UNIVERSITÀ DEGLI STUDI DI MILANO

FACOLTÀ DI SCIENZE E TECNOLOGIE

Corso di Laurea triennale in Fisica

DIPARTIMENTO DI FISICA

"ALDO PONTREMOLI"

Resummation of large infrared logarithms for Thrust distribution in e^+e^- annihilation

Supervisor: Giancarlo Ferrera

Co-supervisor: Wan-Li Ju

Author:

Jiahao Miao

Matr. Nr. 936136

ACADEMIC YEAR 2023-2024

Abstract

This thesis focuses on the Thrust event-shape distribution in electron-positron annihilation, a classical precision QCD observable. We focus on the back-to-back region and perform all-order resummation to address logarithmically enhanced contributions within QCD perturbation theory. Our calculations extend the pioneering work of Catani in this field, aiming to achieve next-to-next-to-next-to-leading logarithmic (N^3LL) accuracy and then consistently combine with the known fixed-order results up to next-to-next-to-leading order (NNLO). Almost all the calculations are done analytically using the software Mathematica.

Contents

Chapter 1

Introduction

In this thesis, we focus on a specific aspect of theoretical high-energy physics and collider physics: the resummation of large logarithms for the Thrust event-shape distribution in electron-positron (e^+e^-) collisions.

High energy physics, often synonymous with particle physics, and collider physics are crucial because they explores the most basic constituents of matter and helps us understand three of the four fundamental forces of nature: the strong, weak, and electromagnetic forces. Collider experiments test the predictions of the Standard Model of particle physics, the most successful theory of particle physics to date. Through high precision theoretical prediction and experimental measurements, it is possible to test the limits of the Standard Model, search for new physics beyond it and gain a better understanding of the fundamental forces.

One key parameter of the Standard Model is the strong coupling constant α_s, which measures the strength of the strong force. Precise measurements of α_s is crucial for accurate predictions in Quantum Chromodynamics (QCD), the theory of the strong force, as all calculations in QCD depends on it.

One example of a process studied in high-energy physics is electron-positron annihilation, as shown in fig. 1. Electron-positron annihilation has been extensively studied, particularly during the operation of the Large Electron-Positron Collider (LEP) at CERN from 1989 to 2000. LEP provided an optimal environment for precision studies in high-energy physics. Unlike hadron colliders, which are complicated by strongly interacting initial states, LEP enabled extremely accurate measurements of Standard Model quantities such as the Z-boson mass. These results tightly constrain beyond-the-Standard Model physics. Precision data from LEP is also used

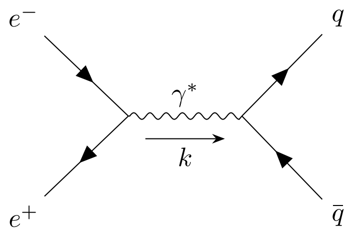


Figure 1: Tree-level Feynman diagram of electron-positron annihilation producing a virtual photon that decays into a quark-antiquark pair.

in Quantum Chromodynamics (QCD) studies, for example, to determine the strong coupling constant, α_S.

When high-energy particles collide, quarks and gluons are produced in the interactions. Due to a phenomenon called color confinement, these quarks and gluons cannot exist freely and thus hadronize, forming jets of particles. A jet is a collimated stream of hadrons (such as protons, pions, and kaons) that originates from the hadronization of a single quark or gluon.

In electron-positron annihilation, the electron and positron interact electromagnetically through the exchange of a photon, which mediates the electromagnetic interaction. This photon can then decay into a quark-antiquark pair in another electromagnetic process, conserving the electrical and color charge of the initial state, as well as the four-momentum.

The quark-antiquark pair produced interacts through both the strong force and the electromagnetic force, as they possess both electric charge and color charge. They can radiate gluons via the strong force and photons via the electromagnetic force. This radiation process continues, creating a cascade of particles known as a parton shower. The parton shower eventually hadronizes into jets when the particles are no longer energetic enough to radiate further, and the final state particles can be detected by the detectors, with their momenta measured to study the underlying processes.

As is typical in physics, the equations governing these interactions are highly complex, finding exact solutions is nearly impossible. Therefore, functions of interest are often expanded perturbatively, meaning they are expressed as a power series in a small parameter.

For the electromagnetic interaction, this small parameter is the fine structure constant (or electromagnetic coupling constant) $\alpha_{em} \sim \frac{1}{137}$.

For interactions involving the strong force, it is natural to use the aformentioned strong coupling constant, α_S. This key parameter becomes small at high energies (or equivalently, short distances) due to the phenomen known as asymptotic freedom of QCD.

Extraction of α_s can be achieved from comparing the distribution of event shape variables such as the thrust and experimental data.

1.1 Thrust variable

Thrust T is defined as:

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|} \stackrel{\text{def}}{=} 1 - \tau \tag{1}$$

where the sum is over all final state particles and $\vec{n}$ is a unit vector. In practice, the sum may be carried over the detected particles only. The thrust distribution represents the probability of observing a given value of T in e^+e^- annihilation, *i.e* the probability of observing a given configuration of momenta of final-state particles with respect to the thrust axis.

It can be seen from this definition that the thrust is an infrared and collinear safe quantity, that is, it is insensitive to the emission of zero momentum particles and to the splitting of one particle into two collinear ones.

In fact, contribution from soft particles with $\vec{p_i} \to 0$ drop out, and collinear splitting does not change the thrust: $|(1-\lambda)\vec{p_i} \cdot \vec{n}| + |\lambda \vec{p_i} \cdot \vec{n}| = |\vec{p_i} \cdot \vec{n}|$ and $|(1-\lambda)\vec{p_i}| + |\lambda \vec{p_i}| = |\vec{p_i}|$

Formally, infrared-safe observables are the one which do not distinguish between (n+1)-partons and n-partons in the soft/collinear limit, *i.e*, are insensitive to what happens at long-distance (non-perturbative) scales.

Infrared safe observables are important in the context of perturbative QCD, because they allow for a meaningful comparison between theory and experiment.

A significant challenge in achieving precise theoretical predictions from QCD lies in the complexity of the relevant fixed-order calculations. While the next-to-leading-order (NLO) results for event shapes have been known since 1980 [11], the relevant next-to-next-leading order (NNLO) calculations were completed only in 2007 [12].

From fig. 2, it is evident that the NNLO prediction agrees well with the data, except in the region near $T = 0.5$ (spherical final state) and $T = 1$ (pencil-like final state).

This is because thrust distribution for $T \simeq 1$ is dominated by two-jet configurations, *i.e.* the final state particles consist of two partons emitted back-to-back like fig. 1. In contrast, the tail of the distribution near $T = 0.5$ is dominated by multijet final states.

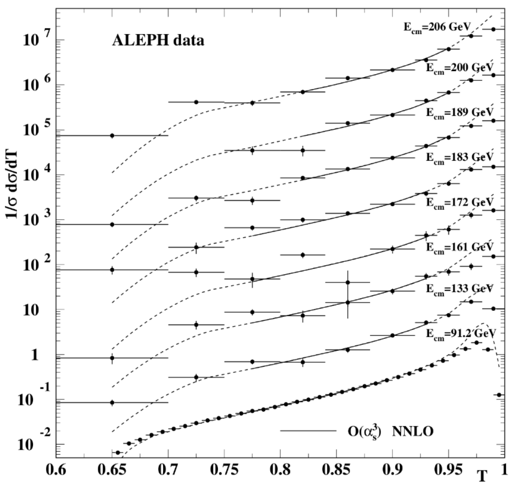


Figure 2: Distributions measured by ALEPH, after correction for backgrounds and detector effects of thrust at energies between 91.2 and 206 GeV together with the fitted NNLO QCD predictions. The error bars correspond to statistical uncertainties.The plotted distributions are scaled by arbitrary factors for presentation. Image taken from [10].

To improve the agreement between theory and experiment, we'll need higher fixed order calculations (challenging task), but this will only improve the agreement in the tail region, to also improve the agreement in the dijet region we need to use resummation techniques.

1.1.1 Thrust distribution

The cross section is defined as the probability of observing a final state with a given thrust value τ and Thrust distribution is expressed in three ways: as a differential cross section, as a cumulative distribution and as its Laplace transform.

$$\sigma(\tau) = \int_0^\tau \mathrm{d}\tau' \, \frac{\mathrm{d}\sigma}{\mathrm{d}\tau'} \tag{2}$$

$$R_T(\tau) = \int_0^\tau \mathrm{d}\tau' \, \frac{1}{\sigma_0} \frac{\mathrm{d}\sigma}{\mathrm{d}\tau'} \tag{3}$$

$$\tilde{\sigma}(\nu) = \int_0^\infty \mathrm{d}\tau \, e^{-\nu\tau} \frac{\mathrm{d}\sigma}{\mathrm{d}\tau} \tag{4}$$

It can be seen that a two-particle final state has fixed $T = 1$, in fact at the zeroth order of the fixed order this corrisponds to a delta distribution for the differential cross section, as shown in eq. (5), consequently the thrust distribution receives its first non-trivial contribution from three-particle final states.

The lower limit on T depends on the number of final-state particles. Neglecting masses, for three particles, $T_{min} = 2/3$, corresponding to a symmetric configuration. For four particles the minimum thrust corresponds to final-state momenta forming the vertices of a regular tetrahedron, each making an angle $\cos^{-1}(1/\sqrt{3})$ with respect to the thrust axis. Thus $T_{min} = 1/\sqrt{3} \approx 0.577$ in this case. For more than four particles, T_{min} approaches $1/2$ from above as the number of particles increases.

At large values of T, however, there are terms in higher order that become enhanced by powers of $\ln(1-T)$. In this kinematical region the real expansion parameter is the large effective coupling $\alpha_s \ln^2(\tau)$ and therefore any finite-order perturbative calculation cannot give an accurate evaluation of the cross section.

For example, at leading order in perturbation theory the thrust distribution has the form:

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \delta(\tau) + \frac{2\alpha_s}{3\pi}\left[\frac{-4\ln\tau - 3}{\tau} + \dots\right] \tag{5}$$

where σ_0 is the born cross section and the ellipsis denotes terms that are regular as $\tau \to 0$. Upon integration over τ, we obtain the cumulative distribution:

$$R_T(\tau) = \int_0^\tau \mathrm{d}\tau' \,\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau'} = 1 + \frac{2\alpha_s}{3\pi}\left[-2\ln^2\tau - 3\ln\tau + \dots\right] \tag{6}$$

Double logarithmic terms of the form $\alpha_s^n \ln^{2n}\tau$ plagues the fixed order expansion in the strong coupling. In the dijet region, higher order terms are as important as lower order ones, necessitating resummation to achieve reliable predictions.

1.1.2 Fixed Order Cross Section

The fixed-order thrust differential distribution has been calculated to leading order analytically and to NLO and NNLO numerically, as mentioned earlier. At a centre-of-mass energy Q and for a renormalization scale μ takes the form:

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}(\tau,Q) = \delta(\tau) + \frac{\alpha_s(\mu)}{2\pi}\frac{\mathrm{d}A}{\mathrm{d}\tau}(\tau) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\frac{\mathrm{d}B}{\mathrm{d}\tau}\left(\tau,\frac{\mu}{Q}\right) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^3\frac{\mathrm{d}C}{\mathrm{d}\tau}\left(\tau,\frac{\mu}{Q}\right) + \mathcal{O}\left(\alpha_s^4\right) \tag{7}$$

where the complete leading order expression for the thrust distribution reads [11]:

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \mathrm{C}_F \left(9\tau - \frac{3}{\tau} + \left(-6 + \frac{4}{(1-\tau)\tau}\right) \log\left(\frac{1-2\tau}{\tau}\right) + 6\right) \tag{8}$$

Where $C_F = \frac{4}{3}$ is the Casimir of the fundamental representation of SU(3). As mentioned earlier, this expansion becomes unreliable near the dijet limit $\tau \to 0$ due to the presence of large logarithms. Resummation of these terms to all orders in α_s is necessary to obtain a reliable prediction.

While the cumulant distribution has the following fixed-order expansion:

$$R_T(\tau) = 1 + A(\tau)\frac{\alpha_s(\mu)}{2\pi} + B(\tau,\mu)\frac{\alpha_s(\mu)}{2\pi}^2 + C(\tau,\mu)\frac{\alpha_s(\mu)}{2\pi}^3 + \mathcal{O}(\alpha_s^4) \tag{9}$$

the fixed-order coeficcient A, B and C can be obtained by integrating the differential cross section eq. (8) to all orderand imposing the normalization condition $R(\tau_{max}) = 1$, where τ_{max} is the maximum kinetically allowed value of τ. At leading order($e^+e^- \to q\bar{q}g$) $\tau_{max} = \frac{1}{3}$, at Next-to-Leading Order is $\tau_{max} = 1 - \frac{1}{\sqrt{3}}$ and from NNLO onwards τ_{max} needs to be estimated numerically.

$$\begin{aligned} A(\tau) = \int_0^\tau \mathrm{d}\tau' \, \frac{\mathrm{d}A}{\mathrm{d}\tau'} = C_F \Big(& \frac{9\tau^2}{2} - 2\log^2(1-\tau) - 2\log^2(\tau) + 6\tau(\log(\tau) + 1) \\ & + 4\log(1-\tau)\log(\tau) - 3\log(\tau) + 3(1-2\tau)\log(1-2\tau) \\ & - 4\mathrm{Li}_2\left(\frac{\tau}{1-\tau}\right)\Big) \end{aligned} \tag{10}$$

Chapter 2

Resummation program

2.1 Soft-gluon effects in QCD cross sections

Soft-gluon effects and the soft-gluon exponentiation are reviewed in [7] and [8], here I summarize the physical motivation and main idea of the resummation of soft gluon effects in QCD.

The finite energy resolution inherent in any particle detector implies that the physical cross sections, those experimentally measured, inherently incorporate all contributions from arbitrarily soft particles produced in the final state. In other words, because we lack the ability to precisely resolve the energy of soft particles, we are unable to distinguish between their presence or absence in our calculations (see fig. 1). Consequently, we must account for the sum over all possible final states.

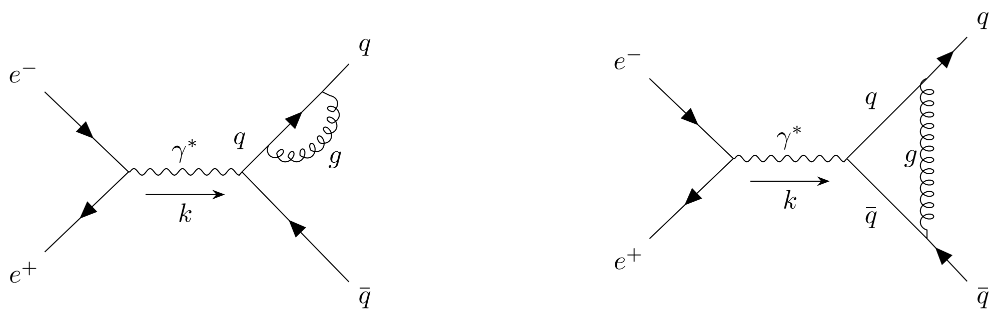


Figure 1: Examples of Next-to-leading order $\mathcal{O}(\alpha_s^2)$ diagrams for $e^+e^- \to q\bar{q}g$ process whose final state is identical to the Tree level process fig. 1 $e^+e^- \to q\bar{q}$

This inclusiveness is essential in QCD calculations. Higher order perturbative contributions due to *virtual* gluons are infrared divergent and the divergences are exactly cancelled by radiation of undetected *real* gluons. In particular kinematic configurations, *e.g* Thrust in the dijet limit $T \to 1$, real and virtual contributions can be highly unbalanced, because the emission of real radiation is inhibited by kinematic constrainsts, spoiling the cancellation mechanism. As a result, soft gluon contribution to QCD cross sections can still be either large or singular.

In these cases, the cancellation of infrared divergences bequeaths higher order contributions of the form:

$$G_{nm}\alpha_s^n \ln^m \frac{1}{\tau}, \quad \text{with } m \leq 2n, \tag{1}$$

that can become large, $\alpha_s \ln^2 \frac{1}{\tau} \gtrsim 1$, even if the QCD coupling α_s is in the perturbative regime $\alpha_s \ll 1$. These logarithmically enhanced terms in eq. (1) are certainly relevant near the dijet limit $\tau \to 0$. In these cases, the theoretical predictions can be improved by evaluating soft gluon contributions to high orders and possibly resumming to all of them in α_s [6, 17].

Certainly! Here's a revised version with improved English and coherence:

The resummation of large logarithms in event shape distributions was described by Catani, Trentadue, Turnock, and Webber (CTTW) [5].

The physical basis for all-order resummation of soft-gluon contributions to QCD cross sections are dynamics and kinmatics factorizations. The first factorization follows from gauge invariance and unitarity while the second factorization is strongly depends on the observable to be computed.

In the appropriate soft limit, if the phase-space for this observable can be written in a factorized way, then resummation is feasible in the form of a generalized exponentiation [18]. However even when phase-space factorization is achievable, it does not always occur in the space where the physical observable x is defined.

Thrust is a good example of this situation, in fact, the thrust distribution admits a factorization in Laplace space, where the observable is the Laplace transform of the thrust distribution.

2.2 CTTW convention

According to general theorems [2],[15],[16], the cumulant cross section $R_T(\tau)$ eq. (3) has a power series expansion in $\alpha_s(Q^2)$ of the form:

$$R_T(\tau) = C\big(\alpha_s(Q^2)\big)\Sigma\big(\tau, \alpha_s(Q^2)\big) + D\big(\tau, \alpha_s(Q^2)\big) \tag{2}$$

where

$$C(\alpha_s) = 1 + \sum_{n=1}^{\infty} C_n \bar{\alpha_s}^n \tag{3}$$

$$\Sigma(\tau, \alpha_s) = \exp\left[\sum_{n=1}^{\infty} \bar{\alpha_s}^n \sum_{m=1}^{2n} G_{nm} \ln^m \tau\right] \tag{4}$$

$$D(\tau, \alpha_s) = \sum_{n=1}^{\infty} \bar{\alpha_s}^n D_n(\tau) \tag{5}$$

Here C_n and G_{nm} are constants and $\bar{\alpha_s} = \frac{\alpha_s}{2\pi}$, while $D_n(\tau)$ is the non-singular part of the fixed-order expansion of $R_T(\tau)$ eq. (9).

