0.1 Renormalization-scale dependece

In this section we consider renormalization-scale dependence. In principle, such scale μ should not appear in the cross sections, as it does not correspond to any fundamental constant or kinematical scale in the problem. The completely resummed perturbative expansion of an observable is indeed formally independent on μ. In practise, truncated perturbative expansions exhibit a residual scale dependence, because of neglected higher orders.

We start with deriving the strong coupling $\alpha_s(Q^2)$ as a function of $\alpha_s(\mu^2)$ and μ^2/Q^2, to do so we expand the explicit **??** in powers of $\alpha_s(\mu^2)$ and obtain:

$$\begin{aligned}
\alpha_s(Q^2) = {} & \alpha_s(\mu^2) + \alpha_s^2(\mu^2) b_0 \ln\left(\frac{\mu^2}{Q^2}\right) + \alpha_s^3(\mu^2)\left[b_1 \ln\left(\frac{\mu^2}{Q^2}\right) + b_0^2 \ln^2\left(\frac{\mu^2}{Q^2}\right)\right] \\
& + \alpha_s^4(\mu^2)\left[b_2 \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{5}{2} b_0 b_1 \ln^2\left(\frac{\mu^2}{Q^2}\right) + b_0^3 \ln^3\left(\frac{\mu^2}{Q^2}\right)\right] \\
& + \alpha_s^5(\mu^2)\left[b_3 \ln\left(\frac{\mu^2}{Q^2}\right) + \left(\frac{3}{2} b_1^2 + 3 b_0 b_2\right) \ln^2\left(\frac{\mu^2}{Q^2}\right)\right. \\
& \left. + \frac{13}{3} b_0^2 b_1 \ln^3\left(\frac{\mu^2}{Q^2}\right) + b_0^4 \ln^4\left(\frac{\mu^2}{Q^2}\right)\right] + \mathcal{O}\big(\alpha_s^6(\mu^2)\big)
\end{aligned} \tag{1}$$

for brevity we'll write:

$$\alpha_s(Q^2) = \alpha_s(\mu^2) + c_1 \alpha_s^2(\mu^2) + c_2 \alpha_s^3(\mu^2) + c_3 \alpha_s^4(\mu^2) + c_4 \alpha_s^5(\mu^2) + \mathcal{O}\big(\alpha_s^6(\mu^2)\big) \tag{2}$$

where c_i are the coefficients of the expansion above.

$$
\begin{aligned}
c_1 &= b_0 \ln\left(\frac{\mu^2}{Q^2}\right) \\
c_2 &= b_1 \ln\left(\frac{\mu^2}{Q^2}\right) + b_0^2 \ln^2\left(\frac{\mu^2}{Q^2}\right) \\
c_3 &= b_2 \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{5}{2} b_0 b_1 \ln^2\left(\frac{\mu^2}{Q^2}\right) + b_0^3 \ln^3\left(\frac{\mu^2}{Q^2}\right) \\
c_4 &= b_3 \ln\left(\frac{\mu^2}{Q^2}\right) + \left(\frac{3}{2} b_1^2 + 3 b_0 b_2\right) \ln^2\left(\frac{\mu^2}{Q^2}\right) + \frac{13}{3} b_0^2 b_1 \ln^3\left(\frac{\mu^2}{Q^2}\right) + b_0^4 \ln^4\left(\frac{\mu^2}{Q^2}\right)
\end{aligned}
\tag{3}
$$

now we substitute eq. (1) into $\lambda = b_0 \alpha_s(Q^2) L$ up to $\mathcal{O}\big(\alpha_s^2(\mu^2)\big)$ and obtain:

$$
\lambda(Q^2) = \lambda(\mu^2)\big(1 + c_1 \alpha_s(\mu^2) + c_2 \alpha_s^2(\mu^2) + c_3 \alpha_s^3(\mu^2) + c_4 \alpha_s^4(\mu^2) + \mathcal{O}\big(\alpha_s^5(\mu^2)\big)\big) \tag{4}
$$

and formally expand in powers of $\alpha_s(\mu^2)$, now $\lambda = \lambda(\mu^2)$:

$$
\begin{aligned}
L f_1(Q^2) &= L f_1(\lambda) + \frac{c_1}{b_0} \lambda^2 f_1'(\lambda) + \frac{\lambda}{b_0}\left(c_2 \lambda f_1'(\lambda) + \frac{1}{2} c_1^2 \lambda^2 f_1''(\lambda)\right) \alpha_s \\
&\quad + \frac{\lambda}{b_0}\left(c_3 \lambda f_1'(\lambda) + c_1 c_2 \lambda^2 f_1''(\lambda) + \frac{1}{6} c_1^3 f_1^{(3)}(\lambda)\right) \alpha_s^2 + \frac{\lambda^2}{b_0}\Big(c_4 f_1'(\lambda) \\
&\quad + \frac{1}{2} c_2^2 \lambda f_1''(\lambda) + c_1 c_3 \lambda f_1''(\lambda) + \frac{1}{2} c_1^2 c_2 \lambda^2 f_1^{(3)}(\lambda) + \frac{1}{24} c_1^4 \lambda^3 f_1^{(4)}(\lambda)\Big) \alpha_s^3 \\
f_2(Q^2) &= f_2(\lambda) + c_1 \lambda f_2'(\lambda) \alpha_s + \left(c_2 \lambda f_2'(\lambda) + \frac{1}{2} c_1^2 \lambda^2 f_2''(\lambda)\right) \alpha_s^2 \\
&\quad + \left(c_3 \lambda f_2'(\lambda) + c_1 c_2 \lambda^2 f_2''(\lambda) + \frac{1}{6} c_1^3 f_2^{(3)}(\lambda)\right) \alpha_s^3 \\
\alpha_s f_3(Q^2) &= \alpha_s f_3(\lambda) + c_1 \lambda f_3'(\lambda) \alpha_s^2 + \left(c_2 \lambda f_3'(\lambda) + \frac{1}{2} c_1^2 \lambda^2 f_3''(\lambda)\right) \alpha_s^3 \\
\alpha_s^2 f_4(Q^2) &= \alpha_s^2 f_4(\lambda) + c_1 \lambda f_4'(\lambda) \alpha_s^3 \\
\alpha_s^3 f_5(Q^2) &= \alpha_s^3 f_5(\lambda)
\end{aligned}
\tag{5}
$$

As usual, the terms from the expansion proportional to α_s corrects f_3, the terms proportional to α_s^2 correct f_4 and so on.

So the additional terms in the functions f_i for $i = 0, 1, 3,$ to (partially) compensate for the scale change $Q^2 \to \mu^2$ therefore read:

$$
\begin{aligned}
\delta f_1\left(\lambda, \frac{\mu^2}{Q^2}\right) &= 0 \\
\delta f_2\left(\lambda, \frac{\mu^2}{Q^2}\right) &= \lambda^2 f_1'(\lambda) \log\left(\frac{\mu^2}{Q^2}\right) \\
\delta f_3\left(\lambda, \frac{\mu^2}{Q^2}\right) &= \frac{1}{2}\lambda^3 f_1''(\lambda) \log\left(\frac{\mu^2}{Q^2}\right) + \lambda^2 f_1'(\lambda)\left(\log^2\left(\frac{\mu^2}{Q^2}\right) + \frac{b1}{b0^2}\log\left(\frac{\mu^2}{Q^2}\right)\right) \\
&\quad + \lambda f_2'(\lambda) \log\left(\frac{\mu^2}{Q^2}\right)
\end{aligned}
\tag{6}
$$

then explicitly calculate those derivatives and obtain the scale dependence.