1 Laplace and Mellin transform in the large N limit

Using the methodology outlined in [4], we will demonstrate that the Mellin transform prescription is also applicable to the Laplace transform. in the large moment $\nu Q^2 = N$ limit, this fact was already known in the literature [2] and we'll show it here for completeness.

We are interested in solving the following integral

$$\int_0^1 \mathrm{d}z\, \frac{e^{-N(1-z)} - 1}{1-z} F(\alpha_s, \ln(1-z)) = \int_0^1 \frac{\mathrm{d}u}{u} \big(e^{-uN} - 1\big) F(\alpha_s, \ln u) \tag{1}$$

Start by considering

$$I_n(N) = \int_0^1 \frac{\mathrm{d}u}{u} \big(e^{-uN} - 1\big) \ln^n(u) \tag{2}$$

the above integral can be evaluated as described in [1]. Using the following identity

$$\ln^n(u) = \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \epsilon}\right)^n u^\epsilon \tag{3}$$

to replace the logarithm term in the integrand eq. (2) and straightforwardly integrate the resulting expression. We obtain

$$
\begin{aligned}
I_n(N) &= \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^n \int_0^1 \mathrm{d}u \left(e^{-uN}-1\right)u^{\epsilon-1} \\
&= \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{-\frac{1}{\epsilon} + N^{-\epsilon}(\Gamma(\epsilon,0)-\Gamma(\epsilon,N))\right\} \\
&= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{-\frac{1}{\epsilon} + N^{-\epsilon}\Gamma(\epsilon)\right\} + e^{-N+\mathcal{O}\left((\frac{1}{N})^2\right)}\mathcal{O}\left(\frac{1}{N}\right) \\
&= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{\frac{1}{\epsilon}(N^{-\epsilon}\epsilon\Gamma(\epsilon)-1)\right\} + e^{-N+\mathcal{O}\left((\frac{1}{N})^2\right)}\mathcal{O}\left(\frac{1}{N}\right) \\
&= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{\frac{1}{\epsilon}(e^{-\epsilon\ln N}\Gamma(1+\epsilon)-1)\right\} + e^{-N+\mathcal{O}\left((\frac{1}{N})^2\right)}\mathcal{O}\left(\frac{1}{N}\right)
\end{aligned} \tag{4}
$$

where $\Gamma(\epsilon,0)=\Gamma(\epsilon)$, $\Gamma(\epsilon,N)$ is the incomplete Gamma function and $\epsilon\Gamma(\epsilon) = \Gamma(1+\epsilon)$

$$
\Gamma(\epsilon,N) = \int_N^\infty \mathrm{d}t\, t^{\epsilon-1}e^{-t} \tag{5}
$$

The last equation in eq. (4) is the same as Eq. (68) obtained in [4] for the Mellin transform. Therefore, we can conclude that the Mellin transform prescription is also applicable to the Laplace transform in the large N limit.

Using the known expansion of the Gamma function for small ϵ

$$
\Gamma(1+\epsilon) = \exp\left\{-\gamma_E\epsilon + \sum_{n=2}^\infty (-1)^n \frac{\zeta(n)\epsilon^n}{n}\right\} \tag{6}
$$

the term in curly brackets in eq. (6) can be expanded in power of ϵ and then derive. The result for $I_n(N)$ is thus a polynomial of degree $n+1$ in the large logarithm $\ln N$:

$$
I_n(N) = \frac{(-1)^n+1}{n+1}(\ln N+\gamma_E)^{n+1} + \frac{(-1)^{n-1}}{2}n\zeta(2)(\ln N+\gamma_E)^{n-1} + \sum_{k=0}^{n-2} a_{nk}(\ln N+\gamma_E)^k + \mathcal{O}\left(\frac{1}{N}\right) \tag{7}
$$

This result can be generalized using the following formal identity:

$$e^{-\epsilon \ln N}\Gamma(1+\epsilon) = \Gamma(1 - \frac{\partial}{\partial \ln N})e^{\epsilon \ln N} \tag{8}$$

then we can perform the n-th derivative with respect to ϵ, and obtain

$$I_n(N) = \Gamma\left(1 - \frac{\partial}{\partial \ln N}\right)\frac{(-\ln N)^n + 1}{n+1} + \mathcal{O}\left(\frac{1}{N}\right) \tag{9}$$

This expression can be regarded as a replacement for eq. (4) to compute the polynomial coefficients a_{nk} in eq. (7). Moreover, by observing that

$$\frac{(-\ln N)^n + 1}{n+1} = -\int_{\frac{1}{N}}^{1} \mathrm{d}u \, \frac{\ln^n(u)}{u} \tag{10}$$

we obtain the all order generalization for of the prescriprion used in [3]:

$$e^{-uN} - 1 = -\Gamma\left(1 - \frac{\partial}{\partial \ln_N}\right)\Theta(u - \frac{1}{N}) + \mathcal{O}\left(\frac{1}{N}\right) \tag{11}$$

$$= -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln_N}\right)\Theta(u - \frac{N_0}{N}) + \mathcal{O}\left(\frac{1}{N}\right) \tag{12}$$

where

$$\tilde{\Gamma}(1-\epsilon) \equiv e^{\gamma_E \epsilon}\Gamma(1+\epsilon) = \exp\left\{\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)\epsilon^n}{n}\right\} \tag{13}$$

It is straightforward to show that the prescription can be applied to as follows:

$$\int_0^1 \frac{\mathrm{d}u}{u}\big(e^{-uN} - 1\big)F(\alpha_s, \ln u) = -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln_N}\right)\int_{\frac{N_0}{N}}^{1} \frac{\mathrm{d}u}{u}F(\alpha_s, \ln u) + \mathcal{O}\left(\frac{1}{N}\right) \tag{14}$$

and to evaluated the ln N-contribution arising from the integration of anu soft-gluon function F that has a generic perturbative expansion of the type

$$F(\alpha_s, \ln u) = \sum_{k=1}^{\infty} \alpha_s^k \sum_{n=0}^{2k-1} F_{kn} \ln^n u \tag{15}$$

The result eq. (14) can be used to obtain **??** as shown in [4].