Thus at small τ (large thrust) it becomes morst important to resum the series of large logarithms in $\Sigma(\tau, \alpha_s)$. These are normally classified as *leading* logarithms when $n < m \leq 2n$, *next-to-leading* when $m = n$ and *subdominant* logarithms when $m < n$.

The cumulant cross section $R(\tau)$, in general, also takes the form (without the $D(\tau, \alpha_s)$ term):

$$R_T(\tau) = \left(1 + \sum_{n=1}^{\infty} C_n \bar{\alpha_s}^n\right) \exp\left[Lg_1(\lambda) + g_2(\lambda) + g_3(\lambda)\alpha_s + g_4(\lambda)\alpha_s^2 + g_5(\lambda)\alpha_s^3 + \mathcal{O}\left(\alpha_s^4\right)\right] \tag{6}$$

where $L = \ln \frac{1}{\tau}$ and $\lambda = \alpha_s b_0 L$. The function g_1 encodes all the leading logarithms, the function g_2 resums all next-to-leading logarithms and so on.

The last equation gives a better prediction of the thrust distribution in the two-jet region, but fails to describe the multijet region $\tau \to \tau_{max}$, where non-singular pieces of the fixed-order prediction become important. To achieve a reliable description of the observable over a broader kinematical range the two expressions eq. (9) and eq. (2) can be matched, taking care of double counting of logarithms appearing in both expressions.

Expanding eq. (2) in powers of α_s we have:

$$R(\tau) = 1 + R^{(1)}(\tau)\bar{\alpha_s} + R^{(2)}(\tau)\bar{\alpha_s}^2 + R^{(3)}(\tau)\bar{\alpha_s}^3 + \dots \tag{7}$$

where

$$
\begin{aligned}
R^{(1)}(\tau) &= C_1 + G_{12}\log^2(\tau) + G_{11}\log(\tau) + D_1(\tau) \\
R^{(2)}(\tau) &= C_2 + \frac{1}{2}G_{12}^2\log^4(\tau) + (G_{11}G_{12} + G_{23})\log^3(\tau) \\
&\quad + \log^2(\tau)\Big(C_1G_{12} + \frac{G_{11}^2}{2} + G_{22}\Big) + \log(\tau)\Big(C_1G_{11} + G_{21}\Big) \\
&\quad + D_2(\tau) \\
R^{(3)}(\tau) &= C_3 + \frac{1}{6}G_{12}^3\log^6(\tau) + \left(\frac{1}{2}G_{11}G_{12}^2 + G_{23}G_{12}\right)\log^5(\tau) \\
&\quad + \log^4(\tau)\left(\frac{1}{2}C_1G_{12}^2 + \frac{1}{2}G_{12}G_{11}^2 + G_{23}G_{11} + G_{12}G_{22} + G_{34}\right) \\
&\quad + \log^3(\tau)\left(C_1G_{12}G_{11} + C_1G_{23} + \frac{G_{11}^3}{6} + G_{22}G_{11} + G_{12}G_{21} + G_{33}\right) \\
&\quad + \log^2(\tau)\left(\frac{1}{2}C_1G_{11}^2 + C_2G_{12} + C_1G_{22} + G_{21}G_{11} + G_{32}\right) \\
&\quad + \log(\tau)\left(C_2G_{11} + C_1G_{21} + G_{31}\right) + D_3(\tau)
\end{aligned} \tag{8}
$$

Observe that written in this form, Fixed-order perturbation theory eq. (7) calculates the cross sections row by row (each $R^{(n)}$ is a row), order by order in α_s, while the resummation approach calculates the cross section column by column, and this is achieved by knowing the resummation coefficients g_i in eq. (6). In fact by expanding $\log\frac{1}{\tau}g_1(\lambda)$ in α_s we get all the coefficients G_{nn+1}, expanding $g_2(\lambda)$ yields the coefficients G_{nn}, expading $g_2\alpha_s$ yields G_{nn-1} and so on.

The difference between the logarithmic part and the full fixed-order series at different orders is given by $D(\tau, \alpha_s)$ above:

$$
\begin{aligned}
D_1(\tau) &= A(\tau) - R^{(1)}(\tau) \\
D_2(\tau) &= B(\tau) - R^{(2)}(\tau) \\
D_3(\tau) &= C(\tau) - R^{(3)}(\tau)
\end{aligned} \tag{9}
$$

It contain the non-logarithmic part of the fixed-order contribution and vanish for $\tau \to 0$.

However, in order to calculate the resummation coefficients $g_i(\lambda)$, whose expansion in α_s yields the G_{nm} coefficients, we need to perform an inverse transformation from Laplace space to real space. This necessity arises because the factorization is carried out in Laplace space. As demonstrated in [5], the problem of resummation can be recast in the form of an integral equation in Laplace space **??**, whose solution directly provides the exponent function.

$$\ln \tilde{J}^q_\nu(Q^2) = \int_0^1 \frac{\mathrm{d}u}{u}\Big(e^{-u\nu Q^2} - 1\Big)\left[\int_{u^2Q^2}^{uQ^2} \frac{1}{q^2} A\big(\alpha_s(q^2)\big)\mathrm{d}q^2 + \frac{1}{2}B\big(\alpha_s(uQ^2)\big)\right] \tag{10}$$

where $\tilde{J}^q_\nu(Q^2)$ is the Laplace transform of the quark jet mass distribution, a quantity related to the thrust distribution.

We'll see explicitly in the next chapter chapter 3 that resolving the above integral gives the exponent:

$$\mathcal{F}(\alpha_s, \ln N) = Lf_1(\lambda) + f_2(\lambda) + f_3(\lambda)\alpha_s + f_4(\lambda)\alpha_s^2 + f_5(\lambda)\alpha_s^3 + \mathcal{O}\big(\alpha_s^4\big) \tag{11}$$

where $L = \ln N = \ln\big(\nu Q^2\big)$ and $\lambda = \alpha_s b_0 L$. We require the functions $f_i(\lambda)$ to be omogeneous,*i.e* $f_i(0) = 0$, so that at N^{n}LL we can write:

$$f_{n+1}(\lambda) = \sum_{k \geq n} \tilde{G}_{k,k+1-n}\alpha_s^k L^{k+1-n} \tag{12}$$

this requirement is automatically satisfied if we choose as variable $L = \ln\Big(\frac{N}{N_0}\Big)$ where $N_0 = e^{-\gamma}$, $\gamma = 0.5772\ldots$ being the Euler-Mascheroni constant. With the latter choice the terms proportional to γ and its powers disappear. The advantage of the variable N is that the total rate is directly reproduced by setting $N = 1$, while in the variable $n = N/N_0$ it's wheb $N = N_0$. These two choices differ only by terms of higher order in γ. It's cleaner to use the variable $n = N/N_0$, but in literature the variable N is more common and in order to compare with other results we'll use the variable N, but we can recover the results in the variable n by setting $\gamma = 0$.

In the presentation of CTTW, N^{k}LL accuracy means how many terms in the exponent of eq. (6) are known.

Chapter 3

Calculations

Next, we turn to the core task: calculating the resummation coefficients $f_i(\lambda)$ of the Sudakov form factor $\exp\{\mathcal{F}\}$. To compute the $f_i(\lambda)$ functions, equation eq. (10) is used, which requires knowledge of the μ-dependence of the QCD running coupling $\alpha_s(\mu)$. We therefore proceed firstly to calculate the running coupling $\alpha_s(\mu)$ from LO up to N^{4}LO because the QCD β-function is known up to five loops [14]. We then use the obtained results to calculate the $f_i(\lambda)$ functions up to $i = 5$.

3.1 QCD running coupling

A surpring effect of the renormalization procedure is that, after renormalizatio, the ”constants” are not constant at all, but depend on the energy.

One way to understand this is the following: Classically, the force between two sources is then given by $F = \frac{\alpha}{r^2}$, characterized by a universal coefficient – the coupling constant α, which quantifies the force between two static bodies of unit “charge” at distance r, *i.e.*, the electric charge for QED, the color charge for QCD, the weak isospin for the weak force, or the mass for gravity. Consequently, the coupling α is defined as being proportional to the elementary charge squared, *e.g.*, $\alpha_{em} \equiv \frac{e^2}{4\pi}$ where e is the elementary electric charge, or $\alpha_s \equiv \frac{g^2}{4\pi}$ where g is the elementary gauge field coupling in QCD. In quantum field theory (QFT), $\frac{1}{r^2}$ is the coordinate-space expression for the propagator of the force carrier (gauge boson) at leading-order in perturbation theory: in momentum space, the analogous propagator is proportional to $\frac{1}{q^2}$, where q is the boson 4-momentum ($Q^2 = -q^2 > 0$).

For sources interacting weakly, the one-boson exchange representation of interactions is a good first approximation. However, when interactions become strong (with "strong" to be defined below), higher orders in perturbation theory become noticeable and the $\frac{1}{r^2}$ law no longer stands. In such cases, it makes good physics sense to fold the extra r-dependence into the coupling, which thereby becomes r, or equivalently Q^2, dependent.

Another way to view this is that the running of the coupling is due to vacuum polarization. The vacuum is not empty, but is filled with virtual particles that are constantly created and annihilated which can interact with the propagating particles, leading to a modification of the interaction strength.

While in QED, the extra r-dependence comes only from the vacuum polarization. In QCD, α_s receives contributions from the vacuum polarization and from gluon self-interactions since the gluon has a color charge.

The two couplings have opposite trends: the QED coupling increases with energy and the theory becomes strongly coupled at high energies, whereas the opposite happens for the QCD coupling as it is large at low energies and decreases with energy. This property of being weakly coupled at high energies is known as *asymptotic freedom* and it means that perturbative calculations in QCD can only be done at high energies where α_s becomes small enough that a power expansion is meaningful.

In the framework of perturbative QCD (*pQCD*), predictions for observables are expressed in terms of the renormalized coupling $\alpha = \alpha(\mu^2)$, a function of an unphysical renormalization scale μ_R. The coupling satisfies the following renormalization group equation (RGE):

$$\mu^2 \frac{d\alpha}{d\mu^2} = \beta(\alpha) = -\left(b_0\alpha^2 + b_1\alpha^3 + b_2\alpha^4 + \ldots\right) \tag{1}$$

where b_0 is the 1-loop β-function coefficient, b_1 is the 2-loop coefficient, b_2 is the 3-loop coefficient. $C_A = 3$ and $C_F = \frac{4}{3}$ are the Casimir operators of the adjoint and fundamental representations of $SU(3)$, $T_R = \frac{1}{2}$ is the trace normalization, n_f is the number of active quark flavors.

It is not possible to solve eq. (1) as it is for two reasons: only the first few b_n coefficients are known (up to b_4); the exact equation becomes more and more complicated as more terms of the series are included, making it impossible to obtain an analytic solution.

In order to solve both problems, the equation is solved in the following way: at first only b_0 is included and the obtained solution is called α_{LO} , as it will only contain a term proportional to α ; then also b_1 is included and only terms up to the second order in α are kept to obtain α_{NLO} ;

this same procedure is used to obtain α_{NNLO} , $\alpha_{\text{N}^3\text{LO}}$, $\alpha_{\text{N}^4\text{LO}}$. There will be a complication in calculating α_{NLO} and higher orders which will be explained and resolved in the following sections.

3.1.1 One-loop running coupling

The one-loop running coupling α_{LO} is obtained by solving the RGE eq. (1) with only the first term of the β-function:

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0 \alpha^2 \tag{2}$$

This equation can be solved by separation of variables and imposing the boundary condition $\alpha(Q^2) = \alpha_s$:

$$\int_{\alpha(Q^2)}^{\alpha(\mu^2)} \frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0 \frac{\mathrm{d}\mu^2}{\mu^2} \tag{3}$$

and one obtains:

$$\alpha_{\text{LO}}(\mu^2) = \frac{\alpha_s}{1 + b_0 \alpha_s \log\left(\frac{\mu^2}{Q^2}\right)} \tag{4}$$

In which one can observe the decreasing with energy trend of the running coupling (asymptotic freedom).

It is useful to define the variable $\lambda_\mu = b_0 \alpha_s \log\left(\frac{\mu^2}{Q^2}\right)$ so that:

$$\alpha_{\text{LO}}(\mu^2) = \frac{\alpha_s}{1 + \lambda_\mu} \tag{5}$$

3.1.2 Two-loop running coupling

In order to obtain the two-loop running coupling α_{NLO}, we need to solve the RGE with the first two terms of the β-function eq. (1):

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0 \alpha^2 - b_1 \alpha^3 \tag{6}$$

but this equation is not solvable in a straightforward way as the one-loop equation, we have to use the perturbative approach. We can rewrite the equation as:

$$\frac{d\alpha}{d\mu^2} = -\frac{b_0 \alpha^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha) \tag{7}$$

and expand the α term in the parenthesis as:

$$\alpha = \alpha_{\mathrm{LO}} + \delta\alpha \tag{8}$$

where α_{LO} is the one-loop running coupling and $\delta\alpha$ contains the higher order correction, one obtains:

$$\frac{d\alpha}{d\mu^2} = -\frac{b_0\alpha^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{LO} - \frac{b_1}{b_0}\delta\alpha) \tag{9}$$

Observe that in parenthesis, by keeping 1 gave us the one-loop running coupling, by keeping $\frac{b_1}{b_0}\alpha_{LO}$ we can obtain the first order corrections and $\delta\alpha$ are needed for higher order corrections. The equation to solve is then:

$$\int_{\alpha_s}^{\alpha(\mu^2)} -\frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0\frac{\mathrm{d}\mu^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{LO}(\mu^2)) \tag{10}$$

Using *Mathematica* to solve this equation, we obtain the two-loop running coupling:

$$\alpha_{\mathrm{NLO}}(\mu^2) = \frac{\alpha_s}{1 + \lambda_\mu + \alpha_s\frac{b_1}{b_0}\log(1 + \lambda_\mu)} \tag{11}$$

in which the expansion in powers of α_s is not explicit. One can expand in powers of α_s by keeping λ_μ fixed and only keeping terms up to $\mathcal{O}(\alpha_s^2)$ by doing so one obtains:

$$\alpha_{\mathrm{NLO}}(\mu^2) = \alpha_{\mathrm{LO}}(\mu^2) - \frac{b_1}{b_0}\alpha_{LO}^2(\mu^2)\log(1 + \lambda_\mu) + \mathcal{O}(\alpha_s^2) \tag{12}$$

We found the correction:

$$\delta\alpha_{\mathrm{NLO}}(\mu^2) = -\frac{b_1}{b_0}\alpha_{\mathrm{LO}}^2(\mu^2)\log(1 + \lambda_\mu) \tag{13}$$

By repeating the same procedure, one can obtain the three-loop running coupling α_{NNLO} and so on.

3.1.3 Higher order corrections

In order to calculate higher order corrections, one need to be careful of the powers of α needed for the desired order, and the contributions to various orders of α_s may not be immediately apparent, but they are straightforward to compute. Expand the running coupling in powers of α_s as:

$$\alpha = \alpha_{\mathrm{LO}} + \delta\alpha_{\mathrm{NLO}} + \delta\alpha_{\mathrm{NNLO}} + \delta\alpha_{\mathrm{N^3LO}} + \delta\alpha_{\mathrm{N^4LO}} + \dots \tag{14}$$

with $\delta\alpha_{\rm NLO} = \mathcal{O}(\alpha_s)$, $\delta\alpha_{\rm NLO} = \mathcal{O}\big(\alpha_s^2\big)$, $\delta\alpha_{\rm NNLO} = \mathcal{O}(\alpha_s^3)$, $\delta\alpha_{\rm N^3LO} = \mathcal{O}(\alpha_s^4)$, $\delta\alpha_{\rm N^4LO} = \mathcal{O}(\alpha_s^5)$, and so on. We present these contributions in the following table:

Power	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$	$\mathcal{O}(\alpha_s^5)$
α	$\alpha_{\rm LO}$	$\delta\alpha_{\rm NLO}$	$\delta\alpha_{\rm NNLO}$	$\delta\alpha_{\rm N^3LO}$	$\delta\alpha_{\rm N^4LO}$
α^2		$\alpha_{\rm LO}^2$	$2\alpha_{\rm LO}\delta\alpha_{\rm NLO}$	$\delta\alpha_{\rm NLO}^2 + 2\alpha_{\rm LO}\delta\alpha_{\rm NNLO}$	$2\alpha_{\rm LO}\delta\alpha_{\rm N^3LO} + 3\alpha_{\rm LO}\delta\alpha_{\rm NLO}^2$
α^3			$\alpha_{\rm LO}^3$	$3\alpha_{\rm LO}^2\delta\alpha_{\rm NLO}$	$3\alpha_{\rm LO}^2\delta\alpha_{\rm NNLO} + 3\alpha_{\rm LO}\delta\alpha_{\rm NLO}^2$
α^4				$\alpha_{\rm LO}^4$	$4\alpha_{\rm LO}^3\delta\alpha_{\rm NLO}$
α^5					$\alpha_{\rm LO}^5$

Table 1: Contributions to different powers of α_s.

For the three-loop running coupling, the equation to solve is:

$$\mu^2\frac{d\alpha}{d\mu^2} = -b_0\alpha^2(1-\frac{b_1}{b_0}\alpha-\frac{b_2}{b_0}\alpha^2) \tag{15}$$

One can substitute the expansion of $\alpha = \alpha_{\rm LO} + \delta\alpha_{\rm NLO} + \mathcal{O}\big(\alpha_s^2\big)$ in powers of α_s and retain only terms up to $\mathcal{O}\big(\alpha_s^2\big)$, with this prescription the equation to solve is:

$$\int_{\alpha_s}^{\alpha(\mu^2)} -\frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0\frac{\mathrm{d}\mu^2}{\mu^2}(1-\frac{b_1}{b_0}\alpha_{\rm NLO}(\mu^2)-\frac{b_2}{b_0}\alpha_{LO}^2(\mu^2)) \tag{16}$$

solving the above integral yields the three-loop running coupling α_{NNLO}:

$$\alpha_{\rm NNLO}(\mu^2) = \alpha_{\rm LO}\big(\mu^2\big) + \delta\alpha_{\rm NLO}\big(\mu^2\big) + \delta\alpha_{\rm NNLO}\big(\mu^2\big) \tag{17}$$

with

$$\delta\alpha_{\rm NNLO}\big(\mu^2\big) = \frac{\alpha_{\rm LO}^3\big(\mu^2\big)}{b_0^2}\big(b_1^2\lambda_\mu - b_0b_2\lambda_\mu + b_1^2\log^2(1+\lambda_\mu) - b_1^2\log(1+\lambda_\mu)\big) \tag{18}$$

Similarly one can obtain the four-loop running coupling α_{N^3LO} and five-loop running coupling α_{N^4LO}.

