

# Chapter 1

## Constants and Ingredients for Resummation

The needed coefficients are often expressed in terms of the following parameters:

$$\begin{aligned} n_f &= 5 && \text{Number of active flavors,} \\ C_F &= \frac{4}{3} && \text{Quadratic Casimir operator for fundamental representation,} \\ C_A &= 3 && \text{Quadratic Casimir operator for adjoint representation,} \\ T_R &= \frac{1}{2} && \text{Trace normalization for fundamental representation.} \end{aligned} \tag{1}$$

The renormalization group equation for the QCD coupling constant reads:

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s) = -\alpha_s^2 \left( b_0 + b_1 \alpha_s + b_2 \alpha_s^2 + \dots \right), \tag{2}$$

where the coefficients of the  $\beta(\alpha_s)$  functions are **Herzog'2017**:

$$\begin{aligned} b_0 &= \frac{11C_A - 4n_f T_R}{12\pi} = \frac{33 - 2n_f}{12\pi}, \\ b_1 &= \frac{17C_A^2 - n_f T_R (10C_A + 6C_F)}{24\pi^2} = \frac{153 - 19n_f}{24\pi^2}, \\ b_2 &= \frac{325n_f^2}{3456\pi^3} - \frac{5033n_f}{1152\pi^3} + \frac{2857}{128\pi^3}, \\ b_3 &= \frac{1093n_f^3}{186624\pi^4} + n_f^2 \left( \frac{809\zeta(3)}{2592\pi^4} + \frac{50065}{41472\pi^4} \right) + n_f \left( -\frac{1627\zeta(3)}{1728\pi^4} - \frac{1078361}{41472\pi^4} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{891\zeta(3)}{64\pi^4} + \frac{149753}{1536\pi^4}, \\
b_4 = n_f^4 & \left( \frac{1205}{2985984\pi^5} - \frac{19\zeta(3)}{10368\pi^5} \right) \\
& + n_f^3 \left( -\frac{24361\zeta(3)}{124416\pi^5} + \frac{115\zeta(5)}{2304\pi^5} + \frac{809}{1244160\pi} - \frac{630559}{5971968\pi^5} \right) \\
& + n_f^2 \left( \frac{698531\zeta(3)}{82944\pi^5} - \frac{5965\zeta(5)}{1296\pi^5} - \frac{5263}{414720\pi} + \frac{25960913}{1990656\pi^5} \right) \\
& + n_f \left( -\frac{1202791\zeta(3)}{20736\pi^5} + \frac{1358995\zeta(5)}{27648\pi^5} + \frac{6787}{110592\pi} - \frac{336460813}{1990656\pi^5} \right) \\
& + \frac{621885\zeta(3)}{2048\pi^5} - \frac{144045\zeta(5)}{512\pi^5} - \frac{9801}{20480\pi} + \frac{8157455}{16384\pi^5}.
\end{aligned} \tag{3}$$

Numerically, for  $n_f = 5$  :

$$\begin{aligned}
b_0 &= 0.875352 - 0.0530516n_f = \frac{23}{12\pi} = 0.610094, \\
b_1 &= 0.645923 - 0.0802126n_f = \frac{29}{12\pi^2} = 0.24486, \\
b_2 &= 0.719864 - 0.140904n_f + 0.00303291n_f^2 = \frac{9769}{3456\pi^3} = 0.0911647, \\
b_3 &= 1.17269 - 0.278557n_f + 0.0162447n_f^2 + 0.0000601247n_f^3 = 0.193536, \\
b_4 &= 1.71413 - 0.594075n_f + 0.0560618n_f^2 - 0.000738048n_f^3 - 5.87966 \cdot 10^{-6}n_f^4 = 0.0493694.
\end{aligned} \tag{4}$$

The A coefficient in the resummation formula are given by:

$$A_1 = C_F, \tag{5}$$

$$A_2 = \frac{1}{2}C_A C_F \left( -\frac{1}{27}10n_f T_R - \frac{\pi^2}{6} + \frac{67}{18} \right), \tag{6}$$

$$\begin{aligned}
A_3 &= \left( -\frac{2051}{1296} - \frac{\pi^2}{18} \right) C_A C_F n_f + \left( -\frac{11\zeta(3)}{4} + \frac{11\pi^4}{720} - \frac{13\pi^2}{432} + \frac{15503}{2592} \right) C_A^2 C_F \\
&+ \left( \frac{\zeta(3)}{2} - \frac{55}{96} \right) C_F^2 n_f + \left( \frac{25}{324} + \frac{\pi^2}{108} \right) C_F n_f^2,
\end{aligned} \tag{7}$$

