0.1 Calculating the resummation coefficients

Now we will calculate the resummation coefficients $f_i(\lambda)$ for the thrust distribution.

In the article by Catani, Turnock, Webber and Trentadue [2], it was observed that for a final state configuration corresponding to a large value of thrust , **??** can be approximated by

$$\tau = 1 - T \approx \frac{k_1^2 + k_2^2}{Q^2} \tag{1}$$

where k_1^2 and k_2^2 are the squared invariant masses of two back-to-back jets and Q^2 is the energy of the center of mass. Thus the key to the evaluation of the thrust distributions is its relation to the quark jet mass distribution $J^q(Q^2, k^2)$, also denoted as $J^q_{k^2}(Q^2)$, which represents the probability of finding a jet originating from quarks, with an invariant mass-squared k^2, produced in collisions with a center-of-mass energy Q^2.

Then the thrust distribution $R_T(\tau, \alpha_s(Q^2))$ **??** takes the form of a convolution of two jet mass distributions $J(Q^2, k_1^2)$ and $J(Q^2, k_2^2)$

$$R_T(\tau, \alpha_s(Q^2)) \underset{\tau \ll 1}{=} \int_0^\infty \mathrm{d}k_1^2 \int_0^\infty \mathrm{d}k_2^2 J^q_{k_1^2}(Q^2) J^q_{k_2^2}(Q^2) \Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) \tag{2}$$

Introducing the Laplace transform of the jet mass distribution:

$$\tilde{J}^q_\nu(Q^2) = \int_0^\infty J^q(Q^2, k^2) e^{-\nu k^2} \mathrm{d}k^2 \tag{3}$$

and using the integral representation of the Heaviside step function:

$$\Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{\mathrm{d}N}{N} e^{N\tau} e^{-N\frac{k_2^2 + k_2^2}{Q^2}} \tag{4}$$

by substituting eq. (4) into eq. (2) and setting $N = \nu Q^2$ we obtain:

$$\begin{aligned} R_T(\tau) &= \int_{C-i\infty}^{C+i\infty} \frac{\mathrm{d}N}{N} \frac{e^{N\tau}}{2\pi i} \left[\int_0^\infty \mathrm{d}k_1^2 e^{-\nu k_1^2} J^q_{k_1^2}(Q^2)\right] \left[\int_0^\infty \mathrm{d}k_2^2 e^{-\nu k_2^2} J^q_{k_2^2}(Q^2)\right] \\ &= \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} e^{N\tau} \left[\tilde{J}^q_\nu(Q^2)\right]^2 \frac{\mathrm{d}N}{N} \end{aligned} \tag{5}$$

where C is a real positive constant to the right of all singularities of the integrand $\tilde{J}_\nu(Q^2)$ in the complex ν plane.

An integral representation for the Laplace transform $\tilde{J}_\nu(Q^2)$ is given by [1]:

$$\ln \tilde{J}^q_\nu(Q^2) = \int_0^1 \frac{\mathrm{d}u}{u}\left(e^{-u\nu Q^2}-1\right)\left[\int_{u^2Q^2}^{uQ^2}\frac{1}{q^2}A\left(\alpha_s(q^2)\right)\mathrm{d}q^2 + \frac{1}{2}B\left(\alpha_s(uQ^2)\right)\right] \tag{6}$$

with

$$A(\alpha_s) = \sum_{n=1}^{\infty} A_n\left(\frac{\alpha_s}{\pi}\right)^n \qquad\qquad B(\alpha_s) = \sum_{n=1}^{\infty} B_n\left(\frac{\alpha_s}{\pi}\right)^n$$

Function $A(\alpha_s)$ is associated with the cusp anomalous dimension and governs the exponentiation of the leading logarithms (LL). It captures the resummation of the soft and collinear gluon emissions that dominate in the limit of large thrust values.

Function $B(\alpha_S)$ includes the next-to-leading logarithmic (NLL) corrections and accounts for subleading contributions from hard collinear emissions. It typically involves the non-cusp part of the anomalous dimensions and running of the coupling constant.

The integral as it is cannot be integrated, the u integration may be performed using the prescription by Paolo Nason in Appendix A of [3] and readapting the formula to the case of Laplace transform instead of Mellin transform.

