Solutions - 21/10/22

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Disclaimer: there could be mistakes of any sort throughout these solutions; please, don't hesitate to contact me if you find one or more!

1 Maximum of Uniform Random Variables

(a) Let's try with some examples. If $n = 1$, then $X_{(1)} = \max_i(\{x_i\}) = x \sim U(a, b)$; on the other hand, if $n \to \infty$, then $X_{(N)} = \max_i(\{x_i\}) \to b$.
The key idea to solve the exercise is to look at the cumulative function of our variable $X_{(N)}$:

$$\begin{aligned} F_{(N)}(x) &= \text{prob}(X(N) < x) \\ &= p(\max(\{x_i\}) < x) \\ &= p(x_1 < x \cap x_2 < x \cap \ldots \cap x_N < x) \\ &= p(x_1 < x) \cdot \ldots \cdot p(x_N < x) \end{aligned} \tag{1}$$

where we used the independence of the variables x_i to split the expression in a product of probabilities. Note also that $p(x_i < x) = F_U(x)$, i.e. the cumulative function of the uniform distribution over the interval $[a, b]$. We conclude that

$$F_{(N)}(x) = F_U^N(x). \tag{2}$$

The cumulative of the uniform distribution is immediately computed and it is

$$F_U(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases} \tag{3}$$

so that the cumulative function of $X_{(N)}$ is

$$F_{(N)}(x) = \begin{cases} 0 & x < a \\ \frac{(x-a)^N}{(b-a)^N} & x \in [a, b] \\ 1 & x > b. \end{cases} \tag{4}$$

Now we just take the derivative to obtain the p.d.f. of $X_{(N)}$:

$$\rho_{(N)}(x) = \begin{cases} \frac{N(x-a)^{N-1}}{(b-a)^N} & x \in [a,b] \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

One can easily check that for $n = 1$ the p.d.f. of the uniform distribution is retrieved. Moreover, note that this distribution tells us that when $n \to \infty$ the probability that $X_{(N)}$ is equal to b approaches 1 (that is exactly what one would expect, see part **c)** of the exercise).

(b) This part of the exercise is basically a painful but rather easy integration session. We must compute

$$M_1 = \langle X_{(N)} \rangle = \int_{-\infty}^{\infty} x\rho_{(N)}(x)dx = \int_a^b x\frac{N(x-a)^{N-1}}{(b-a)^N}dx \tag{6}$$

and

$$M_2 = \langle X_{(N)}^2 \rangle = \int_{-\infty}^{\infty} x^2\rho_{(N)}(x)dx = \int_a^b x^2\frac{N(x-a)^{N-1}}{(b-a)^N}dx. \tag{7}$$

The integrals are easily computed by parts and one gets eventually

$$M_1 = \frac{bN+a}{N+1} \tag{8}$$

and

$$M_2 = \frac{b(bN+1)}{N+1} + \frac{2(b-a)^2}{(N+1)(N+2)} - \frac{b(b-a)}{N+1}. \tag{9}$$

(c) It is trivial to show that when $n \to \infty$ we have $\langle X_{(N)} \rangle = M_1 \sim b$ and $\langle X_{(N)}^2 \rangle = M_2 \sim b^2$, so that

$$\lim_{n\to\infty} \langle X_{(N)}^2 \rangle - \langle X_{(N)} \rangle^2 = 0. \tag{10}$$

This is expected, since when $n \to \infty$ one obtain with probability 1 that $X_{(N)} = b$, the maximum value in the interval $[a, b]$.

2 Waiting for the bus

This exercise explores some classic results concerning the exponential and Poisson distribution.

(a) This is just the survival probability after $t = 10$ minutes:

$$\text{prob}(t > 10) = S_{10}(t) = \int_{10}^{\infty} \lambda e^{-\lambda t} dt = e^{-10\lambda} \tag{11}$$

(b) And this is just the cumulative function in $t = T$:

$$\text{prob}(t > T) = F(t, T) = \int_{0}^{T} \lambda e^{-\lambda t} dt = -e^{-\lambda T} + 1. \tag{12}$$

(c) Now, this question is trickier but trivial if one knows a fundamental property of the exponential distribution, namely the *Memory-lessness* property. It consists of the two following relationships:

$$\text{prob}(T > s + t | T > s) = \text{prob}(T > t), \quad \forall s, t \geq 0 \tag{13}$$

and

$$\text{prob}(T < s + t | T > s) = \text{prob}(T < t), \quad \forall s, t \geq 0. \tag{14}$$

Citing the holy and infallible Wikipedia: " *When T is interpreted as the waiting time for an event to occur relative to some initial time, this relation implies that, if T is conditioned on a failure to observe the event over some initial period of time s, the distribution of the remaining waiting time is the same as the original unconditional distribution*". Let's prove at least the first relationship:

$$\begin{aligned} p(T > t + s | T > s) &= \frac{p(T > t + s \cap T > t)}{p(T > s)} \\ &= \frac{p(T > t + s)}{p(T > s)} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= p(T > t). \end{aligned} \tag{15}$$

The proof of the second relationship is almost identical. So, the answer to this question is: No, the results do not change if we know that 15 minutes from the last bus have already passed. Notice that, in the real world, this is not totally true because the arrival of a bus is not a totally random event. In a perfect world, in fact, it should be a completely deterministic process (the bus arriving according to a precise schedule). Reality is someway in between...