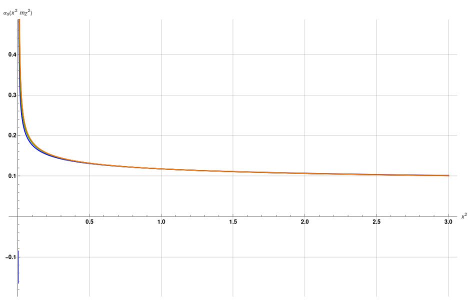
$$\alpha_{\rm N^3LO}(\mu^2) = \alpha_{\rm LO}\big(\mu^2\big) + \delta\alpha_{\rm NLO}\big(\mu^2\big) + \delta\alpha_{\rm NNLO}\big(\mu^2\big) + \delta\alpha_{\rm N^3LO}\big(\mu^2\big) \tag{19}$$

$$\begin{aligned}\delta\alpha_{\rm N^3LO}\big(\mu^2\big) = \frac{\alpha_{\rm LO}^4\big(\mu^2\big)}{2b_0^3}\Big(&-\big(b_1^3 - 2b_0b_2b_1 + b_0^2b_3\big)\,\lambda_\mu^2 \\ &-\big(2b_0^2b_3 - 2b_0b_1b_2\big)\,\lambda_\mu - 2b_1^3\log^3(\lambda_\mu+1) + 5b_1^3\log^2(1+\lambda_\mu) \\ &+\big(2b_0b_1b_2\,(2\lambda_\mu-1) - 4b_1^3\lambda_\mu\big)\log(1+\lambda_\mu)\Big)\end{aligned} \tag{20}$$

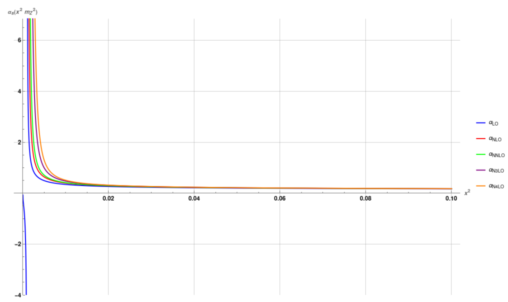
$$\alpha_{\mathrm{N^4LO}}(\mu^2) = \alpha_{\mathrm{LO}}(\mu^2) + \delta\alpha_{\mathrm{NLO}}(\mu^2) + \delta\alpha_{\mathrm{NNLO}}(\mu^2) + \delta\alpha_{\mathrm{N^3LO}}(\mu^2) + \delta\alpha_{\mathrm{N^4LO}}(\mu^2) \tag{21}$$

$$\begin{aligned}
\delta\alpha_{\mathrm{N^4LO}} = \frac{\alpha_{\mathrm{LO}}^5}{6b_0^4}\Big(&\left(2b_1^4 - 6b_0b_2b_1^2 + 4b_0^2b_3b_1 + 2b_0^2b_2^2 - 2b_0^3b_4\right)\lambda_\mu^3 \\
&+ \left(9b_1^4 - 24b_0b_2b_1^2 + 9b_0^2b_3b_1 + 12b_0^2b_2^2 - 6b_0^3b_4\right)\lambda_\mu^2 \\
&+ \left(6b_0^2b_1b_3 - 6b_0^3b_4\right)\lambda_\mu + 6b_1^4\log^4(1+\lambda_\mu) \\
&- 26b_1^4\log^3(\lambda_\mu+1) + 9\left(\left(2b_1^4 - 2b_0b_1^2b_2\right)\lambda_\mu + b_1^4 + 2b_0b_2b_1^2\right)\log^2(1+\lambda_\mu) \\
&+ \left(6b_1\left(b_1^3 - 2b_0b_2b_1 + b_0^2b_3\right)\lambda_\mu^2 + 6b_1\left(-3b_1^3 + b_0b_2b_1 + 2b_0^2b_3\right)\lambda_\mu \right. \\
&\left. - 6b_1b_3b_0^2\right)\log(1+\lambda_\mu)\Big)
\end{aligned} \tag{22}$$

Below we present a plot of the running coupling α_s as a function of the energy scale $(xm_z)^2$ for different orders of perturbation theory, where $m_z = 91.18$ GeV is the mass of the Z boson. The global average value of the strong coupling constant is $\alpha_s(m_z^2) = 0.1179 \pm 0.0009$ [13]. For the plot i use $\alpha_s(m_z^2) = 0.118$



(a) Energy dependence of the strong coupling α_s



(b) Zoomed-in at the low energy region of the strong running coupling α_s at different orders

Figure 1: Running coupling α_s at different orders of perturbation theory.

In fig. 1b wee see that the running coupling α_s is large and diverges at a finite energy (Landau pole), this is a sign of the non-perturbative nature of QCD at low energies. It's very clear for the LO running coupling, for some some low energy scales it becomes negative, which is clearly unphysical.

3.2 Calculating the resummation coefficients

Now we will calculate the resummation coefficients $f_i(\lambda)$ for the thrust distribution.

In the article by Catani, Turnock, Webber and Trentadue [6], it was observed that for a final state configuration corresponding to a large value of thrust , eq. (1) can be approximated by

$$\tau = 1 - T \approx \frac{k_1^2 + k_2^2}{Q^2} \tag{23}$$

where k_1^2 and k_2^2 are the squared invariant masses of two back-to-back jets and Q^2 is the energy of the center of mass. Thus the key to the evaluation of the thrust distributions is its relation to the quark jet mass distribution $J^q(Q^2, k^2)$, also denoted as $J^q_{k^2}(Q^2)$, which represents the probability of finding a jet originating from quarks, with an invariant mass-squared k^2, produced in collisions with a center-of-mass energy Q^2.

Then the thrust distribution $R_T(\tau, \alpha_s(Q^2))$ eq. (3) takes the form of a convolution of two jet mass distributions $J(Q^2, k_1^2)$ and $J(Q^2, k_2^2)$

$$R_T(\tau, \alpha_s(Q^2)) \underset{\tau \ll 1}{=} \int_0^\infty \mathrm{d}k_1^2 \int_0^\infty \mathrm{d}k_2^2 J^q_{k_1^2}(Q^2) J^q_{k_2^2}(Q^2) \Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) \tag{24}$$

Introducing the Laplace transform of the jet mass distribution:

$$\tilde{J}^q_\nu(Q^2) = \int_0^\infty J^q(Q^2, k^2) e^{-\nu k^2} \mathrm{d}k^2 \tag{25}$$

and using the integral representation of the Heaviside step function:

$$\Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) = \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} \frac{\mathrm{d}N}{N} e^{N\tau} e^{-N \frac{k_2^2 + k_2^2}{Q^2}} \tag{26}$$

by substituting eq. (26) into eq. (24) and setting $N = \nu Q^2$ we obtain:

$$\begin{aligned} R_T(\tau) &= \int_{C - i\infty}^{C + i\infty} \frac{\mathrm{d}N}{N} \frac{e^{N\tau}}{2\pi i} \left[\int_0^\infty \mathrm{d}k_1^2 e^{-\nu k_1^2} J^q_{k_1^2}(Q^2)\right] \left[\int_0^\infty \mathrm{d}k_2^2 e^{-\nu k_2^2} J^q_{k_2^2}(Q^2)\right] \\ &= \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} e^{N\tau} \left[\tilde{J}^q_\nu(Q^2)\right]^2 \frac{\mathrm{d}N}{N} \end{aligned} \tag{27}$$

where C is a real positive constant to the right of all singularities of the integrand $\tilde{J}_\nu(Q^2)$ in the complex ν plane.

An integral representation for the Laplace transform $\tilde{J}_\nu(Q^2)$ is given by [5]:

$$\ln \tilde{J}^q_\nu(Q^2) = \int_0^1 \frac{\mathrm{d}u}{u}\Big(e^{-u\nu Q^2} - 1\Big)\left[\int_{u^2Q^2}^{uQ^2} \frac{1}{q^2} A\big(\alpha_s(q^2)\big)\mathrm{d}q^2 + \frac{1}{2} B\big(\alpha_s(uQ^2)\big)\right] \tag{28}$$

with

$$A(\alpha_s) = \sum_{n=1}^{\infty} A_n \left(\frac{\alpha_s}{\pi}\right)^n \qquad\qquad B(\alpha_s) = \sum_{n=1}^{\infty} B_n \left(\frac{\alpha_s}{\pi}\right)^n$$

Function $A(\alpha_s)$ is associated with the cusp anomalous dimension and governs the exponentiation of the leading logarithms (LL). It captures the resummation of the soft and collinear gluon emissions that dominate in the limit of large thrust values.

Function $B(\alpha_S)$ includes the next-to-leading logarithmic (NLL) corrections and accounts for subleading contributions from hard collinear emissions. It typically involves the non-cusp part of the anomalous dimensions and running of the coupling constant.

The integral as it is cannot be integrated, the u integration may be performed using the prescription by Paolo Nason in Appendix A of [9] and readapting the formula to the case of Laplace transform instead of Mellin transform.

This method is a generalization of the prescription to NLL accuracy in [5]

$$e^{-u\nu Q^2} - 1 \simeq -\Theta(u - v) \quad \text{with } v = \frac{N_0}{N} \tag{29}$$

where $N_0 = e^{-\gamma_E}$, $\gamma_E = 0.5772\ldots$ being the Euler-Mascheroni constant

In appendix A we show that the prescription to evaluate the large-N Mellin moments of soft-gluon contributins at an arbitrary logarithmic accuracy, can be used for the Laplace transform as well, then we can use this result to express eq. (10) in an alternative representation:

$$\ln \tilde{J}^q_\nu(Q^2) = -\int_{N_0/N}^1 \frac{\mathrm{d}u}{u}\left[\int_{u^2Q^2}^{uQ^2} \frac{1}{q^2} A\big(\alpha_s(q^2)\big)\mathrm{d}q^2 + \frac{1}{2}\tilde{B}\big(\alpha_s(uQ^2)\big)\right] + \ln \tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) \tag{30}$$

Note that, due to the integration of the running coupling the integral in **??** is singular for all values of $N = \nu Q^2$. However, if we perform the integration up to a fixed logarithmic accuracy N^kLL ($i.e$ we compute the leading $\alpha_s^n \ln^{n+1} N$, next-to-leading $\alpha_s^n \ln^n N$ and so on to $\alpha_s^n \ln^{n+1-k}$ terms), we find the form factor:

$$\ln \tilde{J}^q_\nu(Q^2) = \ln N f_1(\lambda) + f_2(\lambda) + \alpha_s f_3(\lambda) + \alpha_s^2 f_4(\lambda) + \alpha_s^3 f_5(\lambda) + \mathcal{O}\big(\alpha_s^n \ln^{n-4} N\big)$$

$$+\ln\tilde{C}\left(\alpha_s(\mu^2),\frac{\mu^2}{Q^2}\right) \tag{31}$$

From now on i'll always use eq. (30), so i'll drop the ˜ notation for B.

We observe that the N-space formula eq. (30) is finite and uniquely defined up to the very large $N = N_L = \exp\left(\frac{1}{2\alpha_s b_0}\right)$ $(\lambda = \frac{1}{2})$, thanks to the prescription above.

3.2.1 f_i

To calculate explicit expressions for the first few f_i terms, we first write explicitly the internal integral of eq. (10), for now let's forget the $\ln\tilde{C}$ term, it can be absorbed into the definition of A and B:

The q^2 integration also becomes simple if we use the renormalization group equation eq. (1) to change the integration variable to as

$$\begin{aligned}\frac{\mathrm{d}q^2}{q^2} &= -\frac{\mathrm{d}\alpha_s}{b_0\alpha_s^2}\left(1-\frac{b_1}{b_0}\alpha_s+\frac{\left(b_1^2-b_2b_0\right)}{b_0^2}\alpha_s^2+\frac{\left(-b_3b_0^2+2b_2b_1b_0-b_1^3\right)}{b_0^3}\alpha_s^3\right.\\ &\quad\left.+\frac{\left(-b_4b_0^3+2b_3b_1b_0^2+b_2^2b_0^2-3b_2b_1^2b_0+b_1^4\right)}{b_0^4}\alpha_s^4+\mathcal{O}\left(\alpha_s^5\right)\right)\\ &= -\frac{\mathrm{d}\alpha_s}{b_0\alpha_s^2}\left(N_0+N_1\alpha_s+N_2\alpha_s^2+N_3\alpha_s^3+N_4\alpha_s^4+\mathcal{O}\left(\alpha_s^5\right)\right)\end{aligned} \tag{32}$$

where for convenience's sake we have defined:

$$\begin{aligned}&N_0 = 1 \qquad N_1 = \frac{b_1}{b_0} \qquad N_2 = \frac{\left(b_1^2-b_2b_0\right)}{b_0^2}\\ &N_3 = \frac{\left(-b_3b_0^2+2b_2b_1b_0-b_1^3\right)}{b_0^3}\\ &N_4 = \frac{\left(-b_4b_0^3+2b_3b_1b_0^2+b_2^2b_0^2-3b_2b_1^2b_0+b_1^4\right)}{b_0^4}\end{aligned} \tag{33}$$

Subsequently, the integral in eq. (30) can be expressed as:

$$\begin{aligned}&\int_{\alpha_s(u^2Q^2)}^{\alpha_s(uQ^2)}\frac{\mathrm{d}\alpha_s}{b_0\alpha_s^2}\left(N_0+N_1\alpha_s+N_2\alpha_s^2+N_3\alpha_s^3+N_4\alpha_s^4\right)\sum_{n=1}^{\infty}A_n\left(\frac{\alpha_s}{\pi}\right)^n\\ &=\int_{\alpha_s(u^2Q^2)}^{\alpha_s(uQ^2)}\frac{\mathrm{d}\alpha_s}{b_0}\left(\frac{A_1N_0}{\pi\alpha_s}+\frac{\pi A_1N_1+A_2}{\pi^2}+\frac{\left(\pi A_2N_1+\pi^2A_1N_2+A_3\right)}{\pi^3}\alpha_s\right.\\ &\left.+\frac{\left(\pi A_3N_1+\pi^2A_2N_2+\pi^3A_1N_3+A_4\right)}{\pi^4}\alpha_s^2\right.\end{aligned} \tag{34}$$

$$+\frac{\left(\pi A_4 N_1+\pi^2 A_3 N_2+\pi^3 A_2 N_3+\pi^4 A_1 N_4+A_5\right)}{\pi^5}\alpha_s^3+\mathcal{O}\left(\alpha_s^4\right)\Big)$$

and keeping only terms up to N_0,A_1 and α_s^0 we get contributions to f_1, keeping terms up to N_1,A_2 and α_s^1 yields contributions to f_2 and so on. That's because after integration, we evaluate the integrand at $\alpha_s(u^2Q^2)$ and $\alpha_s(uQ^2)$, where α_s is α_{LO} for f_1, α_{NLO} for f_2 and so on.

And there's an easy way to see why it's like this, in fact it's also possible to do the q^2 integration directly, using eqs. (5), (13), (18), (20) and (22) from section 3.1 and keeping in mind table 1.

$$\int_{u^2Q^2}^{uQ^2}\frac{\mathrm{d}q^2}{q^2}\sum_{n=1}^{\infty}\frac{A_n}{\pi^n}\left(\alpha_{\mathrm{LO}}(q^2)+\delta\alpha_{\mathrm{NLO}}(q^2)+\delta\alpha_{\mathrm{NNLO}}(q^2)+\delta\alpha_{\mathrm{N^3LO}}(q^2)+\delta\alpha_{\mathrm{N^4LO}}(q^2)+\ldots\right)^n \tag{35}$$

now we can see that if we consider terms up to α_s^1 only A_1 contributes and this gives f_1, if we consider terms up to α_s^2 we see A_2 starts to contribute together with $A_1\delta\alpha_{\mathrm{NLO}}$ and this gives f_2 , for f_3 we need to consider terms up to α_s^3 and so on.

For the B-term it's similar. However, since the B-term is "already integrated" in q^2, it contributes one order lower in α_s compared to the A-term. Specifically, B_1 starts to contribute from f_2, B_2 from f_3 and so on.

