In order to find the quark jet mass distribution $J^q(Q^2, k^2)$,we have to perform the inverse Laplace transform via the Mellin's inversion formula (or the Bromwich integral) given by the line integral:

$$J^q(Q^2,k^2) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \mathrm{d}\nu e^{\nu k^2} \tilde{J}^q_\nu(Q^2) \tag{1}$$

where C is a real number such that C is at the right of all singularities of the integrand in the complex plane and the function $\tilde{J}^q_\nu(Q^2)$ has to be bounded on the line.

Instead of directly considering the expression in eq. (1), it was pointed in [2] that it is more convenient to work with the mass fraction $R^q(w)$, which gives the fraction of jets with masses less than wQ^2:

$$R^q(w) = \int_0^\infty J^q(Q^2,k^2)\Theta(wQ^2-k^2)\mathrm{d}k^2 \tag{2}$$

and using the integral representation of the Heaviside step function **??**

$$\Theta(wQ^2-k^2) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \frac{\mathrm{d}\nu}{\nu} e^{\nu(wQ^2-k^2)} \tag{3}$$

we recognize the Laplace transform of the quark jet mass distribution **??**

$$\begin{aligned}
R^q(w) &= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \frac{\mathrm{d}\nu}{\nu} e^{w\nu Q^2} \int_0^\infty J^q(Q^2,k^2) e^{-\nu k^2} \mathrm{d}k^2 \\
&= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \frac{\mathrm{d}\nu}{\nu} e^{w\nu Q^2} \tilde{J}^q_\nu(Q^2) \\
&= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \frac{\mathrm{d}\nu}{\nu} e^{w\nu Q^2} e^{\mathcal{F}(\alpha_s, \ln\left(\nu Q^2\right))} \\
&= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C'-iT}^{C'+iT} \frac{\mathrm{d}N}{N} e^{wN} e^{\mathcal{F}(\alpha_s, \ln N)}
\end{aligned} \tag{4}$$

where $N = \nu Q^2$ and $\mathcal{F}$ has the logarithms expansion

$$\mathcal{F}(\alpha_s, \ln N) = f_1(b_0\alpha_s \ln N) \ln N + f_2(b_0\alpha_s \ln N) + f_3(b_0\alpha_s \ln N)\alpha_s \\ + f_4(b_0\alpha_s \ln N)\alpha_s^2 + f_5(b_0\alpha_s \ln N)\alpha_s^3 + \mathcal{O}\big(\alpha_s^4\big) \tag{5}$$

Since the function $\mathcal{F}$ in the exponent varies more slowly with N than wN, we can introduce the integration variable $u = wN$ so that $\ln N = \ln u + \ln \frac{1}{w} = \ln u + L$ and Taylor expand with respect to $\ln u$ around 0, which is equivalent to expanding the original function $\mathcal{F}$ w.r.t $\ln N$ around $\ln N = \ln \frac{1}{w} \equiv L$:

$$\begin{aligned} R^q(w) &= \frac{1}{2\pi i} \int_C \frac{\mathrm{d}u}{u} e^u e^{\mathcal{F}(\alpha_s, \ln u + L)} \\ &\overset{\text{Taylor}}{=} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\mathcal{F}(\alpha_s, L) + \sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \\ &= e^{\mathcal{F}(\alpha_s, L)} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \end{aligned} \tag{6}$$

where the integral is intended as before, along the line C to the right of all singularities of the integrand, and

$$\mathcal{F}^{(n)}(\alpha_s, L) = \left. \frac{\partial^n \mathcal{F}(\alpha_s, \ln u + L)}{\partial \ln u^n} \right|_{\ln u = 0} \tag{7}$$

The first few derivatives are:

$$\mathcal{F}^{(1)}(\alpha_s, L) = f_1(\lambda) + \lambda f_1'(\lambda) + \alpha_s b_0 f_2'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s^3 b_0 f_4'(\lambda) \\ + \alpha_s^4 b_0 f_5'(\lambda) \tag{8}$$

$$\mathcal{F}^{(2)}(\alpha_s, L) = 2\alpha_s b_0 f_1'(\lambda) + \alpha_s b_0 \lambda f_1''(\lambda) + \alpha_s^2 b_0^2 f_2''(\lambda) + \alpha_s^3 b_0^2 f_3''(\lambda) \\ + \alpha_s^4 b_0^2 f_4''(\lambda) + \alpha_s^5 b_0^2 f_5''(\lambda) \tag{9}$$

