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关注 小象学院

Machine Learning

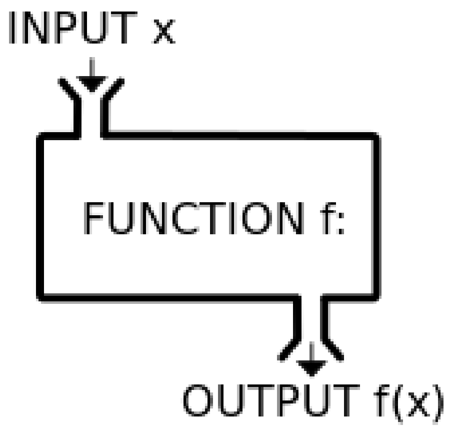
Part 1: Mathematical Foundation of Machine Learning

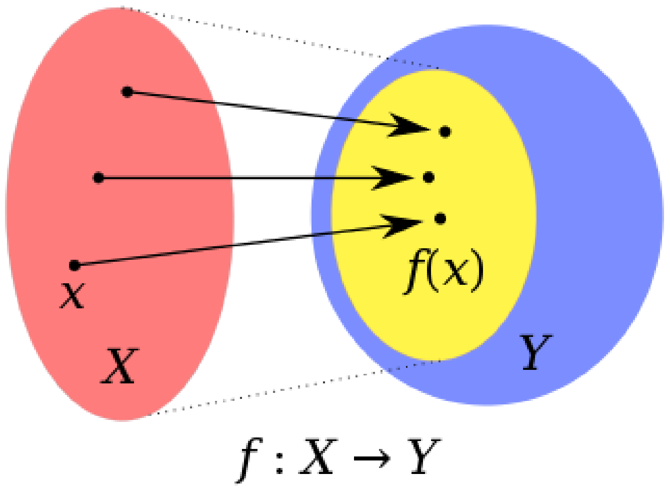
Zengchang Qin (Ph.D.)

Function and Data Generalization

Functions

In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

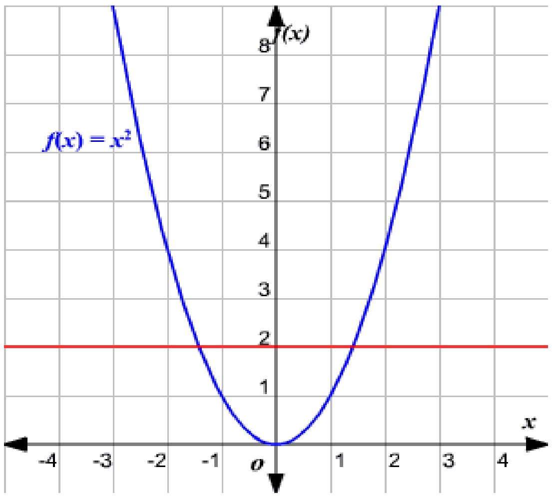


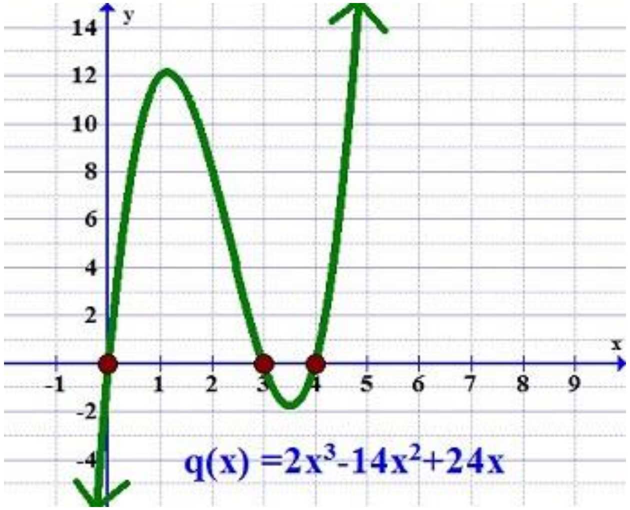


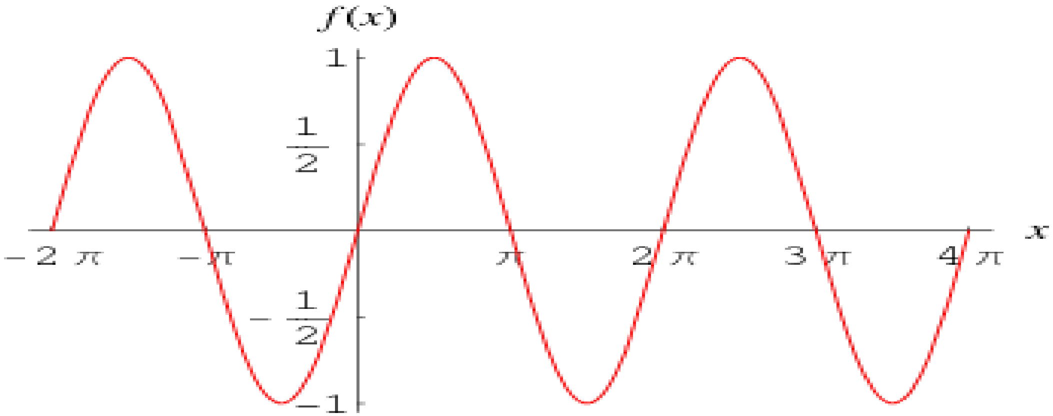
A sample function: f(x) = 2x+3

Functions

We have learned different types of functions.

