1 Benchmarks and Calculations

Armed with the framework for constructing Feynman diagrams and applying Feynman rules, we can now proceed to compute the scattering amplitudes for gluon interactions in (QCD) and verify their properties.

Before diving into the calculations, it's important to understand the expected complexities arising from the generated amplitudes.

1.1 Number of Feynman Diagrams and Color Structures

The total number of tree-level Feynman diagrams for n gluons with 3 and 4 point vertices can be computed using a recurrence relation[1] and shown in **??**. Among these, the number of diagrams featuring only 3-point vertices is given by $(2n-5)!!$ (double factorial) [2].Each of these diagrams corresponds to a unique color substructure. The remaining diagrams, which include at least one 4-point vertex, are generated by adding a 4-point vertex to existing diagrams with only 3-point vertices and they do not introduce any new color substructures.

In general, for n gluons, there are $(2n-5)!!$ color substructures.These takes the form of contractions of $n-2$ structure constants f^{abc} with $n-3$ dummy(summed over) indices a_i and n distinct indices c_i, e.g $f^{c_2c_1a_1}f^{c_4c_3a_1}$ for $n=4$, $f^{c_1a_2a_1}f^{c_3c_3a_1}f^{c_5c_4a_2}$ for $n=5$, and so on.

Table 1: Number of Tree-level Feynman Diagrams for n gluons. The first column indicates the number of external gluons, the second column shows the total number of diagrams, the third column counts the diagrams with only 3-point vertices, and the fourth column counts those with at least one 4-point vertex.

n	Number of Diagrams	3-point Vertices only	with 4-point Vertices
4	4	3	1
5	25	15	10
6	220	105	115
7	2485	945	1540
8	34300	10395	23905

1.2 Amplitude Generation

The process of generating the scattering amplitude is straightforward. We generate the Feynman diagrams using the 'generatediagrams' function (**??**), sum the contributions from each diagram, and then apply the Feynman rules to obtain the amplitude.

The computational resources required for generating the diagrams only, specifically the time and memory, are summarized in Table 2, which shows that time and memory usage grows exponentially with the number of gluons involved in the scattering process. Applying the Feynman rules further increases the computational cost, as shown in the same table.

Table 2: Computational Resources for Feynman Diagram Generation and Amplitude. The Amplitude is generated by applying the operations: Total, FeynmanRules and Expand. The result for 8 gluons is not available (TBD) due to the high computational cost.

Number of Gluons		Generated Diagrams		Amplitude	
n	Number of Diagrams	Time (s)	Memory (MB)	Time (s)	Memory (MB)
4	4	0.0011	0.01	0.004	0.096
5	25	0.0100	0.12	0.051	2.87
6	220	0.1153	1.51	6.503	104.52
7	2485	1.5998	22.99	1112.738	4430.25
8	34300	27.5766	404.70	TBD	TBD

After generating the amplitude, there are several important properties that we can verify to ensure the correctness of our calculations.

1.3 Symmetry under Exchange of External Legs

The scattering amplitude for gluon interactions should be symmetric under the exchange of external legs.

To veriry this property, the following function are defined:

- **swapTwoParticles[amp_,i_,j_]**: This function takes an amplitude and swaps the i-th and j-th external legs, by replacing the corresponding Lorentz, color and momentum labels in the amplitude expression.
- **pairmap[list1_, list2_]** : This function takes two lists and returns a list of pairs, where each pair consists of an element from the first list and the corresponding element from the second list.

In the two to two scattering case ($n = 4$), there are only 3 color substructures and after exchangin external legs, these color substructures are permuted among themselves, their Lorentz coefficients also changes accordingly, but the overall amplitude remains unchanged.

For $n \geq 5$, the permutation are more complex and the function **pairmap** is needed to keep track of which term goes where. Then verify that the amplitude remains unchanged by subtracting the original color substructure Lorentz coefficients from the swapped ones and checking if the result is zero.

From this we can conclude that the amplitude is symmetric under the exchange of external legs and that all the color substructures are permuted among themselves, so only a single color substructure is needed and rest can be generated by substitution rules, which is significatly more efficient than all the diagrams, substituting Feynman rules, summing over all diagrams and then collecting the color substructures.

1.4 Ward Identities

The Ward identities are a set of relations that must be satisfied by the scattering amplitudes in gauge theories. In the case of gluon scattering, the Ward identity states that the amplitude must vanish when any external on-shell gluon polarization is substituted with its momentum.

While the overall scattering amplitude for gluon interactions satisfies the Ward identity, the presence of distinct color substructures within the amplitude poses a unique challenge. Since these color substructures prevent a direct summation of the various Lorentz structures, each individual color substructure must inherently be gauge invariant for the full amplitude to satisfy the identity.

However, the $(2n-5)!!$ color substructures are not all linearly independent, as they are related by the Jacobi identity as shown in Equation eq. (1) for the structure constants f^{abc}, so they are not gauge invariant. These non gauge invariant dependent substructures still vanish when all external polarization vectors are simultaneously substituted with their respective momentum vectors.

$$f^{aeb}f^{ecd} + f^{ade}f^{ecb} - f^{ace}f^{edb} = 0 \tag{1}$$

For n gluons, each of these color substructures has $n-3$ dummy indices, so each term can generate $n-3$ jacobi identities, though these may not all be unique. These color substructures can be mapped to variables using the Mathematica function **MapIndexed** to variables $v[i]$ so that Mathematica can work with them. The Jacobi identities can then be transformed into a system of equations and solved using **Solve**.

