第8章书面作业

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P274. 3(2)

$$\int_0^1 f(x)dx \approx C_0 f(0) + C_1 f(x_1)$$

解: $f(x)=1 \Rightarrow \int_0^1 1dx = 1 = C_0 + C_1$

$f(x) = x \Rightarrow \int_0^1 x dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2} = C_1 x_1$

$f(x) = x^2 \Rightarrow \int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} = C_1 \cdot x_1^2$

解方程 $\begin{cases} C_0 + C_1 = 1 \\ C_1 x_1 = \frac{1}{2} \\ C_1 x_1^2 = \frac{1}{3} \end{cases}$ 解得 $\begin{cases} x_1 = \frac{2}{3} \\ C_1 = \frac{3}{4} \\ C_0 = \frac{1}{4} \end{cases}$

$\therefore \int_0^1 f(x)dx \approx \frac{1}{4}f(0) + \frac{3}{4}f(\frac{2}{3})$

$f(x) = x^3$ 时, $\int_0^1 f(x)dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}$

$\frac{1}{4}f(0) + \frac{3}{4}f(\frac{2}{3}) = \frac{3}{4} \times (\frac{2}{3})^3 = \frac{2}{9} \neq \frac{1}{4}$

$\therefore$ 代数精度为2

P274. 7

$$\int_{0.2}^{1.5} e^{-x^2}dx \quad R(4,4) \quad \text{Romberg}$$

解: $h = b-a = 1.5-0.2 = 1.3$, $R(0,0) = \frac{1.3}{2}(e^{-0.2^2} + e^{-1.5^2}) = 0.6930$

$R(1,0) = R(\frac{b-a}{2^1}) = R(\frac{1.3}{2}) = \frac{1.3}{2\times2}[e^{-0.2^2} + 2e^{-0.85^2} + e^{-1.5^2}] = 0.6621$

$R(1,1) = \dfrac{4R(1,0) - R(0,0)}{4^1-1} = 0.6518$

$R(2,0) = R(\frac{b-a}{2^2}) = R(\frac{1.3}{4}) = \frac{1.3}{2\times4}[e^{-0.2^2} + 2e^{-0.525^2} + 2e^{-0.85^2} + 2e^{-1.175^2} + e^{-1.5^2}] = 0.6595$

$R(2,1) = \dfrac{4^2 \cdot R(2,0) - R(1,0)}{4^2-1} = 0.6593$

$R(2,2) = \dfrac{4^2 R(2,1) - R(1,1)}{4^2-1} = 0.6598$

$$R(3,0)=R\left(\frac{b-a}{2^3}\right)=R\left(\frac{1.3}{8}\right)=0.6590$$

$$R(3,1)=\frac{4^3R(3,0)-R(2,0)}{4^3-1}=0.6590$$

$$R(3,2)=\frac{4^3R(3,1)-R(2,1)}{4^3-1}=0.6590$$

$$R(3,3)=\frac{4^3R(3,2)-R(2,2)}{4^3-1}=0.6590$$

$$R(4,0)=R\left(\frac{b-a}{2^4}\right)=R\left(\frac{1.3}{16}\right)=0.6589$$

$$R(4,1)=\frac{4^4R(4,0)-R(3,0)}{4^4-1}=0.6589$$

$$R(4,2)=\frac{4^4R(4,1)-R(3,1)}{4^4-1}=0.6589$$

$$R(4,3)=\frac{4^4R(4,2)-R(3,2)}{4^4-1}=0.6589$$

$$R(4,4)=\frac{4^4R(4,3)-R(3,3)}{4^4-1}=0.6589$$

$\therefore R(4,4)=0.6589$

P274, 9

构造Gauss型求积公式： $\int_0^1\frac{1}{\sqrt{x}}f(x)dx\approx A_0f(x_0)+A_1f(x_1)$

解：具有最高代数精度的求积公式为Gauss求积公式。

节点 x_0, x_1 为 $[0,1]$ 上以 $\rho(x)=\frac{1}{\sqrt{x}}$ 为权函数的2次正交多项式的零点

不妨设 $\phi_2(x)$ 的首项系数为1

$$\phi_2(x)=x^2+ax+b$$

若 $\int_0^1\frac{1}{\sqrt{x}}\phi_2(x)dx=0$

$\int_0^1\frac{1}{\sqrt{x}}x\phi_2(x)dx=0$

则可知 $\phi_2(x)$ 为 $[0,1]$ 上以 $\frac{1}{\sqrt{x}}$ 为权的二次正交多项式，从而

$$\int_0^1\frac{1}{\sqrt{x}}\phi_2(x)dx=0\Rightarrow\frac{2}{5}+\frac{2}{3}a+2b=0$$

$$\int_0^1\frac{1}{\sqrt{x}}x\phi_2(x)dx=0\Rightarrow\frac{2}{7}+\frac{2}{5}a+\frac{2}{3}b=0$$

可得 $a = -\frac{6}{7}$，$b = \frac{3}{35}$

$\therefore \varphi_2(x) = x^2 - \frac{6}{7}x + \frac{3}{35}$　　　　$35x^2 - 30x + 3$

解得 $x_0 = 0.7416$

$x_1 = 0.1156$

由于二个节点的Gauss求积公式具有3次代数精度，

令 $f(x)=1$，$\int_0^1 \frac{1}{\sqrt{x}}dx = A_0 + A_1$

$f(x)=x$，$\int_0^1 \frac{1}{\sqrt{x}}\cdot x\,dx = A_0x_0 + A_1x_1$

计算得：$\begin{cases} 2 = A_0 + A_1 \\ \frac{2}{3} = 0.7416A_0 + 0.1156A_1 \end{cases}$　解得 $\begin{cases} A_0 = 0.6956 \\ A_1 = 1.3044 \end{cases}$

$\therefore \int_0^1 \frac{1}{\sqrt{x}}f(x)dx \approx 0.6956f(0.7416) + 1.3044f(0.1156)$