Chapter 1

Laplace transform in the large N limit

Using the methodology outlined in [4], we will demonstrate that the Mellin transform prescription is also applicable to the Laplace transform. in the large moment $\nu Q^2 = N$ limit, this fact was already known in the literature [2] and we'll show it here for completeness.

We are interested in solving the following integral

$$\int_0^1 \mathrm{d}z \, \frac{e^{-N(1-z)} - 1}{1-z} F(\alpha_s, \ln(1-z)) = \int_0^1 \frac{\mathrm{d}u}{u} \Big(e^{-uN} - 1 \Big) F(\alpha_s, \ln u) \tag{1}$$

Start by considering

$$I_n(N) = \int_0^1 \frac{\mathrm{d}u}{u} \Big(e^{-uN} - 1 \Big) \ln^n(u) \tag{2}$$

the above integral can be evaluated as described in [1]. Using the following identity

$$\ln^n(u) = \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \epsilon} \right)^n u^\epsilon = \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \epsilon} \right)^n e^{\epsilon \ln u} \tag{3}$$

to replace the logarithm term in the integrand eq. (2) and straightforwardly integrate the resulting expression. We obtain

$$
\begin{aligned}
I_n(N) &= \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^n \int_0^1 \mathrm{d}u\left(e^{-uN}-1\right)u^{\epsilon-1} \\
&= \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{-\frac{1}{\epsilon}+N^{-\epsilon}(\Gamma(\epsilon,0)-\Gamma(\epsilon,N))\right\} \\
&= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{-\frac{1}{\epsilon}+N^{-\epsilon}\Gamma(\epsilon)\right\}+e^{-N+\mathcal{O}\left((\frac{1}{N})^2\right)}\mathcal{O}\left(\frac{1}{N}\right) \\
&= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{\frac{1}{\epsilon}(N^{-\epsilon}\epsilon\Gamma(\epsilon)-1)\right\}+\mathcal{O}\left(\frac{e^{-N}}{N}\right) \\
&= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{\frac{1}{\epsilon}(e^{-\epsilon\ln N}\Gamma(1+\epsilon)-1)\right\}+\mathcal{O}\left(\frac{e^{-N}}{N}\right)
\end{aligned}
\tag{4}
$$

where $\Gamma(\epsilon,0)=\Gamma(\epsilon)$, $\Gamma(\epsilon,N)$ is the incomplete Gamma function and $\epsilon\Gamma(\epsilon)=\Gamma(1+\epsilon)$

$$
\Gamma(\epsilon,N)=\int_N^\infty \mathrm{d}t\, t^{\epsilon-1}e^{-t}
\tag{5}
$$

The last equation in eq. (4) is the same as Eq. (68) obtained in [4] for the Mellin transform. Therefore, we can conclude that the Mellin transform prescription is also applicable to the Laplace transform in the large N limit.

Using the known expansion of the Gamma function for small ϵ

$$
\Gamma(1+\epsilon)=\exp\left\{-\gamma_E\epsilon+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)\epsilon^n}{n}\right\}
\tag{6}
$$

the term in curly brackets in eq. (6) can be expanded in power of ϵ and then derive. The result for $I_n(N)$ is thus a polynomial of degree $n+1$ in the large logarithm $\ln N$:

$$
\begin{aligned}
I_n(N) &= \frac{(-1)^n+1}{n+1}(\ln N+\gamma_E)^{n+1}+\frac{(-1)^{n-1}}{2}n\zeta(2)(\ln N+\gamma_E)^{n-1} \\
&\quad+\sum_{k=0}^{n-2}a_{nk}(\ln N+\gamma_E)^k+\mathcal{O}\left(\frac{e^{-N}}{N}\right)
\end{aligned}
\tag{7}
$$

This result can be generalized using the following formal identity:

$$
e^{-\epsilon\ln N}\Gamma(1+\epsilon)=\Gamma(1-\frac{\partial}{\partial\ln N})e^{\epsilon\ln N}
\tag{8}
$$

then we can perform the n-th derivative with respect to ϵ, and obtain

$$
I_n(N)=\Gamma\left(1-\frac{\partial}{\partial\ln N}\right)\frac{(-\ln N)^n+1}{n+1}+\mathcal{O}\left(\frac{e^{-N}}{N}\right)
\tag{9}
$$

