Chapter 1

Constants and Ingredients for Resummation

The needed coefficients are often expressed in terms of the following parameters:

$$\begin{aligned} n_f &= 5 && \text{Number of active flavors} \\ C_F &= \frac{4}{3} && \text{Quadratic Casimir operator for fundamental representation} \\ C_A &= 3 && \text{Quadratic Casimir operator for adjoint representation} \\ T_R &= \frac{1}{2} && \text{Trace normalization for fundamental representation} \end{aligned} \tag{1}$$

The renormalization group equation for the QCD coupling constant reads:

$$\mu^2 \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu^2} = \beta(\alpha_s) = -\alpha_s^2 \Big(b_0 + b_1 \alpha_s + b_2 \alpha_s^2 + \dots \Big) \tag{2}$$

where the coefficients of the $\beta(\alpha_s)$ functions are [2]:

$$b_0 = \frac{11 C_A - 4 n_f T_R}{12\pi} = \frac{33 - 2 n_f}{12\pi}$$

$$b_1 = \frac{17 C_A^2 - n_f T_R (10 C_A + 6 C_F)}{24\pi^2} = \frac{153 - 19 n_f}{24\pi^2}$$

$$b_2 = \frac{325 n_f^2}{3456\pi^3} - \frac{5033 n_f}{1152\pi^3} + \frac{2857}{128\pi^3}$$

$$b_3 = \frac{1093 n_f^3}{186624\pi^4} + n_f^2 \left(\frac{809\zeta(3)}{2592\pi^4} + \frac{50065}{41472\pi^4} \right) + n_f \left(-\frac{1627\zeta(3)}{1728\pi^4} - \frac{1078361}{41472\pi^4} \right)$$

$$
\begin{aligned}
&+\frac{891\zeta(3)}{64\pi^4}+\frac{149753}{1536\pi^4}\\
b_4 = {}& n_f^4\left(\frac{1205}{2985984\pi^5}-\frac{19\zeta(3)}{10368\pi^5}\right)\\
&+n_f^3\left(-\frac{24361\zeta(3)}{124416\pi^5}+\frac{115\zeta(5)}{2304\pi^5}+\frac{809}{1244160\pi}-\frac{630559}{5971968\pi^5}\right)\\
&+n_f^2\left(\frac{698531\zeta(3)}{82944\pi^5}-\frac{5965\zeta(5)}{1296\pi^5}-\frac{5263}{414720\pi}+\frac{25960913}{1990656\pi^5}\right)\\
&+n_f\left(-\frac{1202791\zeta(3)}{20736\pi^5}+\frac{1358995\zeta(5)}{27648\pi^5}+\frac{6787}{110592\pi}-\frac{336460813}{1990656\pi^5}\right)\\
&+\frac{621885\zeta(3)}{2048\pi^5}-\frac{144045\zeta(5)}{512\pi^5}-\frac{9801}{20480\pi}+\frac{8157455}{16384\pi^5}
\end{aligned}
\tag{3}
$$

Numerically, for $n_f = 5$:

$$
\begin{aligned}
b_0 &= 0.875352 - 0.0530516 n_f = \frac{23}{12\pi} = 0.610094\\
b_1 &= 0.645923 - 0.0802126 n_f = \frac{29}{12\pi^2} = 0.24486\\
b_2 &= 0.719864 - 0.140904 n_f + 0.00303291 n_f^2 = \frac{9769}{3456\pi^3} = 0.0911647\\
b_3 &= 1.17269 - 0.278557 n_f + 0.0162447 n_f^2 + 0.0000601247 n_f^3 = 0.193536\\
b_4 &= 1.71413 - 0.594075 n_f + 0.0560618 n_f^2 - 0.000738048 n_f^3 - 5.87966\cdot 10^{-6} n_f^4 = 0.0493694
\end{aligned}
\tag{4}
$$