(d) Let's start by realizing that an exponential probability distribution implies that the probability for an event to occur in a infinitesimal time dt is $p = \lambda dt$. Now, we take a time interval of length t and we split it in n small intervals of size $dt = t/n$, which are infinitesimal for $n \to \infty$. Then, the probability of having k events distributed over n infinitesimal intervals is clearly a binomial:

$$\begin{aligned} \mathrm{prob}(k,t) &= \binom{n}{k}\left(\lambda\frac{t}{n}\right)^k\left(1-\lambda\frac{t}{n}\right)^{n-k} \\ &= \frac{n!}{k!(n-k)!}\left(\lambda\frac{t}{n}\right)^k\left(1-\lambda\frac{t}{n}\right)^{n-k}. \end{aligned} \tag{16}$$

Using the Stirling approximation for the factorial; ($n! \sim n^{n+\frac{1}{2}}/e^n$) you should be able to prove that in the $n \to \infty$ limit Eq. 16 reduces to

$$p(K,t) = \frac{\lambda^k t^k}{k!} e^{-\lambda t}. \tag{17}$$

where λ is the rate of occurrence of the events. Defining the mean value of events $\mu = \lambda t$ we finally get the well known Poisson distribution:

$$p(K,t) = \frac{\mu^k}{k!} e^{-\mu}. \tag{18}$$

(e) You may realize that the Poisson distribution at $k = 0$ is just the probability of NOT having an event in a time t, i.e. the survival function of a distribution. You get

$$p(K=0,t) = e^{-\lambda t}; \tag{19}$$

the survival function of the exponential distribution is thus retrieved.

3 Return to the origin in Random Walks

(a) Let $S_n \in Z$ be the position of the walker after n steps. We immediately realize that prob$(S_{2n+1} = 0) = 0$; that is, the walker can find itself in the origin only after an even number of steps. To return to the origin, the walker must step $n/2$ times to the 'left' and $n/2$ to the 'right'; it is clearly a binomial distribution:

$$p(S_{2n} = 0) = \binom{2n}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n = \binom{2n}{n} 2^{-2n}. \tag{20}$$

Let's define J_n as the number of returns to the origin after n steps; we want to compute $\langle J_n \rangle$. It is useful to introduce an *occupation function* defined as

$$j_m = \begin{cases} 1 & \text{if} \quad S_m = 0 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

so that

$$J_n = \sum_{m=1}^{n} j_m. \tag{22}$$

One immediately realize that $\langle j_m \rangle = p(S_n = 0)$, that is, the expected value of the occupation function is the probability to be in the origin after m steps. Thus we can write

$$\langle J_n \rangle = \sum_{m=1}^{n} \langle j_m \rangle = \sum_{m=1}^{n} p(S_n = 0) = \sum_{m=1}^{n} \binom{2n}{n} 2^{-2m} \tag{23}$$

and using the identity suggested in the text, we have

$$\langle J_n \rangle = (n+1) \binom{2n+2}{n+1} 2^{-2n-1}. \tag{24}$$

(b) Let ρ be the probability that the RW returns to x=0 at least once. Now, the probability of having k returns when $n \to \infty$ is simply

$$p(J = k) = \rho^k (1-\rho)^k; \tag{25}$$

in fact, the walker must go back to the origin k times and then never return anymore. Then, the average value of the number of returns $\langle J \rangle$ is computed in the usual way:

$$\langle J \rangle = \sum_{k=1}^{\infty} k \rho^k (1-\rho) = \rho(1-\rho) \frac{\partial}{\partial \rho} \sum_{k=1}^{\infty} \rho^k = \frac{\rho}{1-\rho}. \tag{26}$$

We can extract some interesting informations from this result. First, let's compute the scaling of $p(S_{2n} = 0)$ when $n \to \infty$:

$$p(S_{2n} = 0) = \frac{1}{2^{2n}} \binom{2n}{n} \sim \frac{1}{2^{2n}} \frac{2^{2n}}{n^{1/2}} \sim \frac{1}{n^{1/2}}, \tag{27}$$

where we used the Stirling approximation to approximate the binomial coefficient. Let's look at $\langle J \rangle$:

$$\langle J \rangle \sim \sum_{n=0}^{\infty} \frac{1}{n^{1/2}} \to \infty \tag{28}$$

where we used the fact that a p-series converges only for $p > 1$. This tells us that a 1-dimensional RW returns infinitely often to the origin and this means that the probability to return to the origin at least once is $\rho = 1$ (you can draw the same conclusion looking at Eq. 26).

(c) The generalization to d dimensions is easier than one would think. In fact, you immediately realize that to return to the origin, again, you need an even number of steps and that for each dimension you need an equal number of steps to the 'right' and to the 'left' along that direction. Thus, one can write

$$P(S_{2n}^{(d)} = 0) = \left[\left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n \binom{2n}{n} \right]^d = \frac{1}{2^{2nd}} \binom{2n}{n}^d \sim \frac{1}{n^{d/2}}. \tag{29}$$

where we used the scaling behaviour for $n \to \infty$ that we already computed. Finally:

$$\langle J_{(d)} \rangle \sim \sum_{n=0}^{\infty} \frac{1}{n^{d/2}} \tag{30}$$

and one immediately see that the series converges only for $d > 2$. This means that in $d = 1$ and $d = 2$ the RW returns infinitely often to the origin, while this doesn't hold in higher dimensions. In higher dimensions the RW can still go back to the origin but the probability of doing so at least once is less than one.

To summarize, we learned that a drunk 1-dimensional man always manages to go back home wandering in a random fashion, while a normal drunk man - burdened by the curse of 3-dimensionality - is probably doomed to never return. (Actually, this is not even true, because we usually walk on 2-dimensional surfaces filled with obstacles of any sort. But this is a totally different story.)