

1 Laplace transform in the large N limit

Using the methodology outlined in [4], we will demonstrate that the Mellin transform prescription is also applicable to the Laplace transform. in the large moment $\nu Q^2 = N$ limit, this fact was already known in the literature [2] and we'll show it here for completeness.

We are interested in solving the following integral

$$\int_0^1 dz \frac{e^{-N(1-z)} - 1}{1-z} F(\alpha_s, \ln(1-z)) = \int_0^1 \frac{du}{u} (e^{-uN} - 1) F(\alpha_s, \ln u) \quad (1)$$

Start by considering

$$I_n(N) = \int_0^1 \frac{du}{u} (e^{-uN} - 1) \ln^n(u) \quad (2)$$

the above integral can be evaluated as described in [1]. Using the following identity

$$\ln^n(u) = \lim_{\epsilon \rightarrow 0} \left(\frac{\partial}{\partial \epsilon} \right)^n u^\epsilon = \lim_{\epsilon \rightarrow 0} \left(\frac{\partial}{\partial \epsilon} \right)^n e^{\epsilon \ln u} \quad (3)$$

to replace the logarithm term in the integrand eq. (2) and straightforwardly integrate the resulting expression. We obtain

$$\begin{aligned} I_n(N) &= \lim_{\epsilon \rightarrow 0} \left(\frac{\partial}{\partial \epsilon} \right)^n \int_0^1 du (e^{-uN} - 1) u^{\epsilon-1} \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{\partial}{\partial \epsilon} \right)^n \left\{ -\frac{1}{\epsilon} + N^{-\epsilon} (\Gamma(\epsilon, 0) - \Gamma(\epsilon, N)) \right\} \\ &= \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left(\frac{\partial}{\partial \epsilon} \right)^n \left\{ -\frac{1}{\epsilon} + N^{-\epsilon} \Gamma(\epsilon) \right\} + e^{-N + \mathcal{O}((\frac{1}{N})^2)} \mathcal{O}\left(\frac{1}{N}\right) \\ &= \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left(\frac{\partial}{\partial \epsilon} \right)^n \left\{ \frac{1}{\epsilon} (N^{-\epsilon} \Gamma(\epsilon) - 1) \right\} + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \\ &= \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left(\frac{\partial}{\partial \epsilon} \right)^n \left\{ \frac{1}{\epsilon} (e^{-\epsilon \ln N} \Gamma(1 + \epsilon) - 1) \right\} + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \end{aligned} \quad (4)$$

where $\Gamma(\epsilon, 0) = \Gamma(\epsilon)$, $\Gamma(\epsilon, N)$ is the incomplete Gamma function and $\epsilon\Gamma(\epsilon) = \Gamma(1 + \epsilon)$

$$\Gamma(\epsilon, N) = \int_N^\infty dt t^{\epsilon-1} e^{-t} \quad (5)$$

The last equation in eq. (4) is the same as Eq. (68) obtained in [4] for the Mellin transform. Therefore, we can conclude that the Mellin transform prescription is also applicable to the Laplace transform in the large N limit.

Using the known expansion of the Gamma function for small ϵ

$$\Gamma(1 + \epsilon) = \exp \left\{ -\gamma_E \epsilon + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) \epsilon^n}{n} \right\} \quad (6)$$

the term in curly brackets in eq. (6) can be expanded in power of ϵ and then derive. The result for $I_n(N)$ is thus a polynomial of degree $n + 1$ in the large logarithm $\ln N$:

$$\begin{aligned} I_n(N) = & \frac{(-1)^n + 1}{n + 1} (\ln N + \gamma_E)^{n+1} + \frac{(-1)^{n-1}}{2} n \zeta(2) (\ln N + \gamma_E)^{n-1} \\ & + \sum_{k=0}^{n-2} a_{nk} (\ln N + \gamma_E)^k + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \end{aligned} \quad (7)$$

This result can be generalized using the following formal identity:

$$e^{-\epsilon \ln N} \Gamma(1 + \epsilon) = \Gamma\left(1 - \frac{\partial}{\partial \ln N}\right) e^{\epsilon \ln N} \quad (8)$$

then we can perform the n -th derivative with respect to ϵ , and obtain

$$I_n(N) = \Gamma\left(1 - \frac{\partial}{\partial \ln N}\right) \frac{(-\ln N)^n + 1}{n + 1} + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \quad (9)$$