This expression can be regarded as a replacement for eq. (4) to compute the polynomial coefficients a_{nk} in eq. (7). Moreover, by observing that

$$\frac{(-\ln N)^n+1}{n+1} = -\int_{\frac{1}{N}}^{1} \mathrm{d}u\,\frac{\ln^n(u)}{u} \tag{10}$$

we obtain the all order generalization for of the prescriprion used in [3]:

$$e^{-uN}-1 = -\Gamma\left(1-\frac{\partial}{\partial \ln_N}\right)\Theta(u-\frac{1}{N}) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{11}$$

$$= -\tilde{\Gamma}\left(1-\frac{\partial}{\partial \ln_N}\right)\Theta(u-\frac{N_0}{N}) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{12}$$

where

$$\tilde{\Gamma}(1-\epsilon) \equiv e^{\gamma_E \epsilon}\Gamma(1+\epsilon) = \exp\left\{\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)\epsilon^n}{n}\right\} \tag{13}$$

It is straightforward to show that the prescription can be applied to as follows:

$$\int_0^1 \frac{\mathrm{d}u}{u}\left(e^{-uN}-1\right)F(\alpha_s,\ln u) = -\tilde{\Gamma}\left(1-\frac{\partial}{\partial \ln_N}\right)\int_{\frac{N_0}{N}}^{1}\frac{\mathrm{d}u}{u}F(\alpha_s,\ln u) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{14}$$

and to evaluated the $\ln N$-contribution arising from the integration of anu soft-gluon function F that has a generic perturbative expansion of the type

$$F(\alpha_s,\ln u) = \sum_{k=1}^{\infty}\alpha_s^k\sum_{n=0}^{2k-1}F_{kn}\ln^n u \tag{15}$$

The result eq. (14) can be used to obtain **??** as shown in [4].

Chapter 2

Equivalence between resummation formulae

Here i adapt the result in [4] to show the equivalence between the resummation formulae in **??** and **??** in the case of Thrust resummation.

It is straightforward to show that equation (90) in [4] becomes:

$$\begin{aligned}&\int_{N_0/N}^{1}\frac{\mathrm{d}u}{u}\frac{1}{2}\Big(\tilde{B}(\alpha_s(uQ^2))-B(\alpha_s(uQ^2))\Big)-\log\tilde{C}\left(\alpha_s(\mu^2),\frac{\mu^2}{Q^2}\right)=\\&\Gamma_2\left(\frac{\partial}{\partial\log N}\right)\left\{A(\alpha_s(\frac{N_0}{N}Q^2))-\frac{1}{2}\frac{\partial}{\partial\log N}B(\alpha_s(\frac{N_0}{N}Q^2))-2A(\alpha_s(\frac{N_0^2}{N^2}Q^2))\right\}\end{aligned}\tag{1}$$

Observe that using the renormalization group equation **??** and chain rule we can write the following relation:

$$\begin{aligned}\frac{\partial}{\partial\log N}B(\alpha_s(\frac{k}{N}))&=\frac{\partial B(\alpha_s)}{\partial\alpha_s}\frac{\partial\alpha_s(\frac{k}{N})}{\partial\frac{k}{N}}\frac{\partial\frac{k}{N}}{\partial\log N}=-\frac{\partial\alpha_s(\frac{k}{N})}{\partial\log\frac{k}{N}}\frac{\partial B(\alpha_s)}{\partial\alpha_s}\\&=-\beta(\alpha_s)\alpha_s\frac{\partial B(\alpha_s)}{\partial\alpha_s}\\\frac{\partial}{\partial\log N}A(\alpha_s(\frac{k}{N^2}))&=-2\beta(\alpha_s)\alpha_s\frac{\partial A(\alpha_s)}{\partial\alpha_s}\end{aligned}\tag{2}$$

define the differential operator $\partial(\alpha_s)$ as:

$$\partial_{\alpha_s}\equiv-\beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s}\tag{3}$$

Substituting the above relations in the previous equation, we obtain the equivalent of equation (92) in [4]:

$$\begin{aligned}&\int_{N_0/N}^{1}\frac{\mathrm{d}u}{u}\frac{1}{2}\Big(\tilde{B}(\alpha_s(uQ^2))-B(\alpha_s(uQ^2))\Big)-\log\tilde{C}\left(\alpha_s(\mu^2),\frac{\mu^2}{Q^2}\right)=\\&\Gamma_2\big(\partial_{\alpha_s}\big)\left\{A(\alpha_s(\frac{N_0}{N}Q^2))-\frac{1}{2}\partial_{\alpha_s}B(\alpha_s(\frac{N_0}{N}Q^2))\right\}-2\Gamma\big(2\partial_{\alpha_s}\big)A(\alpha_s(\frac{N_0^2}{N^2}Q^2))\end{aligned}\tag{4}$$

