

1 Inversion of the Laplace transform

In order to find the quark jet mass distribution $J^q(Q^2, k^2)$, we have to perform the inverse Laplace transform via the Mellin's inversion formula (or the Bromwich integral) given by the line integral:

$$J^q(Q^2, k^2) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{C-iT}^{C+iT} d\nu e^{\nu k^2} \tilde{J}_\nu^q(Q^2) \quad (1)$$

where C is a real number such that C is at the right of all singularities of the integrand in the complex plane and the function $\tilde{J}_\nu^q(Q^2)$ has to be bounded on the line.

Instead of directly considering the expression in eq. (1), it was pointed in [2] that it is more convenient to work with the mass fraction $R^q(w)$, which gives the fraction of jets with masses less than wQ^2 :

$$R^q(w) = \int_0^\infty J^q(Q^2, k^2) \Theta(wQ^2 - k^2) dk^2 \quad (2)$$

and using the integral representation of the Heaviside step function ??

$$\Theta(wQ^2 - k^2) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{C-iT}^{C+iT} \frac{d\nu}{\nu} e^{\nu(wQ^2 - k^2)} \quad (3)$$

we recognize the Laplace transform of the quark jet mass distribution ??

$$\begin{aligned} R^q(w) &= \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{C-iT}^{C+iT} \frac{d\nu}{\nu} e^{w\nu Q^2} \int_0^\infty J^q(Q^2, k^2) e^{-\nu k^2} dk^2 \\ &= \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{C-iT}^{C+iT} \frac{d\nu}{\nu} e^{w\nu Q^2} \tilde{J}_\nu^q(Q^2) \\ &= \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{C-iT}^{C+iT} \frac{d\nu}{\nu} e^{w\nu Q^2} e^{\mathcal{F}(\alpha_s, \ln(\nu Q^2))} \\ &= \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{C'-iT}^{C'+iT} \frac{dN}{N} e^{wN} e^{\mathcal{F}(\alpha_s, \ln N)} \end{aligned} \quad (4)$$

where $N = \nu Q^2$ and \mathcal{F} has the logarithms expansion

$$\begin{aligned}\mathcal{F}(\alpha_s, \ln N) &= f_1(b_0 \alpha_s \ln N) \ln N + f_2(b_0 \alpha_s \ln N) + f_3(b_0 \alpha_s \ln N) \alpha_s \\ &+ f_4(b_0 \alpha_s \ln N) \alpha_s^2 + f_5(b_0 \alpha_s \ln N) \alpha_s^3 + \mathcal{O}(\alpha_s^4)\end{aligned}\quad (5)$$

Since the function \mathcal{F} in the exponent varies more slowly with N than wN , we can introduce the integration variable $u = wN$ so that $\ln N = \ln u + \ln \frac{1}{w} = \ln u + L$ and Taylor expand with respect to $\ln u$ around 0, which is equivalent to expanding the original function \mathcal{F} w.r.t $\ln N$ around $\ln N = \ln \frac{1}{w} \equiv L$:

$$\begin{aligned}R^q(w) &= \frac{1}{2\pi i} \int_C \frac{du}{u} e^u e^{\mathcal{F}(\alpha_s, \ln u + L)} \\ &\stackrel{\text{Taylor}}{=} \int_C \frac{du}{2\pi i} e^{u - \ln u} e^{\mathcal{F}(\alpha_s, L) + \sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \\ &= e^{\mathcal{F}(\alpha_s, L)} \int_C \frac{du}{2\pi i} e^{u - \ln u} e^{\sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u}\end{aligned}\quad (6)$$

where the integral is intended as before, along the line C to the right of all singularities of the integrand, and

$$\mathcal{F}^{(n)}(\alpha_s, L) = \left. \frac{\partial^n \mathcal{F}(\alpha_s, \ln u + L)}{\partial \ln u^n} \right|_{\ln u=0} \quad (7)$$

As noticed in [2], the n -th derivative of \mathcal{F} w.r.t $\ln u$ evaluated at $\ln u = 0$ is at most of logarithmic order $\alpha_s^{n+k-1} L^k$, so in order to achieve $N^4 LL$ accuracy we need to compute the first four derivatives of \mathcal{F} w.r.t $\ln u$ and neglect the terms of order $\mathcal{O}(\alpha_s^4)$ that appear in the derivation. We obtain the following expressions:

$$\begin{aligned}\mathcal{F}^{(1)}(\alpha_s, L) &= f_1(\lambda) + \lambda f_1'(\lambda) + \alpha_s b_0 f_2'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s^3 b_0 f_4'(\lambda) \\ &+ \mathcal{O}(\alpha_s^n L^{n-3})\end{aligned}\quad (8)$$

$$\begin{aligned}\mathcal{F}^{(2)}(\alpha_s, L) &= 2\alpha_s b_0 f_1'(\lambda) + \alpha_s b_0 \lambda f_1''(\lambda) + \alpha_s^2 b_0^2 f_2''(\lambda) + \alpha_s^3 b_0^2 f_3''(\lambda) \\ &\quad + \mathcal{O}(\alpha_s^n L^{n-3})\end{aligned}\tag{9}$$

$$\mathcal{F}^{(3)}(\alpha_s, L) = 3\alpha_s^2 b_0^2 f_1''(\lambda) + \alpha_s^2 b_0^2 \lambda f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 f_2^{(3)}(\lambda) + \mathcal{O}(\alpha_s^n L^{n-3})\tag{10}$$

$$\mathcal{F}^{(4)}(\alpha_s, L) = 4\alpha_s^3 b_0^3 f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 \lambda f_1^{(4)}(\lambda) + \mathcal{O}(\alpha_s^n L^{n-3})\tag{11}$$