This method is a generalization of the prescription to NLL accuracy in [1]

$$e^{-u\nu Q^2} - 1 \simeq -\Theta(u-v) \quad \text{with } v = \frac{N_0}{N} \tag{7}$$

where $N_0 = e^{-\gamma_E}$, $\gamma_E = 0.5772\ldots$ being the Euler-Mascheroni constant

In **??** we show that the prescription to evaluate the large-N Mellin moments of soft-gluon contributins at an arbitrary logarithmic accuracy, can be used for the Laplace transform as well, then we can use this result to express **??** in an alternative representation:

$$\ln \tilde{J}^q_\nu(Q^2) = -\int_{N_0/N}^1 \frac{\mathrm{d}u}{u}\left[\int_{u^2Q^2}^{uQ^2}\frac{1}{q^2}A\left(\alpha_s(q^2)\right)\mathrm{d}q^2 + \frac{1}{2}\tilde{B}\left(\alpha_s(uQ^2)\right)\right] + \ln\tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) \tag{8}$$

Note that, due to the integration of the running coupling the integral in **??** is singular for all values of $N = \nu Q^2$. However, if we perform the integration up to a fixed logarithmic accuracy N^kLL ($i.e$ we compute the leading $\alpha_s^n \ln^{n+1} N$, next-to-leading $\alpha_s^n \ln^n N$ and so on to $\alpha_s^n \ln^{n+1-k}$ terms), we find the form factor:

$$\ln \tilde{J}^q_\nu(Q^2) = \ln N f_1(\lambda) + f_2(\lambda) + \alpha_s f_3(\lambda) + \alpha_s^2 f_4(\lambda) + \alpha_s^3 f_5(\lambda) + \mathcal{O}\left(\alpha_s^n \ln^{n-4} N\right)$$
$$+ \ln \tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) \tag{9}$$

From now on i'll always use eq. (8), so i'll drop the ˜ notation for B.

We observe that the N-space formula eq. (8) is finite and uniquely defined up to the very large $N = N_L = \exp\left(\frac{1}{2\alpha_s b_0}\right)$ $(\lambda = \frac{1}{2})$, thanks to the prescription above.

0.1.1 f_i

To calculate explicit expressions for the first few f_i terms, we first write explicitly the internal integral of **??**, for now let's forget the $\ln \tilde{C}$ term, it can be absorbed into the definition of A and B:

The q^2 integration also becomes simple if we use the renormalization group equation **??** to change the integration variable to as

$$\begin{aligned}\frac{\mathrm{d}q^2}{q^2} &= -\frac{\mathrm{d}\alpha_s}{b_0\alpha_s^2}\left(1 - \frac{b_1}{b_0}\alpha_s + \frac{\left(b_1^2 - b_2 b_0\right)}{b_0^2}\alpha_s^2 + \frac{\left(-b_3 b_0^2 + 2b_2 b_1 b_0 - b_1^3\right)}{b_0^3}\alpha_s^3\right.\\ &\quad \left. + \frac{\left(-b_4 b_0^3 + 2b_3 b_1 b_0^2 + b_2^2 b_0^2 - 3b_2 b_1^2 b_0 + b_1^4\right)}{b_0^4}\alpha_s^4 + \mathcal{O}\left(\alpha_s^5\right)\right)\\ &= -\frac{\mathrm{d}\alpha_s}{b_0\alpha_s^2}(N_0 + N_1\alpha_s + N_2\alpha_s^2 + N_3\alpha_s^3 + N_4\alpha_s^4 + \mathcal{O}\left(\alpha_s^5\right))\end{aligned} \tag{10}$$

where for convenience's sake we have defined:

$$\begin{aligned}&N_0 = 1 \qquad N_1 = \frac{b_1}{b_0} \qquad N_2 = \frac{\left(b_1^2 - b_2 b_0\right)}{b_0^2}\\ &N_3 = \frac{\left(-b_3 b_0^2 + 2b_2 b_1 b_0 - b_1^3\right)}{b_0^3}\\ &N_4 = \frac{\left(-b_4 b_0^3 + 2b_3 b_1 b_0^2 + b_2^2 b_0^2 - 3b_2 b_1^2 b_0 + b_1^4\right)}{b_0^4}\end{aligned} \tag{11}$$