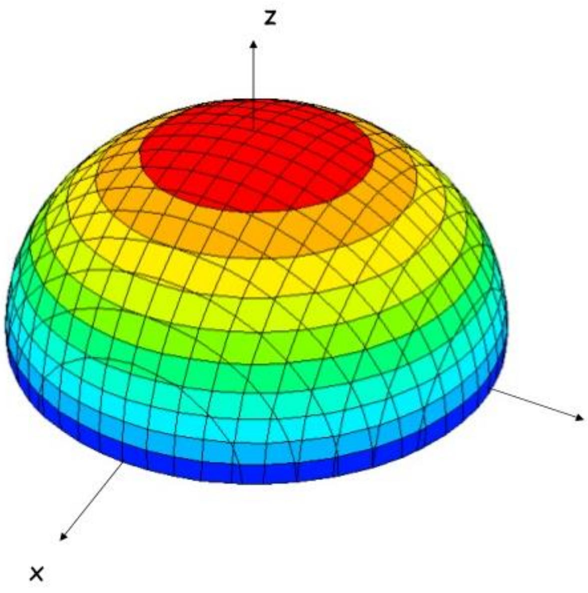


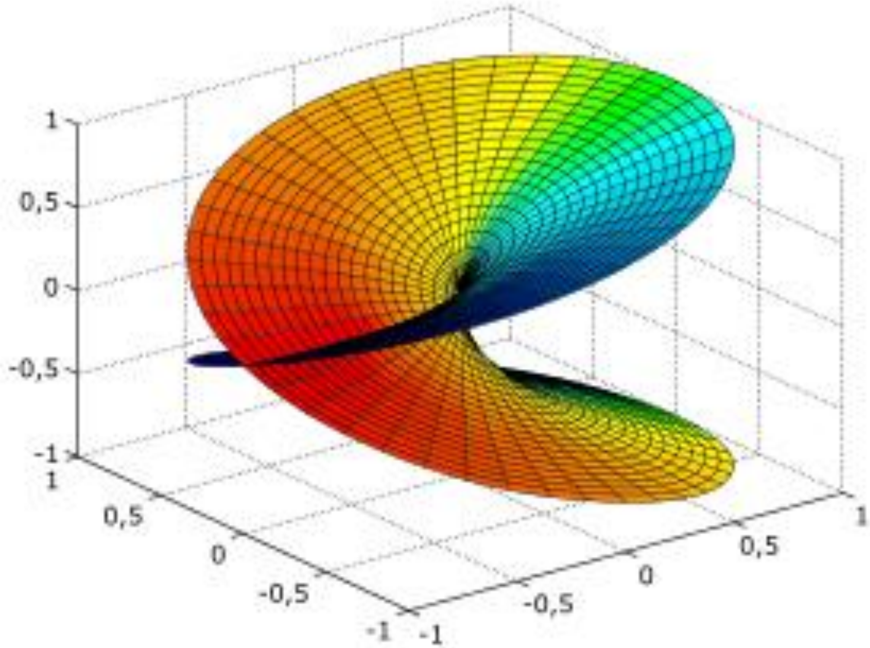




Functions

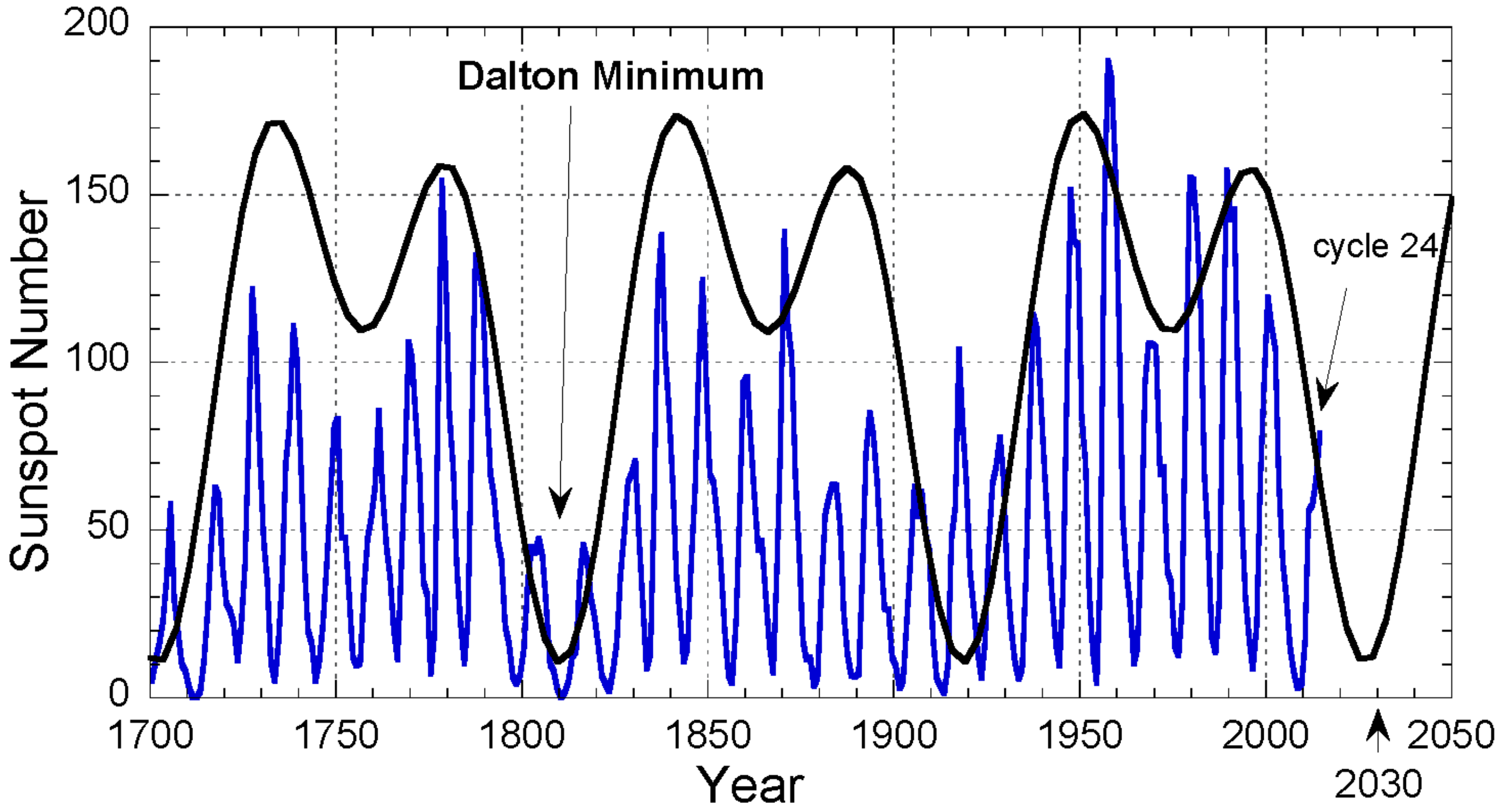
We have learned different types of functions.



