$$\therefore R_{YY}(\tau) = 1 - |\tau|, \qquad \tau| \le 1$$

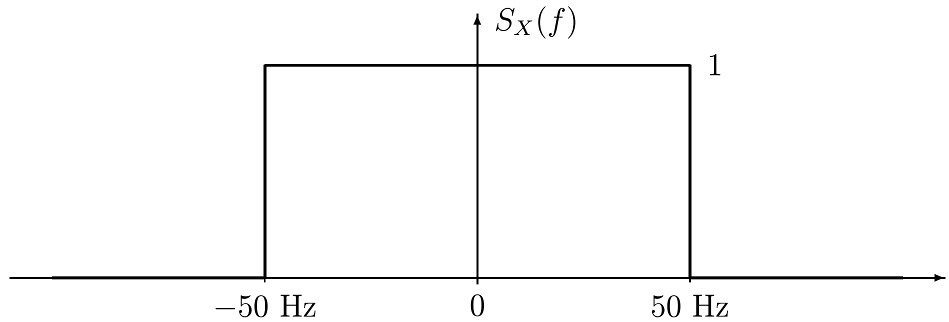


EE 600 SOLUTION SET # 11

Professor Edward J. Delp **April 21, 1998**

11.1 a)

$$\overline{X^2} = 100 = 100(1)$$



$$R_X(\tau) = 100\,\mathrm{sinc}\,100\tau$$
$$R_Y(k) = R_X(kT_s) = 100\,\mathrm{sinc}(k)$$

$$R_Y(k) = \begin{cases} 100, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

11.2 Since $Y(t) = \alpha X(\beta t)$,

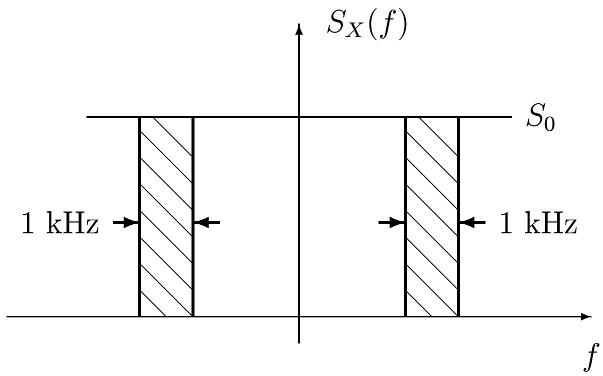
a)

$$R_Y(\tau) = E\big[Y_t Y_{t+\tau}\big] = \alpha^2 E\left[X(\beta t)X\big(\beta(t+\tau)\big)\right] = \alpha^2 R_X(\beta\tau)$$

b)

$$S_Y(\omega) = \int_{-\infty}^{\infty} R_Y(\tau)e^{-i\omega\tau}\,d\tau = \frac{\alpha^2}{|\beta|}\int_{-\infty}^{\infty} R_X(\mu)e^{-\frac{i\omega}{\beta}\mu}\,d\omega = \frac{\alpha^2}{|\beta|}S_X\left(\frac{\omega}{\beta}\right)$$

11.3 Since the input is white noise, we know $R_X(\tau) = S_0\delta(\tau)$, thus $2 \times 10^3 S_0 = 100$, which gives $S_0 = 50 \times 10^{-3}$.



In general, $R_{XY}(\tau) = R_X(\tau) * h(\tau)$, and for a white noise input $R_{XY}(\tau) = S_0 h(\tau)$. Substituting, we have $25e^{-6\tau} = 50 \times 10^{-3} h(\tau) \Longrightarrow h(t) = 500e^{-6t}u(t)$. Thus

$$H(\omega) = \frac{500}{6 + j\omega} \Longrightarrow |H(\omega)|^2 = \frac{(500)^2}{36 + \omega^2}$$

$$\therefore S_Y(\omega) = S_X(\omega)|H(\omega)|^2 = 50 \times 10^{-3} \frac{(500)^2}{36 + \omega^2} = \frac{12.5 \times 10^3}{36 + \omega^2}$$

$$\therefore R_Y(\tau) = \frac{12.5}{12} \times 10^3 e^{-6|\tau|}$$

11.4 Let's look at the more general problem: let $X_1(t)$ and $X_2(t)$ be jointly wide-sense stationary.

$$X_1(t) \longrightarrow \boxed{h_1(t)} \longrightarrow Y_1(t)$$

$$X_2(t) \longrightarrow \boxed{h_2(t)} \longrightarrow Y_2(t)$$

Let's find an expression for the cross-correlation $R_{Y_1Y_2}(t)$ and the cross spectral density $S_{Y_1Y_2}(\omega)$.