$$\begin{aligned}
A_4 &= \left( -\frac{\zeta(3)^2}{16} + \frac{11\pi^2\zeta(3)}{32} - \frac{24461\zeta(3)}{864} + \frac{20513\pi^2}{5184} + \frac{1925\zeta(5)}{288} - \frac{253\pi^4}{1920} - \frac{313\pi^6}{90720} \right. \\
&+ \left. \frac{4311229}{186624} \right) C_F C_A^3 + \left( \frac{7\pi^2\zeta(3)}{72} + \frac{43\zeta(3)}{144} + \frac{451\pi^4}{4320} + \frac{131\zeta(5)}{144} \right. \\
&+ \left. T_R \left( -\frac{1}{72}11\pi^2\zeta(3) + \frac{685\zeta(3)}{48} - 3\zeta(5) - \frac{11\pi^4}{144} - \frac{1123}{162} - \frac{12247\pi^2}{15552} \right) \right)
\end{aligned} \tag{8}$$

$$\begin{aligned}
& -\frac{64421\pi^2}{31104} - \frac{260731}{62208} \Big) C_F n_f C_A^2 + \left( \left( \frac{34\zeta(3)}{9} + \frac{11\pi^4}{720} + \frac{55\pi^2}{576} - \frac{7351}{1152} \right) T_R - \frac{\pi^2\zeta(3)}{12} \right. \\
& + \frac{29\zeta(3)}{18} + \frac{5\zeta(5)}{8} - \frac{55\pi^2}{1152} - \frac{11\pi^4}{1440} - \frac{17033}{10368} \Big) n_f C_F^2 \\
& + \left( \left( -\frac{7\zeta(3)}{54} + \frac{\pi^4}{60} - \frac{481\pi^2}{1944} + \frac{1747}{3888} \right) T_R^2 + \left( -\frac{143\zeta(3)}{108} + \frac{847\pi^2}{1296} + \frac{55}{486} \right) T_R + \frac{803\pi^2}{3888} \right. \\
& - \frac{\zeta(3)}{144} - \frac{49\pi^4}{4320} + \frac{19889}{62208} \Big) n_f^2 C_F \Big) C_A + \frac{31\pi^6}{60480} + \left( \left( -\frac{7\zeta(3)}{27} + \frac{5\pi^2}{324} + \frac{130}{729} \right) T_R^3 \right. \\
& + \left( \frac{13\zeta(3)}{108} - \frac{5}{486} - \frac{77\pi^2}{1296} \right) T_R + \frac{\zeta(3)}{108} - \frac{1}{648} \Big) C_F n_f^3 + \frac{\pi^2}{96} + \left( \left( -\frac{19\zeta(3)}{18} + \frac{215}{96} - \frac{5\pi^2}{144} \right. \right. \\
& - \frac{\pi^4}{180} \Big) T_R^2 + \frac{\pi^4}{720} + \frac{5\pi^2}{192} - \frac{5\zeta(3)}{18} + \frac{299}{2592} \Big) C_F^2 n_f^2 + \frac{3\zeta(3)^2}{16} \\
& + 81 \left( -\frac{\zeta(3)^2}{32} + \frac{\zeta(3)}{288} + \frac{55\zeta(5)}{576} - \frac{\pi^2}{576} - \frac{31\pi^6}{362880} \right) \\
& + 9 \left( -\frac{5\zeta(3)^2}{32} + \frac{5\zeta(3)}{288} + \frac{275\zeta(5)}{576} - \frac{5\pi^2}{576} - \frac{31\pi^6}{72576} \right) + \left( \left( \frac{37\zeta(3)}{48} - \frac{5\zeta(5)}{4} + \frac{143}{576} \right) C_F^3 \right. \\
& + \frac{1}{3} \left( -\frac{\zeta(3)}{12} + \frac{\pi^2}{24} - \frac{5\zeta(5)}{12} \right) + \frac{1}{27} \left( \frac{\zeta(3)}{16} + \frac{5\zeta(5)}{16} - \frac{\pi^2}{32} \right) + 3 \left( \frac{7\zeta(3)}{288} + \frac{35\zeta(5)}{288} - \frac{7\pi^2}{576} \right) \\
& \left. + 27 \left( -\frac{\zeta(3)}{288} + \frac{\pi^2}{576} - \frac{5\zeta(5)}{288} \right) \right) n_f - \frac{\zeta(3)}{48} - \frac{55\zeta(5)}{96},
\end{aligned}$$

