1 Inversion of the Laplace transform

In order to find the quark jet mass distribution $J^q(Q^2, k^2)$, we have to perform the inverse Laplace transform via the Mellin's inversion formula (or the Bromwich integral) given by the line integral:

$$J^q(Q^2, k^2) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{C-iT}^{C+iT} d\nu \, e^{\nu k^2} \tilde{J}_\nu^q(Q^2) \tag{1}$$

where C is a real number such that C is at the right of all singularities of the integrand in the complex plane and the function $\tilde{J}_\nu^q(Q^2)$ has to be bounded on the line.

Instead of directly considering the expression in eq. (1), it was pointed in [2] that it is more convenient to work with the mass fraction $R^q(w)$, which gives the fraction of jets with masses less than wQ^2:

$$R^q(w) = \int_0^\infty J^q(Q^2, k^2) \Theta(wQ^2 - k^2) dk^2 \tag{2}$$

and using the integral representation of the Heaviside step function **??**

$$\Theta(wQ^2 - k^2) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{C-iT}^{C+iT} \frac{d\nu}{\nu} e^{\nu(wQ^2 - k^2)} \tag{3}$$

we recognize the Laplace transform of the quark jet mass distribution **??**

$$\begin{aligned}
R^q(w) &= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{C-iT}^{C+iT} \frac{d\nu}{\nu} e^{w\nu Q^2} \int_0^\infty J^q(Q^2, k^2) e^{-\nu k^2} dk^2 \\
&= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{C-iT}^{C+iT} \frac{d\nu}{\nu} e^{w\nu Q^2} \tilde{J}_\nu^q(Q^2) \\
&= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{C-iT}^{C+iT} \frac{d\nu}{\nu} e^{w\nu Q^2} e^{\mathcal{F}(\alpha_s, \ln(\nu Q^2))} \\
&= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{C'-iT}^{C'+iT} \frac{dN}{N} e^{wN} e^{\mathcal{F}(\alpha_s, \ln N)}
\end{aligned} \tag{4}$$

where $N = \nu Q^2$ and $\mathcal{F}$ has the logarithms expansion

$$
\begin{aligned}
\mathcal{F}(\alpha_s, \ln N) = {} & f_1(b_0 \alpha_s \ln N) \ln N + f_2(b_0 \alpha_s \ln N) + f_3(b_0 \alpha_s \ln N)\alpha_s \\
& + f_4(b_0 \alpha_s \ln N)\alpha_s^2 + f_5(b_0 \alpha_s \ln N)\alpha_s^3 + \mathcal{O}\!\left(\alpha_s^4\right)
\end{aligned}
\tag{5}
$$

Since the function $\mathcal{F}$ in the exponent varies more slowly with N than wN, we can introduce the integration variable $u = wN$ so that $\ln N = \ln u + \ln \frac{1}{w} = \ln u + L$ and Taylor expand with respect to $\ln u$ around 0, which is equivalent to expanding the original function $\mathcal{F}$ w.r.t $\ln N$ around $\ln N = \ln \frac{1}{w} \equiv L$:

$$
\begin{aligned}
R^q(w) = {} & \frac{1}{2\pi i} \int_C \frac{\mathrm{d}u}{u} e^u e^{\mathcal{F}(\alpha_s, \ln u + L)} \\
& \overset{\text{Taylor}}{=} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\mathcal{F}(\alpha_s, L) + \sum_{n=1}^\infty \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \\
& = e^{\mathcal{F}(\alpha_s, L)} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\sum_{n=1}^\infty \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u}
\end{aligned}
\tag{6}
$$

where the integral is intended as before, along the line C to the right of all singularities of the integrand, and

$$
\mathcal{F}^{(n)}(\alpha_s, L) = \left. \frac{\partial^n \mathcal{F}(\alpha_s, \ln u + L)}{\partial \ln u^n} \right|_{\ln u = 0}
\tag{7}
$$

As noticed in [2], the n-th derivative of $\mathcal{F}$ w.r.t $\ln u$ evaluated at $\ln u = 0$ is at most of logarithmic order $\alpha_s^{n+k-1} L^k$, so in order to achieve $N^4 LL$ accuracy we need to compute the first four derivatives of $\mathcal{F}$ w.r.t $\ln u$ and neglect the terms of order $\mathcal{O}\!\left(\alpha_s^4\right)$ that appear in the derivation. We obtain the following expressions:

$$
\begin{aligned}
\mathcal{F}^{(1)}(\alpha_s, L) = {} & f_1(\lambda) + \lambda f_1'(\lambda) + \alpha_s b_0 f_2'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s^3 b_0 f_4'(\lambda) \\
& + \mathcal{O}\!\left(\alpha_s^n L^{n-3}\right)
\end{aligned}
\tag{8}
$$