$$Y_1(t) = \int_{-\infty}^{\infty} h_1(\alpha) X_1(t - \alpha)\, d\alpha \qquad Y_2(t) = \int_{-\infty}^{\infty} h_2(\beta) X_2(t - \beta)\, d\beta$$

$$\begin{aligned} R_{Y_1Y_2}(\tau) &= E\big[Y_1(t)Y_2(t+\tau)\big] = E\left[Y_1(t) \int_{-\infty}^{\infty} h_2(\beta) X_2(t + \tau - \beta)\, d\beta\right] \\ &= \int_{-\infty}^{\infty} R_{Y_1X_2}(\tau - \beta) h_2(\beta)\, d\beta \end{aligned}$$

$$\therefore S_{Y_1Y_2}(\omega) = S_{Y_1X_2}(\omega) H_2(\omega)$$

$$\begin{aligned} R_{Y_1X_2}(\tau) &= E\big[Y_1(t)X_2(t+\tau)\big] = E\left[\left(\int_{-\infty}^{\infty} X_1(t - \alpha) h_1(\alpha)\, d\alpha\right) X_2(t+\tau)\right] \\ &= \int_{-\infty}^{\infty} R_{X_1X_2}(\tau + \alpha) h(\alpha)\, d\alpha \end{aligned}$$

$$\therefore S_{Y_1X_2}(\omega) = S_{X_1X_2}(\omega) H_1^*(\omega) = H_1^*(\omega) H_2(\omega) S_{X_1X_2}(\omega)$$

Now let us apply this general equation to the problem, and we now have:

$$S_{Y_1Y_2}(\omega) = H_1^*(\omega)H_2(\omega)S_X(\omega)$$

if $H_1^*(\omega)H_2(\omega) = 0$, $\forall\omega$, then $S_{Y_1Y_2}(\omega) = 0$ and if $E\big[X_1(t)\big] = 0$, and/or $E\big[X_2(t)\big] = 0$, then $Y_1(t)$ and $Y_2(t)$ are uncorrelated.

The condition $H_1^*(\omega)H_2(\omega) = 0$, $\forall\omega$ implies that systems 1 and 2 do not have *overlapping frequency response.*

11.5 Let $Z(t) = X_0(t)+Y_0(t)$, where $X_0(t)$ is the output of system 1 and $Y_0(t)$ is the output of system 2.

$$\begin{aligned} R_Z(\tau) &= E\Big[\big(X_0(t)+Y_0(t)\big)\big(X_0(t+\tau)+Y_0(t+\tau)\big)\Big] \\ &= R_{X_0}(\tau) + R_{Y_0}(\tau) + R_{X_0Y_0}(\tau) + R_{Y_0X_0}(\tau) \end{aligned}$$

$$R_{X_0Y_0}(\tau) = E\big[X_0(t)Y_0(t+\tau)\big]$$

$$X_0(t) = \int_0^\infty X(t-\alpha)h(\alpha)\,d\alpha \qquad Y_0(t) = \int_0^\infty Y(t-\beta)h(\beta)\,d\beta$$

$$\begin{aligned} R_{X_0Y_0}(\tau) &= E\left[\iint_0^\infty X(t-\alpha)Y(t+\tau-\beta)h(\alpha)h(\beta)\,dx\,d\beta\right] \\ &= \iint_0^\infty E\big[X(t-\alpha)Y(t+\tau-\beta)h(\alpha)h(\beta)\,dx\,d\beta = 0 \end{aligned}$$

$$R_Z(\tau) = R_{X_0}(\tau) + R_{Y_0}(\tau) \qquad S_Z(\omega) = S_{X_0}(\omega) + S_{Y_0}(\omega)$$

$$S_{X_0}(\omega) = |H_1(\omega)|^2 S_X(\omega)$$

Substituting,

$$S_X(\omega) = \frac{18}{81+\omega^2} \qquad H_1(\omega) = \frac{1}{1+j\omega} \qquad |H_1(\omega)|^2 = \frac{1}{1+\omega^2}$$

$$S_{X_0}(\omega) = \frac{18}{(81+\omega^2)(1+\omega^2)} = \frac{-18/80}{(81+\omega^2)} + \frac{18/80}{(1+\omega^2)}$$

$$R_{X_0}(\tau) = -\frac{1}{80}e^{-9|\tau|} + \frac{9}{80}e^{-|\tau|}$$

Similarly,

$$S_{Y_0}(\omega) = \frac{32}{256+\omega^2}|H_2(\omega)|^2 \qquad H_2(\omega) = \frac{1}{2+j\omega} \qquad |H_2(\omega)|^2 = \frac{1}{4+\omega^2}$$

$$S_{Y_0}(\omega) = \frac{32}{(256+\omega^2)(4+\omega^2)} = \frac{-32/252}{256+\omega^2} + \frac{32/252}{4+\omega^2}$$

$$R_{Y_0}(\tau) = -\frac{1}{252}e^{-16|\tau|} + \frac{8}{252}e^{2|\tau|}$$

11.6 a)

$$E[X_n] = 0$$

$$R_X(k) = E[X_n X_{n+k}] = \begin{cases} 1, \text{if } k = 0 \\ 0, \text{if } k \neq 0 \end{cases}$$