$$A_5 = 14541.099. \quad (9)$$

The  $B$  coefficients are given by:

$$B_1 = -\frac{1}{2} (3C_F), \quad (10)$$

$$B_2 = C_A C_F \left( \frac{3\zeta(3)}{2} - \frac{57}{16} + \frac{11\pi^2}{24} \right) + C_F n_f \left( \frac{5}{8} - \frac{\pi^2}{12} \right) + C_F^2 \left( -3\zeta(3) - \frac{3}{16} + \frac{\pi^2}{4} \right), \quad (11)$$

$$\begin{aligned}
B_3 = & C_A C_F n_f \left( \left( \frac{34\zeta(3)}{27} - \frac{485\pi^2}{432} - \frac{\pi^4}{720} + \frac{3683}{864} \right) T_R + \frac{31\zeta(3)}{54} + \frac{131\pi^4}{4320} + \frac{5261}{1728} \right. \\
& - \frac{2657\pi^2}{2592} \Big) + C_A C_F^2 \left( -\frac{\pi^2\zeta(3)}{12} - \frac{89\zeta(3)}{12} - \frac{15\zeta(5)}{4} + \frac{287\pi^2}{192} - \frac{23}{16} - \frac{17\pi^4}{360} \right) \\
& + C_A^2 C_F \left( \frac{241\zeta(3)}{108} - \frac{5\zeta(5)}{4} + \frac{22841\pi^2}{5184} - \frac{5951}{432} - \frac{713\pi^4}{4320} \right) + C_F^2 n_f \left( \left( \frac{17\zeta(3)}{6} + \frac{23}{16} - \right. \right. \\
& \left. \left. \frac{5\pi^2}{72} - \frac{29\pi^4}{1080} \right) T_R - \frac{\zeta(3)}{4} + \frac{41\pi^4}{2160} + \frac{31}{64} - \frac{71\pi^2}{288} \right) + C_F n_f^2 \left( \left( \frac{2\zeta(3)}{9} + \frac{17}{72} - \frac{5\pi^2}{81} \right) T_R^2 + \right.
\end{aligned} \quad (12)$$

$$\left(-\frac{13\zeta(3)}{27} + \frac{193\pi^2}{648} - \frac{433}{432}\right) T_R) + \left(\frac{\pi^2\zeta(3)}{6} - \frac{17\zeta(3)}{8} + \frac{15\zeta(5)}{2} - \frac{\pi^4}{20} - \frac{3\pi^2}{32} - \frac{29}{64}\right) C_F^3,$$

$$B_4 = 3817.42. \quad (13)$$

By expanding in  $\alpha_s$  the exponent of ?? with ????????? substituted, we can compare to the expression ?? and obtain:

$$\begin{aligned} G_{12} &= -2A_1, \\ G_{11} &= -2B_1, \\ G_{23} &= -4\pi A_1 b_0, \\ G_{22} &= \frac{2}{3} \left( -2\pi^2 A_1^2 - 6A_2 - 3\pi b_0 B_1 \right), \\ G_{21} &= \frac{2}{3} \left( -A_1 \left( 24A_1 \zeta(3) - 3\pi^3 b_0 + 2\pi^2 B_1 \right) - 6B_2 \right), \\ G_{34} &= \frac{1}{3} (-28)\pi^2 A_1 b_0^2, \\ G_{33} &= \frac{4}{45} \left( -90\pi^3 A_1^2 b_0 - 90\pi^2 A_1 b_1 - 180\pi A_2 b_0 + 240A_1^3 \zeta(3) - 30\pi^2 b_0^2 B_1 \right), \\ G_{32} &= \frac{4}{45} \left( -75\pi^3 A_1 b_0 B_1 - 1620\pi A_1^2 b_0 \zeta(3) + 105\pi^4 A_1 b_0^2 + 360A_1^2 B_1 \zeta(3) + 2\pi^4 A_1^3 - 60\pi^2 A_2 A_1 \right. \\ &\quad \left. - 90A_3 - 45\pi^2 b_1 B_1 - 90\pi b_0 B_2 \right), \\ G_{31} &= \frac{4}{45} \left( -900\pi A_1 b_0 B_1 \zeta(3) + 840\pi^2 A_1 b_0^2 \zeta(3) + 45\pi^4 A_1 b_1 + 90\pi^3 A_2 b_0 + 180A_1 B_1^2 \zeta(3) \right. \\ &\quad \left. + 2\pi^4 A_1^2 B_1 - 30\pi^2 A_1 B_2 - 30\pi^2 A_2 B_1 - 240\pi^2 A_1^3 \zeta(3) + 2160A_1^3 \zeta(5) - 720A_2 A_1 \zeta(3) \right. \\ &\quad \left. - 15\pi^3 b_0 B_1^2 + 15\pi^4 b_0^2 B_1 - 90B_3 \right), \\ G_{45} &= -24\pi^3 A_1 b_0^3, \\ G_{44} &= 192\pi A_1^3 b_0 \zeta(3) - \frac{332}{9} \pi^4 A_1^2 b_0^2 - \frac{140}{3} \pi^3 A_1 b_0 b_1 - 56\pi^2 A_2 b_0^2 + \frac{1}{45} (-32)\pi^4 A_1^4 - 4\pi^3 b_0^3 B_1, \\ G_{43} &= 256\pi A_1^2 b_0 B_1 \zeta(3) - \frac{232}{9} \pi^4 A_1 b_0^2 B_1 - \frac{2656}{3} \pi^2 A_1^2 b_0^2 \zeta(3) + \frac{32}{15} \pi^5 A_1^3 b_0 \\ &\quad - 16\pi^4 A_1^2 b_1 + 40\pi^5 A_1 b_0^3 - 48\pi^3 A_2 A_1 b_0 - 16\pi^3 A_1 b_2 - 48\pi A_3 b_0 - 32\pi^2 A_2 b_1 - \frac{64}{45} \pi^4 A_1^3 B_1 \\ &\quad + \frac{128}{3} \pi^2 A_1^4 \zeta(3) - 512A_1^4 \zeta(5) + 128A_2 A_1^2 \zeta(3) - \frac{40}{3} \pi^3 b_0 b_1 B_1 - 16\pi^2 b_0^2 B_2, \\ G_{42} &= 112\pi A_1 b_0 B_1^2 \zeta(3) - 464\pi^2 A_1 b_0^2 B_1 \zeta(3) + \frac{32}{15} \pi^5 A_1^2 b_0 B_1 - \frac{40}{3} \pi^4 A_1 b_1 B_1 \\ &\quad - \frac{56}{3} \pi^3 A_1 b_0 B_2 - \frac{64}{3} \pi^3 A_2 b_0 B_1 - 416\pi^3 A_1^3 b_0 \zeta(3) + 3648\pi A_1^3 b_0 \zeta(5) - 288\pi^2 A_1^2 b_1 \zeta(3) \\ &\quad + 480\pi^3 A_1 b_0^3 \zeta(3) - 864\pi A_2 A_1 b_0 \zeta(3) + \frac{56}{45} \pi^6 A_1^2 b_0^2 + \frac{140}{3} \pi^5 A_1 b_0 b_1 + 56\pi^4 A_2 b_0^2 \end{aligned}$$

$$\begin{aligned}
& + 64\pi^2 A_1^3 B_1 \zeta(3) - 768 A_1^3 B_1 \zeta(5) + 64 A_1^2 B_2 \zeta(3) + 128 A_2 A_1 B_1 \zeta(3) - \frac{16}{15} \pi^4 A_1^2 B_1^2 \\
& + 512 A_1^4 \zeta(3)^2 + \frac{1}{945} (-464) \pi^6 A_1^4 + \frac{16}{15} \pi^4 A_2 A_1^2 - \frac{32}{3} \pi^2 A_3 A_1 - \frac{16}{3} \pi^2 A_2^2 \\
& - 16 A_4 - 4 \pi^4 b_0^2 B_1^2 + 4 \pi^5 b_0^3 B_1 - 8 \pi^3 b_2 B_1 - 16 \pi^2 b_1 B_2 - 24 \pi b_0 B_3,
\end{aligned}$$

