Chapter 1

Resummed Calculations for Thrust

We now turn to one of the many results of the present work: the calculation of the resummation coefficients $f_i(\lambda)$ of the Sudakov form factor $\exp\{\mathcal{F}\}$. To compute the $f_i(\lambda)$ functions, equation **??** is used, which requires knowledge of the μ-dependence of the QCD running coupling $\alpha_s(\mu)$. We therefore proceed firstly to calculate the running coupling $\alpha_s(\mu)$ from LO up to N^{4}LO because the QCD β-function is known up to five loops [1]. We then use the obtained results to calculate the $f_i(\lambda)$ functions up to $i = 5$.

1.1 QCD running coupling

A surpring effect of the renormalization procedure is that, after renormalization, the coupling "constants" are not constant at all, but depend on the energy.

One way to understand this is the following: Classically, the Coulomb potential between two sources is then given by $V = \frac{\alpha}{r}$, characterized by a universal coefficient – the coupling constant α, which quantifies the force between two static bodies of unit "charge" at distance r, *i.e.*, the electric charge for QED, the color charge for QCD, the weak isospin for the weak force, or the mass for gravity. Consequently, the coupling α is defined as being proportional to the elementary charge squared, *e.g.*, $\alpha_{em} \equiv \frac{e^2}{4\pi}$ where e is the elementary electric charge, or $\alpha_s \equiv \frac{g^2}{4\pi}$ where g is the elementary gauge field coupling in QCD. In the non-relativistic limit of QCD, $\frac{1}{r}$ is the coordinate-space representation for the propagator of the gluon (force carrier) at leading-order in perturbation theory: in momentum space, the analogous propagator is proportional to $\frac{1}{q^2}$, where q is the boson 4-momentum ($Q^2 = -q^2 > 0$).

For sources interacting weakly, the one-boson exchange representation of interactions is a good approximation. However, when interactions become strong, higher orders in perturbation theory become noticeable and the $\frac{1}{r}$ law no longer stands. In such cases, it makes good physics sense to fold the extra r-dependence into the coupling, which thereby becomes r, or equivalently Q^2, dependent.

Another way to view this is that the running of the coupling is due to vacuum polarization. The vacuum is not empty, but is filled with virtual particles that are constantly created and annihilated which can interact with the propagating particles, leading to a modification of the interaction strength.

While in QED, the extra r-dependence comes only from the vacuum polarization. In QCD, α_s receives contributions from the vacuum polarization and from gluon self-interactions since the gluon has a color charge.

The two couplings have opposite trends: the QED coupling increases with energy and the theory could eventually become strongly coupled at extremely high energies, whereas the opposite happens for the QCD coupling as it is large at low energies and decreases with energy. This property of being weakly coupled at high energies is known as *asymptotic freedom* and it means that perturbative calculations in QCD can be used at high energies where α_s becomes small enough that a power expansion is meaningful.

In the framework of perturbative QCD (*pQCD*), predictions for observables are expressed in terms of the renormalized coupling $\alpha = \alpha(\mu^2)$, a function of an unphysical renormalization scale μ_R. The coupling satisfies the following renormalization group equation (RGE):

$$\mu^2 \frac{d\alpha}{d\mu^2} = \beta(\alpha) = -\left(b_0\alpha^2 + b_1\alpha^3 + b_2\alpha^4 + \ldots\right), \tag{1}$$

where b_0 is the 1-loop β-function coefficient, b_1 is the 2-loop coefficient, b_2 is the 3-loop coefficient. $C_A = 3$ and $C_F = \frac{4}{3}$ are the Casimir operators of the adjoint and fundamental representations of $SU(3)$, $T_R = \frac{1}{2}$ is the trace normalization, n_f is the number of active quark flavors.

It is not possible to solve eq. (1) as it is for two reasons: only the first few b_n coefficients are known (up to b_4); the exact equation becomes more and more complicated as more terms of the series are included, making it impossible to obtain an analytic solution.

In order to solve both problems, the equation is solved in the following way: at first only b_0 is included and the obtained solution is called α_{LO} , as it will only contain a term proportional to α ; then also b_1 is included and only terms up to the second order in α are kept to obtain α_{NLO} ; this same procedure is used to obtain α_{NNLO} , $\alpha_{\mathrm{N^3LO}}$, $\alpha_{\mathrm{N^4LO}}$. There will be a complication in calculating α_{NLO} and higher orders which will be explained and resolved in the following sections.