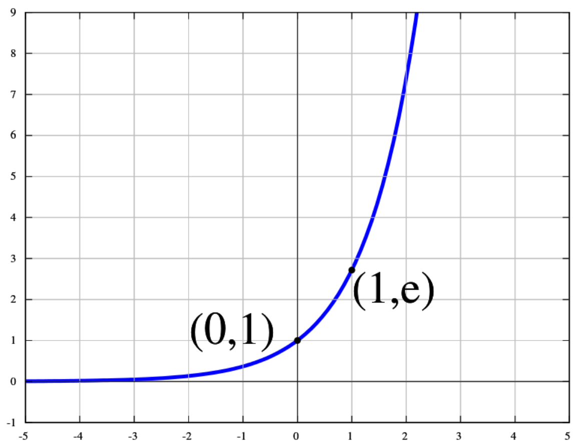


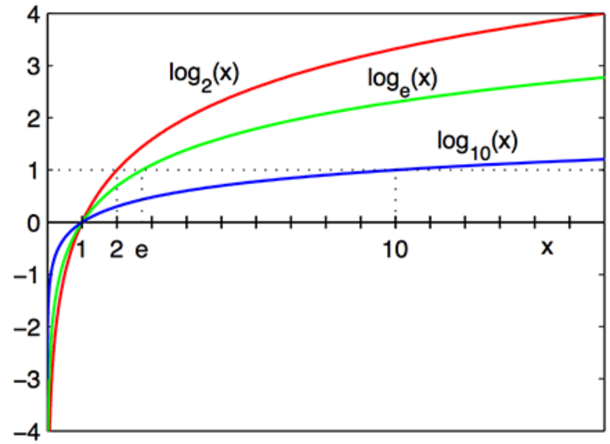
The Real-World Data

In the real-world, when we are investigating relations, we may find the following:

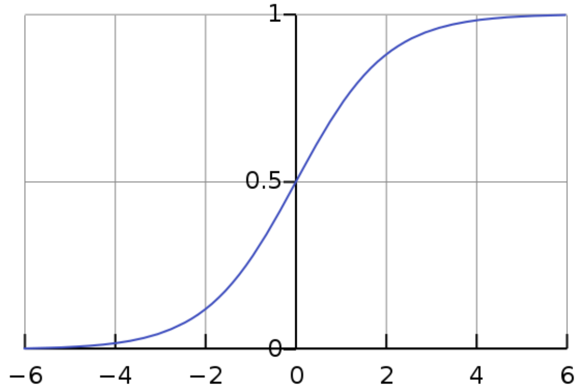


Some Functions

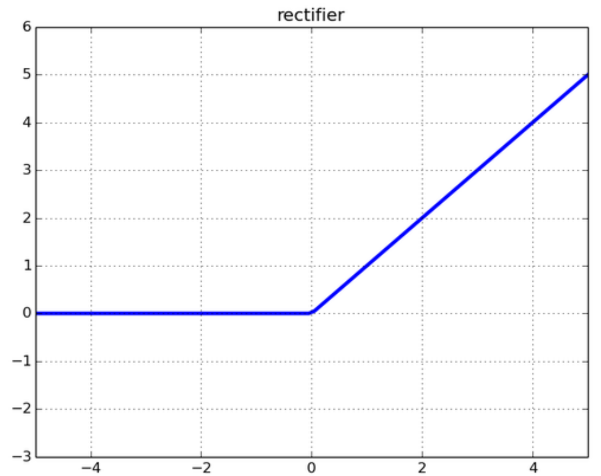




$y = e^x$ http://setosa.io/ev/exponentiation/



$$S(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$$



$$f(x) = x^+ = \max(0, x)$$

Function Decomposition

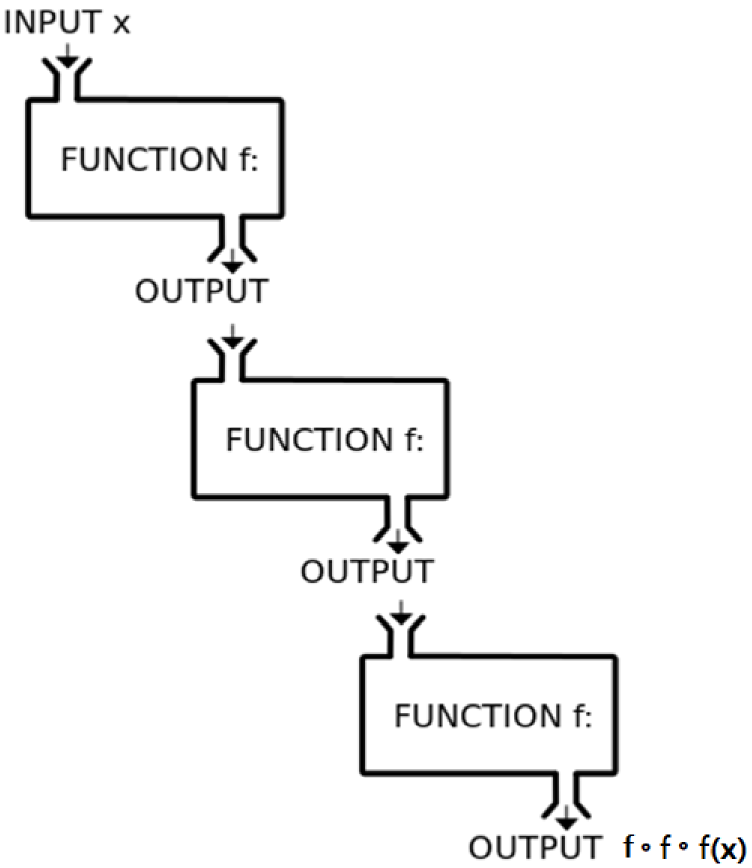
"Function Composition" is applying one function to the results of another:
The result of f() is sent through g()

