Question: Let
$$f(z) = \frac{e^{iz}}{z}$$
, and consider the contour integral $\int_{\mathbb{R}} f(z) \, dz$ over the real line. Assume the contour is closed by a semicircular arc in the upper half-plane. Prove or disprove that the

value of the improper integral
$$\iint_0^\infty \frac{\sin x}{x} dx$$
 is given by:

Final Answer: True

O3-mini Solution: To prove this theorem, we start by writing $\int_{0}^{\infty} \frac{\sin x}{x} dx = \Im \left(\int_{0}^{\infty} \frac{e^{ix}}{x} dx \right)$ Because the integrand $\frac{e^{iz}}{z}$ is meromorphic with a simple pole at z=0,

we consider the principal value of the integral along the whole real axis. That is, we consider p.v. $\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$

However, due to the pole at z = 0 lying on the real axis,

Thus, the theorem statement $\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ has been proven. boxed{True}