The A coefficient in the resummation formula are given by:

$$A_1 = C_F \tag{5}$$

$$A_2 = \frac{1}{2} C_A C_F \left(-\frac{1}{27} 10 n_f T_R - \frac{\pi^2}{6} + \frac{67}{18}\right) \tag{6}$$

$$
\begin{aligned}
A_3 = {}& \left(-\frac{2051}{1296}-\frac{\pi^2}{18}\right) C_A C_F n_f + \left(-\frac{11\zeta(3)}{4}+\frac{11\pi^4}{720}-\frac{13\pi^2}{432}+\frac{15503}{2592}\right) C_A^2 C_F\\
&+\left(\frac{\zeta(3)}{2}-\frac{55}{96}\right) C_F^2 n_f + \left(\frac{25}{324}+\frac{\pi^2}{108}\right) C_F n_f^2
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
A_4 = {}& \Big(-\frac{\zeta(3)^2}{16}+\frac{11\pi^2\zeta(3)}{32}-\frac{24461\zeta(3)}{864}+\frac{20513\pi^2}{5184}+\frac{1925\zeta(5)}{288}-\frac{253\pi^4}{1920}-\frac{313\pi^6}{90720}\\
&+\frac{4311229}{186624}\Big) C_F C_A^3 + \Big(\frac{7\pi^2\zeta(3)}{72}+\frac{43\zeta(3)}{144}+\frac{451\pi^4}{4320}+\frac{131\zeta(5)}{144}\\
&+T_R\left(-\frac{1}{72}11\pi^2\zeta(3)+\frac{685\zeta(3)}{48}-3\zeta(5)-\frac{11\pi^4}{144}-\frac{1123}{162}-\frac{12247\pi^2}{15552}\right)
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&- \frac{64421\pi^2}{31104} - \frac{260731}{62208}\Big)C_F n_f C_A^2 + \Big(\Big(\Big(\frac{34\zeta(3)}{9} + \frac{11\pi^4}{720} + \frac{55\pi^2}{576} - \frac{7351}{1152}\Big)T_R - \frac{\pi^2\zeta(3)}{12} \\
&+ \frac{29\zeta(3)}{18} + \frac{5\zeta(5)}{8} - \frac{55\pi^2}{1152} - \frac{11\pi^4}{1440} - \frac{17033}{10368}\Big)n_f C_F^2 \\
&+ \Big(\Big(-\frac{7\zeta(3)}{54} + \frac{\pi^4}{60} - \frac{481\pi^2}{1944} + \frac{1747}{3888}\Big)T_R^2 + \Big(-\frac{143\zeta(3)}{108} + \frac{847\pi^2}{1296} + \frac{55}{486}\Big)T_R + \frac{803\pi^2}{3888} \\
&- \frac{\zeta(3)}{144} - \frac{49\pi^4}{4320} + \frac{19889}{62208}\Big)n_f^2 C_F\Big)C_A + \frac{31\pi^6}{60480} + \Big(\Big(-\frac{7\zeta(3)}{27} + \frac{5\pi^2}{324} + \frac{130}{729}\Big)T_R^3 \\
&+ \Big(\frac{13\zeta(3)}{108} - \frac{5}{486} - \frac{77\pi^2}{1296}\Big)T_R + \frac{\zeta(3)}{108} - \frac{1}{648}\Big)C_F n_f^3 + \frac{\pi^2}{96} + \Big(\Big(-\frac{19\zeta(3)}{18} + \frac{215}{96} - \frac{5\pi^2}{144} \\
&- \frac{\pi^4}{180}\Big)T_R^2 + \frac{\pi^4}{720} + \frac{5\pi^2}{192} - \frac{5\zeta(3)}{18} + \frac{299}{2592}\Big)C_F^2 n_f^2 + \frac{3\zeta(3)^2}{16} \\
&+ 81\Big(-\frac{\zeta(3)^2}{32} + \frac{\zeta(3)}{288} + \frac{55\zeta(5)}{576} - \frac{\pi^2}{576} - \frac{31\pi^6}{362880}\Big) \\
&+ 9\Big(-\frac{5\zeta(3)^2}{32} + \frac{5\zeta(3)}{288} + \frac{275\zeta(5)}{576} - \frac{5\pi^2}{576} - \frac{31\pi^6}{72576}\Big) + \Big(\Big(\frac{37\zeta(3)}{48} - \frac{5\zeta(5)}{4} + \frac{143}{576}\Big)C_F^3 \\
&+ \frac{1}{3}\Big(-\frac{\zeta(3)}{12} + \frac{\pi^2}{24} - \frac{5\zeta(5)}{12}\Big) + \frac{1}{27}\Big(\frac{\zeta(3)}{16} + \frac{5\zeta(5)}{16} - \frac{\pi^2}{32}\Big) + 3\Big(\frac{7\zeta(3)}{288} + \frac{35\zeta(5)}{288} - \frac{7\pi^2}{576}\Big) \\
&+ 27\Big(-\frac{\zeta(3)}{288} + \frac{\pi^2}{576} - \frac{5\zeta(5)}{288}\Big)\Big)n_f - \frac{\zeta(3)}{48} - \frac{55\zeta(5)}{96}
\end{aligned}
$$