Now armed with this knowledge we can calculate eq. (30) and find:

$$\ln\tilde{J}_\nu^q(Q^2)=\ln N f_1(\lambda)+f_2(\lambda)+\alpha_s f_3(\lambda)+\alpha_s^2 f_4(\lambda)+\alpha_s^3 f_5(\lambda)+\mathcal{O}\left(\alpha_s^n\ln^{n-4}N\right) \tag{36}$$

where $\lambda=\alpha_s b_0\ln N$, $N=\nu Q^2$, $\alpha_s=\alpha_s(Q^2)$ and

$$f_1(\lambda)=-\frac{A_1}{2\pi b_0\lambda}[(1-2\lambda)\log(1-2\lambda)-2(1-\lambda)\log(1-\lambda)] \tag{37}$$

$$\begin{aligned}f_2(\lambda)=&-\frac{A_2}{2\pi^2 b_0^2}[2\log(1-\lambda)-\log(1-2\lambda)]+\frac{B_1\log(1-\lambda)}{2\pi b_0}+\frac{\gamma A_1}{\pi b_0}[\log(1-2\lambda)-\log(1-\lambda)]\\&-\frac{A_1 b_1}{2\pi b_0^3}\left[-\log^2(1-\lambda)+\frac{1}{2}\log^2(1-2\lambda)-2\log(1-\lambda)+\log(1-2\lambda)\right]\end{aligned} \tag{38}$$

$$f_3(\lambda)=-\frac{A_3}{2\pi^3 b_0^2(\lambda-1)(2\lambda-1)}\lambda^2+\frac{B_2}{2\pi^2 b_0(\lambda-1)}\lambda+\frac{B_2\lambda}{2\pi^2 b_0(\lambda-1)}$$

$$
\begin{aligned}
&+\frac{b_1 A_2}{2\pi^2 b_0^3(\lambda-1)(2\lambda-1)}\left[3\lambda^2+(1-\lambda)\log(1-2\lambda)-2(1-2\lambda)\log(1-\lambda)\right]\\
&-\frac{\gamma A_2\lambda}{\pi^2 b_0(\lambda-1)(2\lambda-1)}-\frac{B_1 b_1}{2\pi b_0^2(\lambda-1)}[\lambda+\log(1-\lambda)]+\frac{\gamma B_1\lambda}{2\pi(\lambda-1)}+\frac{\gamma^2 A_1\lambda(2\lambda-3)}{2\pi(\lambda-1)(2\lambda-1)}\\
&+\frac{b_1\gamma A_1}{\pi b_0^2(\lambda-1)(2\lambda-1)}[-\lambda+(1-2\lambda)\log(1-\lambda)-(1-\lambda)\log(1-2\lambda)] \qquad (39)\\
&-\frac{b_1^2 A_1}{2\pi b_0^4(\lambda-1)(2\lambda-1)}\Big[\left(\lambda^2+(2\lambda-1)\log(1-\lambda)(2\lambda+\log(1-\lambda))\right)\\
&+\frac{1}{2}\left((1-\lambda)\log^2(1-2\lambda)\right)-2(\lambda-1)\lambda\log(1-2\lambda)\Big]\\
&-\frac{b_0 b_2 A_1}{2\pi b_0^4(\lambda-1)(2\lambda-1)}\left[\lambda^2+(\lambda-1)(2\lambda-1)(2\log(1-\lambda)+\log(1-2\lambda))\right]
\end{aligned}
$$

$$
\begin{aligned}
f_4(\lambda)=&-\frac{A_4\lambda^2\left(2\lambda^2-6\lambda+3\right)}{6\pi^4 b_0^2(\lambda-1)^2(2\lambda-1)^2}+\frac{B_3(\lambda-2)\lambda}{4\pi^3 b_0(\lambda-1)^2}+\frac{b_1 A_3}{12\pi^3 b_0^3(\lambda-1)^2(2\lambda-1)^2}\Big[15\lambda^2\\
&+10(\lambda-3)\lambda^3+3(\lambda-1)^2\log(1-2\lambda)-6(1-2\lambda)^2\log(1-\lambda)\Big]\\
&+\frac{\gamma A_3\lambda(3\lambda-2)}{2\pi^3 b_0(\lambda-1)^2(2\lambda-1)^2}+\frac{\gamma B_2(\lambda-2)\lambda}{2\pi^2(\lambda-1)^2}-\frac{b_1 B_2\left[\lambda^2-2\lambda-2\log(1-\lambda)\right]}{4\pi^2 b_0^2(\lambda-1)^2}\\
&+\frac{\gamma^2 A_2\lambda\left(4\lambda^3-12\lambda^2+15\lambda-6\right)}{2\pi^2(\lambda-1)^2(2\lambda-1)^2}+\frac{b_1\gamma A_2}{2\pi^2 b_0^2(\lambda-1)^2(2\lambda-1)^2}\Big[\lambda(2-3\lambda)\\
&+2(\lambda-1)^2\log(1-2\lambda)-2(1-2\lambda)^2\log(1-\lambda)\Big]+\frac{b_2 A_2\lambda^3(4\lambda-3)}{3\pi^2 b_0^3(\lambda-1)^2(2\lambda-1)^2}\\
&-\frac{b_1^2 A_2}{12\pi^2 b_0^4(\lambda-1)^2(2\lambda-1)^2}\Big[\lambda^2\left(22\lambda^2-30\lambda+9\right)+3(\lambda-1)^2\log^2(1-2\lambda)\\
&+3(\lambda-1)^2\log(1-2\lambda)-6(1-2\lambda)^2\log(1-\lambda)(\log(1-\lambda)+1)\Big] \qquad (40)\\
&+\frac{b_0\gamma^2 B_1(\lambda-2)\lambda}{4\pi(\lambda-1)^2}+\frac{b_1\gamma B_1\log(1-\lambda)}{2\pi b_0(\lambda-1)^2}+\frac{b_2 B_1\lambda^2}{4\pi b_0^2(\lambda-1)^2}\\
&+\frac{b_1^2 B_1(\lambda-\log(1-\lambda))(\lambda+\log(1-\lambda))}{4\pi b_0^3(\lambda-1)^2}+\frac{b_0\gamma^3 A_1\lambda\left(12\lambda^3-36\lambda^2+39\lambda-14\right)}{6\pi(\lambda-1)^2(2\lambda-1)^2}\\
&+\frac{b_1\gamma^2 A_1}{2\pi b_0(\lambda-1)^2(2\lambda-1)^2}\left[2(\lambda-1)^2\log(1-2\lambda)-(1-2\lambda)^2\log(1-\lambda)\right]\\
&-\frac{b_1^2\gamma A_1}{2\pi b_0^3 A_1(\lambda-1)^2(2\lambda-1)^2}\left[(4\lambda-3)\lambda^2-(1-2\lambda)^2\log^2(1-\lambda)+(\lambda-1)^2\log^2(1-2\lambda)\right]\\
&-\frac{b_1^3 A_1}{12\pi b_0^5(\lambda-1)^2(2\lambda-1)^2}\Big[4(3-4\lambda)\lambda^3+2(1-2\lambda)^2\log(1-\lambda)\left(\log^2(1-\lambda)-3\lambda^2\right)\\
&+12(\lambda-1)^2\lambda^2\log(1-2\lambda)-(\lambda-1)^2\log^3(1-2\lambda)\Big]+\frac{b_2\gamma A_1\lambda^2(4\lambda-3)}{2\pi b_0^2(\lambda-1)^2(2\lambda-1)^2}\\
&\frac{b_1 b_2 A_1}{12\pi b_0^4(\lambda-1)^2(2\lambda-1)^2}\Big[\lambda^2(2\lambda(3-7\lambda)+3)+3\left(8\lambda^2-4\lambda+1\right)(\lambda-1)^2\log(1-2\lambda)\\
&-6(1-2\lambda)^2(2(\lambda-1)\lambda+1)\log(1-\lambda)\Big]+-\frac{b_3 A_1}{12\pi b_0^3(\lambda-1)^2(2\lambda-1)^2}\Big[(2(\lambda-3)\lambda+3)\lambda^2
\end{aligned}
$$

$$- 6\left(2\lambda^2 - 3\lambda + 1\right)^2 \log(1-\lambda) + 3\left(2\lambda^2 - 3\lambda + 1\right)^2 \log(1-2\lambda)\Big]$$

$$\begin{aligned}
f_5(\lambda) = &-\frac{A_5\lambda^2\left(4\lambda^4 - 18\lambda^3 + 33\lambda^2 - 24\lambda + 6\right)}{12\pi^5 b_0^2(\lambda-1)^3(2\lambda-1)^3} + \frac{B_4\lambda\left(\lambda^2 - 3\lambda + 3\right)}{6\pi^4 b_0(\lambda-1)^3} \qquad (41)\\
&- \frac{\gamma A_4\lambda\left(7\lambda^2 - 9\lambda + 3\right)}{3\pi^4 b_0(\lambda-1)^3(2\lambda-1)^3} + \frac{b_1 A_4}{36\pi^4 b_0^3(\lambda-1)^3(2\lambda-1)^3}\Big[7\lambda^2(\lambda(\lambda(2\lambda(2\lambda-9)+33)-24)\\
&+6) - 6(\lambda-1)^3\log(1-2\lambda) + 12(2\lambda-1)^3\log(1-\lambda)\Big] + \frac{\gamma B_3\lambda\left(\lambda^2 - 3\lambda + 3\right)}{2\pi^3(\lambda-1)^3} +\\
&\frac{b_1 B_3\left(\lambda^3 - 3\lambda^2 + 3\lambda + 3\log(1-\lambda)\right)}{6\pi^3 b_0^2(\lambda-1)^3} + \frac{\gamma^2 A_3\lambda\left(8\lambda^5 - 36\lambda^4 + 66\lambda^3 - 69\lambda^2 + 39\lambda - 9\right)}{2\pi^3(\lambda-1)^3(2\lambda-1)^3}\\
&- \frac{b_1\gamma A_3}{3\pi^3 b_0^2(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda((9-7\lambda)\lambda-3) + 3(\lambda-1)^3\log(1-2\lambda)\\
&- 3(2\lambda-1)^3\log(1-\lambda)\Big] + \frac{b_2 A_3\lambda^3\left(4\lambda^3 - 18\lambda^2 + 19\lambda - 6\right)}{4\pi^3 b_0^3(\lambda-1)^3(2\lambda-1)^3} -\\
&\frac{b_1^2 A_3}{36\pi^3 b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^3(\lambda(26\lambda(2\lambda-9)+303)-150) - 9(\lambda-1)^3\log^2(1-2\lambda)\\
&+ 24\lambda^2 - 6(\lambda-1)^3\log(1-2\lambda) + 6(2\lambda-1)^3\log(1-\lambda)(3\log(1-\lambda)+2)\Big]\\
&+ \frac{b_0\gamma^2 B_2\lambda\left(\lambda^2 - 3\lambda + 3\right)}{2\pi^2(\lambda-1)^3} - \frac{b_1\gamma B_2\log(1-\lambda)}{\pi^2 b_0(\lambda-1)^3} + \frac{b_1^2 B_2\left(\lambda^3 - 3\lambda^2 + 3\log^2(1-\lambda)\right)}{6\pi^2 b_0^3(\lambda-1)^3}\\
&- \frac{b_2 B_2(\lambda-3)\lambda^2}{6\pi^2 b_0^2(\lambda-1)^3} + \frac{b_0\gamma^3 A_2\lambda\left(24\lambda^5 - 108\lambda^4 + 198\lambda^3 - 193\lambda^2 + 99\lambda - 21\right)}{3\pi^2(\lambda-1)^3(2\lambda-1)^3}\\
&+ \frac{b_1^2\gamma A_2}{3\pi^2 b_0^3(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(18\lambda-25)+9) + 3(\lambda-1)^3\log^2(1-2\lambda)\\
&- 3(2\lambda-1)^3\log^2(1-\lambda)\Big] - \frac{b_2\gamma A_2\lambda^2\left(18\lambda^2 - 25\lambda + 9\right)}{3\pi^2 b_0^2(\lambda-1)^3(2\lambda-1)^3}\\
&+ \frac{b_1^3 A_2}{36\pi^2 b_0^5(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(2\lambda(50\lambda-171)+339)-114)+6)\\
&- 6(\lambda-1)^3\log^3(1-2\lambda) + 12(2\lambda-1)^3\log(1-\lambda)\left(-3\lambda + \log^2(1-\lambda) + 1\right)\\
&+ 6(6\lambda-1)(\lambda-1)^3\log(1-2\lambda)\Big] + \frac{b_3 A_2\lambda^3\left(20\lambda^3 - 54\lambda^2 + 45\lambda - 12\right)}{12\pi^2 b_0^3(\lambda-1)^3(2\lambda-1)^3}\\
&- \frac{b_1\gamma^2 A_2}{\pi^2 b_0(\lambda-1)^3(2\lambda-1)^3}\Big[2(\lambda-1)^3\log(1-2\lambda) - (2\lambda-1)^3\log(1-\lambda)\Big]\\
&- \frac{b_2 b_1 A_2}{18\pi^2 b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(4\lambda(20\lambda-63)+237)-75)+3)\\
&+ 3(6\lambda-1)(\lambda-1)^3\log(1-2\lambda) - 6(2\lambda-1)^3(3\lambda-1)\log(1-\lambda)\Big]\\
&+ \frac{b_0^2\gamma^3 B_1\lambda\left(\lambda^2 - 3\lambda + 3\right)}{6\pi(\lambda-1)^3} - \frac{b_1^2\gamma B_1\left(\lambda - \log^2(1-\lambda) + \log(1-\lambda)\right)}{2\pi b_0^2(\lambda-1)^3} -
\end{aligned}$$

$$
\begin{aligned}
&\frac{b_1^3 B_1}{12\pi b_0^4(\lambda-1)^3}(\lambda+\log(1-\lambda))\left(2\lambda^2-3\lambda+2\log^2(1-\lambda)-2\lambda\log(1-\lambda)-3\log(1-\lambda)\right) \\
&+\frac{b_2\gamma B_1\lambda}{2\pi b_0(\lambda-1)^3}-\frac{b_3 B_1\lambda^2(2\lambda-3)}{12\pi b_0^2(\lambda-1)^3}+\frac{b_1\gamma^2 B_1\left(\lambda^3-3\lambda^2+3\lambda-2\log(1-\lambda)\right)}{4\pi(\lambda-1)^3} \\
&+\frac{b_2 b_1 B_1\lambda\left(2\lambda^2-3\lambda-3\log(1-\lambda)\right)}{6\pi b_0^3(\lambda-1)^3}+\frac{b_0^2\gamma^4 A_1}{12\pi(\lambda-1)^3(2\lambda-1)^3}\lambda(2\lambda-3)\Big[28\lambda^4 \\
&-84\lambda^3+105\lambda^2-63\lambda+15\Big]+\frac{b_1^3\gamma A_1}{6\pi b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(4\lambda(3(\lambda-3)\lambda+8)-9) \\
&-2(\lambda-1)^3\log^3(1-2\lambda)+3(\lambda-1)^3\log^2(1-2\lambda)+12\lambda(\lambda-1)^3\log(1-2\lambda) \\
&+(2\lambda-1)^3\log(1-\lambda)(\log(1-\lambda)(2\log(1-\lambda)-3)-6\lambda)\Big] \\
&+\frac{b_1^4 A_1}{24\pi b_0^6(\lambda-1)^3(2\lambda-1)^3}\Big[8(\lambda-1)^3\lambda^2(4\lambda-3)\log(1-2\lambda) \\
&-2(2\lambda-1)^3\log(1-\lambda)\left(2(2\lambda-3)\lambda^2+\log(1-\lambda)((\log(1-\lambda)-2)\log(1-\lambda)-6\lambda)\right) \\
&+2\lambda^3\left(\lambda\left(-28\lambda^2+54\lambda-33\right)+6\right)+(\lambda-1)^3\log^4(1-2\lambda)-2(\lambda-1)^3\log^3(1-2\lambda)- \\
&12(\lambda-1)^3\lambda\log^2(1-2\lambda)\Big]-\frac{b_2\gamma^2 A_1\lambda\left(4\lambda^3-6\lambda+3\right)}{2\pi b_0(\lambda-1)^3(2\lambda-1)^3}+\frac{b_3\gamma\lambda^2\left(12\lambda^3-36\lambda^2+32\lambda-9\right)}{6\pi b_0^2(\lambda-1)^3(2\lambda-1)^3} \\
&+\frac{b_2^2 A_1}{36\pi b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(6-\lambda(\lambda(2\lambda(10\lambda+9)-69)+42)) \\
&-12\left(2\lambda^2-3\lambda+1\right)^3\log(1-\lambda)+6\left(2\lambda^2-3\lambda+1\right)^3\log(1-2\lambda)\Big] \\
&+\frac{b_1^2\gamma^2 A_1}{2\pi b_0^2(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda\left(4\lambda^3-6\lambda+3\right)+2(\lambda-1)^3\log^2(1-2\lambda) \\
&-2(\lambda-1)^3\log(1-2\lambda)-(2\lambda-1)^3(\log(1-\lambda)-1)\log(1-\lambda)\Big] \\
&-\frac{b_2 b_1^2 A_1}{36\pi b_0^5(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(2(153-82\lambda)\lambda-165)+12)+6) \\
&-18\lambda(\lambda-1)^3\log^2(1-2\lambda)+6\left(6(1-2\lambda)^2\lambda-1\right)(\lambda-1)^3\log(1-2\lambda) \\
&-6(2\lambda-1)^3\log(1-\lambda)\left(6\lambda(\lambda-1)^2-3\lambda\log(1-\lambda)-2\right)\Big] \\
&+\frac{b_1\gamma^3 A_1}{6\pi(\lambda-1)^3(2\lambda-1)^3}\Big[24\lambda^6-108\lambda^5+198\lambda^4-193\lambda^3+16\lambda^3\log(1-\lambda) \\
&-8\lambda^3\log(1-2\lambda)+99\lambda^2-24\lambda^2\log(1-\lambda)+24\lambda^2\log(1-2\lambda)-21\lambda \\
&+12\lambda\log(1-\lambda)-24\lambda\log(1-2\lambda)-2\log(1-\lambda)+8\log(1-2\lambda)\Big] \\
&-\frac{b_2 b_1\gamma A_1}{3\pi b_0^3(\lambda-1)^3(2\lambda-1)^3}\lambda\Big[\lambda(4\lambda(3(\lambda-3)\lambda+8)-9)+6(\lambda-1)^3\log(1-2\lambda) \\
&-3(2\lambda-1)^3\log(1-\lambda)\Big]+\frac{b_3 b_1 A_1}{18\pi b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^3(\lambda(4(18-7\lambda)\lambda-51)+6) \\
&+3\lambda^2+3(2\lambda(\lambda(8\lambda-9)+3)-1)(\lambda-1)^3\log(1-2\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&- 3(2\lambda-1)^3(\lambda(\lambda(4\lambda-9)+6)-2)\log(1-\lambda)\Big] \\
&- \frac{b_4 A_1}{36\pi b_0^3(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(2\lambda(2\lambda-9)+33)-24)+6) \\
&- 12\left(2\lambda^2-3\lambda+1\right)^3\log(1-\lambda)+6\left(2\lambda^2-3\lambda+1\right)^3\log(1-2\lambda)\Big]
\end{aligned}
$$

In eqs. (39) to (41) we have removed some constant terms in order to make them homogeneous, *i.e.* $f_i(0)=0$, those constant can be reabsorbed in the C-term eq. (4). All the relevant constant can be found in appendix C.