It is written: $(g \circ f)(x)$
Which means: $g(f(x))$

$f(x) = 2x+3$

$f \circ f(x) = ?$
$f \circ f \circ f\,(x) = ?$

$f \circ f \circ f\,(2) =$

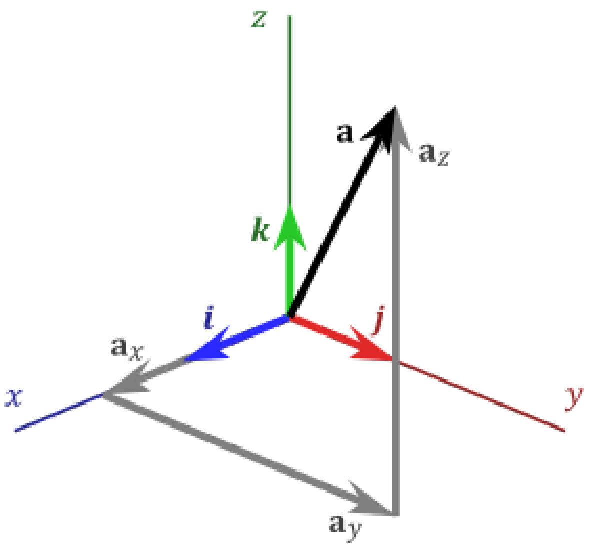


Linear Algebra

Vector

A vector space V is a set (the elements of which are called vectors) on which two operations are defined: vectors can be added together, and vectors can be multiplied by real numbers called scalars.

Can be written in column form or row form – Column form is conventional!

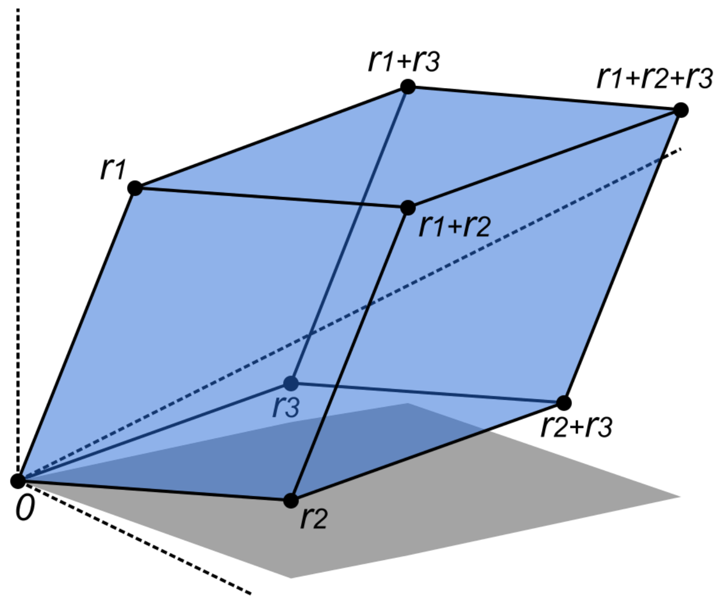


$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}, \qquad \alpha\mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Vector Space

- Euclidean space is used to mathematically represent physical space, with notions such as distance, length, and angles.

- Although it becomes hard to visualize for n > 3, these concepts generalize mathematically in obvious ways.
-
- Linear relations hold in high dimensional space.



Norm of Vectors

A **norm** on a real vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ that satisfies

(i) $\|\mathbf{x}\| \geq 0$, with equality if and only if $\mathbf{x} = \mathbf{0}$

(ii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$

(iii) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the **triangle inequality** again)

We will typically only be concerned with a few specific norms on $\mathbb{R}^n$:

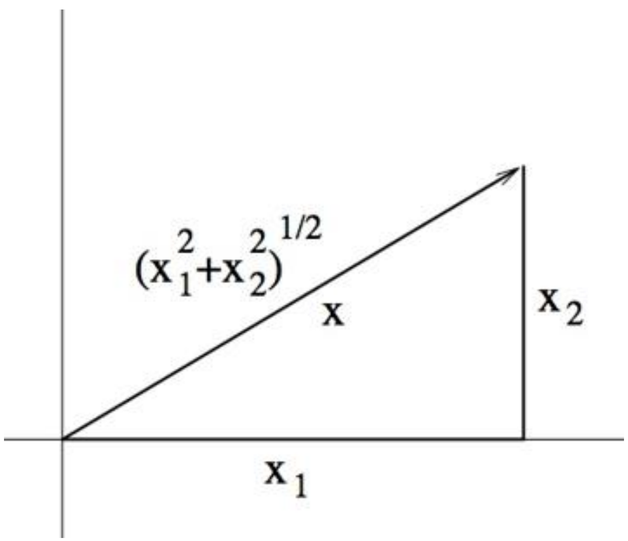
$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \qquad (p \geq 1)$$

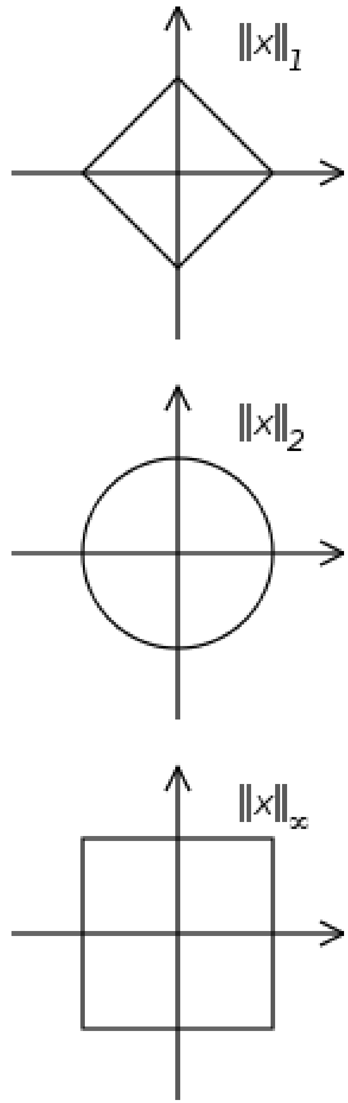
$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \qquad \|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$$