$$\mathcal{F}^{(2)}(\alpha_s, L) = 2\alpha_s b_0 f_1'(\lambda) + \alpha_s b_0 \lambda f_1''(\lambda) + \alpha_s^2 b_0^2 f_2''(\lambda) + \alpha_s^3 b_0^2 f_3''(\lambda)$$
$$+ \mathcal{O}(\alpha_s^n L^{n-3}) \tag{9}$$

$$\mathcal{F}^{(3)}(\alpha_s, L) = 3\alpha_s^2 b_0^2 f_1''(\lambda) + \alpha_s^2 b_0^2 \lambda f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 f_2^{(3)}(\lambda) + \mathcal{O}(\alpha_s^n L^{n-3}) \tag{10}$$

$$\mathcal{F}^{(4)}(\alpha_s, L) = 4\alpha_s^3 b_0^3 f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 \lambda f_1^{(4)}(\lambda) + \mathcal{O}(\alpha_s^n L^{n-3}) \tag{11}$$

Here $\lambda = \alpha_s b_0 L$ and derivative w.r.t $\ln u$ and then evaluated at $\ln u = 0$, or equivalently derivative w.r.t L gives the same result.

After recasting the expansion presented in eq. (6) using the expression $\gamma(\alpha_s, L) = f_1(\lambda) + \lambda f_1'(\lambda)$ from [2], and defining $\mathcal{F}_{res}^{(1)}(\alpha_s, L) \equiv \mathcal{F}^{(1)}(\alpha_s, L) - \gamma(\alpha_s, L)$, we proceed to expand the second exponential with respect to $\ln u$ around 0, following the approach outlined in [1]. This yields the subsequent expansion:

$$\begin{aligned}
R^q(w) &= e^{\mathcal{F}(\alpha_s, L)} \int_C \frac{du}{2\pi i} e^{u - (1 - \gamma(\alpha_s, L)) \ln u} e^{\mathcal{F}_{res}^{(1)}(\alpha_s, L) \ln u + \sum_{n=2}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \\
&= \int_C \frac{du}{2\pi i} e^{u - (1 - \gamma(\alpha_s, L)) \ln u} \left(1 + \mathcal{F}_{res}^{(1)} \ln u + \frac{1}{2} \left(\mathcal{F}^{(2)} + (\mathcal{F}_{res}^{(1)})^2 \right) \ln^2 u \right. \\
&+ \frac{1}{6} \left(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)} \mathcal{F}_{res}^{(1)} + (\mathcal{F}_{res}^{(1)})^3 \right) \ln^3 u \\
&+ \frac{1}{24} \left(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)} \mathcal{F}_{res}^{(1)} + 6\mathcal{F}^{(2)} (\mathcal{F}_{res}^{(1)})^2 + (\mathcal{F}_{res}^{(1)})^4 \right) \ln^4 u \\
&\left. + \mathcal{O}(\ln^5 u) \right)
\end{aligned} \tag{12}$$

Lastly, we utilize the following result to evaluate the integral presented in eq. (2).

$$\int_C \frac{du}{2\pi i} \ln^k u \, e^{u - (1 - \gamma(\alpha_s, L)) \ln u} = \frac{d^k}{d\gamma^k} \frac{1}{\Gamma(1 - \gamma(\alpha_s, L))} \tag{13}$$

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where Γ is the Euler Γ-function, notice that for $k = 0$ the integral is the Hankel integral representation of the Γ-function:

$$\frac{1}{2\pi i} \int_C \mathrm{d}u e^u u^{-(1-\gamma(\alpha_s, L))} = \frac{1}{\Gamma(1 - \gamma(\alpha_s, L))} \tag{14}$$

The reciprocal of the Euler Γ-function is an entire function, meaning that it has no singularities in the complex plane, it's holomorphic everywhere. The above integral eq. (13) can be evaluated by differentiating with respect to γ k times, and it is straightforward to see that each time i derive inside the integral i will get a factor of $\ln u$. Interchaning the order of derivation and integration is justified by the fact that the integral:

- is a contour integral, and the function is holomorphic in the complex plane for each value of $\gamma \in [-2, 0]$

- is dominated by the exponential factor $e^{u-(1-gamma)\ln u}$ as long as $\mathrm{Re}\{u\} > 0$