To obtain the laplace trasform of the thrust distribution eq. (4), we perform the laplace transform of the convolution eq. (24). By applying the convolution theorem, we find that it is twice the integral given by eq. (10) that we have just calculated. Therefore, we multiply by 2 the $f_i(\lambda)$ we just obtained eqs. (37) to (41).

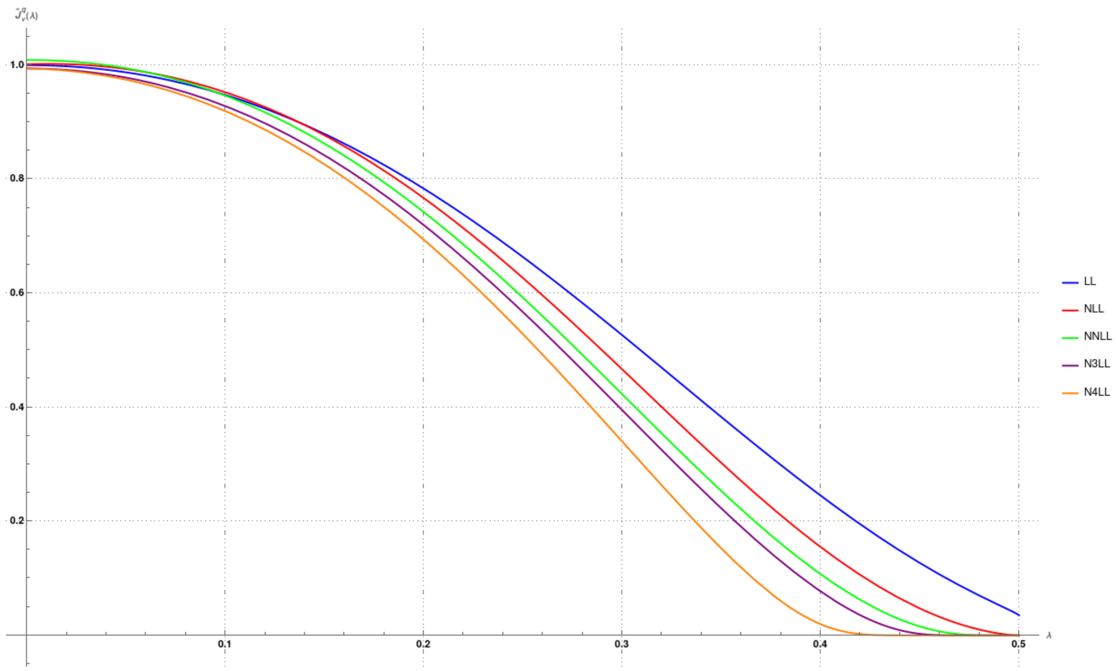


Figure 2: Plot of eq. (31) at different logarithmin orders.

3.2.2 Renormalization-scale dependece

In this section we consider renormalization-scale dependence. In principle, such scale μ should not appear in the cross sections, as it does not correspond to any fundamental constant or kinematical scale in the problem. The completely resummed perturbative expansion of an observable is indeed formally independent on μ. In practise, truncated perturbative expansions exhibit a residual scale dependence, because of neglected higher orders.

We start with deriving the strong coupling $\alpha_s(Q^2)$ as a function of $\alpha_s(\mu^2)$ and μ^2/Q^2, to do so we expand the explicit eq. (21) in powers of $\alpha_s(\mu^2)$ and obtain:

$$\begin{aligned}\alpha_s(Q^2) = {} & \alpha_s(\mu^2) + \alpha_s^2(\mu^2) b_0 \ln\left(\frac{\mu^2}{Q^2}\right) + \alpha_s^3(\mu^2)\left[b_1 \ln\left(\frac{\mu^2}{Q^2}\right) + b_0^2 \ln^2\left(\frac{\mu^2}{Q^2}\right)\right] \\ & + \alpha_s^4(\mu^2)\left[b_2 \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{5}{2} b_0 b_1 \ln^2\left(\frac{\mu^2}{Q^2}\right) + b_0^3 \ln^3\left(\frac{\mu^2}{Q^2}\right)\right] \\ & + \alpha_s^5(\mu^2)\left[b_3 \ln\left(\frac{\mu^2}{Q^2}\right) + \left(\frac{3}{2} b_1^2 + 3 b_0 b_2\right) \ln^2\left(\frac{\mu^2}{Q^2}\right)\right. \\ & \left. + \frac{13}{3} b_0^2 b_1 \ln^3\left(\frac{\mu^2}{Q^2}\right) + b_0^4 \ln^4\left(\frac{\mu^2}{Q^2}\right)\right] + \mathcal{O}\big(\alpha_s^6(\mu^2)\big)\end{aligned} \tag{42}$$

for brevity we'll write:

$$\alpha_s(Q^2) = \alpha_s(\mu^2) + c_1 \alpha_s^2(\mu^2) + c_2 \alpha_s^3(\mu^2) + c_3 \alpha_s^4(\mu^2) + c_4 \alpha_s^5(\mu^2) + \mathcal{O}\big(\alpha_s^6(\mu^2)\big) \tag{43}$$

where c_i are the coefficients of the expansion above.

$$\begin{aligned} c_1 &= b_0 \ln\left(\frac{\mu^2}{Q^2}\right) \\ c_2 &= b_1 \ln\left(\frac{\mu^2}{Q^2}\right) + b_0^2 \ln^2\left(\frac{\mu^2}{Q^2}\right) \\ c_3 &= b_2 \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{5}{2} b_0 b_1 \ln^2\left(\frac{\mu^2}{Q^2}\right) + b_0^3 \ln^3\left(\frac{\mu^2}{Q^2}\right) \\ c_4 &= b_3 \ln\left(\frac{\mu^2}{Q^2}\right) + \left(\frac{3}{2} b_1^2 + 3 b_0 b_2\right) \ln^2\left(\frac{\mu^2}{Q^2}\right) + \frac{13}{3} b_0^2 b_1 \ln^3\left(\frac{\mu^2}{Q^2}\right) + b_0^4 \ln^4\left(\frac{\mu^2}{Q^2}\right) \end{aligned} \tag{44}$$

now we substitute eq. (42) into $\lambda = b_0 \alpha_s(Q^2) L$ and obtain:

$$\lambda(Q^2) = \lambda(\mu^2)\big(1 + c_1 \alpha_s(\mu^2) + c_2 \alpha_s^2(\mu^2) + c_3 \alpha_s^3(\mu^2) + c_4 \alpha_s^4(\mu^2) + \mathcal{O}\big(\alpha_s^5(\mu^2)\big)\big) \tag{45}$$

and formally expand in powers of $\alpha_s(\mu^2)$ all the relevant functions, now $\lambda = \lambda(\mu^2)$:

$$
\begin{aligned}
Lf_1(Q^2) &= Lf_1(\lambda) + \frac{c_1}{b_0}\lambda^2 f_1'(\lambda) + \frac{\lambda}{b_0}\left(c_2\lambda f_1'(\lambda) + \frac{1}{2}c_1^2\lambda^2 f_1''(\lambda)\right)\alpha_s \\
&+ \frac{\lambda}{b_0}\left(c_3\lambda f_1'(\lambda) + c_1c_2\lambda^2 f_1''(\lambda) + \frac{1}{6}c_1^3 f_1^{(3)}(\lambda)\right)\alpha_s^2 + \frac{\lambda^2}{b_0}\left(c_4 f_1'(\lambda)\right. \\
&\left. + \frac{1}{2}c_2^2\lambda f_1''(\lambda) + c_1c_3\lambda f_1''(\lambda) + \frac{1}{2}c_1^2c_2\lambda^2 f_1^{(3)}(\lambda) + \frac{1}{24}c_1^4\lambda^3 f_1^{(4)}(\lambda)\right)\alpha_s^3 \\
f_2(Q^2) &= f_2(\lambda) + c_1\lambda f_2'(\lambda)\alpha_s + \left(c_2\lambda f_2'(\lambda) + \frac{1}{2}c_1^2\lambda^2 f_2''(\lambda)\right)\alpha_s^2 \\
&+ \left(c_3\lambda f_2'(\lambda) + c_1c_2\lambda^2 f_2''(\lambda) + \frac{1}{6}c_1^3 f_2^{(3)}(\lambda)\right)\alpha_s^3 \\
\alpha_s f_3(Q^2) &= \alpha_s f_3(\lambda) + c_1\lambda f_3'(\lambda)\alpha_s^2 + \left(c_2\lambda f_3'(\lambda) + \frac{1}{2}c_1^2\lambda^2 f_3''(\lambda)\right)\alpha_s^3 \\
\alpha_s^2 f_4(Q^2) &= \alpha_s^2 f_4(\lambda) + c_1\lambda f_4'(\lambda)\alpha_s^3 \\
\alpha_s^3 f_5(Q^2) &= \alpha_s^3 f_5(\lambda)
\end{aligned}
\tag{46}
$$

As usual, the terms from the expansion proportional to α_s corrects f_3, the terms proportional to α_s^2 corrects f_4 and so on.

So the additional terms in the functions f_i for $i = 0, 1, 3$, to partially compensate for the scale change $Q^2 \to \mu^2$ therefore read:

$$
\begin{aligned}
\delta f_1\left(\lambda, \frac{\mu^2}{Q^2}\right) &= 0 \\
\delta f_2\left(\lambda, \frac{\mu^2}{Q^2}\right) &= \lambda^2 f_1'(\lambda)\log\left(\frac{\mu^2}{Q^2}\right) \\
\delta f_3\left(\lambda, \frac{\mu^2}{Q^2}\right) &= \frac{1}{2}\lambda^3 f_1''(\lambda)\log\left(\frac{\mu^2}{Q^2}\right) + \lambda^2 f_1'(\lambda)\left(\log^2\left(\frac{\mu^2}{Q^2}\right) + \frac{b1}{b0^2}\log\left(\frac{\mu^2}{Q^2}\right)\right) \\
&+ \lambda f_2'(\lambda)\log\left(\frac{\mu^2}{Q^2}\right)
\end{aligned}
\tag{47}
$$

then explicitly calculate those derivatives and obtain the scale dependence.

3.3 Inversion of the Laplace transform

In order to find the quark jet mass distribution $J^q(Q^2, k^2)$ or $R_T(\tau)$, we have to perform the inverse Laplace transform via the Mellin's inversion formula (or the Bromwich integral) given by the line integral:

$$
J^q(Q^2, k^2) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT} \mathrm{d}\nu e^{\nu k^2}\tilde{J}_\nu^q(Q^2) \tag{48}
$$

where C is a real number such that C is at the right of all singularities of the integrand in the complex plane and the function $\tilde{J}^q_\nu(Q^2)$ has to be bounded on the line.

Instead of working with the differential quark jet mass distribution J^q, it more convenient to deal with the mass fraction $R^q(w)$, which gives the fraction of jets with masses less than wQ^2 (cumulant distribution):

$$R^q(w) = \int_0^\infty J^q(Q^2, k^2)\Theta(wQ^2 - k^2)\mathrm{d}k^2 \tag{49}$$

and using the integral representation of the Heaviside step function eq. (26):

$$\Theta(wQ^2 - k^2) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{\nu(wQ^2-k^2)} \tag{50}$$

we recognize the Laplace transform of the quark jet mass distribution eq. (25)

$$\begin{aligned}
R^q(w) &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{w\nu Q^2}\int_0^\infty J^q(Q^2,k^2)e^{-\nu k^2}\mathrm{d}k^2 \\
&= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{w\nu Q^2}\tilde{J}^q_\nu(Q^2) \\
&= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{w\nu Q^2}e^{\mathcal{F}(\alpha_s,\ln\left(\nu Q^2\right))} \\
&= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C'-iT}^{C'+iT}\frac{\mathrm{d}N}{N}e^{wN}e^{\mathcal{F}(\alpha_s,\ln N)}
\end{aligned} \tag{51}$$

where $N = \nu Q^2$ and $\mathcal{F}$ has the logarithms expansion:

$$\begin{aligned}
\mathcal{F}(\alpha_s, \ln N) &= f_1(b_0\alpha_s\ln N)\ln N + f_2(b_0\alpha_s\ln N) + f_3(b_0\alpha_s\ln N)\alpha_s \\
&+ f_4(b_0\alpha_s\ln N)\alpha_s^2 + f_5(b_0\alpha_s\ln N)\alpha_s^3 + \mathcal{O}\big(\alpha_s^4\big)
\end{aligned} \tag{52}$$

Since the function $\mathcal{F}$ in the exponent varies more slowly with N than wN, we can introduce the integration variable $u = wN$ so that $\ln N = \ln u + \ln\frac{1}{w} = \ln u + L$ and Taylor expand with respect to $\ln u$ around 0, which is equivalent to expanding the original function $\mathcal{F}$ w.r.t $\ln N$ around $\ln N = \ln\frac{1}{w} \equiv L$:

$$\begin{aligned}
R^q(w) &= \frac{1}{2\pi i}\int_C\frac{\mathrm{d}u}{u}e^u e^{\mathcal{F}(\alpha_s,\ln u+L)} \\
&\overset{\text{Taylor}}{=}\int_C\frac{\mathrm{d}u}{2\pi i}e^{u-\ln u}e^{\mathcal{F}(\alpha_s,L)+\sum_{n=1}^\infty\frac{\mathcal{F}^{(n)}(\alpha_s,L)}{n!}\ln^n u} \\
&= e^{\mathcal{F}(\alpha_s,L)}\int_C\frac{\mathrm{d}u}{2\pi i}e^{u-\ln u}e^{\sum_{n=1}^\infty\frac{\mathcal{F}^{(n)}(\alpha_s,L)}{n!}\ln^n u}
\end{aligned} \tag{53}$$

where the integral is intended as before, along the line C to the right of all singularities of the integrand, and

$$\mathcal{F}^{(n)}(\alpha_s, L) = \frac{\partial^n \mathcal{F}(\alpha_s, \ln u + L)}{\partial(\ln u)^n}\bigg|_{\ln u=0} \tag{54}$$

The n-th derivative of $\mathcal{F}$ w.r.t $\ln u$ evaluated at $\ln u = 0$ is at most of logarithmic order $\alpha_s^{n+k-1}L^k$ [5], so in order to achieve N^{4}LL accuracy we need to compute the first four derivatives of $\mathcal{F}$ w.r.t $\ln u$ and neglect the terms of order $\mathcal{O}(\alpha_s^4)$ that appear in the derivation. We obtain the following expressions:

$$\begin{aligned}\mathcal{F}^{(1)}(\alpha_s, L) = f_1(\lambda) + \lambda f_1'(\lambda) + \alpha_s b_0 f_2'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s^3 b_0 f_4'(\lambda)\\ + \mathcal{O}(\alpha_s^n L^{n-3})\end{aligned} \tag{55}$$

$$\begin{aligned}\mathcal{F}^{(2)}(\alpha_s, L) = 2\alpha_s b_0 f_1'(\lambda) + \alpha_s b_0 \lambda f_1''(\lambda) + \alpha_s^2 b_0^2 f_2''(\lambda) + \alpha_s^3 b_0^2 f_3''(\lambda)\\ + \mathcal{O}(\alpha_s^n L^{n-3})\end{aligned} \tag{56}$$

$$\mathcal{F}^{(3)}(\alpha_s, L) = 3\alpha_s^2 b_0^2 f_1''(\lambda) + \alpha_s^2 b_0^2 \lambda f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 f_2^{(3)}(\lambda) + \mathcal{O}(\alpha_s^n L^{n-3}) \tag{57}$$

$$\mathcal{F}^{(4)}(\alpha_s, L) = 4\alpha_s^3 b_0^3 f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 \lambda f_1^{(4)}(\lambda) + \mathcal{O}(\alpha_s^n L^{n-3}) \tag{58}$$

Here $\lambda = \alpha_s b_0 L$ and derivative w.r.t $\ln u$ and then evaluated at $\ln u = 0$, or equivalently derivative w.r.t L gives the same result.