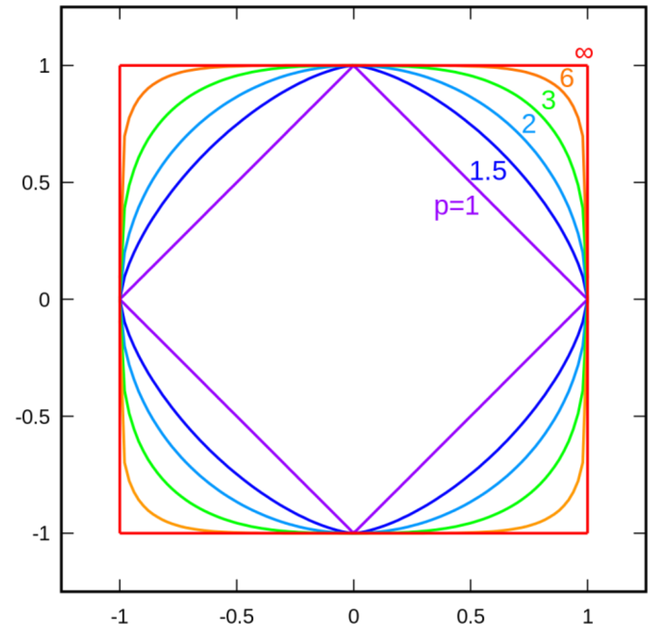
L-0 to L-infinity Norms

a **norm** is a function that assigns a strictly *positive length* to a vector。

A simple example is two dimensional Euclidean space R2 equipped with the "Euclidean norm"







$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Matrix

A vector can be regarded as special case of a matrix, where one of matrix dimensions = 1.

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Matrix transpose (denoted T)

$$\mathbf{A} = \begin{pmatrix} 2 & 7 & -1 & 0 & 3 \\ 4 & 6 & -3 & 1 & 8 \end{pmatrix} \qquad \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 7 & 6 \\ -1 & -3 \\ 0 & 1 \\ 3 & 8 \end{pmatrix}$$

$$C = AB \qquad \Leftrightarrow \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

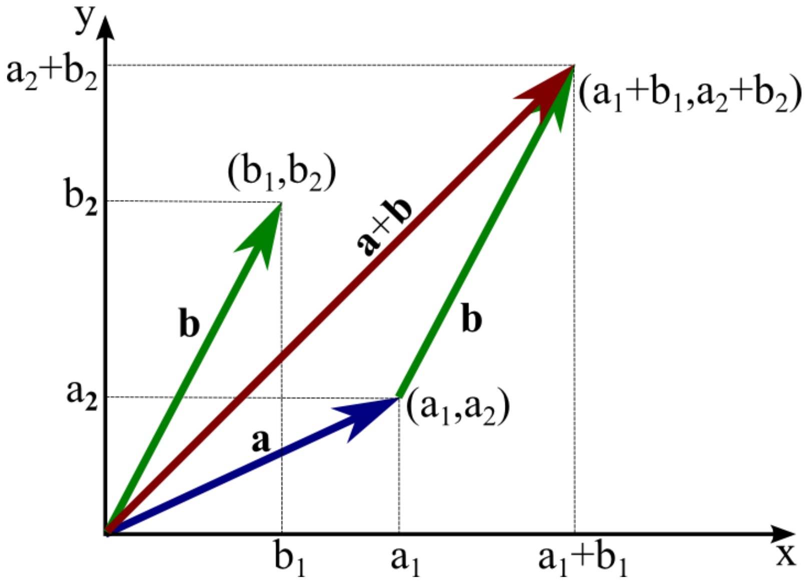
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

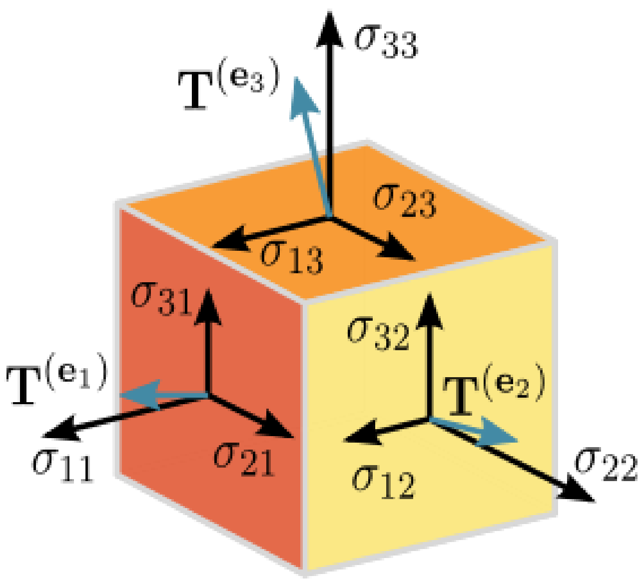
Vector to Tensor

https://www.quora.com/What-is-a-tensor

Columns are the stresses (forces per unit area) acting on the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ faces of the cube.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$