Subsequently, the integral in eq. (8) can be expressed as:

$$
\begin{aligned}
&\int_{\alpha_s(u^2Q^2)}^{\alpha_s(uQ^2)} \frac{\mathrm{d}\alpha_s}{b_0\alpha_s^2}(N_0 + N_1\alpha_s + N_2\alpha_s^2 + N_3\alpha_s^3 + N_4\alpha_s^4)\sum_{n=1}^{\infty} A_n\Big(\frac{\alpha_s}{\pi}\Big)^n \\
&= \int_{\alpha_s(u^2Q^2)}^{\alpha_s(uQ^2)} \frac{\mathrm{d}\alpha_s}{b_0}\Bigg(\frac{A_1N_0}{\pi\alpha_s} + \frac{\pi A_1N_1 + A_2}{\pi^2} + \frac{\Big(\pi A_2N_1 + \pi^2 A_1N_2 + A_3\Big)}{\pi^3}\alpha_s \\
&+ \frac{\Big(\pi A_3N_1 + \pi^2 A_2N_2 + \pi^3 A_1N_3 + A_4\Big)}{\pi^4}\alpha_s^2 \\
&+ \frac{\Big(\pi A_4N_1 + \pi^2 A_3N_2 + \pi^3 A_2N_3 + \pi^4 A_1N_4 + A_5\Big)}{\pi^5}\alpha_s^3 + \mathcal{O}\Big(\alpha_s^4\Big)\Bigg)
\end{aligned}
\tag{12}
$$

and keeping only terms up to N_0,A_1 and α_s^0 we get contributions to f_1, keeping terms up to N_1,A_2 and α_s^1 yields contributions to f_2 and so on. That's because after integration, we evaluate the integrand at $\alpha_s(u^2Q^2)$ and $\alpha_s(uQ^2)$, where α_s is α_{LO} for f_1, α_{NLO} for f_2 and so on.

And there's an easy way to see why it's like this, in fact it's also possible to do the q^2 integration directly, using **??????????** from **??** and keeping in mind **??**.

$$
\int_{u^2Q^2}^{uQ^2} \frac{\mathrm{d}q^2}{q^2}\sum_{n=1}^{\infty}\frac{A_n}{\pi^n}\Big(\alpha_{\mathrm{LO}}(q^2) + \delta\alpha_{\mathrm{NLO}}(q^2) + \delta\alpha_{\mathrm{NNLO}}(q^2) + \delta\alpha_{\mathrm{N^3LO}}(q^2) + \delta\alpha_{\mathrm{N^4LO}}(q^2) + \dots\Big)^n
\tag{13}
$$

now we can see that if we consider terms up to α_s^1 only A_1 contributes and this gives f_1, if we consider terms up to α_s^2 we see A_2 starts to contribute together with $A_1\delta\alpha_{\mathrm{NLO}}$ and this gives f_2 , for f_3 we need to consider terms up to α_s^3 and so on.

For the B-term it's similar. However, since the B-term is "already integrated" in q^2, it contributes one order lower in α_s compared to the A-term. Specifically, B_1 starts to contribute from f_2, B_2 from f_3 and so on.

Now armed with this knowledge we can calculate eq. (8) and find:

$$
\ln \tilde{J}^q_\nu(Q^2) = \ln N f_1(\lambda) + f_2(\lambda) + \alpha_s f_3(\lambda) + \alpha_s^2 f_4(\lambda) + \alpha_s^3 f_5(\lambda) + \mathcal{O}\Big(\alpha_s^n \ln^{n-4} N\Big)
\tag{14}
$$

where $\lambda = \alpha_s b_0 \ln N$, $N = \nu Q^2$, $\alpha_s = \alpha_s(Q^2)$ and

$$
f_1(\lambda) = -\frac{A_1}{2\pi b_0\lambda}[(1-2\lambda)\log(1-2\lambda) - 2(1-\lambda)\log(1-\lambda)]
\tag{15}
$$