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s, L)}}{\Gamma(1-\gamma)} \Bigg[& 1 + \mathcal{F}_{res}^{(1)} \psi_0 (1 - \gamma) + \frac{1}{2} \left(\mathcal{F}^{(2)} + (\mathcal{F}_{res}^{(1)})^2 \right) \left(\psi_0^2 - \psi_1 \right)(1 - \gamma) \\
& + \frac{1}{6} \left(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)} \mathcal{F}_{res}^{(1)} + (\mathcal{F}_{res}^{(1)})^3 \right) \left(\psi_0^3 - 3\psi_0 \psi_1 + \psi_2 \right)(1 - \gamma) \\
& + \frac{1}{24} \left(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)} \mathcal{F}_{res}^{(1)} + 6\mathcal{F}^{(2)} (\mathcal{F}_{res}^{(1)})^2 + (\mathcal{F}_{res}^{(1)})^4 \right) \\
& \left(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0 \psi_2 - \psi_3 \right)(1 - \gamma) + \mathcal{O}\left(\ln^5 u \right) \Bigg]
\end{aligned}
\tag{15}
$$

where $\psi_n(z)$ are the polygamma functions, defined as:

$$\psi_n(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} \ln \Gamma(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} \frac{\Gamma'(z)}{\Gamma(z)} = \frac{\mathrm{d}^n}{\mathrm{d}z^n} \psi_0(z) \tag{16}$$

Substituting the expressions eqs. (8) to (11) into eq. (15) we obtain:

$$R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Bigg[1 + \Big(\alpha_s^3 b_0 f_4'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s b_0 f_2'(\lambda)\Big)\psi_0(1-\gamma)$$

$$+ \frac{1}{2}\Big(\alpha_s^3\big(2b_0^2 f_2'(\lambda) f_3'(\lambda) + b_0^2 f_3''(\lambda)\big) + \frac{1}{2}\alpha_s^2\big(b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2\big)$$

$$+ \frac{1}{2}\alpha_s\big(b_0\lambda f_1''(\lambda) + 2b_0 f_1'(\lambda)\big)\Big)\big(\psi_0^2 - \psi_1\big)(1-\gamma)$$

$$+ \frac{1}{6}\Big(\alpha_s^3\big(b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda) f_2''(\lambda) + 3b_0^2\lambda f_1''(\lambda) f_3'(\lambda)$$

$$+ 6b_0^2 f_1'(\lambda) f_3'(\lambda)\big) + \frac{1}{6}\alpha_s^2\big(b_0^2\lambda f_1^{(3)}(\lambda) + 3b_0^2\lambda f_1''(\lambda) f_2'(\lambda) + 3b_0^2 f_1''(\lambda)$$

$$+ 6b_0^2 f_1'(\lambda) f_2'(\lambda)\big)\Big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma)$$

$$+ \frac{1}{24}\Big(\alpha_s^3\big(b_0^3\lambda f_1^{(4)}(\lambda) + 4b_0^3\lambda f_1^{(3)}(\lambda) f_2'(\lambda)$$

$$+ 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3\lambda f_1''(\lambda) f_2''(\lambda) + 6b_0^3\lambda f_1''(\lambda) f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda) f_2'(\lambda)$$

$$+ 12b_0^3 f_1'(\lambda) f_2''(\lambda) + 12b_0^3 f_1'(\lambda) f_2'(\lambda)^2\big) + \frac{1}{8}\alpha_s^2\big(b_0^2\lambda^2 f_1''(\lambda)^2 + 4b_0^2 f_1'(\lambda)^2$$

$$+ 4b_0^2\lambda f_1'(\lambda) f_1''(\lambda)\big)\Big)\big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Bigg]$$

$$(17)$$

and reorganize as a power series of α_s:

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)} \bigg[& 1 + \alpha_s b_0 \Big(\psi_0(1-\gamma) f_2'(\lambda) + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)\big(\lambda f_1''(\lambda) \\
& + 2f_1'(\lambda)\big) \Big) + \alpha_s^2 \Big(\frac{1}{8}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1-\gamma)\big(b_0^2\lambda^2 f_1''(\lambda)^2 \\
& + 4b_0^2 f_1'(\lambda)^2 + 4b_0^2\lambda f_1'(\lambda) f_1''(\lambda)\big) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1-\gamma)\big(b_0^2\lambda f_1^{(3)}(\lambda) \\
& + 3b_0^2\lambda f_1''(\lambda) f_2'(\lambda) + 3b_0^2 f_1''(\lambda) + 6b_0^2 f_1'(\lambda) f_2'(\lambda)\big) + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma) \\
& \big(b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2\big) + b_0\psi_0(1-\gamma) f_3'(\lambda) \Big) + \alpha_s^3 \Big(\frac{1}{24}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 \\
& - \psi_3)(1-\gamma)\big(b_0^3\lambda f_1^{(4)}(\lambda) + 4b_0^3\lambda f_1^{(3)}(\lambda) f_2'(\lambda) + 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3\lambda f_1''(\lambda) f_2'(\lambda) \\
& + 6b_0^3\lambda f_1''(\lambda) f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda) f_2'(\lambda) + 12b_0^3 f_1'(\lambda) f_2''(\lambda) + 12b_0^3 f_1'(\lambda) f_2'(\lambda)^2\big) \\
& + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)\big(2b_0^2 f_2'(\lambda) f_3'(\lambda) + b_0^2 f_3''(\lambda)\big) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 \\
& + \psi_2)(1-\gamma)\big(b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda) f_2''(\lambda) + 3b_0^2\lambda f_1''(\lambda) f_3'(\lambda) \\
& + 6b_0^2 f_1'(\lambda) f_3'(\lambda)\big) + b_0\psi_0(1-\gamma) f_4'(\lambda) \Big) + \mathcal{O}\big(\alpha_s^4\big) \bigg]
\end{aligned}
$$