$\therefore X_n$ is wide-sense stationary. We cannot say if X_n is strictly stationary or not.

b) $Y_n = \dfrac{X_n + X_{n-1}}{2}$

$$E[Y_n] = 0$$

$$R_Y(k) = E[Y_n Y_{n+k}] = \frac{1}{4}\big[2R_X(k) + R_X(k-1) + R_X(k+1)\big] = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{4} & k = \pm 1 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore Y_n$ is wide-sense stationary.

c) $Y_n = \alpha Y_{n-1} + Y_n$

$$E[Y_n] = 0$$

$$E[Y_n^2] = E\big[(\alpha Y_{n-1} + Y_n)^2\big] = E\big[\alpha^2 Y_{n-1} + 2\alpha Y_{n-1} X_n + X_n^2\big].$$

but $E[Y_{n-1}X_n] = 0$.

$$\overline{Y_n^2} = \alpha^2 \overline{Y_{n-1}^2} + 1 \Longrightarrow \overline{Y_n^2} = \frac{1}{1-\alpha^2}$$

$$R_Y(k) = E[Y_{n-k}Y_n] = \alpha R_Y(k-1) \Longrightarrow R_Y(k) = \frac{\alpha^{|k|}}{1-\alpha^2}$$

$\therefore Y_n$ is wide-sense stationary.

11.7 a)

$$F_X(x \mid Y=1) = F_X(x \mid X>1) = \frac{P(X \leq x, X>1)}{P(X>1)} = \frac{P(1<X\leq x)}{P(X>1)}$$
$$= \frac{F_X(x)-F_X(1)}{1-F_X(1)}$$

$$f_X(x \mid X>1) = \frac{d\,F_X(x \mid X>1)}{dx} = \begin{cases} \dfrac{f_X(x)}{Pr(X>1)} & x>1 \\ 0 & \text{elsewhere} \end{cases}$$
$$= \begin{cases} \dfrac{\frac{1}{2}e^{-x}}{P(X>1)} & x>1 \\ 0 & \text{elsewhere} \end{cases}$$

$$Pr(X>1) = \int_1^\infty \frac{1}{2}e^{-x}\,dx = \frac{1}{2}e^{-1}$$

$$\therefore f_X(x/Y=1) = \begin{cases} e^{1-x} & x>1 \\ 0 & \text{elsewhere} \end{cases}$$

b) $Q = Y - X$ if $Y = 1 \Longrightarrow Q = 1 - X$ (simple linear transformation!)

$$f_Q(q \mid Y=1) = f_Q(q \mid X>1) = \frac{f_Q(q)}{P(X>1)}$$
$$= \frac{\frac{1}{2}e^{-(1-q)}}{\frac{1}{2}e^{-1}} = e^q \qquad -\infty < q < 0$$

(note limits because $x > 1$.)

Hence

$$f_Q(q \mid Y=1) = \begin{cases} e^q & -\infty<q<0 \\ 0 & \text{elsewhere} \end{cases}$$

c)

$$f_Q(q) = f_Q(q \mid Y=-1)P(Y=-1) + f_Q(q/Y=0)P(Y=0)$$
$$+ f_Q(q/Y=1)P(Y=1)$$

From symmetry

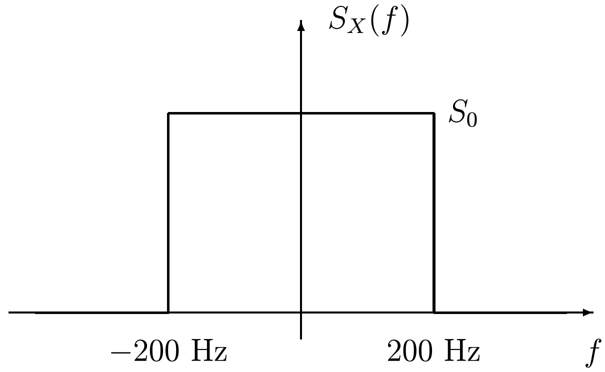
$$f_Q(q \mid Y=-1) = \begin{cases} e^{-q} & 0 < q < \infty \\ 0 & \text{elswhere} \end{cases}$$

$$f_Q(q \mid Y=0) = \begin{cases} \dfrac{\frac{1}{2}e^{-|q|}}{1-e^{-1}} & -1 < q < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$Pr(Y=1) = \frac{1}{2}e^{-1} \qquad Pr(Y=0) = 1-e^{-1} \qquad Pr(Y=-1) = \frac{1}{2}e^{-1}$$

Substitute into above.

11.8



$$\overline{X^2} = 2S_0(200) = 1 \qquad S_0 = \frac{1}{400}$$

$$S_Y(f) = f^2 S_X(t) \qquad \overline{Y^2} = \int_{-200}^{200} f^2 \frac{1}{400}\, df = \frac{200^2}{3}$$