$$
\begin{aligned}
f_2(\lambda) = &-\frac{A_2}{2\pi^2 b_0^2}[2\log(1-\lambda)-\log(1-2\lambda)] + \frac{B_1\log(1-\lambda)}{2\pi b_0} + \frac{\gamma A_1}{\pi b_0}[\log(1-2\lambda)-\log(1-\lambda)] \\
&- \frac{A_1 b_1}{2\pi b_0^3}\left[-\log^2(1-\lambda)+\frac{1}{2}\log^2(1-2\lambda)-2\log(1-\lambda)+\log(1-2\lambda)\right]
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
f_3(\lambda) = &-\frac{A_3}{2\pi^3 b_0^2(\lambda-1)(2\lambda-1)}\lambda^2 + \frac{B_2}{2\pi^2 b_0(\lambda-1)}\lambda + \frac{B_2\lambda}{2\pi^2 b_0(\lambda-1)} \\
&+ \frac{b_1 A_2}{2\pi^2 b_0^3(\lambda-1)(2\lambda-1)}\left[3\lambda^2+(1-\lambda)\log(1-2\lambda)-2(1-2\lambda)\log(1-\lambda)\right] \\
&- \frac{\gamma A_2\lambda}{\pi^2 b_0(\lambda-1)(2\lambda-1)} - \frac{B_1 b_1}{2\pi b_0^2(\lambda-1)}[\lambda+\log(1-\lambda)] + \frac{\gamma B_1\lambda}{2\pi(\lambda-1)} + \frac{\gamma^2 A_1\lambda(2\lambda-3)}{2\pi(\lambda-1)(2\lambda-1)} \\
&+ \frac{b_1\gamma A_1}{\pi b_0^2(\lambda-1)(2\lambda-1)}[-\lambda+(1-2\lambda)\log(1-\lambda)-(1-\lambda)\log(1-2\lambda)] \\
&- \frac{b_1^2 A_1}{2\pi b_0^4(\lambda-1)(2\lambda-1)}\left[\left(\lambda^2+(2\lambda-1)\log(1-\lambda)(2\lambda+\log(1-\lambda))\right)\right. \\
&\left.+\frac{1}{2}\left((1-\lambda)\log^2(1-2\lambda)\right)-2(\lambda-1)\lambda\log(1-2\lambda)\right] \\
&- \frac{b_0 b_2 A_1}{2\pi b_0^4(\lambda-1)(2\lambda-1)}\left[\lambda^2+(\lambda-1)(2\lambda-1)(2\log(1-\lambda)+\log(1-2\lambda))\right]
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
f_4(\lambda) = &-\frac{A_4\lambda^2\left(2\lambda^2-6\lambda+3\right)}{6\pi^4 b_0^2(\lambda-1)^2(2\lambda-1)^2} + \frac{B_3(\lambda-2)\lambda}{4\pi^3 b_0(\lambda-1)^2} + \frac{b_1 A_3}{12\pi^3 b_0^3(\lambda-1)^2(2\lambda-1)^2}\left[15\lambda^2\right. \\
&\left.+10(\lambda-3)\lambda^3+3(\lambda-1)^2\log(1-2\lambda)-6(1-2\lambda)^2\log(1-\lambda)\right] \\
&+ \frac{\gamma A_3\lambda(3\lambda-2)}{2\pi^3 b_0(\lambda-1)^2(2\lambda-1)^2} + \frac{\gamma B_2(\lambda-2)\lambda}{2\pi^2(\lambda-1)^2} - \frac{b_1 B_2\left[\lambda^2-2\lambda-2\log(1-\lambda)\right]}{4\pi^2 b_0^2(\lambda-1)^2} \\