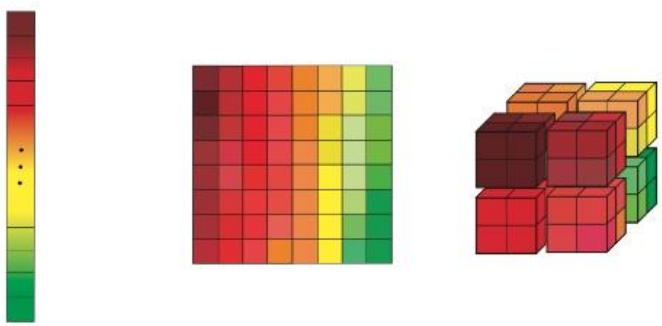


https://www.wukong.com/question/6531498435785261325/

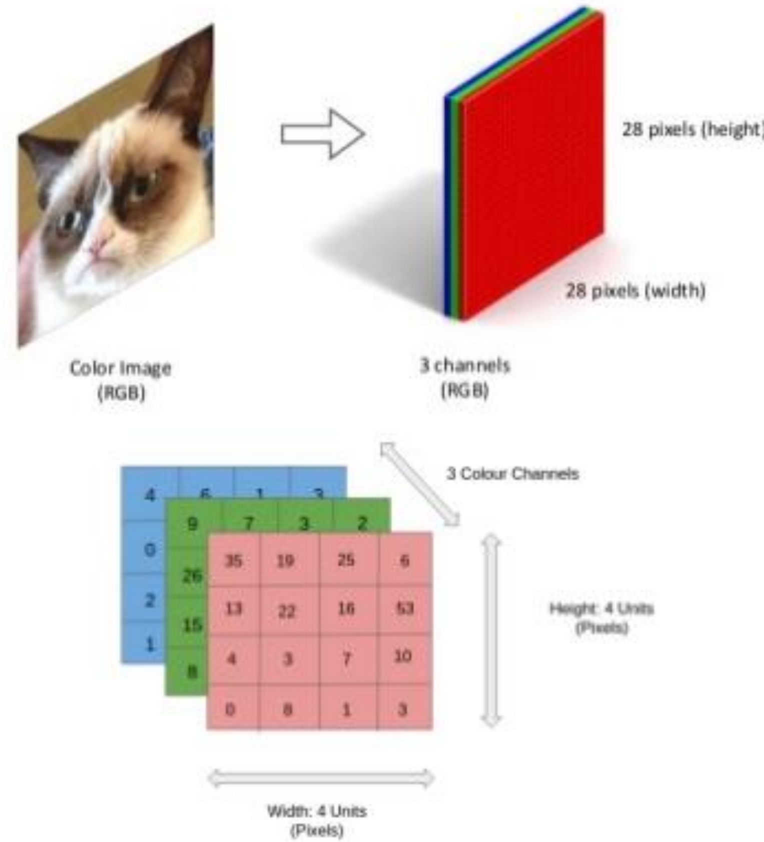
Tensor

tensor = multidimensional array

vector matrix tensor

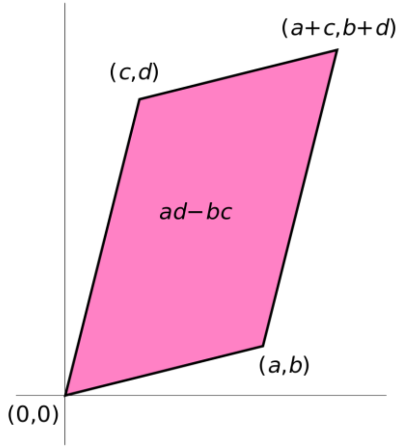


$\mathbf{v} \in \mathbb{R}^{64}$ $X \in \mathbb{R}^{8x8}$ $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{4x4x4}$

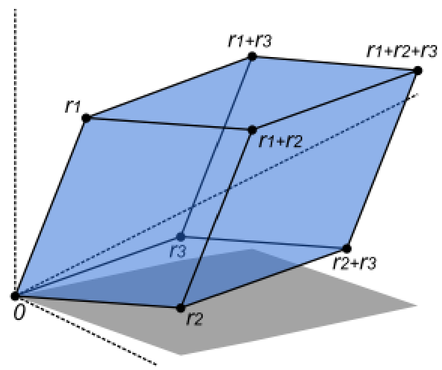


Determinant

In linear algebra, the determinant is a useful value that can be computed from the elements of a square matrix. The determinant of a matrix A is denoted det(A), det A, or |A|. It can be viewed as the scaling factor of the transformation described by the matrix.



$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$



$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= aei + bfg + cdh - ceg - bdi - afh.$$

Eigenvector and Eigenvalue

For a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, there may be vectors which, when $\mathbf{A}$ is applied to them, are simply scaled by some constant. We say that a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ is an **eigenvector** of $\mathbf{A}$ corresponding to **eigenvalue** λ if

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

The zero vector is excluded from this definition because $\mathbf{A0} = \mathbf{0} = \lambda\mathbf{0}$ for every λ.

We now give some useful results about how eigenvalues change after various manipulations.

The **trace** of a square matrix is the sum of its diagonal entries:

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} A_{ii}$$

http://setosa.io/ev/eigenvectors-and-eigenvalues/

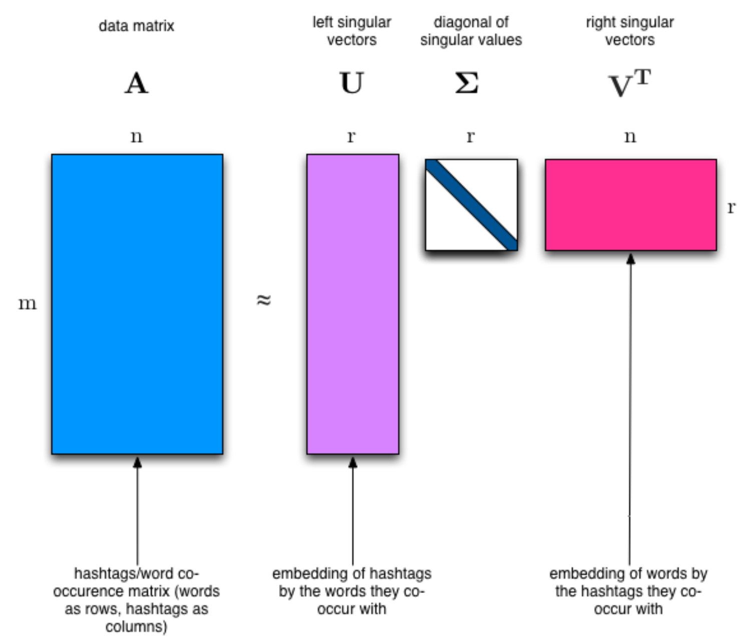
Singular Value Decomposition

Singular Value Decomposition:

Formally, the SVD of a real m $\times$ n matrix A is a factorization of the form $A = U \Sigma V^T$, where U is an m $\times$ m orthogonal matrix of left singular vectors, Σ is an m $\times$ n diagonal matrix of singular values, and V^T is an n $\times$ n orthogonal matrix of right singular vectors.

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{\Sigma} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{V}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$



Jacobian and Hessian Matrices

The **Jacobian** of $f : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix of first-order partial derivatives:

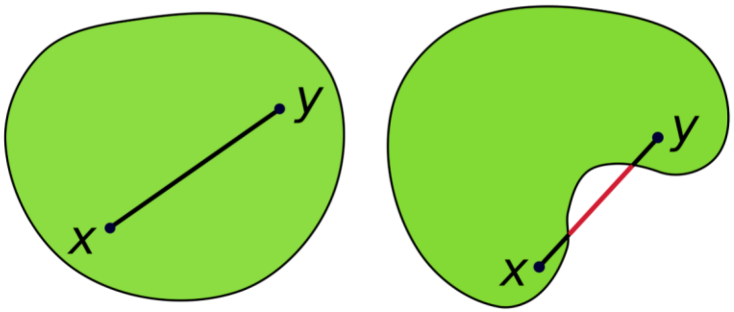
$$\mathbf{J}_f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad \text{i.e.} \quad [\mathbf{J}_f]_{ij} = \frac{\partial f_i}{\partial x_j}$$

Note the special case $m = 1$, where $\nabla f = \mathbf{J}_f^\top$.