1.1.1 One-loop running coupling and higher order corrections

The one-loop running coupling α_{LO} is obtained by solving the RGE eq. (1) with only the first term of the β-function:

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0\alpha^2. \tag{2}$$

This equation can be solved by separation of variables and imposing the boundary condition $\alpha(Q^2) = \alpha_s$:

$$\int_{\alpha(Q^2)}^{\alpha(\mu^2)} \frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0 \frac{\mathrm{d}\mu^2}{\mu^2}, \tag{3}$$

and one obtains:

$$\alpha_{\mathrm{LO}}(\mu^2) = \frac{\alpha_s}{1 + b_0\alpha_s \log\left(\frac{\mu^2}{Q^2}\right)}, \tag{4}$$

in which one can observe the decreasing with energy trend of the running coupling (asymptotic freedom).

It is useful to define the variable $\lambda_\mu = b_0\alpha_s \log\left(\frac{\mu^2}{Q^2}\right)$ so that:

$$\alpha_{\mathrm{LO}}(\mu^2) = \frac{\alpha_s}{1 + \lambda_\mu}. \tag{5}$$

In order to obtain the two-loop running coupling α_{NLO}, we need to solve the RGE with the first two terms of the β-function eq. (1):

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0\alpha^2 - b_1\alpha^3, \tag{6}$$

but this equation is not solvable in a straightforward way as the one-loop equation, we have to use the perturbative approach. We can rewrite the equation as:

$$\frac{d\alpha}{d\mu^2} = -\frac{b_0\alpha^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha), \tag{7}$$

and expand the α term in the parenthesis as:

$$\alpha = \alpha_{\mathrm{LO}} + \delta\alpha, \tag{8}$$

where α_{LO} is the one-loop running coupling and $\delta\alpha$ contains the higher order correction, one obtains:

$$\frac{d\alpha}{d\mu^2} = -\frac{b_0\alpha^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{LO} - \frac{b_1}{b_0}\delta\alpha). \tag{9}$$

Observe that in parenthesis, by keeping 1 gave us the one-loop running coupling, by keeping $\frac{b_1}{b_0}\alpha_{LO}$ we can obtain the first order corrections and $\delta\alpha$ are needed for higher order corrections. The equation to solve is then:

$$\int_{\alpha_s}^{\alpha(\mu^2)} -\frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0\frac{\mathrm{d}\mu^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{LO}(\mu^2)). \tag{10}$$

Using *Mathematica* [2] to solve this equation, we obtain the two-loop running coupling:

$$\alpha_{\mathrm{NLO}}(\mu^2) = \frac{\alpha_s}{1 + \lambda_\mu + \alpha_s\frac{b_1}{b_0}\log\left(1 + \lambda_\mu\right)}, \tag{11}$$

in which the expansion in powers of α_s is not explicit. One can expand in powers of α_s by keeping λ_μ fixed and only keeping terms up to $\mathcal{O}\left(\alpha_s^2\right)$ by doing so one obtains:

$$\alpha_{\mathrm{NLO}}\left(\mu^2\right) = \alpha_{\mathrm{LO}}\left(\mu^2\right) - \frac{b_1}{b_0}\alpha_{LO}^2\left(\mu^2\right)\log\left(1 + \lambda_\mu\right) + \mathcal{O}(\alpha_s^2). \tag{12}$$

We found the correction:

$$\delta\alpha_{\mathrm{NLO}}\left(\mu^2\right) = -\frac{b_1}{b_0}\alpha_{\mathrm{LO}}^2\left(\mu^2\right)\log\left(1 + \lambda_\mu\right). \tag{13}$$

By repeating the same procedure, one can obtain the three-loop running coupling α_{NNLO} and so on.