&+ \frac{\gamma^2 A_2\lambda\left(4\lambda^3-12\lambda^2+15\lambda-6\right)}{2\pi^2(\lambda-1)^2(2\lambda-1)^2} + \frac{b_1\gamma A_2}{2\pi^2 b_0^2(\lambda-1)^2(2\lambda-1)^2}\left[\lambda(2-3\lambda)\right. \\
&\left.+2(\lambda-1)^2\log(1-2\lambda)-2(1-2\lambda)^2\log(1-\lambda)\right] + \frac{b_2 A_2\lambda^3(4\lambda-3)}{3\pi^2 b_0^3(\lambda-1)^2(2\lambda-1)^2} \\
&- \frac{b_1^2 A_2}{12\pi^2 b_0^4(\lambda-1)^2(2\lambda-1)^2}\left[\lambda^2\left(22\lambda^2-30\lambda+9\right)+3(\lambda-1)^2\log^2(1-2\lambda)\right. \\
&\left.+3(\lambda-1)^2\log(1-2\lambda)-6(1-2\lambda)^2\log(1-\lambda)(\log(1-\lambda)+1)\right] \\
&+ \frac{b_0\gamma^2 B_1(\lambda-2)\lambda}{4\pi(\lambda-1)^2} + \frac{b_1\gamma B_1\log(1-\lambda)}{2\pi b_0(\lambda-1)^2} + \frac{b_2 B_1\lambda^2}{4\pi b_0^2(\lambda-1)^2} \\
&+ \frac{b_1^2 B_1(\lambda-\log(1-\lambda))(\lambda+\log(1-\lambda))}{4\pi b_0^3(\lambda-1)^2} + \frac{b_0\gamma^3 A_1\lambda\left(12\lambda^3-36\lambda^2+39\lambda-14\right)}{6\pi(\lambda-1)^2(2\lambda-1)^2}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&+\frac{b_1\gamma^2 A_1}{2\pi b_0(\lambda-1)^2(2\lambda-1)^2}\left[2(\lambda-1)^2\log(1-2\lambda)-(1-2\lambda)^2\log(1-\lambda)\right]\\
&-\frac{b_1^2\gamma A_1}{2\pi b_0^3 A_1(\lambda-1)^2(2\lambda-1)^2}\left[(4\lambda-3)\lambda^2-(1-2\lambda)^2\log^2(1-\lambda)+(\lambda-1)^2\log^2(1-2\lambda)\right]\\
&-\frac{b_1^3 A_1}{12\pi b_0^5(\lambda-1)^2(2\lambda-1)^2}\Big[4(3-4\lambda)\lambda^3+2(1-2\lambda)^2\log(1-\lambda)\left(\log^2(1-\lambda)-3\lambda^2\right)\\
&+12(\lambda-1)^2\lambda^2\log(1-2\lambda)-(\lambda-1)^2\log^3(1-2\lambda)\Big]+\frac{b_2\gamma A_1\lambda^2(4\lambda-3)}{2\pi b_0^2(\lambda-1)^2(2\lambda-1)^2}\\
&\frac{b_1b_2A_1}{12\pi b_0^4(\lambda-1)^2(2\lambda-1)^2}\Big[\lambda^2(2\lambda(3-7\lambda)+3)+3\left(8\lambda^2-4\lambda+1\right)(\lambda-1)^2\log(1-2\lambda)\\
&-6(1-2\lambda)^2(2(\lambda-1)\lambda+1)\log(1-\lambda)\Big]+-\frac{b_3A_1}{12\pi b_0^3(\lambda-1)^2(2\lambda-1)^2}\Big[(2(\lambda-3)\lambda+3)\lambda^2\\
&-6\left(2\lambda^2-3\lambda+1\right)^2\log(1-\lambda)+3\left(2\lambda^2-3\lambda+1\right)^2\log(1-2\lambda)\Big]
\end{aligned}
$$