After recasting the expansion presented in eq. (53) using the expression $\gamma(\alpha_s, L) = f_1(\lambda) + \lambda f_1'(\lambda)$ from [5], and defining $\mathcal{F}_{res}^{(1)}(\alpha_s, L) \equiv \mathcal{F}^{(1)}(\alpha_s, L) - \gamma(\alpha_s, L)$, we proceed to expand the second exponential with respect to $\ln u$ around 0, following the approach outlined in [1]. This yields the subsequent expansion:

$$\begin{aligned}R^q(w) &= e^{\mathcal{F}(\alpha_s,L)} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u} e^{\mathcal{F}_{res}^{(1)}(\alpha_s,L)\ln u + \sum_{n=2}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s,L)}{n!}\ln^n u}\\ &= \int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u}\Big(1 + \mathcal{F}_{res}^{(1)}\ln u + \frac{1}{2}\Big(\mathcal{F}^{(2)} + (\mathcal{F}_{res}^{(1)})^2\Big)\ln^2 u\\ &+ \frac{1}{6}\Big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}_{res}^{(1)} + (\mathcal{F}_{res}^{(1)})^3\Big)\ln^3 u\\ &+ \frac{1}{24}\Big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}_{res}^{(1)} + 6\mathcal{F}^{(2)}(\mathcal{F}_{res}^{(1)})^2 + (\mathcal{F}_{res}^{(1)})^4\Big)\ln^4 u\\ &+ \mathcal{O}(\ln^5 u)\Big)\end{aligned} \tag{59}$$

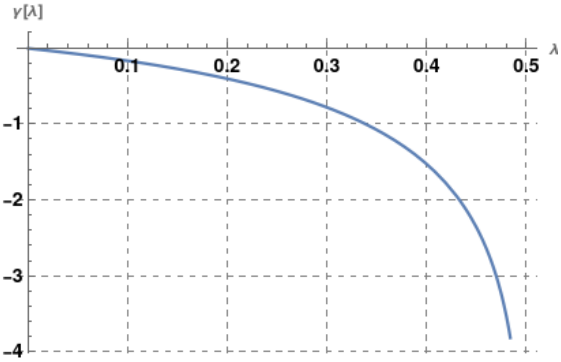


Figure 3: Plot of $\gamma(\lambda)$ in the range of interest $0 < \lambda < \frac{1}{2}$

Lastly, we utilize the following result to evaluate the integral presented in eq. (49).

$$\int_C \frac{\mathrm{d}u}{2\pi i} \ln^k u e^{u-(1-\gamma(\alpha_s,L))\ln u} = \frac{\mathrm{d}^k}{\mathrm{d}\gamma^k} \frac{1}{\Gamma(1-\gamma(\alpha_s,L))} \tag{60}$$

where Γ is the Euler Γ-function, notice that for $k = 0$ the integral is the Hankel integral representation of the Γ-function:

$$\frac{1}{2\pi i} \int_C \mathrm{d}u e^u u^{-(1-\gamma(\alpha_s,L))} = \frac{1}{\Gamma(1-\gamma(\alpha_s,L))} \tag{61}$$

The reciprocal of the Euler Γ-function is an entire function, meaning that it has no singularities in the complex plane, it's holomorphic everywhere. The above integral eq. (60) can be evaluated by differentiating with respect to γ k times, and it is straightforward to see that each time i derive inside the integral i will get a factor of $\ln u$. Interchaning the order of derivation and integration is justified by the fact that the integral:

- is a contour integral whose contour doesn't depent on γ, and the function is holomorphic in the complex plane for each value of $\gamma \in [-4, 0] fig.$ 3
- is dominated by the exponential factor $e^{u-(1-\gamma)\ln u}$ as long as $\mathrm{Re}\{u\} > 0$

$$\begin{aligned} R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)} \Big[& 1 + \mathcal{F}^{(1)}_{res}\psi_0(1-\gamma) + \frac{1}{2}\Big(\mathcal{F}^{(2)} + (\mathcal{F}^{(1)}_{res})^2\Big)\big(\psi_0^2 - \psi_1\big)(1-\gamma) \\ & + \frac{1}{6}\Big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}^{(1)}_{res} + (\mathcal{F}^{(1)}_{res})^3\Big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma) \\ & + \frac{1}{24}\Big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}^{(1)}_{res} + 6\mathcal{F}^{(2)}(\mathcal{F}^{(1)}_{res})^2 + (\mathcal{F}^{(1)}_{res})^4\Big) \\ & \big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Big] \end{aligned} \tag{62}$$

where $\psi_n(z)$ are the polygamma functions, defined as:

$$\psi_n(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} \ln\Gamma(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{\Gamma'(z)}{\Gamma(z)} = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\psi_0(z) \tag{63}$$

Substituting the expressions eqs. (55) to (58) into eq. (62) we obtain:

$$\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Big[& 1 + \Big(\alpha_s^3 b_0 f_4'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s b_0 f_2'(\lambda)\Big)\psi_0(1-\gamma) \\
& + \frac{1}{2}\Big(\alpha_s^3(2b_0^2 f_2'(\lambda) f_3'(\lambda) + b_0^2 f_3''(\lambda)) + \frac{1}{2}\alpha_s^2\big(b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2\big) \\
& + \frac{1}{2}\alpha_s(b_0\lambda f_1''(\lambda) + 2b_0 f_1'(\lambda))\Big)\big(\psi_0^2 - \psi_1\big)(1-\gamma) \\
& + \frac{1}{6}\Big(\alpha_s^3(b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda) f_2''(\lambda) + 3b_0^2\lambda f_1''(\lambda) f_3'(\lambda) \\
& + 6b_0^2 f_1'(\lambda) f_3'(\lambda)) + \frac{1}{6}\alpha_s^2(b_0^2\lambda f_1^{(3)}(\lambda) + 3b_0^2\lambda f_1''(\lambda) f_2'(\lambda) + 3b_0^2 f_1''(\lambda) \\
& + 6b_0^2 f_1'(\lambda) f_2'(\lambda))\Big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma) \\
& + \frac{1}{24}\Big(\alpha_s^3(b_0^3\lambda f_1^{(4)}(\lambda) + 4b_0^3\lambda f_1^{(3)}(\lambda) f_2'(\lambda) \\
& + 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3\lambda f_1''(\lambda) f_2''(\lambda) + 6b_0^3\lambda f_1''(\lambda) f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda) f_2'(\lambda) \\
& + 12b_0^3 f_1'(\lambda) f_2''(\lambda) + 12b_0^3 f_1'(\lambda) f_2'(\lambda)^2) + \frac{1}{8}\alpha_s^2(b_0^2\lambda^2 f_1''(\lambda)^2 + 4b_0^2 f_1'(\lambda)^2 \\
& + 4b_0^2\lambda f_1'(\lambda) f_1''(\lambda))\Big)\big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Big]
\end{aligned} \tag{64}$$

and reorganize as a power series of α_s:

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Big[&1+\alpha_s b_0\Big(\psi_0(1-\gamma)f_2'(\lambda)+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)(\lambda f_1''(\lambda)\\
&+2f_1'(\lambda))\Big)+\alpha_s^2\Big(\frac{1}{8}(\psi_0^4-6\psi_1+3\psi_1^3+4\psi_0\psi_2-\psi_3)(1-\gamma)(b_0^2\lambda^2 f_1''(\lambda)^2\\
&+4b_0^2 f_1'(\lambda)^2+4b_0^2\lambda f_1'(\lambda)f_1''(\lambda))+\frac{1}{6}(\psi_0^3-3\psi_0\psi_1+\psi_2)(1-\gamma)(b_0^2\lambda f_1^{(3)}(\lambda)\\
&+3b_0^2\lambda f_1''(\lambda)f_2'(\lambda)+3b_0^2 f_1''(\lambda)+6b_0^2 f_1'(\lambda)f_2'(\lambda))+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\\
&(b_0^2 f_2''(\lambda)+b_0^2 f_2'(\lambda)^2)+b_0\psi_0(1-\gamma)f_3'(\lambda)\Big)+\alpha_s^3\Big(\frac{1}{24}(\psi_0^4-6\psi_1+3\psi_1^3+4\psi_0\psi_2\\
&-\psi_3)(1-\gamma)(b_0^3\lambda f_1^{(4)}(\lambda)+4b_0^3\lambda f_1^{(3)}(\lambda)f_2'(\lambda)+4b_0^3 f_1^{(3)}(\lambda)+6b_0^3\lambda f_1''(\lambda)f_2''(\lambda)\\
&+6b_0^3\lambda f_1''(\lambda)f_2'(\lambda)^2+12b_0^3 f_1''(\lambda)f_2'(\lambda)+12b_0^3 f_1'(\lambda)f_2''(\lambda)+12b_0^3 f_1'(\lambda)f_2'(\lambda)^2)\\
&+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)(2b_0^2 f_2'(\lambda)f_3'(\lambda)+b_0^2 f_3''(\lambda))+\frac{1}{6}(\psi_0^3-3\psi_0\psi_1\\
&+\psi_2)(1-\gamma)(b_0^3 f_2^{(3)}(\lambda)+b_0^3 f_2'(\lambda)^3+3b_0^3 f_2'(\lambda)f_2''(\lambda)+3b_0^2\lambda f_1''(\lambda)f_3'(\lambda)\\
&+6b_0^2 f_1'(\lambda)f_3'(\lambda))+b_0\psi_0(1-\gamma)f_4'(\lambda)\Big)+\mathcal{O}\big(\alpha_s^4\big)\Big]
\end{aligned}
\tag{65}
$$

By comparing eq. (27) and eq. (51) we see that to obtain the Thrust cross section, we simply multiply by 2 all the f_i eqs. (37) to (41) obtained in the previous section. The final resummed expression $R_T(\tau)$ can be written as one exponential eq. (6),

$$
R_T(\tau) = \left(1+\sum_{n=1}^{\infty} C_n \bar{\alpha_s}^n\right)\exp\big[Lg_1(\lambda)+g_2(\lambda)+g_3(\lambda)\alpha_s+g_4(\lambda)\alpha_s^2+g_5(\lambda)\alpha_s^3+\mathcal{O}\big(\alpha_s^4\big)\big] \tag{66}
$$

to do so we observe that $\frac{1}{\Gamma(1-\gamma)} = \exp\{-\ln(\Gamma(1-\gamma))\}$ corrects f_2, while the expression in square parenthesis in eq. (65) can be seen as the expansion of an exponential for $\alpha_s \to 0$:

$$
\begin{aligned}
e^{\alpha_s g_3(\lambda)+\alpha_s^2 g_4(\lambda)+\alpha_s^3 g_5(\lambda)+\mathcal{O}\left(\alpha_s^4\right)} &= 1+\alpha_s g_3(\lambda)+\frac{1}{2}\alpha_s^2\big(g_3^2(\lambda)+2g_4(\lambda)\big)\\
&\quad+\frac{1}{6}\alpha_s^3\big(g_3^3(\lambda)+6g_3(\lambda)g_4(\lambda)+6g_5(\lambda)\big)+\mathcal{O}\big(\alpha_s^4\big)
\end{aligned}
\tag{67}
$$

to obtain $g_3(\lambda)$, $g_4(\lambda)$ and $g_5(\lambda)$ we match eq. (65) with eq. (67) and obtain the following expressions:

$$
g_1(\lambda) = f_1(\lambda) \tag{68}
$$

$$
g_2(\lambda) = f_2(\lambda) - \ln\Gamma(1-f_1(\lambda)-\lambda f_1'(\lambda)) \tag{69}
$$

$$g_3(\lambda) = f_3(\lambda) + \Big(\psi_0(1-\gamma)f_2'(\lambda) + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)(\lambda f_1''(\lambda) + 2f_1'(\lambda))\Big) \tag{70}$$

$$\begin{aligned} g_4(\lambda) = f_4(\lambda) + \Big(&\frac{1}{8}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1-\gamma)(b_0^2\lambda^2 f_1''(\lambda)^2 \\ &+ 4b_0^2 f_1'(\lambda)^2 + 4b_0^2\lambda f_1'(\lambda)f_1''(\lambda)) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1-\gamma)(b_0^2\lambda f_1^{(3)}(\lambda) \\ &+ 3b_0^2\lambda f_1''(\lambda)f_2'(\lambda) + 3b_0^2 f_1''(\lambda) + 6b_0^2 f_1'(\lambda)f_2'(\lambda)) + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma) \\ &(b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2) + b_0\psi_0(1-\gamma)f_3'(\lambda)\Big) \end{aligned} \tag{71}$$

$$\begin{aligned} g_5(\lambda) = f_5(\lambda) + \Big(&\frac{1}{24}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1-\gamma) \\ &(b_0^3\lambda f_1^{(4)}(\lambda) + 4b_0^3\lambda f_1^{(3)}(\lambda)f_2'(\lambda) + 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3\lambda f_1''(\lambda)f_2''(\lambda) \\ &+ 6b_0^3\lambda f_1''(\lambda)f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda)f_2'(\lambda) + 12b_0^3 f_1'(\lambda)f_2''(\lambda) + 12b_0^3 f_1'(\lambda)f_2'(\lambda)^2) \\ &+ \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)(2b_0^2 f_2'(\lambda)f_3'(\lambda) + b_0^2 f_3''(\lambda)) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1-\gamma) \\ &(b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda)f_2''(\lambda) + 3b_0^2\lambda f_1''(\lambda)f_3'(\lambda) + 6b_0^2 f_1'(\lambda)f_3'(\lambda)) \\ &+ b_0\psi_0(1-\gamma)f_4'(\lambda)\Big) \end{aligned} \tag{72}$$

Observe in fig. 4, the peak of the distribution gets near the experimental data peak as we increase the logarithmic accuracy order, this is expected since the resummation of the logarithms is supposed to improve the accuracy of the prediction. However, the peak also becomes lower while the tail of the distribution becomes higher, this means that if we set the value of the coupling to be $\alpha_s = 0.118$, there's too much radiation. To fix this, we would need to decrease the value of the coupling, which then would lead us to extract a lower value of α_s with respect to the world average.

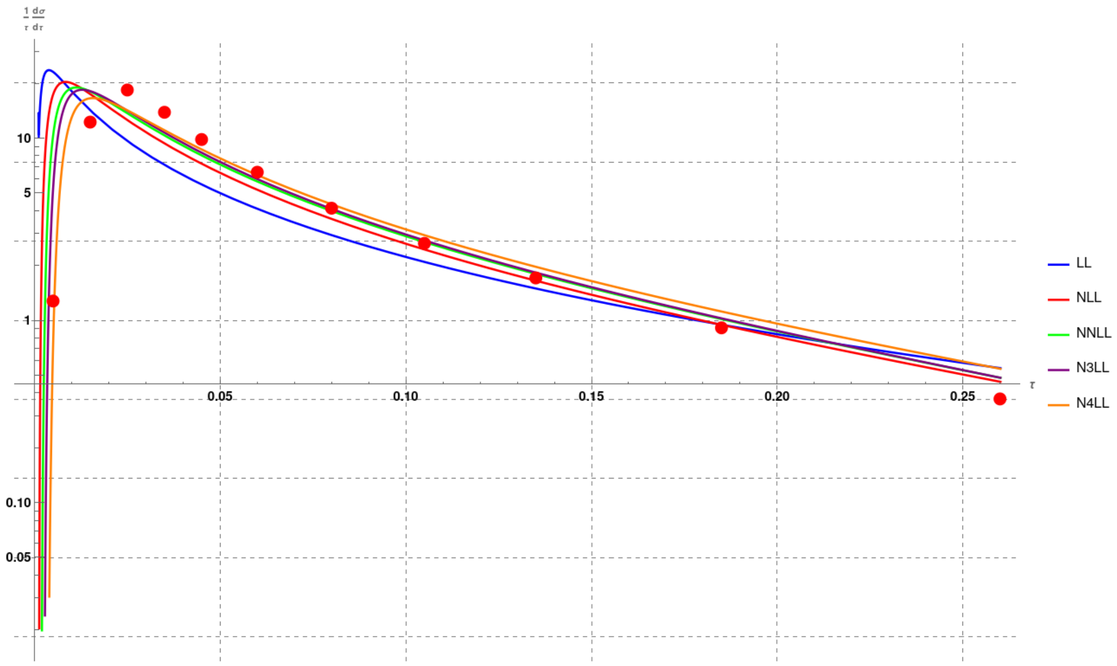


Figure 4: Log Plot of eq. (6) for different logarithmic accuracy orders, with fixed scale $\alpha_s = 0.118$. The plot also shows experimental data from Opal Experiment at LEP.

Appendix A

Laplace transform in the large N limit

Using the methodology outlined in [9], we will demonstrate that the Mellin transform prescription is also applicable to the Laplace transform. in the large moment $\nu Q^2 = N$ limit, this fact was already known in the literature [4] and we'll show it here for completeness.

We are interested in solving the following integral

$$\int_0^1 \mathrm{d}z\,\frac{e^{-N(1-z)}-1}{1-z}F(\alpha_s,\ln(1-z)) = \int_0^1 \frac{\mathrm{d}u}{u}\big(e^{-uN}-1\big)F(\alpha_s,\ln u) \tag{1}$$

Start by considering

$$I_n(N) = \int_0^1 \frac{\mathrm{d}u}{u}\big(e^{-uN}-1\big)\ln^n(u) \tag{2}$$

the above integral can be evaluated as described in [3]. Using the following identity

$$\ln^n(u) = \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^n u^\epsilon = \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^n e^{\epsilon\ln u} \tag{3}$$

to replace the logarithm term in the integrand eq. (2) and straightforwardly integrate the resulting expression. We obtain

$$
\begin{aligned}
I_n(N) &= \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^n \int_0^1 \mathrm{d}u\,\big(e^{-uN}-1\big)u^{\epsilon-1} \\
&= \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{-\frac{1}{\epsilon}+N^{-\epsilon}(\Gamma(\epsilon,0)-\Gamma(\epsilon,N))\right\} \\
&= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{-\frac{1}{\epsilon}+N^{-\epsilon}\Gamma(\epsilon)\right\}+e^{-N+\mathcal{O}\left((\frac{1}{N})^2\right)}\mathcal{O}\left(\frac{1}{N}\right) \\
&= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{\frac{1}{\epsilon}(N^{-\epsilon}\epsilon\Gamma(\epsilon)-1)\right\}+\mathcal{O}\left(\frac{e^{-N}}{N}\right) \\
&= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\frac{\partial}{\partial\epsilon}\right)^n \left\{\frac{1}{\epsilon}(e^{-\epsilon\ln N}\Gamma(1+\epsilon)-1)\right\}+\mathcal{O}\left(\frac{e^{-N}}{N}\right)
\end{aligned} \tag{4}
$$

where $\Gamma(\epsilon,0)=\Gamma(\epsilon)$, $\Gamma(\epsilon,N)$ is the incomplete Gamma function and $\epsilon\Gamma(\epsilon)=\Gamma(1+\epsilon)$

$$
\Gamma(\epsilon,N)=\int_N^\infty \mathrm{d}t\, t^{\epsilon-1}e^{-t} \tag{5}
$$

The last equation in eq. (4) is the same as Eq. (68) obtained in [9] for the Mellin transform. Therefore, we can conclude that the Mellin transform prescription is also applicable to the Laplace transform in the large N limit.

