Ex. PageRank as a random walk

since

$$p_{ij} = \begin{cases} (1-\eta)\frac{1}{N} + \eta\frac{1}{d_i} & \text{, if } i \to j \\ (1-\eta)\frac{1}{N} & \text{, otherwise} \end{cases}$$

we have that

$$\begin{aligned}
\sum_{j=1}^{N} p_{ij} &= \sum_{j \in [N], i \to j} p_{ij} + \sum_{j \in [N], i \nrightarrow j} p_{ij} \\
&= \sum_{j \in [N], i \to j} (1-\eta)\frac{1}{N} + \eta\frac{1}{d_i} + \sum_{j \in [N], i \nrightarrow j} (1-\eta)\frac{1}{N} \\
&= \sum_{k=1}^{d_i} (1-\eta)\frac{1}{N} + \eta\frac{1}{d_i} + \sum_{k=1}^{N-d_i} (1-\eta)\frac{1}{N} \\
&= d_i \cdot \left[(1-\eta)\frac{1}{N} + \eta\frac{1}{d_i}\right] + (N-d_i) \cdot \left[(1-\eta)\frac{1}{N}\right] \\
&= d_i(1-\eta)\frac{1}{N} + \eta + (N-d_i)(1-\eta)\frac{1}{N} \\
&= N \cdot \frac{1}{N}(1-\eta) + \eta \\
&= 1 - \eta + \eta \\
&= 1
\end{aligned}$$

Thus this complete the proof that

$$\sum_{j=1}^{N} p_{ij} = 1$$