$$
\begin{aligned}
f_5(\lambda)=&-\frac{A_5\lambda^2\left(4\lambda^4-18\lambda^3+33\lambda^2-24\lambda+6\right)}{12\pi^5 b_0^2(\lambda-1)^3(2\lambda-1)^3}+\frac{B_4\lambda\left(\lambda^2-3\lambda+3\right)}{6\pi^4 b_0(\lambda-1)^3}\\
&-\frac{\gamma A_4\lambda\left(7\lambda^2-9\lambda+3\right)}{3\pi^4 b_0(\lambda-1)^3(2\lambda-1)^3}+\frac{b_1A_4}{36\pi^4 b_0^3(\lambda-1)^3(2\lambda-1)^3}\Big[7\lambda^2(\lambda(\lambda(2\lambda(2\lambda-9)+33)-24)\\
&+6)-6(\lambda-1)^3\log(1-2\lambda)+12(2\lambda-1)^3\log(1-\lambda)\Big]+\frac{\gamma B_3\lambda\left(\lambda^2-3\lambda+3\right)}{2\pi^3(\lambda-1)^3}+\\
&\frac{b_1B_3\left(\lambda^3-3\lambda^2+3\lambda+3\log(1-\lambda)\right)}{6\pi^3 b_0^2(\lambda-1)^3}+\frac{\gamma^2A_3\lambda\left(8\lambda^5-36\lambda^4+66\lambda^3-69\lambda^2+39\lambda-9\right)}{2\pi^3(\lambda-1)^3(2\lambda-1)^3}\\
&-\frac{b_1\gamma A_3}{3\pi^3 b_0^2(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda((9-7\lambda)\lambda-3)+3(\lambda-1)^3\log(1-2\lambda)\\
&-3(2\lambda-1)^3\log(1-\lambda)\Big]+\frac{b_2A_3\lambda^3\left(4\lambda^3-18\lambda^2+19\lambda-6\right)}{4\pi^3 b_0^3(\lambda-1)^3(2\lambda-1)^3}-\\
&\frac{b_1^2A_3}{36\pi^3 b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^3(\lambda(26\lambda(2\lambda-9)+303)-150)-9(\lambda-1)^3\log^2(1-2\lambda)\\
&+24\lambda^2-6(\lambda-1)^3\log(1-2\lambda)+6(2\lambda-1)^3\log(1-\lambda)(3\log(1-\lambda)+2)\Big]\\
&+\frac{b_0\gamma^2B_2\lambda\left(\lambda^2-3\lambda+3\right)}{2\pi^2(\lambda-1)^3}-\frac{b_1\gamma B_2\log(1-\lambda)}{\pi^2 b_0(\lambda-1)^3}+\frac{b_1^2B_2\left(\lambda^3-3\lambda^2+3\log^2(1-\lambda)\right)}{6\pi^2 b_0^3(\lambda-1)^3}\\
&-\frac{b_2B_2(\lambda-3)\lambda^2}{6\pi^2 b_0^2(\lambda-1)^3}+\frac{b_0\gamma^3A_2\lambda\left(24\lambda^5-108\lambda^4+198\lambda^3-193\lambda^2+99\lambda-21\right)}{3\pi^2(\lambda-1)^3(2\lambda-1)^3}\\
&+\frac{b_1^2\gamma A_2}{3\pi^2 b_0^3(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(18\lambda-25)+9)+3(\lambda-1)^3\log^2(1-2\lambda)
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&- 3(2\lambda-1)^3\log^2(1-\lambda)\Big] - \frac{b_2\gamma A_2\lambda^2\left(18\lambda^2-25\lambda+9\right)}{3\pi^2 b_0^2(\lambda-1)^3(2\lambda-1)^3} \\
&+ \frac{b_1^3 A_2}{36\pi^2 b_0^5(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(2\lambda(50\lambda-171)+339)-114)+6) \\
&- 6(\lambda-1)^3\log^3(1-2\lambda) + 12(2\lambda-1)^3\log(1-\lambda)\left(-3\lambda+\log^2(1-\lambda)+1\right) \\
&+ 6(6\lambda-1)(\lambda-1)^3\log(1-2\lambda)\Big] + \frac{b_3 A_2\lambda^3\left(20\lambda^3-54\lambda^2+45\lambda-12\right)}{12\pi^2 b_0^3(\lambda-1)^3(2\lambda-1)^3} \\
&- \frac{b_1\gamma^2 A_2}{\pi^2 b_0(\lambda-1)^3(2\lambda-1)^3}\Big[2(\lambda-1)^3\log(1-2\lambda) - (2\lambda-1)^3\log(1-\lambda)\Big] \\
&- \frac{b_2 b_1 A_2}{18\pi^2 b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(4\lambda(20\lambda-63)+237)-75)+3) \\
&+ 3(6\lambda-1)(\lambda-1)^3\log(1-2\lambda) - 6(2\lambda-1)^3(3\lambda-1)\log(1-\lambda)\Big] \\
&+ \frac{b_0^2\gamma^3 B_1\lambda\left(\lambda^2-3\lambda+3\right)}{6\pi(\lambda-1)^3} - \frac{b_1^2\gamma B_1\left(\lambda-\log^2(1-\lambda)+\log(1-\lambda)\right)}{2\pi b_0^2(\lambda-1)^3} - \\
&\frac{b_1^3 B_1}{12\pi b_0^4(\lambda-1)^3}(\lambda+\log(1-\lambda))\left(2\lambda^2-3\lambda+2\log^2(1-\lambda)-2\lambda\log(1-\lambda)-3\log(1-\lambda)\right) \\