$$\mathcal{F}^{(3)}(\alpha_s, L) = 3\alpha_s^2 b_0^2 f_1''(\lambda) + \alpha_s^2 b_0^2 \lambda f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 f_2^{(3)}(\lambda) + \alpha_s^4 b_0^3 f_3^{(3)}(\lambda) \\ + \alpha_s^5 b_0^3 f_4^{(3)}(\lambda) + \alpha_s^6 b_0^3 f_5^{(3)}(\lambda) \tag{10}$$

$$\begin{aligned}\mathcal{F}^{(4)}(\alpha_s, L) = 4\alpha_s^3 b_0^3 f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 \lambda f_1^{(4)}(\lambda) + \alpha_s^4 b_0^4 f_2^{(4)}(\lambda) + \alpha_s^5 b_0^4 f_3^{(4)}(\lambda) \\ + \alpha_s^6 b_0^4 f_4^{(4)}(\lambda) + \alpha_s^7 b_0^4 f_5^{(4)}(\lambda)\end{aligned} \tag{11}$$

After recasting the expansion presented in eq. (6) using the expression $\gamma(\alpha_s, L) = f_1(\lambda) + \lambda f_1'(\lambda)$ from [2], and defining $\mathcal{F}_{res}^{(1)}(\alpha_s, L) \equiv \mathcal{F}^{(1)}(\alpha_s, L) - \gamma(\alpha_s, L)$, we proceed to expand the second exponential with respect to $\ln u$ around 0, following the approach outlined in [1]. This yields the subsequent expansion:

$$\begin{aligned}R^q(w) &= e^{\mathcal{F}(\alpha_s,L)} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u} e^{\mathcal{F}_{res}^{(1)}(\alpha_s,L)\ln u + \sum_{n=2}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s,L)}{n!} \ln^n u} \\ &= \int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u} \Bigg(1 + \mathcal{F}_{res}^{(1)} \ln u + \frac{1}{2}\Big(\mathcal{F}^{(2)} + (\mathcal{F}_{res}^{(1)})^2\Big) \ln^2 u \\ &+ \frac{1}{6}\Big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}_{res}^{(1)} + (\mathcal{F}_{res}^{(1)})^3\Big) \ln^3 u \\ &+ \frac{1}{24}\Big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}_{res}^{(1)} + 6\mathcal{F}^{(2)}(\mathcal{F}_{res}^{(1)})^2 + (\mathcal{F}_{res}^{(1)})^4\Big) \ln^4 u \\ &+ \mathcal{O}\big(\ln^5 u\big)\Bigg)\end{aligned} \tag{12}$$

Lastly, we utilize the following result to evaluate the integral presented in eq. (2).

$$\int_C \frac{\mathrm{d}u}{2\pi i} \ln^k u e^{u-(1-\gamma(\alpha_s,L))\ln u} = \frac{\mathrm{d}^k}{\mathrm{d}\gamma^k} \frac{1}{\Gamma(1-\gamma(\alpha_s, L))} \tag{13}$$

where Γ is the Euler Γ-function.

$$\begin{aligned}R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)} \Bigg[&1 + \mathcal{F}_{res}^{(1)}\psi_0(1-\gamma) + \frac{1}{2}\Big(\mathcal{F}^{(2)} + (\mathcal{F}_{res}^{(1)})^2\Big)\big(\psi_0^2 - \psi_1\big)(1-\gamma) \\ &+ \frac{1}{6}\Big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}_{res}^{(1)} + (\mathcal{F}_{res}^{(1)})^3\Big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma) \\ &+ \frac{1}{24}\Big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}_{res}^{(1)} + 6\mathcal{F}^{(2)}(\mathcal{F}_{res}^{(1)})^2 + (\mathcal{F}_{res}^{(1)})^4\Big) \\ &\big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Bigg]\end{aligned} \tag{14}$$

where

$$\psi_n(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} \ln \Gamma(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} \frac{\Gamma'(z)}{\Gamma(z)} = \frac{\mathrm{d}^n}{\mathrm{d}z^n} \psi_0(z) \tag{15}$$

are the polygamma functions.

References

[1] Ugo Aglietti and Giulia Ricciardi. "Approximate NNLO threshold resummation in heavy flavor decays". In: *Phys. Rev. D* 66 (2002), p. 074003. DOI: 10.1103/PhysRevD.66.074003. arXiv: hep-ph/0204125.

[2] S. Catani et al. "Resummation of large logarithms in e+e− event shape distributions". In: *Nuclear Physics B* 407.1 (1993), pp. 3–42. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(93)90271-P. URL: https://www.sciencedirect.com/science/article/pii/055032139390271P.