The **Hessian** matrix of $f : \mathbb{R}^d \to \mathbb{R}$ is a matrix of second-order partial derivatives:

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \quad \text{i.e.} \quad [\nabla^2 f]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Convex Set and Function



A function f is **convex** if

$$\boxed{f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y})}$$

for all $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$ and all $t \in [0, 1]$.

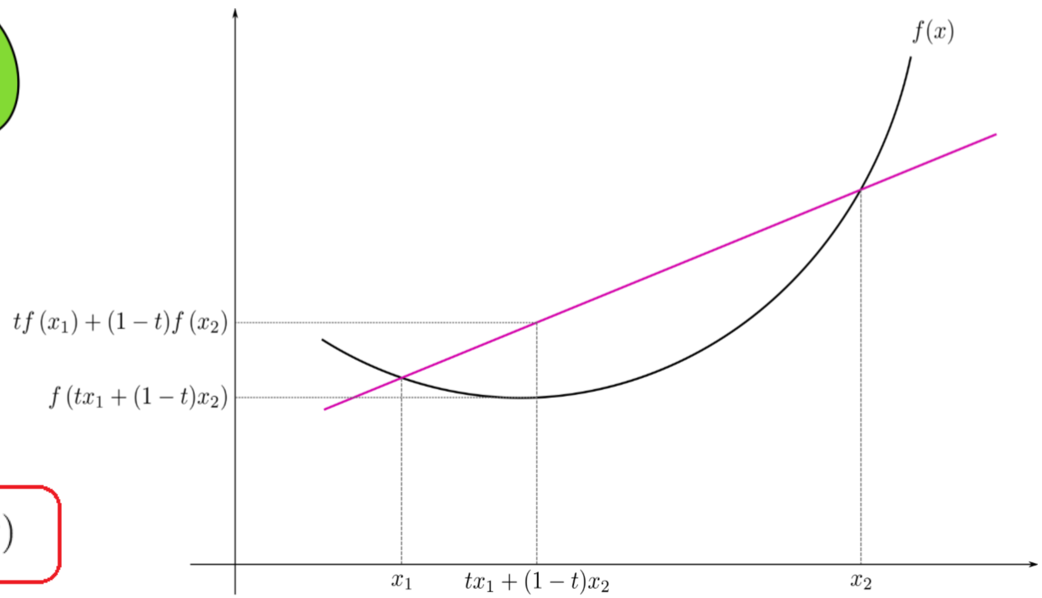


Figure 2: What convex functions look like

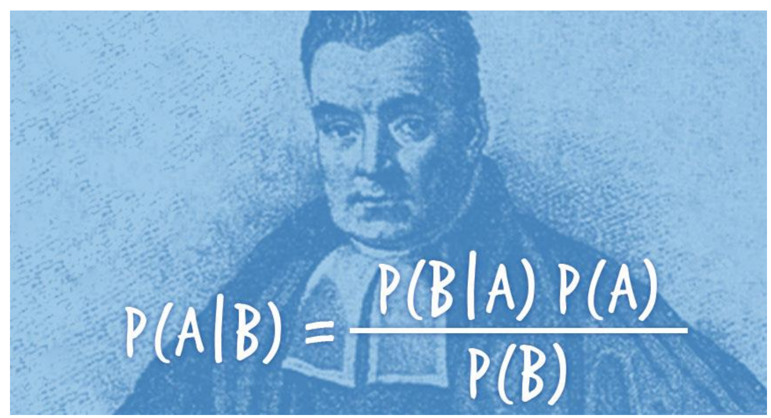
Probability & Statistics

Probability (Objective and Subjective)

The first approach is to define probability in terms of frequency of occurrence, as a percentage of successes in a moderately large number of similar situations.

Such an interpretation is often natural. For example, when we say that a perfectly manufactured coin lands on heads “with probability 50%,” we typically mean “roughly half of the time.”

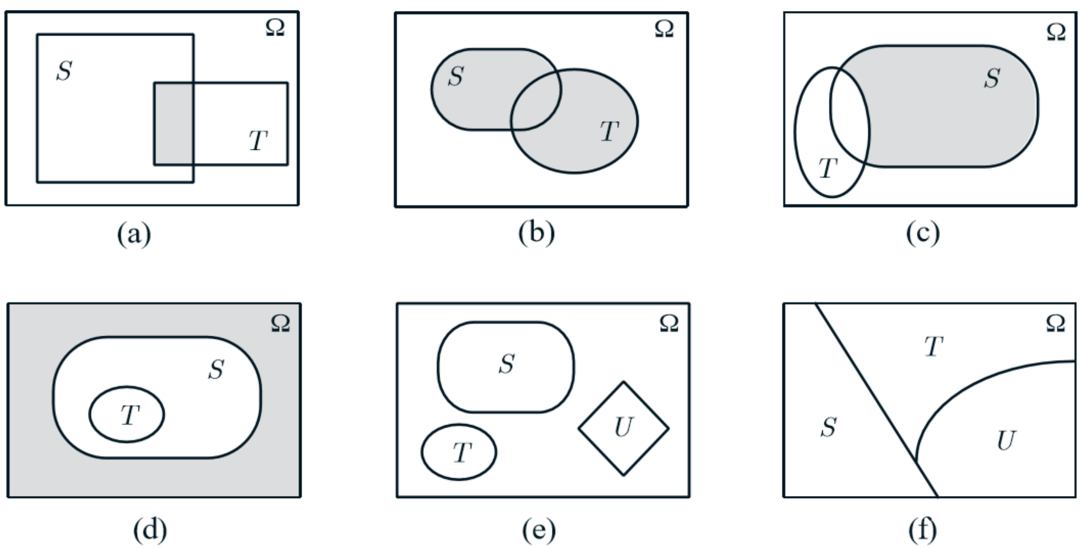
Consider, for example, a scholar who asserts that the Lliad and the Odyssey were composed by the same person, with probability 90%. Such an assertion conveys some information, but not in terms of frequencies, since the subject is a one-time event. Rather, it is an expression of the scholar’s subjective belief.