&+ \frac{b_2\gamma B_1\lambda}{2\pi b_0(\lambda-1)^3} - \frac{b_3 B_1\lambda^2(2\lambda-3)}{12\pi b_0^2(\lambda-1)^3} + \frac{b_1\gamma^2 B_1\left(\lambda^3-3\lambda^2+3\lambda-2\log(1-\lambda)\right)}{4\pi(\lambda-1)^3} \\
&+ \frac{b_2 b_1 B_1\lambda\left(2\lambda^2-3\lambda-3\log(1-\lambda)\right)}{6\pi b_0^3(\lambda-1)^3} + \frac{b_0^2\gamma^4 A_1}{12\pi(\lambda-1)^3(2\lambda-1)^3}\lambda(2\lambda-3)\Big[28\lambda^4 \\
&- 84\lambda^3+105\lambda^2-63\lambda+15\Big] + \frac{b_1^3\gamma A_1}{6\pi b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(4\lambda(3(\lambda-3)\lambda+8)-9) \\
&- 2(\lambda-1)^3\log^3(1-2\lambda) + 3(\lambda-1)^3\log^2(1-2\lambda) + 12\lambda(\lambda-1)^3\log(1-2\lambda) \\
&+ (2\lambda-1)^3\log(1-\lambda)(\log(1-\lambda)(2\log(1-\lambda)-3)-6\lambda)\Big] \\
&+ \frac{b_1^4 A_1}{24\pi b_0^6(\lambda-1)^3(2\lambda-1)^3}\Big[8(\lambda-1)^3\lambda^2(4\lambda-3)\log(1-2\lambda) \\
&- 2(2\lambda-1)^3\log(1-\lambda)\left(2(2\lambda-3)\lambda^2+\log(1-\lambda)((\log(1-\lambda)-2)\log(1-\lambda)-6\lambda)\right) \\
&+ 2\lambda^3\left(\lambda\left(-28\lambda^2+54\lambda-33\right)+6\right) + (\lambda-1)^3\log^4(1-2\lambda) - 2(\lambda-1)^3\log^3(1-2\lambda) - \\
&12(\lambda-1)^3\lambda\log^2(1-2\lambda)\Big] - \frac{b_2\gamma^2 A_1\lambda\left(4\lambda^3-6\lambda+3\right)}{2\pi b_0(\lambda-1)^3(2\lambda-1)^3} + \frac{b_3\gamma\lambda^2\left(12\lambda^3-36\lambda^2+32\lambda-9\right)}{6\pi b_0^2(\lambda-1)^3(2\lambda-1)^3} \\
&+ \frac{b_2^2 A_1}{36\pi b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(6-\lambda(\lambda(2\lambda(10\lambda+9)-69)+42)) \\
&- 12\left(2\lambda^2-3\lambda+1\right)^3\log(1-\lambda) + 6\left(2\lambda^2-3\lambda+1\right)^3\log(1-2\lambda)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{b_1^2\gamma^2 A_1}{2\pi b_0^2(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda\left(4\lambda^3-6\lambda+3\right)+2(\lambda-1)^3\log^2(1-2\lambda)\\
&-2(\lambda-1)^3\log(1-2\lambda)-(2\lambda-1)^3(\log(1-\lambda)-1)\log(1-\lambda)\Big]\\
&-\frac{b_2 b_1^2 A_1}{36\pi b_0^5(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(2(153-82\lambda)\lambda-165)+12)+6)\\
&-18\lambda(\lambda-1)^3\log^2(1-2\lambda)+6\left(6(1-2\lambda)^2\lambda-1\right)(\lambda-1)^3\log(1-2\lambda)\\
&-6(2\lambda-1)^3\log(1-\lambda)\left(6\lambda(\lambda-1)^2-3\lambda\log(1-\lambda)-2\right)\Big]\\
&+\frac{b_1\gamma^3 A_1}{6\pi(\lambda-1)^3(2\lambda-1)^3}\Big[24\lambda^6-108\lambda^5+198\lambda^4-193\lambda^3+16\lambda^3\log(1-\lambda)\\
&-8\lambda^3\log(1-2\lambda)+99\lambda^2-24\lambda^2\log(1-\lambda)+24\lambda^2\log(1-2\lambda)-21\lambda\\
&+12\lambda\log(1-\lambda)-24\lambda\log(1-2\lambda)-2\log(1-\lambda)+8\log(1-2\lambda)\Big]\\
&-\frac{b_2 b_1\gamma A_1}{3\pi b_0^3(\lambda-1)^3(2\lambda-1)^3}\lambda\Big[\lambda(4\lambda(3(\lambda-3)\lambda+8)-9)+6(\lambda-1)^3\log(1-2\lambda)\\
&-3(2\lambda-1)^3\log(1-\lambda)\Big]+\frac{b_3 b_1 A_1}{18\pi b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^3(\lambda(4(18-7\lambda)\lambda-51)+6)\\
&+3\lambda^2+3(2\lambda(\lambda(8\lambda-9)+3)-1)(\lambda-1)^3\log(1-2\lambda)\\
&-3(2\lambda-1)^3(\lambda(\lambda(4\lambda-9)+6)-2)\log(1-\lambda)\Big]\\
&-\frac{b_4 A_1}{36\pi b_0^3(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(2\lambda(2\lambda-9)+33)-24)+6)\\
&-12\left(2\lambda^2-3\lambda+1\right)^3\log(1-\lambda)+6\left(2\lambda^2-3\lambda+1\right)^3\log(1-2\lambda)\Big]
\end{aligned}
$$