This expression can be regarded as a replacement for eq. (4) to compute the polynomial coefficients a_{nk} in eq. (7). Moreover, by observing that

$$\frac{(-\ln N)^n + 1}{n+1} = - \int_{\frac{1}{N}}^1 du \frac{\ln^n(u)}{u} \quad (10)$$

we obtain the all order generalization for of the prescription used in [3]:

$$e^{-uN} - 1 = -\Gamma\left(1 - \frac{\partial}{\partial \ln N}\right) \Theta\left(u - \frac{1}{N}\right) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \quad (11)$$

$$= -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln N}\right) \Theta\left(u - \frac{N_0}{N}\right) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \quad (12)$$

where

$$\tilde{\Gamma}(1 - \epsilon) \equiv e^{\gamma_E \epsilon} \Gamma(1 + \epsilon) = \exp\left\{\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) \epsilon^n}{n}\right\} \quad (13)$$

It is straightforward to show that the prescription can be applied to as follows:

$$\int_0^1 \frac{du}{u} (e^{-uN} - 1) F(\alpha_s, \ln u) = -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln N}\right) \int_{\frac{N_0}{N}}^1 \frac{du}{u} F(\alpha_s, \ln u) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \quad (14)$$

and to evaluated the $\ln N$ -contribution arising from the integration of anu soft-gluon function F that has a generic perturbative expansion of the type

$$F(\alpha_s, \ln u) = \sum_{k=1}^{\infty} \alpha_s^k \sum_{n=0}^{2k-1} F_{kn} \ln^n u \quad (15)$$

The result eq. (14) can be used to obtain ?? as shown in [4].

2 Equivalence between resummation formulae

Here i adapt the result in [4] to show the equivalence between the resummation formulae in ?? and ?? in the case of Thrust resummation.

It is straightforward to show that equation (90) in [4] becomes:

$$\int_{N_0/N}^1 \frac{du}{u} \frac{1}{2} \left(\tilde{B}(\alpha_s(uQ^2)) - B(\alpha_s(uQ^2)) \right) - \log \tilde{C} \left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2} \right) = \Gamma_2 \left(\frac{\partial}{\partial \log N} \right) \left\{ A(\alpha_s(\frac{N_0}{N}Q^2)) - \frac{1}{2} \frac{\partial}{\partial \log N} B(\alpha_s(\frac{N_0}{N}Q^2)) - 2A(\alpha_s(\frac{N_0^2}{N^2}Q^2)) \right\} \quad (16)$$

Observe that using the renormalization group equation ?? and chain rule we can write the following relation:

$$\begin{aligned} \frac{\partial}{\partial \log N} B(\alpha_s(\frac{k}{N})) &= \frac{\partial B(\alpha_s)}{\partial \alpha_s} \frac{\partial \alpha_s(\frac{k}{N})}{\partial \frac{k}{N}} \frac{\partial \frac{k}{N}}{\partial \log N} = -\frac{\partial \alpha_s(\frac{k}{N})}{\partial \log \frac{k}{N}} \frac{\partial B(\alpha_s)}{\partial \alpha_s} \\ &= -\beta(\alpha_s) \alpha_s \frac{\partial B(\alpha_s)}{\partial \alpha_s} \\ \frac{\partial}{\partial \log N} A(\alpha_s(\frac{k}{N^2})) &= -2\beta(\alpha_s) \alpha_s \frac{\partial A(\alpha_s)}{\partial \alpha_s} \end{aligned} \quad (17)$$

define the differential operator $\partial(\alpha_s)$ as:

$$\partial_{\alpha_s} \equiv -\beta(\alpha_s) \alpha_s \frac{\partial}{\partial \alpha_s} \quad (18)$$

Substituting the above relations in the previous equation, we obtain the equivalent of equation (92) in [4]:

$$\int_{N_0/N}^1 \frac{du}{u} \frac{1}{2} \left(\tilde{B}(\alpha_s(uQ^2)) - B(\alpha_s(uQ^2)) \right) - \log \tilde{C} \left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2} \right) = \Gamma_2(\partial_{\alpha_s}) \left\{ A(\alpha_s(\frac{N_0}{N}Q^2)) - \frac{1}{2} \partial_{\alpha_s} B(\alpha_s(\frac{N_0}{N}Q^2)) \right\} - 2\Gamma(2\partial_{\alpha_s}) A(\alpha_s(\frac{N_0^2}{N^2}Q^2)) \quad (19)$$