In order to calculate higher order corrections, one need to be careful of the powers of α needed for the desired order, and the contributions to various orders of α_s may not be immediately apparent, but they are straightforward to compute. Expand the running coupling in powers of α_s as:

$$\alpha = \alpha_{\mathrm{LO}} + \delta\alpha_{\mathrm{NLO}} + \delta\alpha_{\mathrm{NNLO}} + \delta\alpha_{\mathrm{N^3LO}} + \delta\alpha_{\mathrm{N^4LO}} + \ldots, \tag{14}$$

with $\delta\alpha_{\mathrm{NLO}} = \mathcal{O}(\alpha_s)$, $\delta\alpha_{\mathrm{NLO}} = \mathcal{O}\left(\alpha_s^2\right)$, $\delta\alpha_{\mathrm{NNLO}} = \mathcal{O}(\alpha_s^3)$, $\delta\alpha_{\mathrm{N^3LO}} = \mathcal{O}(\alpha_s^4)$, $\delta\alpha_{\mathrm{N^4LO}} = \mathcal{O}(\alpha_s^5)$, and so on.

We present these contributions in the following table:

For the three-loop running coupling, the equation to solve is:

$$\mu^2\frac{d\alpha}{d\mu^2} = -b_0\alpha^2(1 - \frac{b_1}{b_0}\alpha - \frac{b_2}{b_0}\alpha^2). \tag{15}$$

One can substitute the expansion of $\alpha = \alpha_{\mathrm{LO}} + \delta\alpha_{\mathrm{NLO}} + \mathcal{O}\left(\alpha_s^2\right)$ in powers of α_s and retain only terms up to $\mathcal{O}\left(\alpha_s^2\right)$, with this prescription the equation to solve is:

Table 1: Contributions to different powers of α_s.

Power	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$	$\mathcal{O}(\alpha_s^5)$
α	α_{LO}	$\delta\alpha_{\mathrm{NLO}}$	$\delta\alpha_{\mathrm{NNLO}}$	$\delta\alpha_{\mathrm{N^3LO}}$	$\delta\alpha_{\mathrm{N^4LO}}$
α^2		α_{LO}^2	$2\alpha_{\mathrm{LO}}\delta\alpha_{\mathrm{NLO}}$	$\delta\alpha_{\mathrm{NLO}}^2+2\alpha_{\mathrm{LO}}\delta\alpha_{\mathrm{NNLO}}$	$2\alpha_{\mathrm{LO}}\delta\alpha_{\mathrm{N^3LO}}+3\alpha_{\mathrm{LO}}\delta\alpha_{\mathrm{NLO}}^2$
α^3			α_{LO}^3	$3\alpha_{\mathrm{LO}}^2\delta\alpha_{\mathrm{NLO}}$	$3\alpha_{\mathrm{LO}}^2\delta\alpha_{\mathrm{NNLO}}+3\alpha_{\mathrm{LO}}\delta\alpha_{\mathrm{NLO}}^2$
α^4				α_{LO}^4	$4\alpha_{\mathrm{LO}}^3\delta\alpha_{\mathrm{NLO}}$
α^5					α_{LO}^5

$$\int_{\alpha_s}^{\alpha(\mu^2)} -\frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0\frac{\mathrm{d}\mu^2}{\mu^2}(1-\frac{b_1}{b_0}\alpha_{\mathrm{NLO}}(\mu^2)-\frac{b_2}{b_0}\alpha_{LO}^2(\mu^2)), \tag{16}$$

solving the above integral yields the three-loop running coupling α_{NNLO}:

$$\alpha_{\mathrm{NNLO}}(\mu^2) = \alpha_{\mathrm{LO}}\Big(\mu^2\Big)+\delta\alpha_{\mathrm{NLO}}\Big(\mu^2\Big)+\delta\alpha_{\mathrm{NNLO}}\Big(\mu^2\Big), \tag{17}$$

with

$$\delta\alpha_{\mathrm{NNLO}}\Big(\mu^2\Big) = \frac{\alpha_{\mathrm{LO}}^3\Big(\mu^2\Big)}{b_0^2}\Big(b_1^2\lambda_\mu - b_0b_2\lambda_\mu + b_1^2\log^2(1+\lambda_\mu) - b_1^2\log\big(1+\lambda_\mu\big)\Big). \tag{18}$$