In eqs. (17) to (19) we have removed some constant terms in order to make them homogeneous, *i.e.* $f_i(0)=0$, those constant can be reabsorbed in the C-term **??**. All the relevant constant can be found in **??**.

To obtain the laplace trasform of the thrust distribution **??**, we perform the laplace transform of the convolution eq. (2). By applying the convolution theorem, we find that it is twice the integral given by **??** that we have just calculated. Therefore, we multiply by 2 the $f_i(\lambda)$ we just obtained eqs. (15) to (19).

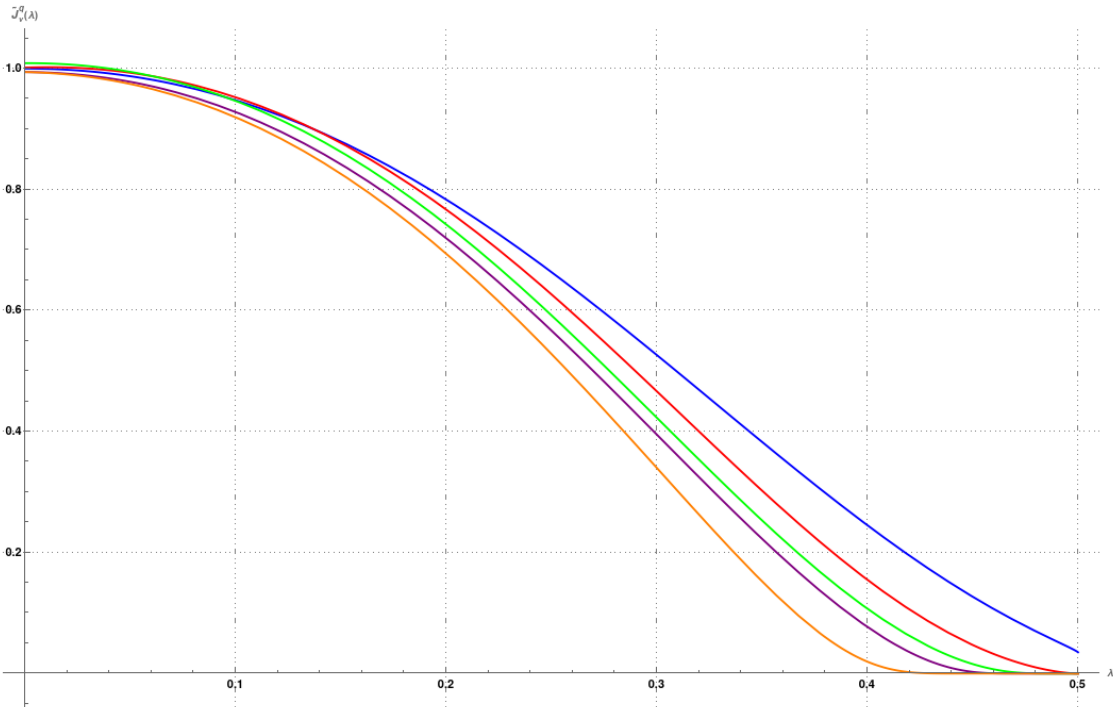

Figure 1: Plot of eq. (9) at different logarithmin orders.