1 Calculation of the f functions

In the two-jet region the fixed-order thrust distribution is enhanced by large infrared logarithms which spoil the convergence of the perturbative series. The convergence can be restored by resumming the logarithms to all orders in the coupling constant α_s.

According to general theorems [1],[5],[6], the cross section **??** has a power series expansion in $\alpha_s(Q^2)$ of the form:

$$\frac{\sigma(\tau, Q^2)}{\sigma_t} = C\big(\alpha_s(Q^2)\big)\Sigma\big(\tau, \alpha_s(Q^2)\big) + F\big(\tau, \alpha_s(Q^2)\big) \tag{1}$$

where σ_t is the total hadronic cross section and

$$C(\alpha_s) = 1 + \sum_{n=1}^{\infty} C_n \alpha_s^n \tag{2}$$

$$\Sigma(\tau, \alpha_s) = \exp\left[\sum_{n=1}^{\infty} \alpha_s^n \sum_{m=1}^{2n} G_{nm} \ln^m \tau\right] \tag{3}$$

$$F(\tau, \alpha_s) = \sum_{n=1}^{\infty} \alpha_s^n F_n(\tau) \tag{4}$$

Here C_n and G_{nm} are constants, while $F_n(\tau)$ are perturbatively computable functions that vanish at small τ. Thus at small τ (large thrust) it becomes morst important to resum the series of large logarithms in $\Sigma(\tau, \alpha_s)$. These are normally classified as *leading* logarithms when $n < m \leq 2n$, *next-to-leading* when $m = n$ and *subdominant* logarithms when $m < n$.

The term eq. (3) can be re-written as

$$\Sigma(\tau, \alpha_s) = \exp\left\{L f_1(\lambda) + \sum_{i=0}^{\infty} f_{i+1}(\lambda)\alpha_s^i\right\} \tag{5}$$

where $L = \ln N = \ln\big(\nu Q^2\big)$ and $\lambda = \alpha_s b_0 L$. We require the functions $f_i(\lambda)$ to be omogeneous,$i.e$ $f_i(0) = 0$, so that at $N^n LL$ we can write:

$$f_{n+1}(\lambda) = \sum_{k \geq n} \tilde{G}_{k,k+1-n} \alpha_s^k L^{k+1-n} \tag{6}$$

this requirement is automatically satisfied if we choose as variable $L = \ln\left(\frac{N}{N_0}\right)$ where $N_0 = e^{-\gamma_E}$, $\gamma_E = 0.5772\ldots$ being the Euler-Mascheroni constant. With the latter choice the terms proportional to γ_E and its powers disappear. The advantage of the variable N is that the total rate is directly reproduced by setting $N = 1$, while in the variable $n = N/N_0$ it is given $f_{n=1/N_0}$. These two choices differ only by terms of higher order in γ_E.

In the article by Catani, Turnock, Webber and Trentadue [3], it was observed that for a final state configuration corresponding to a large value of thrust , **??** can be approximated by

$$\tau = 1 - T \approx \frac{k_1^2 + k_2^2}{Q^2} \tag{7}$$

where k_1^2 and k_2^2 are the invariant masses squared of two back-to-back jets and Q^2 is the energy of the center of mass. Thus the key to the evaluation of the thrust distributions is its relation to the quark jet mass distribution $J^q(Q^2, k^2) = J^q_{k^2}(Q^2)$ which denotes the probability of jet invariant mass-squared k^2 at scale Q^2, then the thrust fraction

$$R_T(\tau, \alpha_s(Q^2)) = \frac{\sigma(\tau, Q^2)}{\sigma_t} = \frac{1}{\sigma_t} \int_0^1 \frac{\mathrm{d}\sigma(\tau', Q^2)}{\mathrm{d}\tau'} \Theta\big(\tau - \tau'\big) \mathrm{d}\tau' \tag{8}$$

takes the form of a convolution of two jet mass distributions $J(Q^2, k_1^2)$ and $J(Q^2, k_2^2)$

$$R_T(\tau, \alpha_s(Q^2)) \underset{\tau \ll 1}{=} \int_0^\infty \mathrm{d}k_1^2 \int_0^\infty \mathrm{d}k_2^2 J^q_{k_1^2}(Q^2) J^q_{k_2^2}(Q^2) \Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) \tag{9}$$

Introducing the Laplace transform of the jet mass distribution

$$\tilde{J}^q_\nu(Q^2) = \int_0^\infty J^q(Q^2, k^2) e^{-\nu k^2} \mathrm{d}k^2 \tag{10}$$

and using the integral representation of the Heaviside step function

$$\Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{\mathrm{d}N}{N} e^{N\tau} e^{-N\frac{k_2^2+k_2^2}{Q^2}} \tag{11}$$

by substituting eq. (11) into eq. (9) and $N = \nu Q^2$:

$$\begin{aligned} R_T(\tau) &= \int_{C-i\infty}^{C+i\infty} \frac{\mathrm{d}N}{N} \frac{e^{N\tau}}{2\pi i} \left[\int_0^\infty \mathrm{d}k_1^2 e^{-\nu k_1^2} J^q_{k_1^2}(Q^2)\right] \left[\int_0^\infty \mathrm{d}k_2^2 e^{-\nu k_2^2} J^q_{k_2^2}(Q^2)\right] \\ &= \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} e^{N\tau} \left[\tilde{J}^q_\nu(Q^2)\right]^2 \frac{\mathrm{d}N}{N} \end{aligned} \tag{12}$$

where C is a real positive constant to the right of all singularities of the integrand $\tilde{J}_\nu(Q^2)$ in the complex ν plane.

An integral representation for the Laplace transform $\tilde{J}_\nu(Q^2)$ is given by

$$\ln \tilde{J}^q_\nu(Q^2) = \int_0^1 \frac{\mathrm{d}u}{u} \left(e^{-u\nu Q^2} - 1\right) \left[\int_{u^2Q^2}^{uQ^2} \frac{1}{q^2} A\big(\alpha_s(q^2)\big) \mathrm{d}q^2 + \frac{1}{2} B\big(\alpha_s(uQ^2)\big)\right] \tag{13}$$

with

$$A(\alpha_s) = \sum_{n=1}^{\infty} \frac{A_n}{\pi^n} \alpha_s^n \qquad\qquad B(\alpha_s) = \sum_{n=1}^{\infty} \frac{B_n}{\pi^n} \alpha_s^n$$

The integral as it is cannot be integrated, the u integration may be performed using the prescription in Appendix A of [4] and readapting the formula to the case of Laplace transform instead of Mellin transform.

In **??** we show that the prescription, to evaluate the large-N Mellin moments of soft-gluon contributins at an arbitrary logarithmic accuracy, can be used for the Laplace transform as well, we can use this result to express eq. (13) in an alternative representation:

The method is a generalization of the prescription to NLL accuray in [2]

$$e^{-u\nu Q^2} - 1 \simeq -\Theta(u - v) \quad \text{with } v = \frac{N_0}{N} \tag{14}$$

where $N_0 = e^{-\gamma_E}$, $\gamma_E = 0.5772\ldots$ being the Euler-Mascheroni constant

$$\ln \tilde{J}^q_\nu(Q^2) = -\int_{N_0/N}^{1} \frac{\mathrm{d}u}{u} \left[\int_{u^2Q^2}^{uQ^2} \frac{1}{q^2} A\big(\alpha_s(q^2)\big) \mathrm{d}q^2 + \frac{1}{2} \tilde{B}\big(\alpha_s(uQ^2)\big) \right] + \ln \tilde{C}(Q^2) \tag{15}$$