The coefficients $\tilde{B}$ and $\tilde{C}$ are related to the coefficients A and B in the following way:

$$\tilde{B}(\alpha_s) = B(\alpha_s) + 4\partial_\alpha \Gamma_2(\partial_\alpha)\left[A(\alpha_s) - \frac{1}{4}B(\alpha_s)\right] \tag{16}$$

$$\tilde{C}(\alpha_s) = \exp\left\{-4\Gamma_2(\partial_\alpha)\left[A(\alpha_s 1 - \frac{1}{4}B(\alpha_s))\right]\right\} \tag{17}$$

where

$$\Gamma_2(\partial_\alpha) = \frac{1}{\epsilon^2}\left[1 - e^{-\gamma_E \epsilon \Gamma(1-\epsilon)}\right] = \frac{1}{\epsilon^2}\left\{1 - \exp\left[\sum_{n=2}^{\infty} \frac{\zeta(n)}{n}\epsilon^n\right]\right\} \tag{18}$$

$$\partial_\alpha \equiv -2\beta(\alpha_s)\alpha_s \frac{\partial}{\partial \alpha_s} \tag{19}$$

$\beta(\alpha_s)$ is the QCD beta function **??**

by inserting the expansion

$$\begin{aligned}\Gamma_2(\epsilon) = \frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3)\epsilon - \frac{9}{16}\zeta(4)\epsilon^2 - \left(\frac{1}{6}\zeta(2)\zeta(3) + \frac{1}{5}\zeta(5)\right)\epsilon^3 \\ - \left(\frac{1}{18}\zeta(3)^2 - \frac{61}{128}\zeta(6)\right)\epsilon^4 + \mathcal{O}(\epsilon^5)\end{aligned} \tag{20}$$

in eq. (16) and eq. (17), we can obtain the coefficients $\tilde{B}$ and $\tilde{C}$ in terms of the coefficients A and B up to N^4LL accuracy:

$$
\begin{aligned}
\tilde{B}(\alpha_s) - B(\alpha_s) = &-\frac{4A_1 b_0}{\pi}(\zeta(2))\alpha_s^2 \\
&+ \left(-\frac{8A_2 b_0}{\pi^2}\zeta(2) + \frac{4b_0^2 B_1}{\pi}\zeta(2) - \frac{4A_1}{3\pi}(8b_0^2\zeta(3) + 3b_1\zeta(2))\right)\alpha_s^3 \\
&+ \left(-\frac{12A_3 b_0}{\pi^3}\zeta(2) + \frac{12b_0^2 B_2}{\pi^2}\zeta(2) - \frac{8A_2}{\pi^2}(4b_0^2\zeta(3) + b_1\zeta(2))\right. \\
&+ \frac{B_1}{3\pi}(48b_0^3\zeta(3) + 30b_0 b_1\zeta(2)) - \frac{4A_1}{(3\pi)}(20b_0 b_1\zeta(3) + 3b_2\zeta(2) \\
&\left. + 81b_0^3\zeta(4))\right)\alpha_s^4 + \mathcal{O}\big(\alpha_s^5\big)
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\ln\tilde{C}(\alpha_s) = &\frac{2A_1\zeta(2)}{\pi}\alpha_s \\
&+ \left(\frac{2A_2}{\pi^2}\zeta(2) - \frac{b_0 B_1}{\pi}\zeta(2) + \frac{A_1}{3\pi}(8b_0\zeta(3) - 6b_0\ln\left[\frac{\mu^2}{Q^2}\right]\zeta(2))\right)\alpha_s^2 \\
&+ \left(\frac{2A_3}{\pi^3}\zeta(2) - \frac{2b_0 B2}{\pi^2}\zeta(2) + \frac{A_2}{3\pi^2}(16b_0\zeta(3) - 12b_0\ln\left[\frac{\mu^2}{Q^2}\right]\zeta(2))\right. \\
&+ \frac{B_1}{3\pi}\Big(-8b_0^2\zeta(3) - 3b_1\zeta(2) + 6b_0^2\ln\left[\frac{\mu^2}{Q^2}\right]\zeta(2)\Big) + \frac{A_1}{3\pi}\Big(8b_1\zeta(3) \\
&\left. + 6b_0^2\ln^2\left[\frac{\mu^2}{Q^2}\right]\zeta(2) - 2\ln\left[\frac{\mu^2}{Q^2}\right](8b_0^2\zeta(3) + 3b_1\zeta(2)) + 54b_0^2\zeta(4)\Big)\right)\alpha_s^3 \\
&+ \mathcal{O}\big(\alpha_s^4\big)
\end{aligned} \tag{22}
$$

We note that $\tilde{B}$ corrects the B terms so it has to be expanded up to α_s^4 to achieve N^4LL accuracy while $\ln\tilde{C}$ corrects the f_i functions so they have to be expanded up to α_s^3 to achieve N^4LL accuracy, these corrections are necessary only for NNLL accuracy and beyond, consistent with the results in [3].