Now by setting $N = N_0$ or applying $\frac{\partial}{\partial \log N}$ one obtains respectively the functions \tilde{C} and \tilde{B} as functions of A and B :

$$\begin{aligned} \tilde{C}(\alpha_s) &= \exp \left\{ -\Gamma_2(\partial_{\alpha_s}) \left[A(\alpha_s) - \frac{1}{2} B(\alpha_s) \right] \right\} \\ &\quad - 2\Gamma(2\partial_{\alpha_s}) A(\alpha_s(\frac{N_0^2}{N^2}Q^2)) \end{aligned} \quad (20)$$

$$\begin{aligned}\frac{\tilde{B}(\alpha_s)}{2} &= \frac{B(\alpha_s)}{2} + \partial_{\alpha_s} \left\{ \Gamma_2(\partial_{\alpha_s}) \left[A(\alpha_s) - \frac{1}{2} \partial_{\alpha_s} B(\alpha_s) \right] \right\} \\ &\quad - 4 \partial_{\alpha_s} \left\{ \Gamma_s(2\partial_{\alpha_s}) A(\alpha_s(\frac{N_0^2}{N^2} Q^2)) \right\}\end{aligned}\tag{21}$$

by inserting the expansion

$$\begin{aligned}\Gamma_2(\epsilon) &= -\frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3)\epsilon - \frac{9}{16}\zeta(4)\epsilon^2 - \left(\frac{1}{6}\zeta(2)\zeta(3) + \frac{1}{5}\zeta(5) \right) \epsilon^3 \\ &\quad - \left(\frac{1}{18}\zeta(3)^2 - \frac{61}{128}\zeta(6) \right) \epsilon^4 + \mathcal{O}(\epsilon^5)\end{aligned}\tag{22}$$

in eq. (21) and eq. (20), we can obtain the coefficients \tilde{B} and \tilde{C} in terms of the coefficients A and B up to N^4LL accuracy:

$$\tilde{B}(\alpha_s(uQ^2)) = B(\alpha_s(uQ^2))\tag{23}$$

$$\begin{aligned}\log \tilde{C}\left(\alpha_s(mu^2), \frac{\mu^2}{Q^2}\right) &= \frac{A_1}{\pi}(-\zeta(2) - 1)\alpha_s + \left(\frac{-2A_2\zeta(2) - 2A_2 + \pi b_0 B_1}{2\pi^2} \right. \\ &\quad \left. + \frac{A_1 b_0 \left(3\zeta(2) \log\left(\frac{\mu^2}{Q^2}\right) + 3 \log\left(\frac{\mu^2}{Q^2}\right) - 4\zeta(3) \right)}{3\pi} \right) \alpha_s^2 \\ &\quad + \left(\frac{A_1}{3\pi} \left(-27b_0^2\zeta(4) - 3b_0^2\zeta(2) \log^2\left(\frac{\mu^2}{Q^2}\right) - 3b_0^2 \log^2\left(\frac{\mu^2}{Q^2}\right) \right. \right. \\ &\quad \left. \left. + 8b_0^2\zeta(3) \log\left(\frac{\mu^2}{Q^2}\right) + 3b_1\zeta(2) \log\left(\frac{\mu^2}{Q^2}\right) + 3b_1 \log\left(\frac{\mu^2}{Q^2}\right) \right. \right. \\ &\quad \left. \left. - 4b_1\zeta(3) \right) - \frac{1}{6\pi^3} \left(-12\pi A_2 b_0 \zeta(2) \log\left(\frac{\mu^2}{Q^2}\right) \right. \right. \\ &\quad \left. \left. - 12\pi A_2 b_0 \log\left(\frac{\mu^2}{Q^2}\right) + 16\pi A_2 b_0 \zeta(3) + 6A_3 \zeta(2) + 6A_3 \right. \right. \\ &\quad \left. \left. + 6\pi^2 b_0^2 B_1 \log\left(\frac{\mu^2}{Q^2}\right) - 6\pi b_0 B_2 - 3\pi^2 b_1 B_1 \right) \right) \alpha_s^3 + \mathcal{O}(\alpha_s^4)\end{aligned}\tag{24}$$

We note that \tilde{B} corrects the B terms so it has to be expanded up to α_s^4 to achieve N^4LL accuracy while $\ln \tilde{C}$ corrects the f_i functions so they have to be expanded up to α_s^3 to achieve N^4LL accuracy, these corrections are necessary only for NNLL accuracy and beyond, consistent with the results in [\[3\]](#).