Optimization with Inequality Constraints

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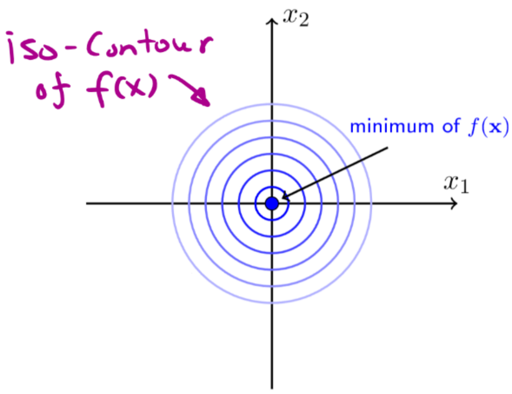
Optimization with Inequality Constraints

1) $\begin{cases} \text{MIN } f(x) \\ \text{s.t. } g(x) \leq 0 \end{cases}$

For example:

$f(x) = x_1^2 + x_2^2$

$g(x) = x_1^2 + x_2^2 - 1$

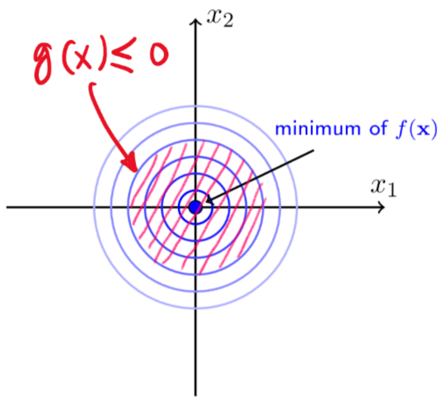


As we can see from the figures. The minimum of $f(x)$ lies in the feasible region of $g(x) \leq 0$.

Therefore, the Constraint is NOT active.

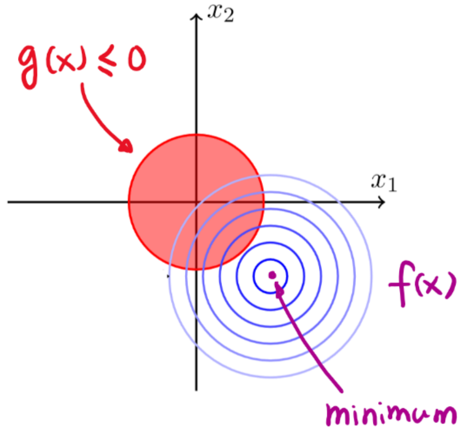
$$\nabla_x f(x) = 0$$

to solve x: $x^* = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



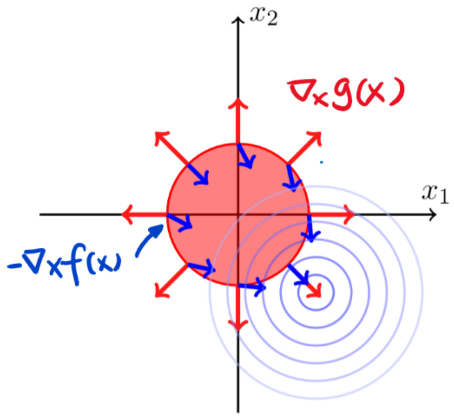
2) If given $\begin{cases} f(x) = (x_1 - 1.1)^2 + (x_2 + 1.1)^2 \\ g(x) = x_1^2 + x_2^2 - 1 \end{cases}$

unconstrained local minimum of f(x) lies outside of the feasible region.



The problem becomes an optimization problem with an equality constraint $g(x) = 0$

$$\begin{cases} -\nabla_x f(x) = \lambda \nabla_x g(x) \\ \lambda > 0 \end{cases}$$



In summary, there are two cases:

Case 1 (minimum in the feasible region)

$$\begin{cases} 1.\ g(x^*) < 0 \\ 2.\ \nabla_x f(x^*) = 0 \end{cases}$$

Case 2 (minimum is outside of the feasible region)

$$\begin{cases} 1.\ g(x^*) = 0 \\ 2.\ -\nabla_x f(x^*) = \lambda \nabla_x g(x) \\ 3.\ \lambda > 0 \end{cases}$$

3) By introducing Lagrangian multiplier, the original optimization problem becomes:

$$L(x,\lambda) = f(x) + \lambda g(x)$$

$$\text{s.t.} \left.\begin{array}{ll} (1) & \nabla_x L(x^*, \lambda^*) = 0 \\ (2) & \lambda^* \geq 0 \\ (3) & \lambda^* g(x^*) = 0 \\ (4) & g(x^*) \leq 0 \end{array}\right\} \begin{array}{l} \text{Karush} \\ \text{Kuhn} \\ \text{Tuker} \\ \text{conditions} \end{array}$$

Consider the condition (3):

either $\lambda^* = 0$, $g(x^*) = 0$

when $\lambda^* = 0$: $\boxed{\begin{array}{l} \nabla_x L(x^*, \lambda^*) = \nabla_x f(x^*) = 0 \\ g(x^*) < 0 \end{array}}$

Case 1

when $g(x^*) = 0$ $\boxed{\begin{array}{l} \lambda^* > 0 \\ \nabla_x L(x^*, \lambda^*) = 0 \Leftrightarrow \\ -\nabla_x f(x^*) = \lambda \nabla_x g(x^*) \\ g(x^*) = 0 \end{array}}$ Case 2

4) Given an example $f(x) = x^2$

s.t. $b - x \leq 0$

Lagrangian multiplier α :

$b - x \leq 0$
$\alpha \geq 0$

$$L(x, \alpha) = x^2 + \alpha(b - x)$$

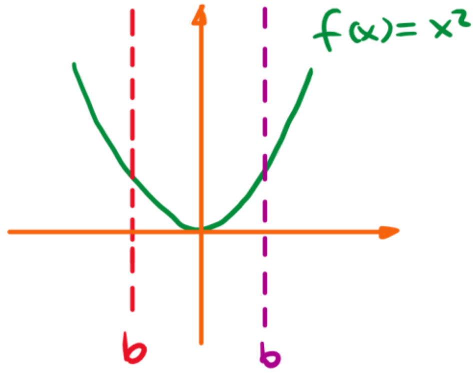
$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2x - \alpha = 0 \Rightarrow x = \frac{\alpha}{2}$$

$$\alpha = 0 \Rightarrow \frac{\partial L}{\partial x} = 0 \Rightarrow 2x = 0 \Rightarrow x^* = 0$$

$$\alpha > 0 \Rightarrow \frac{\partial L}{\partial \alpha} = 0 \Rightarrow b - x = 0 \Rightarrow b - \frac{\alpha}{2} = 0$$

$$\Rightarrow \alpha^* = 2b, \quad x^* = b$$

$$\alpha^* = \max(0, 2b)$$



$$\text{Solve}\begin{cases}\min\limits_{x}\max\limits_{\alpha} L(x,\alpha)\\ \text{s.t. } \alpha \geq 0\end{cases}$$