$$A_5 = 14541.099 \tag{9}$$

The B coefficients are given by:

$$B_1 = -\frac{1}{2}\left(3C_F\right) \tag{10}$$

$$B_2 = \left(2\left(\frac{3\zeta(3)}{4} - \frac{57}{32}\right) + \frac{11\pi^2}{24}\right)C_A C_F + \left(\frac{5}{8} - \frac{\pi^2}{12}\right)C_F n_f + \left(2\left(-\frac{3\zeta(3)}{2} - \frac{3}{32}\right) + \frac{\pi^2}{4}\right)C_F^2 \tag{11}$$

$$
\begin{aligned}
B_3 = C_A \Bigg(& C_F n_f \left(\left(\frac{34\zeta(3)}{27} - \frac{485\pi^2}{432} - \frac{\pi^4}{720} + \frac{3683}{864} \right) T_R + \frac{31\zeta(3)}{54} + \frac{131\pi^4}{4320} + \frac{5261}{1728} - \frac{2657\pi^2}{2592} \right) \\
& + \left(-\frac{\pi^2\zeta(3)}{12} - \frac{89\zeta(3)}{12} - \frac{15\zeta(5)}{4} + \frac{287\pi^2}{192} - \frac{23}{16} - \frac{17\pi^4}{360} \right) C_F^2 \Bigg) \\
& + \left(\frac{241\zeta(3)}{108} - \frac{5\zeta(5)}{4} + \frac{22841\pi^2}{5184} - \frac{5951}{432} - \frac{713\pi^4}{4320} \right) C_A^2 C_F \\
& + C_F^2 n_f \left(\left(\frac{17\zeta(3)}{6} + \frac{23}{16} - \frac{5\pi^2}{72} - \frac{29\pi^4}{1080} \right) T_R - \frac{\zeta(3)}{4} + \frac{41\pi^4}{2160} + \frac{31}{64} - \frac{71\pi^2}{288} \right) \\
& + C_F n_f^2 \left(\left(\frac{2\zeta(3)}{9} + \frac{17}{72} - \frac{5\pi^2}{81} \right) T_R^2 + \left(-\frac{13\zeta(3)}{27} + \frac{193\pi^2}{648} - \frac{433}{432} \right) T_R \right) \\
& + \left(\frac{\pi^2\zeta(3)}{6} - \frac{17\zeta(3)}{8} + \frac{15\zeta(5)}{2} - \frac{\pi^4}{20} - \frac{3\pi^2}{32} - \frac{29}{64} \right) C_F^3
\end{aligned}
\tag{12}
$$

$$
B_4 = 3817.42 \tag{13}
$$

A_1, A_2 and B_1 were already known in [1] and were obtained from the one and two loop splitting functions. A_3 and B_2 were obtained by comparing **??** with the equivalent expression in Soft Collinear Effective Theory (SCET) [3] (Eq(4.17)) by absorbing the jet and soft terms into the A and B term. A_4, A_5 and B_3, B_4 were obtained in a similar fashion but the SCET expressions were gently provided by my co-supervisor Wan-Li.