In conclusion, the initial set of $(2n-5)!!$ color substructures reduces to $(n-2)!$ independent color substructures [3]. These independent color substructures are each accompanied by their respective Lorentz structures, which are inherently gauge invariant and collectively satisfy the Ward identity.

Thanks to the symmetry under exchange, it is sufficient to verify the Ward identity for just one of these independent color substructures.

Necessary conditions for the Ward identity are momentum conservation and the transversality of the polarization vectors.

All the steps above are implemented in the module `VerifyWard[ngluons]`, in table 3 we summarize the performance of the verification of the Ward identity for $n = 4, 5, 6$ gluons. Notice that times and memory usage scales exponentially with the number of gluons and simplification times depends on the size of the expressions,

Table 3: Performance Metrics for Ward Identity Verification in Gluon Scattering

Contracted with	Contraction Time (s)	Memory (MB)	Simplification Time (s)
$n = 4$ Gluons			
4p 0ϵ	0.002294	0.000016	0.000087
3p 1ϵ	0.003764	0.003496	0.000080
2p 2ϵ	0.006850	0.021032	0.000159
1p 3ϵ	0.009851	0.049232	0.000204
$n = 5$ Gluons			
5p 0ϵ	0.092677	0.076912	0.000343
4p 1ϵ	0.134000	0.162256	0.000622
3p 2ϵ	0.202662	0.514672	0.001603
2p 3ϵ	0.281269	0.951728	0.003277
1p 4ϵ	0.376129	1.561320	0.076727
$n = 6$ Gluons			
6p 0ϵ	4.042710	2.666150	0.232839
5p 1ϵ	5.078180	4.516330	0.788833
4p 2ϵ	7.033140	11.482300	9.227526
3p 3ϵ	9.641990	22.105400	30.048647
2p 4ϵ	12.593800	37.025000	151.343383
1p 5ϵ	15.696300	54.530100	183.856281

* The ”Memory” column indicates the memory (in Megabytes) required to store the expression just before the simplification step. The ”Contraction Time” includes applying momentum conservation, the transverse condition, and contractions. The ”Simplification Time” reflects the duration of the `Simplify` operation.

2 Modulus Squared of the Amplitude

Calculating the modulus squared of the amplitude is typically the most computationally intensive step in scattering amplitude computations. For gluon scattering, various strategies can be employed to manage this complexity, primarily by organizing the amplitude and its color substructures differently.

1. **Direct Squaring of Feynman Diagrams:** This is the most straightforward method, involving the direct squaring of the sum of all Feynman diagrams.
2. **Collecting Non-Independent Color Substructures:** A more efficient approach involves collecting all $(2n-5)!!$ non-independent color substructures before squaring their sum. This strategy significantly improves efficiency by allowing for early cancellations, thereby substantially reducing the size of intermediate expressions and overall memory footprint.
3. **Utilizing Jacobi Identities for Independent Color Substructures:** This advanced method further reduces the number of color substructures by employing Jacobi identities to derive a basis of independent color structures. While this approach minimizes the number of terms that need to be computed, it can lead to coefficients with larger individual expressions. This presents a trade-off between the number of calculations performed and the symbolic size of the resulting terms.

To illustrate the computational characteristics of these approaches, Table 4 presents performance metrics, including memory usage and computation time, for $n = 4$ and $n = 5$ gluon scattering processes. The memory usage is detailed for various stages of the calculation: the size of a single selected term, its size after contraction with polarization sums (using specific choices for polarization vectors n^μ aligned with external momenta for simplification), the size after expanding against a conjugate term, the size after subsequent contraction, after renaming Lorentz invariants, and finally, after full simplification. Lastly, the total time taken to compute the modulus squared is also recorded.

Table 4: Performance Metrics for Modulus Squared Calculation in Gluon Scattering

Operation Stage	**Feynman Diagrams**	**Non-Independent Basis**	**Independent Basis**
		$n = 4$ Gluons	
Total Terms	10	6	3
Single Term (MB)	0.028	0.025	0.050
Times Polarization (MB)	0.945	0.693	1.512
Times Conjugate (MB)	15.449	4.186	32.592
Contract (MB)	1.364	0.358	2.220
Mandelstam (MB)	0.389	0.116	0.622
Simplify (MB)	0.005	0.003	0.006
All Contributions (MB)	0.026	0.023	0.018
Time Taken (s)	3.318	1.106	3.184
		$n = 5$ Gluons	
Total Terms	325	120	21
Single Term (MB)	0.134	0.133	0.633
Times Polarization (MB)	10.455	8.970	56.295
Times Conjugate (MB)	4893.640	3695.500	TBD
Contract (MB)	6.325	1.553	TBD
Mandelstam (MB)	7.760	1.681	TBD
Simplify (MB)	0.072	0.059	TBD
All Contributions (MB)	TBD	6.867	TBD
Time Taken (min)	TBD	$\sim$50	TBD

Note: the most memory-intensive operation is the expansion against a conjugate term, which is necessary for Contraction to act on the expression.