$$\begin{aligned}
G_{41} = & -120 \pi A_1^3 b_0 \gamma^6 + \frac{64}{3} \pi^2 A_1^4 \gamma^5 + \frac{368}{3} \pi^2 A_1^2 b_0^2 \gamma^5 - 16 A_1^2 A_2 \gamma^5 - 56 \pi A_1^2 b_0 B_1 \gamma^5 + \frac{460}{3} \pi^3 A_1^3 b_0 \gamma^4 \\
& + \frac{40}{3} \pi^2 A_1^3 B_1 \gamma^4 - 480 A_1^4 \zeta(3) \gamma^4 - \frac{32}{9} \pi^4 A_1^4 \gamma^3 - \frac{1840}{9} \pi^4 A_1^2 b_0^2 \gamma^3 + \frac{80}{3} \pi^2 A_1^2 A_2 \gamma^3 \\
& + \frac{280}{3} \pi^3 A_1^2 b_0 B_1 \gamma^3 - 1280 \pi A_1^3 b_0 \zeta(3) \gamma^3 - 320 A_1^3 B_1 \zeta(3) \gamma^3 - \frac{2}{3} \pi^5 A_1^3 b_0 \gamma^2 - \frac{8}{3} \pi^4 A_1^3 B_1 \gamma^2 \\
& + \frac{800}{3} \pi^2 A_1^4 \zeta(3) \gamma^2 + \frac{7360}{3} \pi^2 A_1^2 b_0^2 \zeta(3) \gamma^2 - 320 A_1^2 A_2 \zeta(3) \gamma^2 - 1120 \pi A_1^2 b_0 B_1 \zeta(3) \gamma^2 \\
& - 1920 A_1^4 \zeta(5) \gamma^2 + \frac{92}{9} \pi^6 A_1^2 b_0^2 \gamma - \frac{4}{3} \pi^4 A_1^2 A_2 \gamma - \frac{14}{3} \pi^5 A_1^2 b_0 B_1 \gamma - \frac{1600}{3} \pi^3 A_1^3 b_0 \zeta(3) \gamma \\
& + \frac{800}{3} \pi^2 A_1^3 B_1 \zeta(3) \gamma + 3840 \pi A_1^3 b_0 \zeta(5) \gamma - 1920 A_1^3 B_1 \zeta(5) \gamma - 2 \pi^7 A_1 b_0^3 - \frac{16}{45} \pi^4 A_1 B_1^3 \\
& + \frac{28}{45} \pi^5 A_1 b_0 B_1^2 - \frac{8}{3} \pi^4 b_1 B_1^2 + 128 \pi A_1^3 b_0 \zeta(3)^2 - 128 A_1^3 B_1 \zeta(3)^2 - \frac{1}{15} \pi^7 A_1^3 b_0 \\
& + 24 \pi^3 A_3 b_0 + 16 \pi^4 A_2 b_1 + 8 \pi^5 A_1 b_2 - \frac{2}{135} \pi^6 A_1^3 B_1 + \frac{56}{45} \pi^6 A_1 b_0^2 B_1 + \frac{32}{45} \pi^4 A_1 A_2 B_1 \\
& - \frac{16}{3} \pi^2 A_3 B_1 + \frac{20}{3} \pi^5 b_0 b_1 B_1 + \frac{16}{45} \pi^4 A_1^2 B_2 + 8 \pi^4 b_0^2 B_2 - \frac{16}{3} \pi^2 A_2 B_2 - 8 \pi^3 b_0 B_1 B_2 \\
& - \frac{16}{3} \pi^2 A_1 B_3 - 16 B_4 + \frac{344}{45} \pi^4 A_1^4 \zeta(3) + 16 \pi b_0 B_1^3 \zeta(3) - 64 A_2^2 \zeta(3) + \frac{752}{9} \pi^4 A_1^2 b_0^2 \zeta(3) \\
& + 448 \pi^2 A_2 b_0^2 \zeta(3) + 32 \pi^2 A_1^2 B_1^2 \zeta(3) - 48 \pi^2 b_0^2 B_1^2 \zeta(3) + 32 A_2 B_1^2 \zeta(3) - \frac{224}{3} \pi^2 A_1^2 A_2 \zeta(3) \\
& - 128 A_1 A_3 \zeta(3) + \frac{1120}{3} \pi^3 A_1 b_0 b_1 \zeta(3) + 32 \pi^3 b_0^3 B_1 \zeta(3) - \frac{304}{3} \pi^3 A_1^2 b_0 B_1 \zeta(3) \\
& - 256 \pi A_2 b_0 B_1 \zeta(3) - 160 \pi^2 A_1 b_1 B_1 \zeta(3) - 224 \pi A_1 b_0 B_2 \zeta(3) + 64 A_1 B_1 B_2 \zeta(3) \\
& - 64 \pi^2 A_1^4 \zeta(5) - 896 \pi^2 A_1^2 b_0^2 \zeta(5) - 384 A_1^2 B_1^2 \zeta(5) + 768 A_1^2 A_2 \zeta(5) + 1152 \pi A_1^2 b_0 B_1 \zeta(5).
\end{aligned}$$

(14)