$$(18)$$

By comparing **??** and eq. (4) we see that to obtain the Thrust cross section, we simply multiply by 2 the inverse Laplace transform obtained. Following the convention of [2], the final resummed expression $R_T(\tau)$ can be written as one exponential eq. (19),

$$
R_T(\tau) = \exp\big\{ L g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \alpha_s^3 g_5(\lambda) + \mathcal{O}\big(\alpha_s^4\big) \big\} \qquad (19)
$$

to do so we observe that $\frac{1}{\Gamma(1-\gamma)} = \exp\{-\ln(\Gamma(1-\gamma))\}$ corrects f_2, while the expression in parenthesis starting with 1 can be seen as the expansion of an exponential for $\alpha_s \to 0$:

$$e^{\alpha_s g_3(\lambda)+\alpha_s^2 g_4(\lambda)+\alpha_s^3 g_5(\lambda)+\mathcal{O}\left(\alpha_s^4\right)} = 1 + \alpha_s g_3(\lambda) + \frac{1}{2}\alpha_s^2\left(g_3^2(\lambda) + 2g_4(\lambda)\right)$$
$$+ \frac{1}{6}\alpha_s^3\left(g_3^3(\lambda) + 6g_3(\lambda)g_4(\lambda) + 6g_5(\lambda)\right) + \mathcal{O}\left(\alpha_s^4\right) \tag{20}$$

to obtain $g_3(\lambda)$, $g_4(\lambda)$ and $g_5(\lambda)$ we match eq. (18) with eq. (20) and obtain the following expressions:

$$g_1(\lambda) = 2f_1(\lambda) \tag{21}$$

$$g_2(\lambda) = 2f_2(\lambda) - \ln\Gamma(1 - 2f_1(\lambda) - 2\lambda f_1{}^{'}(\lambda)) \tag{22}$$

$$g_3(\lambda) = 2f_3(\lambda) + 2\left(\psi_0(1-\gamma)f_2'(\lambda) + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)\left(\lambda f_1''(\lambda) + 2f_1'(\lambda)\right)\right) \tag{23}$$

$$g_4(\lambda) = 2f_4(\lambda) + 2\Bigg(\frac{1}{8}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1-\gamma)\big(b_0^2\lambda^2 f_1''(\lambda)^2$$
$$+ 4b_0^2 f_1'(\lambda)^2 + 4b_0^2\lambda f_1'(\lambda)f_1''(\lambda)\big) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1-\gamma)\big(b_0^2\lambda f_1^{(3)}(\lambda)$$
$$+ 3b_0^2\lambda f_1''(\lambda)f_2'(\lambda) + 3b_0^2 f_1''(\lambda) + 6b_0^2 f_1'(\lambda)f_2'(\lambda)\big) + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)$$
$$\big(b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2\big) + b_0\psi_0(1-\gamma)f_3'(\lambda)\Bigg) \tag{24}$$

$$g_5(\lambda) = 2f_5(\lambda) + 2\Bigg(\frac{1}{24}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1-\gamma)$$
$$\big(b_0^3\lambda f_1^{(4)}(\lambda) + 4b_0^3\lambda f_1^{(3)}(\lambda)f_2'(\lambda) + 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3\lambda f_1''(\lambda)f_2''(\lambda)$$
$$+ 6b_0^3\lambda f_1''(\lambda)f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda)f_2'(\lambda) + 12b_0^3 f_1'(\lambda)f_2''(\lambda) + 12b_0^3 f_1'(\lambda)f_2'(\lambda)^2\big)$$
$$+ \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)\big(2b_0^2 f_2'(\lambda)f_3'(\lambda) + b_0^2 f_3''(\lambda)\big) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1-\gamma)$$
$$\big(b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda)f_2''(\lambda) + 3b_0^2\lambda f_1''(\lambda)f_3'(\lambda) + 6b_0^2 f_1'(\lambda)f_3'(\lambda)\big)$$
$$+ b_0\psi_0(1-\gamma)f_4'(\lambda)\Bigg) \tag{25}$$