Set Operation

Examples of Venn diagrams.

(a) The shaded region is S ∩ T.
(b) The shaded region is S ∪ T.
(c) The shaded region is S ∩c(T) .
(d) Here, T ⊂ S. The shaded region is the complement of S.
(e) The sets S, T, and U are disjoint.
(f) The sets S, T, and U form a partition of the set **Ω**

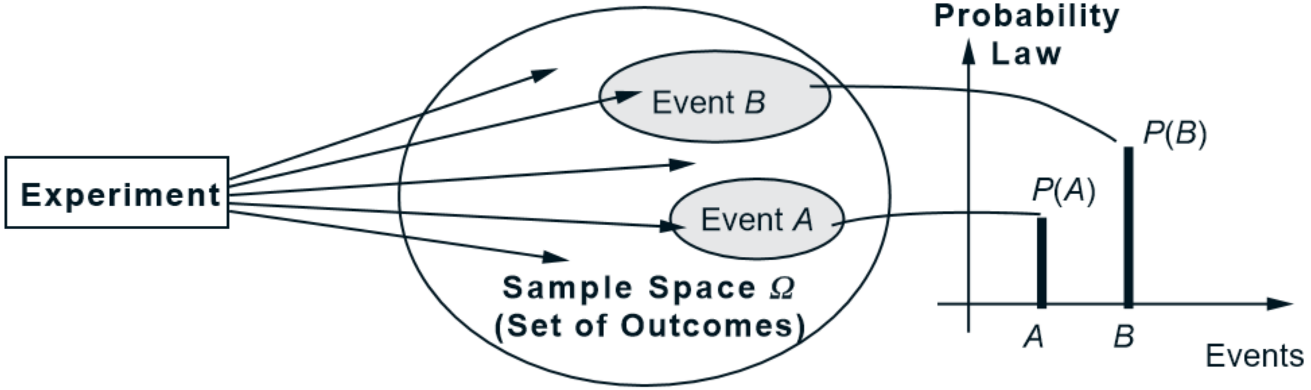


Probabilistic Models

Elements of a Probabilistic Model

The sample space Ω, which is the set of all possible outcomes of an experiment.

The **probability law,** which assigns to a set A of possible outcomes (also called an event) anonnegative number P(A) (called the probability of A) that encodes our knowledge or belief about the collective “likelihood” of the elements of A. The probability law must satisfy certain properties to be introduced shortly.



Probability Axioms

Probability Axioms

1. **(Nonnegativity)** $\mathbf{P}(A) \geq 0$, for every event A.
2. **(Additivity)** If A and B are two disjoint events, then the probability of their union satisfies

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B).$$

   Furthermore, if the sample space has an infinite number of elements and $A_1, A_2, \ldots$ is a sequence of disjoint events, then the probability of their union satisfies

$$\mathbf{P}(A_1 \cup A_2 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots$$

3. **(Normalization)** The probability of the entire sample space Ω is equal to 1, that is, $\mathbf{P}(\Omega) = 1$.

Conditional Probability

Properties of Conditional Probability

- The conditional probability of an event A, given an event B with $\mathbf{P}(B) > 0$, is defined by

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)},$$

  and specifies a new (conditional) probability law on the same sample space Ω. In particular, all known properties of probability laws remain valid for conditional probability laws.
- Conditional probabilities can also be viewed as a probability law on a new universe B, because all of the conditional probability is concentrated on B.
- In the case where the possible outcomes are finitely many and equally likely, we have

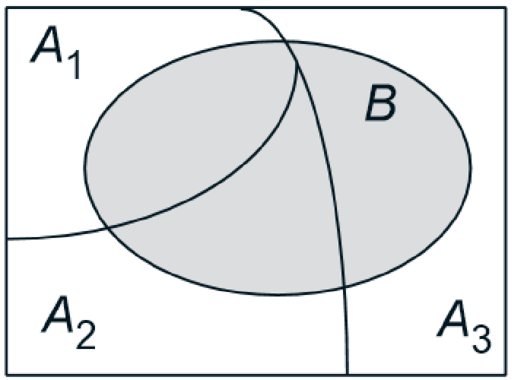
$$\mathbf{P}(A \mid B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B}.$$

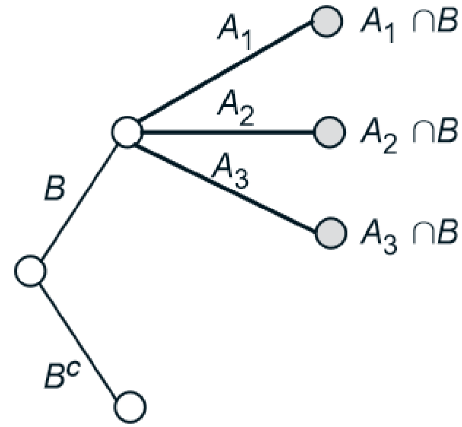
My neighbor John has two kids.

1. He told me that one of his two kids is a boy, what is the probability that the other one is a girl.

2. If I saw one's kids is playing outside, that is a boy, what is the probability that the other one is a girl.

Total Probability Theorem





Total Probability Theorem

Let $A_1, \ldots, A_n$ be disjoint events that form a partition of the sample space (each possible outcome is included in one and only one of the events $A_1, \ldots, A_n$) and assume that $\mathbf{P}(A_i) > 0$, for all $i = 1, \ldots, n$. Then, for any event B, we have

$$\begin{aligned} \mathbf{P}(B) &= \mathbf{P}(A_1 \cap B) + \cdots + \mathbf{P}(A_n \cap B) \\ &= \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n). \end{aligned}$$

Acknowledgement

This slide of this class is modified from Lecture Notes of Dimitri P. Bertsekas and John N. Tsitsiklis – Introduction to Probability, MIT, 2000. & Wikipedia.
UNC Lecture Notes on Ecological Stats:
https://www.unc.edu/courses/2008fall/ecol/563/001/docs/lectures/lecture3.htm
Jeff Howbert Introduction to Machine Learning Winter 2012
Mathematics for Machine Learning Garrett Thomas
http:// gwthomas.github.io/docs/math4ml.pdf
https://rorasa.wordpress.com/2012/05/13/l0-norm-l1-norm-l2-norm-l-infinity-norm/

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