Similarly, one can obtain the four-loop running coupling α_{N^3LO} and five-loop running coupling α_{N^4LO}

$$\alpha_{\mathrm{N^3LO}}(\mu^2) = \alpha_{\mathrm{LO}}\Big(\mu^2\Big)+\delta\alpha_{\mathrm{NLO}}\Big(\mu^2\Big)+\delta\alpha_{\mathrm{NNLO}}\Big(\mu^2\Big)+\delta\alpha_{\mathrm{N^3LO}}\Big(\mu^2\Big), \tag{19}$$

$$\begin{aligned}\delta\alpha_{\mathrm{N^3LO}}\Big(\mu^2\Big) = \frac{\alpha_{\mathrm{LO}}^4\Big(\mu^2\Big)}{2b_0^3}\Big(&-\Big(b_1^3-2b_0b_2b_1+b_0^2b_3\Big)\lambda_\mu^2\\ &-\Big(2b_0^2b_3-2b_0b_1b_2\Big)\lambda_\mu - 2b_1^3\log^3\left(\lambda_\mu+1\right)+5b_1^3\log^2\left(1+\lambda_\mu\right)\\ &+\Big(2b_0b_1b_2\left(2\lambda_\mu-1\right)-4b_1^3\lambda_\mu\Big)\log\left(1+\lambda_\mu\right)\Big),\end{aligned} \tag{20}$$

$$\alpha_{\mathrm{N^4LO}}(\mu^2) = \alpha_{\mathrm{LO}}\Big(\mu^2\Big)+\delta\alpha_{\mathrm{NLO}}\Big(\mu^2\Big)+\delta\alpha_{\mathrm{NNLO}}\Big(\mu^2\Big)+\delta\alpha_{\mathrm{N^3LO}}\Big(\mu^2\Big)+\delta\alpha_{\mathrm{N^4LO}}\Big(\mu^2\Big), \tag{21}$$

$$
\begin{aligned}
\delta\alpha_{\mathrm{N^4LO}} = \frac{\alpha_{\mathrm{LO}}^5}{6b_0^4}\Big(&\left(\left(2b_1^4 - 6b_0b_2b_1^2 + 4b_0^2b_3b_1 + 2b_0^2b_2^2 - 2b_0^3b_4\right)\lambda_\mu^3\right. \\
&+ \left(9b_1^4 - 24b_0b_2b_1^2 + 9b_0^2b_3b_1 + 12b_0^2b_2^2 - 6b_0^3b_4\right)\lambda_\mu^2 \\
&+ \left(6b_0^2b_1b_3 - 6b_0^3b_4\right)\lambda_\mu + 6b_1^4\log^4(1+\lambda_\mu) \\
&- 26b_1^4\log^3\left(\lambda_\mu+1\right) + 9\left(\left(2b_1^4 - 2b_0b_1^2b_2\right)\lambda_\mu + b_1^4 + 2b_0b_2b_1^2\right)\log^2(1+\lambda_\mu) \\
&+ (6b_1\left(b_1^3 - 2b_0b_2b_1 + b_0^2b_3\right)\lambda_\mu^2 + 6b_1\left(-3b_1^3 + b_0b_2b_1 + 2b_0^2b_3\right)\lambda_\mu \\
&- 6b_1b_3b_0^2)\log\big(1+\lambda_\mu\big)\Big).
\end{aligned}
\tag{22}
$$

The plot of the running coupling α_s as a function of the energy scale $(xm_z)^2$ for different orders of perturbation theory, where $m_z = 91.18$ GeV is the mass of the Z boson, is shown in fig. 1. The global average value of the strong coupling constant is $\alpha_s(m_z^2) = 0.1179 \pm 0.0009$ [3]. For the plots $\alpha_s(m_z^2) = 0.118$ is used.