Using the known expansion of the Gamma function for small ϵ

$$
\Gamma(1+\epsilon)=\exp\left\{-\gamma_E\epsilon+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)\epsilon^n}{n}\right\} \tag{6}
$$

the term in curly brackets in eq. (6) can be expanded in power of ϵ and then derive. The result for $I_n(N)$ is thus a polynomial of degree $n+1$ in the large logarithm $\ln N$:

$$
\begin{aligned}
I_n(N) = &\frac{(-1)^n+1}{n+1}(\ln N+\gamma_E)^{n+1}+\frac{(-1)^{n-1}}{2}n\zeta(2)(\ln N+\gamma_E)^{n-1} \\
&+\sum_{k=0}^{n-2}a_{nk}(\ln N+\gamma_E)^k+\mathcal{O}\left(\frac{e^{-N}}{N}\right)
\end{aligned} \tag{7}
$$

This result can be generalized using the following formal identity:

$$
e^{-\epsilon\ln N}\Gamma(1+\epsilon)=\Gamma(1-\frac{\partial}{\partial\ln N})e^{\epsilon\ln N} \tag{8}
$$

then we can perform the n-th derivative with respect to ϵ, and obtain

$$
I_n(N)=\Gamma\left(1-\frac{\partial}{\partial\ln N}\right)\frac{(-\ln N)^n+1}{n+1}+\mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{9}
$$

This expression can be regarded as a replacement for eq. (4) to compute the polynomial coefficients a_{nk} in eq. (7). Moreover, by observing that

$$\frac{(-\ln N)^n + 1}{n+1} = -\int_{\frac{1}{N}}^{1} \mathrm{d}u \, \frac{\ln^n(u)}{u} \tag{10}$$

we obtain the all order generalization for of the prescriprion used in [5]:

$$e^{-uN} - 1 = -\Gamma\left(1 - \frac{\partial}{\partial \ln_N}\right)\Theta(u - \frac{1}{N}) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{11}$$

$$= -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln_N}\right)\Theta(u - \frac{N_0}{N}) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{12}$$

where

$$\tilde{\Gamma}(1-\epsilon) \equiv e^{\gamma_E \epsilon}\Gamma(1+\epsilon) = \exp\left\{\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)\epsilon^n}{n}\right\} \tag{13}$$

It is straightforward to show that the prescription can be applied to as follows:

$$\int_0^1 \frac{\mathrm{d}u}{u}\left(e^{-uN} - 1\right)F(\alpha_s, \ln u) = -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln_N}\right)\int_{\frac{N_0}{N}}^{1} \frac{\mathrm{d}u}{u} F(\alpha_s, \ln u) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{14}$$

and to evaluated the $\ln N$-contribution arising from the integration of anu soft-gluon function F that has a generic perturbative expansion of the type

$$F(\alpha_s, \ln u) = \sum_{k=1}^{\infty} \alpha_s^k \sum_{n=0}^{2k-1} F_{kn} \ln^n u \tag{15}$$

The result eq. (14) can be used to obtain eq. (30) as shown in [9].

Appendix B

Equivalence between resummation formulae

Here i adapt the result in [9] to show the equivalence between the resummation formulae in **??** and eq. (30) in the case of Thrust resummation.

It is straightforward to show that equation (90) in [9] becomes:

$$\int_{N_0/N}^{1} \frac{\mathrm{d}u}{u}\frac{1}{2}\Big(\tilde{B}(\alpha_s(uQ^2)) - B(\alpha_s(uQ^2))\Big) - \log\tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) = \\ \Gamma_2\left(\frac{\partial}{\partial\log N}\right)\left\{A(\alpha_s(\frac{N_0}{N}Q^2)) - \frac{1}{2}\frac{\partial}{\partial\log N}B(\alpha_s(\frac{N_0}{N}Q^2)) - 2A(\alpha_s(\frac{N_0^2}{N^2}Q^2))\right\} \tag{1}$$

Observe that using the renormalization group equation eq. (1) and chain rule we can write the following relation:

$$\begin{aligned} \frac{\partial}{\partial\log N}B(\alpha_s(\frac{k}{N})) &= \frac{\partial B(\alpha_s)}{\partial\alpha_s}\frac{\partial\alpha_s(\frac{k}{N})}{\partial\frac{k}{N}}\frac{\partial\frac{k}{N}}{\partial\log N} = -\frac{\partial\alpha_s(\frac{k}{N})}{\partial\log\frac{k}{N}}\frac{\partial B(\alpha_s)}{\partial\alpha_s} \\ &= -\beta(\alpha_s)\alpha_s\frac{\partial B(\alpha_s)}{\partial\alpha_s} \\ \frac{\partial}{\partial\log N}A(\alpha_s(\frac{k}{N^2})) &= -2\beta(\alpha_s)\alpha_s\frac{\partial A(\alpha_s)}{\partial\alpha_s} \end{aligned} \tag{2}$$

define the differential operator $\partial(\alpha_s)$ as:

$$\partial_{\alpha_s} \equiv -\beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s} \tag{3}$$

Substituting the above relations in the previous equation, we obtain the equivalent of equation (92) in [9]:

$$\begin{aligned}&\int_{N_0/N}^{1}\frac{\mathrm{d}u}{u}\frac{1}{2}\Big(\tilde{B}(\alpha_s(uQ^2))-B(\alpha_s(uQ^2))\Big)-\log\tilde{C}\left(\alpha_s(\mu^2),\frac{\mu^2}{Q^2}\right)=\\&\Gamma_2\big(\partial_{\alpha_s}\big)\Big\{A(\alpha_s(\frac{N_0}{N}Q^2))-\frac{1}{2}\partial_{\alpha_s}B(\alpha_s(\frac{N_0}{N}Q^2))\Big\}-2\Gamma\big(2\partial_{\alpha_s}\big)A(\alpha_s(\frac{N_0^2}{N^2}Q^2))\end{aligned}\tag{4}$$

Now by setting $N=N_0$ or applying $\frac{\partial}{\partial\log N}$ one obtains respectively the functions $\tilde{C}$ and $\tilde{B}$ as functions of A and B:

$$\tilde{C}(\alpha_s)=\exp\Big\{-\Gamma_2(\partial_{\alpha_s})\Big[A(\alpha_s)-\frac{1}{2}\partial_{\alpha_s}B(\alpha_s)\Big]-2\Gamma\big(2\partial_{\alpha_s}\big)A(\alpha_s)\Big\}\Big|_{\alpha_s=\alpha_s\left(Q^2\right)}\tag{5}$$

$$\begin{aligned}\frac{\tilde{B}(\alpha_s)}{2}=\frac{B(\alpha_s)}{2}&+\partial_{\alpha_s}\Big\{\Gamma_2(\partial_{\alpha_s})\Big[A(\alpha_s)-\frac{1}{2}\partial_{\alpha_s}B(\alpha_s)\Big]\Big\}\Big|_{\alpha_s=\alpha_s\left(\frac{N_0}{N}Q^2\right)}\\&-4\partial_{\alpha_s}\big\{\Gamma_s\big(2\partial_{\alpha_s}\big)A(\alpha_s)\big\}\Big|_{\alpha_s=\alpha_s\left(\frac{N_0^2}{N^2}Q^2\right)}\end{aligned}\tag{6}$$

by inserting the expansion

$$\begin{aligned}\Gamma_2(\epsilon)=-\frac{1}{2}\zeta(2)-\frac{1}{3}\zeta(3)\epsilon-\frac{9}{16}\zeta(4)\epsilon^2-\left(\frac{1}{6}\zeta(2)\zeta(3)+\frac{1}{5}\zeta(5)\right)\epsilon^3\\-\left(\frac{1}{18}\zeta(3)^2-\frac{61}{128}\zeta(6)\right)\epsilon^4+\mathcal{O}\big(\epsilon^5\big)\end{aligned}\tag{7}$$

in eq. (6) and eq. (5), we can obtain the coefficients $\tilde{B}$ and $\tilde{C}$ in terms of the coefficients A and B up to N^4LL accuracy:

$$\tilde{B}\big(\alpha_s\big(uQ^2\big)\big)=B\big(\alpha_s\big(uQ^2\big)\big)+\dots\tag{8}$$

$$
\begin{aligned}
\log \tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) = {} & \frac{A_1}{\pi}(-\zeta(2)-1)\alpha_s + \left(\frac{-2A_2\zeta(2) - 2A_2 + \pi b_0 B_1}{2\pi^2}\right. \\
& \left. + \frac{A_1 b_0 \left(3\zeta(2)\log\left(\frac{\mu^2}{Q^2}\right) + 3\log\left(\frac{\mu^2}{Q^2}\right) - 4\zeta(3)\right)}{3\pi}\right)\alpha_s^2 \\
& + \left(\frac{A_1}{3\pi}\left(-27b_0^2\zeta(4) - 3b_0^2\zeta(2)\log^2\left(\frac{\mu^2}{Q^2}\right) - 3b_0^2\log^2\left(\frac{\mu^2}{Q^2}\right)\right.\right. \\
& + 8b_0^2\zeta(3)\log\left(\frac{\mu^2}{Q^2}\right) + 3b_1\zeta(2)\log\left(\frac{\mu^2}{Q^2}\right) + 3b_1\log\left(\frac{\mu^2}{Q^2}\right) \\
& \left. - 4b_1\zeta(3)\right) - \frac{1}{6\pi^3}\left(-12\pi A_2 b_0\zeta(2)\log\left(\frac{\mu^2}{Q^2}\right)\right. \\
& - 12\pi A_2 b_0\log\left(\frac{\mu^2}{Q^2}\right) + 16\pi A_2 b_0\zeta(3) + 6A_3\zeta(2) + 6A_3 \\
& \left.\left. + 6\pi^2 b_0^2 B_1\log\left(\frac{\mu^2}{Q^2}\right) - 6\pi b_0 B_2 - 3\pi^2 b_1 B_1\right)\right)\alpha_s^3 + \mathcal{O}\big(\alpha_s^4\big)
\end{aligned}
\tag{9}
$$

We note that $\tilde{B}$ corrects the B terms so it has to be expanded up to α_s^4 to achieve N^4LL accuracy while $\ln\tilde{C}$ corrects the f_i functions so they have to be expanded up to α_s^3 to achieve N^4LL accuracy, these corrections are necessary only for NNLL accuracy and beyond, consistent with the results in [5].

Appendix C

Constants and Ingredients for Resummation

The needed coefficients are often expressed in terms of the following parameters:

$$\begin{aligned} n_f &= 5 && \text{Number of active flavors} \\ C_F &= \frac{4}{3} && \text{Quadratic Casimir operator for fundamental representation} \\ C_A &= 3 && \text{Quadratic Casimir operator for adjoint representation} \\ T_R &= \frac{1}{2} && \text{Trace normalization for fundamental representation} \end{aligned} \tag{1}$$

The renormalization group equation for the QCD coupling constant reads:

$$\mu^2 \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu^2} = \beta(\alpha_s) = -\alpha_s^2 \big(b_0 + b_1 \alpha_s + b_2 \alpha_s^2 + \dots\big) \tag{2}$$

where the coefficients of the $\beta(\alpha_s)$ functions are [14]:

$$\begin{aligned} b_0 &= \frac{11C_A - 4n_f T_R}{12\pi} = \frac{33 - 2n_f}{12\pi} \\ b_1 &= \frac{17C_A^2 - n_f T_R(10C_A + 6C_F)}{24\pi^2} = \frac{153 - 19n_f}{24\pi^2} \\ b_2 &= \frac{325n_f^2}{3456\pi^3} - \frac{5033n_f}{1152\pi^3} + \frac{2857}{128\pi^3} \\ b_3 &= \frac{1093n_f^3}{186624\pi^4} + n_f^2 \left(\frac{809\zeta(3)}{2592\pi^4} + \frac{50065}{41472\pi^4}\right) + n_f \left(-\frac{1627\zeta(3)}{1728\pi^4} - \frac{1078361}{41472\pi^4}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{891\zeta(3)}{64\pi^4}+\frac{149753}{1536\pi^4}\\
b_4 =\ & n_f^4\left(\frac{1205}{2985984\pi^5}-\frac{19\zeta(3)}{10368\pi^5}\right) \qquad (3)\\
&+n_f^3\left(-\frac{24361\zeta(3)}{124416\pi^5}+\frac{115\zeta(5)}{2304\pi^5}+\frac{809}{1244160\pi}-\frac{630559}{5971968\pi^5}\right)\\
&+n_f^2\left(\frac{698531\zeta(3)}{82944\pi^5}-\frac{5965\zeta(5)}{1296\pi^5}-\frac{5263}{414720\pi}+\frac{25960913}{1990656\pi^5}\right)\\
&+n_f\left(-\frac{1202791\zeta(3)}{20736\pi^5}+\frac{1358995\zeta(5)}{27648\pi^5}+\frac{6787}{110592\pi}-\frac{336460813}{1990656\pi^5}\right)\\
&+\frac{621885\zeta(3)}{2048\pi^5}-\frac{144045\zeta(5)}{512\pi^5}-\frac{9801}{20480\pi}+\frac{8157455}{16384\pi^5}
\end{aligned}
$$

Numerically, for $n_f = 5$:

$$
\begin{aligned}
b_0 &= 0.875352-0.0530516n_f=\frac{23}{12\pi}=0.610094\\
b_1 &= 0.645923-0.0802126n_f=\frac{29}{12\pi^2}=0.24486\\
b_2 &= 0.719864-0.140904n_f+0.00303291n_f^2=\frac{9769}{3456\pi^3}=0.0911647\\
b_3 &= 1.17269-0.278557n_f+0.0162447n_f^2+0.0000601247n_f^3=0.193536\\
b_4 &= 1.71413-0.594075n_f+0.0560618n_f^2-0.000738048n_f^3-5.87966\cdot 10^{-6}n_f^4=0.0493694
\end{aligned}
\qquad (4)
$$

The A coefficient in the resummation formula are given by:

$$
\begin{aligned}
A_1 &= C_F \qquad (5)\\
A_2 &= \frac{1}{2}C_AC_F\left(-\frac{1}{27}10n_fT_R-\frac{\pi^2}{6}+\frac{67}{18}\right) \qquad (6)\\
A_3 &= \left(-\frac{2051}{1296}-\frac{\pi^2}{18}\right)C_AC_Fn_f+\left(-\frac{11\zeta(3)}{4}+\frac{11\pi^4}{720}-\frac{13\pi^2}{432}+\frac{15503}{2592}\right)C_A^2C_F \qquad (7)\\
&+\left(\frac{\zeta(3)}{2}-\frac{55}{96}\right)C_F^2n_f+\left(\frac{25}{324}+\frac{\pi^2}{108}\right)C_Fn_f^2\\
A_4 &= \Big(-\frac{\zeta(3)^2}{16}+\frac{11\pi^2\zeta(3)}{32}-\frac{24461\zeta(3)}{864}+\frac{20513\pi^2}{5184}+\frac{1925\zeta(5)}{288}-\frac{253\pi^4}{1920}-\frac{313\pi^6}{90720} \qquad (8)\\
&+\frac{4311229}{186624}\Big)C_FC_A^3+\Big(\frac{7\pi^2\zeta(3)}{72}+\frac{43\zeta(3)}{144}+\frac{451\pi^4}{4320}+\frac{131\zeta(5)}{144}\\
&+T_R\left(-\frac{1}{72}11\pi^2\zeta(3)+\frac{685\zeta(3)}{48}-3\zeta(5)-\frac{11\pi^4}{144}-\frac{1123}{162}-\frac{12247\pi^2}{15552}\right)\\
&-\frac{64421\pi^2}{31104}-\frac{260731}{62208}\Big)C_Fn_fC_A^2+\left(\left(\left(\frac{34\zeta(3)}{9}+\frac{11\pi^4}{720}+\frac{55\pi^2}{576}-\frac{7351}{1152}\right)T_R-\frac{\pi^2\zeta(3)}{12}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{29\zeta(3)}{18}+\frac{5\zeta(5)}{8}-\frac{55\pi^2}{1152}-\frac{11\pi^4}{1440}-\frac{17033}{10368}\Big)n_f C_F^2\\
&+\Bigg(\bigg(\Big(-\frac{7\zeta(3)}{54}+\frac{\pi^4}{60}-\frac{481\pi^2}{1944}+\frac{1747}{3888}\Big)T_R^2+\Big(-\frac{143\zeta(3)}{108}+\frac{847\pi^2}{1296}+\frac{55}{486}\Big)T_R+\frac{803\pi^2}{3888}\\
&-\frac{\zeta(3)}{144}-\frac{49\pi^4}{4320}+\frac{19889}{62208}\bigg)n_f^2 C_F\Bigg)C_A+\frac{31\pi^6}{60480}+\bigg(\bigg(\Big(-\frac{7\zeta(3)}{27}+\frac{5\pi^2}{324}+\frac{130}{729}\Big)T_R^3\\
&+\Big(\frac{13\zeta(3)}{108}-\frac{5}{486}-\frac{77\pi^2}{1296}\Big)T_R+\frac{\zeta(3)}{108}-\frac{1}{648}\bigg)C_F n_f^3+\frac{\pi^2}{96}+\bigg(\Big(\Big(-\frac{19\zeta(3)}{18}+\frac{215}{96}-\frac{5\pi^2}{144}\\
&-\frac{\pi^4}{180}\Big)T_R^2+\frac{\pi^4}{720}+\frac{5\pi^2}{192}-\frac{5\zeta(3)}{18}+\frac{299}{2592}\Big)C_F^2 n_f^2+\frac{3\zeta(3)^2}{16}\\
&+81\left(-\frac{\zeta(3)^2}{32}+\frac{\zeta(3)}{288}+\frac{55\zeta(5)}{576}-\frac{\pi^2}{576}-\frac{31\pi^6}{362880}\right)\\
&+9\left(-\frac{5\zeta(3)^2}{32}+\frac{5\zeta(3)}{288}+\frac{275\zeta(5)}{576}-\frac{5\pi^2}{576}-\frac{31\pi^6}{72576}\right)+\left(\left(\frac{37\zeta(3)}{48}-\frac{5\zeta(5)}{4}+\frac{143}{576}\right)C_F^3\right.\\
&+\frac{1}{3}\left(-\frac{\zeta(3)}{12}+\frac{\pi^2}{24}-\frac{5\zeta(5)}{12}\right)+\frac{1}{27}\left(\frac{\zeta(3)}{16}+\frac{5\zeta(5)}{16}-\frac{\pi^2}{32}\right)+3\left(\frac{7\zeta(3)}{288}+\frac{35\zeta(5)}{288}-\frac{7\pi^2}{576}\right)\\
&\left.+27\left(-\frac{\zeta(3)}{288}+\frac{\pi^2}{576}-\frac{5\zeta(5)}{288}\right)\right)n_f-\frac{\zeta(3)}{48}-\frac{55\zeta(5)}{96}
\end{aligned}
$$

$$
A_5 = 14541.099 \tag{9}
$$

The B coefficients are given by:

$$
B_1 = -\frac{1}{2}\left(3C_F\right) \tag{10}
$$

$$
B_2 = \left(2\left(\frac{3\zeta(3)}{4}-\frac{57}{32}\right)+\frac{11\pi^2}{24}\right)C_A C_F+\left(\frac{5}{8}-\frac{\pi^2}{12}\right)C_F n_f+\left(2\left(-\frac{3\zeta(3)}{2}-\frac{3}{32}\right)+\frac{\pi^2}{4}\right)C_F^2 \tag{11}
$$

$$
\begin{aligned}
B_3 = C_A\Bigg(&C_F n_f\bigg(\Big(\Big(\frac{34\zeta(3)}{27}-\frac{485\pi^2}{432}-\frac{\pi^4}{720}+\frac{3683}{864}\Big)T_R+\frac{31\zeta(3)}{54}+\frac{131\pi^4}{4320}+\frac{5261}{1728}-\frac{2657\pi^2}{2592}\Big)\\
&+\left(-\frac{\pi^2\zeta(3)}{12}-\frac{89\zeta(3)}{12}-\frac{15\zeta(5)}{4}+\frac{287\pi^2}{192}-\frac{23}{16}-\frac{17\pi^4}{360}\right)C_F^2\bigg)\\
&+\left(\frac{241\zeta(3)}{108}-\frac{5\zeta(5)}{4}+\frac{22841\pi^2}{5184}-\frac{5951}{432}-\frac{713\pi^4}{4320}\right)C_A^2 C_F\\
&+C_F^2 n_f\left(\left(\frac{17\zeta(3)}{6}+\frac{23}{16}-\frac{5\pi^2}{72}-\frac{29\pi^4}{1080}\right)T_R-\frac{\zeta(3)}{4}+\frac{41\pi^4}{2160}+\frac{31}{64}-\frac{71\pi^2}{288}\right)\\
&+C_F n_f^2\left(\left(\frac{2\zeta(3)}{9}+\frac{17}{72}-\frac{5\pi^2}{81}\right)T_R^2+\left(-\frac{13\zeta(3)}{27}+\frac{193\pi^2}{648}-\frac{433}{432}\right)T_R\right)\\
&+\left(\frac{\pi^2\zeta(3)}{6}-\frac{17\zeta(3)}{8}+\frac{15\zeta(5)}{2}-\frac{\pi^4}{20}-\frac{3\pi^2}{32}-\frac{29}{64}\right)C_F^3
\end{aligned}
\tag{12}
$$

$$B_4 = 3817.42 \tag{13}$$

A_1, A_2 and B_1 were already known in [5] and were obtained from the one and two loop splitting functions. A_3 and B_2 were obtained by comparing eq. (39) with the equivalent expression in Soft Collinear Effective Theory (SCET) [17] (Eq(4.17)) by absorbing the jet and soft terms into the A and B term. A_4, A_5 and B_3, B_4 were obtained in a similar fashion but the SCET expressions were gently provided by my co-supervisor Wan-Li.

By expanding in α_s the exponent of eq. (6) with eqs. (68) to (72) substituted, we can compare to the expression eq. (4) and obtain:

$$
\begin{aligned}
G_{12} &= -2A_1 \\
G_{11} &= -2B_1 \\
G_{23} &= -4\pi A_1 b_0 \\
G_{22} &= \frac{2}{3}\left(-2\pi^2 A_1^2 - 6A_2 - 3\pi b_0 B_1\right) \\
G_{21} &= \frac{2}{3}\left(-A_1\left(24A_1\zeta(3) - 3\pi^3 b_0 + 2\pi^2 B_1\right) - 6B_2\right) \\
G_{34} &= \frac{1}{3}(-28)\pi^2 A_1 b_0^2 \\
G_{33} &= \frac{4}{45}\left(-90\pi^3 A_1^2 b_0 - 90\pi^2 A_1 b_1 - 180\pi A_2 b_0 + 240 A_1^3\zeta(3) - 30\pi^2 b_0^2 B_1\right) \\
G_{32} &= \frac{4}{45}\Big(-75\pi^3 A_1 b_0 B_1 - 1620\pi A_1^2 b_0\zeta(3) + 105\pi^4 A_1 b_0^2 + 360 A_1^2 B_1\zeta(3) + 2\pi^4 A_1^3 - 60\pi^2 A_2 A_1 \\
&\quad - 90A_3 - 45\pi^2 b_1 B_1 - 90\pi b_0 B_2\Big) \\
G_{31} &= \frac{4}{45}\Big(-900\pi A_1 b_0 B_1\zeta(3) + 840\pi^2 A_1 b_0^2\zeta(3) + 45\pi^4 A_1 b_1 + 90\pi^3 A_2 b_0 + 180 A_1 B_1^2\zeta(3) \\
&\quad + 2\pi^4 A_1^2 B_1 - 30\pi^2 A_1 B_2 - 30\pi^2 A_2 B_1 - 240\pi^2 A_1^3\zeta(3) + 2160 A_1^3\zeta(5) - 720 A_2 A_1\zeta(3) \\
&\quad - 15\pi^3 b_0 B_1^2 + 15\pi^4 b_0^2 B_1 - 90B_3\Big) \\
G_{45} &= -24\pi^3 A_1 b_0^3 \\
G_{44} &= 192\pi A_1^3 b_0\zeta(3) - \frac{332}{9}\pi^4 A_1^2 b_0^2 - \frac{140}{3}\pi^3 A_1 b_0 b_1 - 56\pi^2 A_2 b_0^2 + \frac{1}{45}(-32)\pi^4 A_1^4 - 4\pi^3 b_0^3 B_1 \\
G_{43} &= 256\pi A_1^2 b_0 B_1\zeta(3) - \frac{232}{9}\pi^4 A_1 b_0^2 B_1 - \frac{2656}{3}\pi^2 A_1^2 b_0^2\zeta(3) + \frac{32}{15}\pi^5 A_1^3 b_0 \\
&\quad - 16\pi^4 A_1^2 b_1 + 40\pi^5 A_1 b_0^3 - 48\pi^3 A_2 A_1 b_0 - 16\pi^3 A_1 b_2 - 48\pi A_3 b_0 - 32\pi^2 A_2 b_1 - \frac{64}{45}\pi^4 A_1^3 B_1 \\
&\quad + \frac{128}{3}\pi^2 A_1^4\zeta(3) - 512 A_1^4\zeta(5) + 128 A_2 A_1^2\zeta(3) - \frac{40}{3}\pi^3 b_0 b_1 B_1 - 16\pi^2 b_0^2 B_2
\end{aligned}
$$

$$
\begin{aligned}
G_{42} = {}& 112\pi A_1 b_0 B_1^2 \zeta(3) - 464\pi^2 A_1 b_0^2 B_1 \zeta(3) + \frac{32}{15}\pi^5 A_1^2 b_0 B_1 - \frac{40}{3}\pi^4 A_1 b_1 B_1 \\
& - \frac{56}{3}\pi^3 A_1 b_0 B_2 - \frac{64}{3}\pi^3 A_2 b_0 B_1 - 416\pi^3 A_1^3 b_0 \zeta(3) + 3648\pi A_1^3 b_0 \zeta(5) - 288\pi^2 A_1^2 b_1 \zeta(3) \\
& + 480\pi^3 A_1 b_0^3 \zeta(3) - 864\pi A_2 A_1 b_0 \zeta(3) + \frac{56}{45}\pi^6 A_1^2 b_0^2 + \frac{140}{3}\pi^5 A_1 b_0 b_1 + 56\pi^4 A_2 b_0^2 \\
& + 64\pi^2 A_1^3 B_1 \zeta(3) - 768 A_1^3 B_1 \zeta(5) + 64 A_1^2 B_2 \zeta(3) + 128 A_2 A_1 B_1 \zeta(3) - \frac{16}{15}\pi^4 A_1^2 B_1^2 \\
& + 512 A_1^4 \zeta(3)^2 + \frac{1}{945}(-464)\pi^6 A_1^4 + \frac{16}{15}\pi^4 A_2 A_1^2 - \frac{32}{3}\pi^2 A_3 A_1 - \frac{16}{3}\pi^2 A_2^2 \\
& - 16 A_4 - 4\pi^4 b_0^2 B_1^2 + 4\pi^5 b_0^3 B_1 - 8\pi^3 b_2 B_1 - 16\pi^2 b_1 B_2 - 24\pi b_0 B_3
\end{aligned}
$$

$$
\begin{aligned}
G_{41} = {}& -120\pi A_1^3 b_0 \gamma^6 + \frac{64}{3}\pi^2 A_1^4 \gamma^5 + \frac{368}{3}\pi^2 A_1^2 b_0^2 \gamma^5 - 16 A_1^2 A_2 \gamma^5 - 56\pi A_1^2 b_0 B_1 \gamma^5 + \frac{460}{3}\pi^3 A_1^3 b_0 \gamma^4 \\
& + \frac{40}{3}\pi^2 A_1^3 B_1 \gamma^4 - 480 A_1^4 \zeta(3)\gamma^4 - \frac{32}{9}\pi^4 A_1^4 \gamma^3 - \frac{1840}{9}\pi^4 A_1^2 b_0^2 \gamma^3 + \frac{80}{3}\pi^2 A_1^2 A_2 \gamma^3 \\
& + \frac{280}{3}\pi^3 A_1^2 b_0 B_1 \gamma^3 - 1280\pi A_1^3 b_0 \zeta(3)\gamma^3 - 320 A_1^3 B_1 \zeta(3)\gamma^3 - \frac{2}{3}\pi^5 A_1^3 b_0 \gamma^2 - \frac{8}{3}\pi^4 A_1^3 B_1 \gamma^2 \\
& + \frac{800}{3}\pi^2 A_1^4 \zeta(3)\gamma^2 + \frac{7360}{3}\pi^2 A_1^2 b_0^2 \zeta(3)\gamma^2 - 320 A_1^2 A_2 \zeta(3)\gamma^2 - 1120\pi A_1^2 b_0 B_1 \zeta(3)\gamma^2 \\
& - 1920 A_1^4 \zeta(5)\gamma^2 + \frac{92}{9}\pi^6 A_1^2 b_0^2 \gamma - \frac{4}{3}\pi^4 A_1^2 A_2 \gamma - \frac{14}{3}\pi^5 A_1^2 b_0 B_1 \gamma - \frac{1600}{3}\pi^3 A_1^3 b_0 \zeta(3)\gamma \\
& + \frac{800}{3}\pi^2 A_1^3 B_1 \zeta(3)\gamma + 3840\pi A_1^3 b_0 \zeta(5)\gamma - 1920 A_1^3 B_1 \zeta(5)\gamma - 2\pi^7 A_1 b_0^3 - \frac{16}{45}\pi^4 A_1 B_1^3 \\
& + \frac{28}{45}\pi^5 A_1 b_0 B_1^2 - \frac{8}{3}\pi^4 b_1 B_1^2 + 128\pi A_1^3 b_0 \zeta(3)^2 - 128 A_1^3 B_1 \zeta(3)^2 - \frac{1}{15}\pi^7 A_1^3 b_0 \\
& + 24\pi^3 A_3 b_0 + 16\pi^4 A_2 b_1 + 8\pi^5 A_1 b_2 - \frac{2}{135}\pi^6 A_1^3 B_1 + \frac{56}{45}\pi^6 A_1 b_0^2 B_1 + \frac{32}{45}\pi^4 A_1 A_2 B_1 \\
& - \frac{16}{3}\pi^2 A_3 B_1 + \frac{20}{3}\pi^5 b_0 b_1 B_1 + \frac{16}{45}\pi^4 A_1^2 B_2 + 8\pi^4 b_0^2 B_2 - \frac{16}{3}\pi^2 A_2 B_2 - 8\pi^3 b_0 B_1 B_2 \\
& - \frac{16}{3}\pi^2 A_1 B_3 - 16 B_4 + \frac{344}{45}\pi^4 A_1^4 \zeta(3) + 16\pi b_0 B_1^3 \zeta(3) - 64 A_2^2 \zeta(3) + \frac{752}{9}\pi^4 A_1^2 b_0^2 \zeta(3) \\
& + 448\pi^2 A_2 b_0^2 \zeta(3) + 32\pi^2 A_1^2 B_1^2 \zeta(3) - 48\pi^2 b_0^2 B_1^2 \zeta(3) + 32 A_2 B_1^2 \zeta(3) - \frac{224}{3}\pi^2 A_1^2 A_2 \zeta(3) \\
& - 128 A_1 A_3 \zeta(3) + \frac{1120}{3}\pi^3 A_1 b_0 b_1 \zeta(3) + 32\pi^3 b_0^3 B_1 \zeta(3) - \frac{304}{3}\pi^3 A_1^2 b_0 B_1 \zeta(3) \\
& - 256\pi A_2 b_0 B_1 \zeta(3) - 160\pi^2 A_1 b_1 B_1 \zeta(3) - 224\pi A_1 b_0 B_2 \zeta(3) + 64 A_1 B_1 B_2 \zeta(3) \\
& - 64\pi^2 A_1^4 \zeta(5) - 896\pi^2 A_1^2 b_0^2 \zeta(5) - 384 A_1^2 B_1^2 \zeta(5) + 768 A_1^2 A_2 \zeta(5) + 1152\pi A_1^2 b_0 B_1 \zeta(5)
\end{aligned}
\tag{14}
$$

Bibliography

[1] Ugo Aglietti and Giulia Ricciardi. "Approximate NNLO threshold resummation in heavy flavor decays". In: *Phys. Rev. D* 66 (2002), p. 074003. DOI: 10.1103/PhysRevD.66.074003. arXiv: hep-ph/0204125.

[2] F. Bloch and A. Nordsieck. "Note on the Radiation Field of the Electron". In: *Phys. Rev.* 52 (2 July 1937), pp. 54–59. DOI: 10.1103/PhysRev.52.54.

[3] S. Catani and L. Trentadue. "Resummation of the QCD perturbative series for hard processes". In: *Nuclear Physics B* 327.2 (1989), pp. 323–352. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(89)90273-3. URL: https://www.sciencedirect.com/science/article/pii/0550321389902733.

[4] S. Catani, B.R. Webber, and G. Marchesini. "QCD coherent branching and semi-inclusive processes at large χ". In: *Nuclear Physics B* 349.3 (1991), pp. 635–654. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(91)90390-J. URL: https://www.sciencedirect.com/science/article/pii/055032139190390J.

[5] S. Catani et al. "Resummation of large logarithms in e+e− event shape distributions". In: *Nuclear Physics B* 407.1 (1993), pp. 3–42. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(93)90271-P. URL: https://www.sciencedirect.com/science/article/pii/055032139390271P.

[6] S. Catani et al. "Thrust distribution in e+e− annihilation". In: *Physics Letters B* 263.3 (1991), pp. 491–497. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(91)90494-B. URL: https://www.sciencedirect.com/science/article/pii/037026939190494B.

[7] Stefano Catani. "Higher-order QCD corrections in hadron collisions: Soft-gluon resummation and exponentiation". In: *Nuclear Physics B - Proceedings Supplements* 54.1–2 (Mar.

1997), pp. 107–113. ISSN: 0920-5632. DOI: 10.1016/s0920-5632(97)00024-8. URL: http://dx.doi.org/10.1016/S0920-5632(97)00024-8.

[8] Stefano Catani. *Soft-Gluon Resummation: a Short Review*. 1997. arXiv: hep-ph/9709503 [hep-ph].

[9] Stefano Catani et al. "Soft-gluon resummation for Higgs boson production at hadron colliders". In: *Journal of High Energy Physics* 2003.07 (July 2003), pp. 028–028. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2003/07/028. URL: http://dx.doi.org/10.1088/1126-6708/2003/07/028.

[10] G Dissertori et al. "First determination of the strong coupling constant using NNLO predictions for hadronic event shapes in e+e−annihilations". In: *Journal of High Energy Physics* 2008.02 (Feb. 2008), pp. 040–040. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/02/040. URL: http://dx.doi.org/10.1088/1126-6708/2008/02/040.

[11] R. Keith Ellis, D. A. Ross, and A. E. Terrano. "The Perturbative Calculation of Jet Structure in e+ e- Annihilation". In: *Nucl. Phys. B* 178 (1981), pp. 421–456. DOI: 10.1016/0550-3213(81)90165-6.

[12] A. Gehrmann-De Ridder et al. "Second-order QCD corrections to the thrust distribution". In: *Phys. Rev. Lett.* 99 (2007), p. 132002. DOI: 10.1103/PhysRevLett.99.132002. arXiv: 0707.1285 [hep-ph].

[13] Particle Data Group et al. "Review of Particle Physics". In: *Progress of Theoretical and Experimental Physics* 2022.8 (Aug. 2022), p. 083C01. ISSN: 2050-3911. DOI: 10.1093/ptep/ptac097. eprint: https://academic.oup.com/ptep/article-pdf/2022/8/083C01/49175539/ptac097.pdf. URL: https://doi.org/10.1093/ptep/ptac097.

[14] F. Herzog et al. "The five-loop beta function of Yang-Mills theory with fermions". In: *Journal of High Energy Physics* 2017.2 (Feb. 2017). ISSN: 1029-8479. DOI: 10.1007/jhep02(2017)090. URL: http://dx.doi.org/10.1007/JHEP02(2017)090.

[15] T. Kinoshita. "Mass singularities of Feynman amplitudes". In: *J. Math. Phys.* 3 (1962), pp. 650–677. DOI: 10.1063/1.1724268.

[16] T. D. Lee and M. Nauenberg. "Degenerate Systems and Mass Singularities". In: *Phys. Rev.* 133 (1964). Ed. by G. Feinberg, B1549–B1562. DOI: 10.1103/PhysRev.133.B1549.

[17] Pier Francesco Monni, Thomas Gehrmann, and Gionata Luisoni. “Two-Loop Soft Corrections and Resummation of the Thrust Distribution in the Dijet Region”. In: *JHEP* 08 (2011), p. 010. DOI: 10.1007/JHEP08(2011)010. arXiv: 1105.4560 [hep-ph].

[18] George F. Sterman. “Summation of Large Corrections to Short Distance Hadronic Cross-Sections”. In: *Nucl. Phys. B* 281 (1987), pp. 310–364. DOI: 10.1016/0550-3213(87)90258-6.