Now by setting $N=N_0$ or applying $\frac{\partial}{\partial\log N}$ one obtains respectively the functions $\tilde{C}$ and $\tilde{B}$ as functions of A and B:

$$\tilde{C}(\alpha_s)=\exp\left\{-\Gamma_2(\partial_{\alpha_s})\left[A(\alpha_s)-\frac{1}{2}\partial_{\alpha_s}B(\alpha_s)\right]-2\Gamma\big(2\partial_{\alpha_s}\big)A(\alpha_s)\right\}\Big|_{\alpha_s=\alpha_s\left(Q^2\right)}\tag{5}$$

$$\begin{aligned}\frac{\tilde{B}(\alpha_s)}{2}&=\frac{B(\alpha_s)}{2}+\partial_{\alpha_s}\left\{\Gamma_2(\partial_{\alpha_s})\left[A(\alpha_s)-\frac{1}{2}\partial_{\alpha_s}B(\alpha_s)\right]\right\}\Big|_{\alpha_s=\alpha_s\left(\frac{N_0}{N}Q^2\right)}\\&\quad-4\partial_{\alpha_s}\left\{\Gamma_s\big(2\partial_{\alpha_s}\big)A(\alpha_s)\right\}\Big|_{\alpha_s=\alpha_s\left(\frac{N_0^2}{N^2}Q^2\right)}\end{aligned}\tag{6}$$

by inserting the expansion

$$\begin{aligned}\Gamma_2(\epsilon)=&-\frac{1}{2}\zeta(2)-\frac{1}{3}\zeta(3)\epsilon-\frac{9}{16}\zeta(4)\epsilon^2-\left(\frac{1}{6}\zeta(2)\zeta(3)+\frac{1}{5}\zeta(5)\right)\epsilon^3\\&-\left(\frac{1}{18}\zeta(3)^2-\frac{61}{128}\zeta(6)\right)\epsilon^4+\mathcal{O}\left(\epsilon^5\right)\end{aligned}\tag{7}$$

in eq. (6) and eq. (5), we can obtain the coefficients $\tilde{B}$ and $\tilde{C}$ in terms of the coefficients A and B up to N^4LL accuracy:

$$\tilde{B}\Big(\alpha_s\Big(uQ^2\Big)\Big)=B\Big(\alpha_s\Big(uQ^2\Big)\Big)+\ldots\tag{8}$$

$$
\begin{aligned}
\log \tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) = {} & \frac{A_1}{\pi}(-\zeta(2)-1)\alpha_s + \left(\frac{-2A_2\zeta(2) - 2A_2 + \pi b_0 B_1}{2\pi^2}\right. \\
& \left. + \frac{A_1 b_0 \left(3\zeta(2)\log\left(\frac{\mu^2}{Q^2}\right) + 3\log\left(\frac{\mu^2}{Q^2}\right) - 4\zeta(3)\right)}{3\pi}\right)\alpha_s^2 \\
& + \left(\frac{A_1}{3\pi}\left(-27b_0^2\zeta(4) - 3b_0^2\zeta(2)\log^2\left(\frac{\mu^2}{Q^2}\right) - 3b_0^2\log^2\left(\frac{\mu^2}{Q^2}\right)\right.\right. \\
& + 8b_0^2\zeta(3)\log\left(\frac{\mu^2}{Q^2}\right) + 3b_1\zeta(2)\log\left(\frac{\mu^2}{Q^2}\right) + 3b_1\log\left(\frac{\mu^2}{Q^2}\right) \\
& \left. - 4b_1\zeta(3)\right) - \frac{1}{6\pi^3}\left(-12\pi A_2 b_0\zeta(2)\log\left(\frac{\mu^2}{Q^2}\right)\right. \\
& - 12\pi A_2 b_0 \log\left(\frac{\mu^2}{Q^2}\right) + 16\pi A_2 b_0\zeta(3) + 6A_3\zeta(2) + 6A_3 \\
& \left.\left. + 6\pi^2 b_0^2 B_1 \log\left(\frac{\mu^2}{Q^2}\right) - 6\pi b_0 B_2 - 3\pi^2 b_1 B_1\right)\right)\alpha_s^3 + \mathcal{O}\left(\alpha_s^4\right)
\end{aligned} \tag{9}
$$

We note that $\tilde{B}$ corrects the B terms so it has to be expanded up to α_s^4 to achieve N^4LL accuracy while $\ln\tilde{C}$ corrects the f_i functions so they have to be expanded up to α_s^3 to achieve N^4LL accuracy, these corrections are necessary only for NNLL accuracy and beyond, consistent with the results in [3].