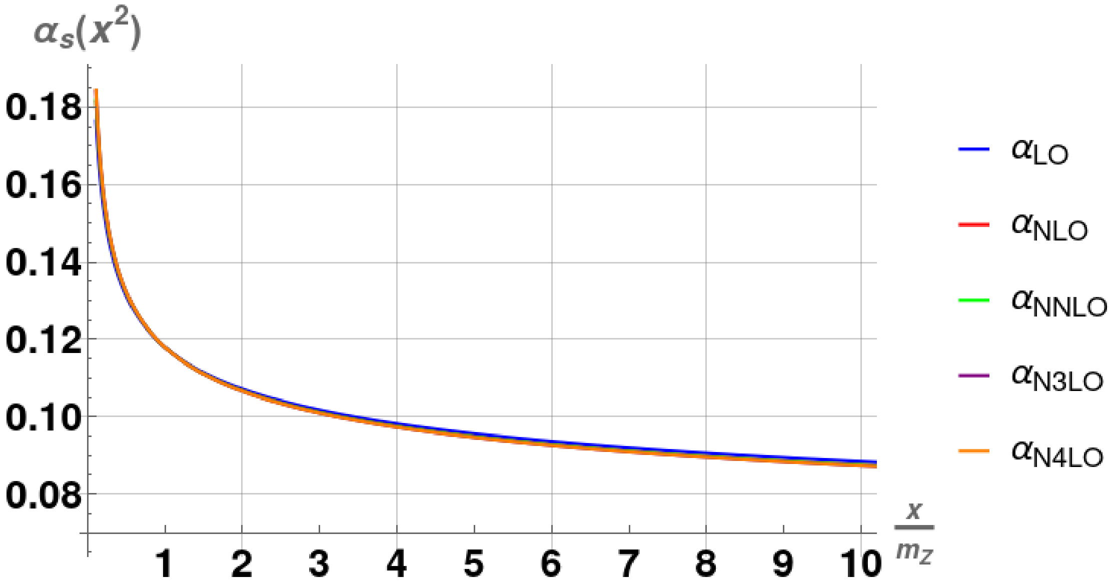

Figure 1: Energy dependence of the strong coupling α_s

In fig. 2 wee see that at low energies the running coupling α_s is large and diverges at a finite energy (the so called Landau pole), this is a sign of the non-perturbative nature of QCD at low energies.

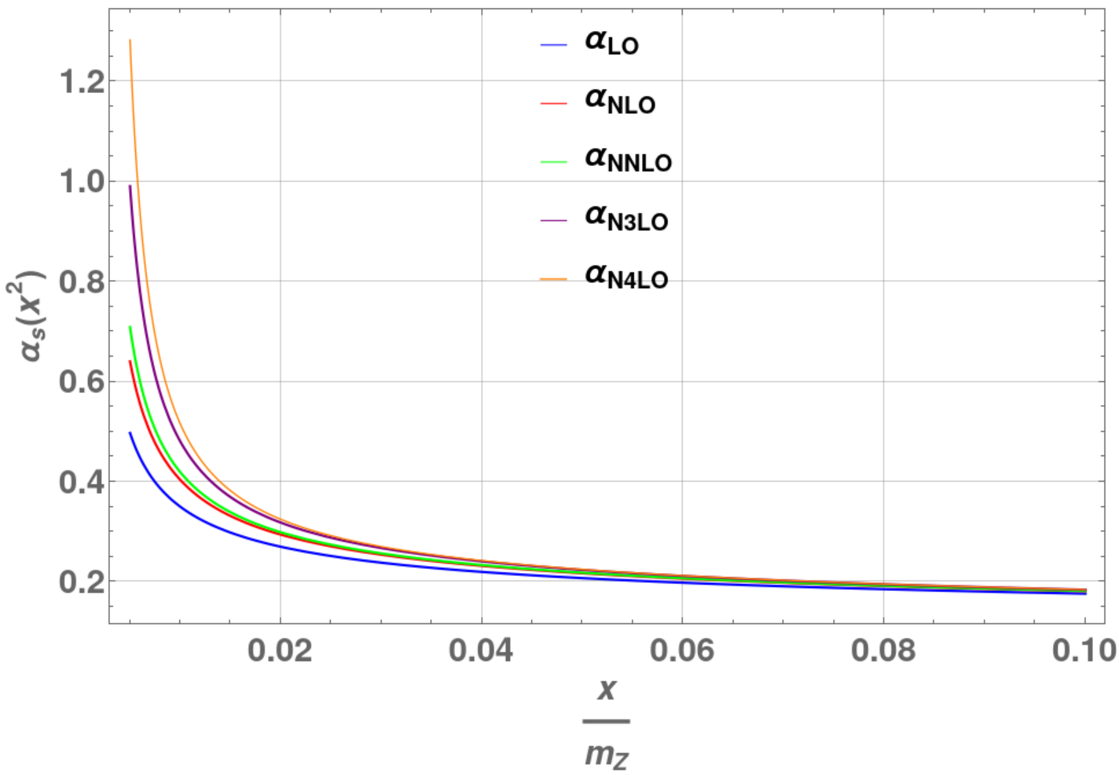

Figure 2: Zoom-in at low energies of the strong running coupling α_s at different orders