Here $\lambda = \alpha_s b_0 L$ and derivative w.r.t $\ln u$ and then evaluated at $\ln u = 0$, or equivalently derivative w.r.t L gives the same result.

After recasting the expansion presented in eq. (6) using the expression $\gamma(\alpha_s, L) = f_1(\lambda) + \lambda f_1'(\lambda)$ from [2], and defining $\mathcal{F}_{res}^{(1)}(\alpha_s, L) \equiv \mathcal{F}^{(1)}(\alpha_s, L) - \gamma(\alpha_s, L)$, we proceed to expand the second exponential with respect to $\ln u$ around 0, following the approach outlined in [1]. This yields the subsequent expansion:

$$\begin{aligned}R^q(w) &= e^{\mathcal{F}(\alpha_s, L)} \int_C \frac{du}{2\pi i} e^{u-(1-\gamma(\alpha_s, L)) \ln u} e^{\mathcal{F}_{res}^{(1)}(\alpha_s, L) \ln u + \sum_{n=2}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \\ &= \int_C \frac{du}{2\pi i} e^{u-(1-\gamma(\alpha_s, L)) \ln u} \left(1 + \mathcal{F}_{res}^{(1)} \ln u + \frac{1}{2} \left(\mathcal{F}^{(2)} + (\mathcal{F}_{res}^{(1)})^2 \right) \ln^2 u \right. \\ &\quad + \frac{1}{6} \left(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)} \mathcal{F}_{res}^{(1)} + (\mathcal{F}_{res}^{(1)})^3 \right) \ln^3 u \\ &\quad + \frac{1}{24} \left(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)} \mathcal{F}_{res}^{(1)} + 6\mathcal{F}^{(2)} (\mathcal{F}_{res}^{(1)})^2 + (\mathcal{F}_{res}^{(1)})^4 \right) \ln^4 u \\ &\quad \left. + \mathcal{O}(\ln^5 u) \right)\end{aligned}\tag{12}$$

Lastly, we utilize the following result to evaluate the integral presented in eq. (2).

$$\int_C \frac{du}{2\pi i} \ln^k u e^{u-(1-\gamma(\alpha_s, L)) \ln u} = \frac{d^k}{d\gamma^k} \frac{1}{\Gamma(1-\gamma(\alpha_s, L))}\tag{13}$$

where Γ is the Euler Γ -function.

$$\begin{aligned}
R^q(w) = & \frac{e^{\mathcal{F}(\alpha_s, L)}}{\Gamma(1-\gamma)} \left[1 + \mathcal{F}_{res}^{(1)} \psi_0 (1-\gamma) + \frac{1}{2} \left(\mathcal{F}^{(2)} + (\mathcal{F}_{res}^{(1)})^2 \right) (\psi_0^2 - \psi_1) (1-\gamma) \right. \\
& + \frac{1}{6} \left(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)} \mathcal{F}_{res}^{(1)} + (\mathcal{F}_{res}^{(1)})^3 \right) (\psi_0^3 - 3\psi_0 \psi_1 + \psi_2) (1-\gamma) \\
& + \frac{1}{24} \left(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)} \mathcal{F}_{res}^{(1)} + 6\mathcal{F}^{(2)} (\mathcal{F}_{res}^{(1)})^2 + (\mathcal{F}_{res}^{(1)})^4 \right) \\
& \left. (\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0 \psi_2 - \psi_3) (1-\gamma) + \mathcal{O}(\ln^5 u) \right] \tag{14}
\end{aligned}$$

where $\psi_n(z)$ are the polygamma functions, defined as:

$$\psi_n(z) = \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z) = \frac{d^n}{dz^n} \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d^n}{dz^n} \psi_0(z) \tag{15}$$

Substituting the expressions eqs. (8) to (11) into eq. (14) we obtain:

$$\begin{aligned}
R^q(w) = & \frac{e^{\mathcal{F}(\alpha_s, L)}}{\Gamma(1-\gamma)} \left[1 + \left(\alpha_s^3 b_0 f_4'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s b_0 f_2'(\lambda) \right) \psi_0 (1-\gamma) \right. \\
& + \frac{1}{2} \left(\alpha_s^3 (2b_0^2 f_2'(\lambda) f_3'(\lambda) + b_0^2 f_3''(\lambda)) + \frac{1}{2} \alpha_s^2 (b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2) \right. \\
& + \frac{1}{2} \alpha_s (b_0 \lambda f_1''(\lambda) + 2b_0 f_1'(\lambda)) \left. \left. \right) (\psi_0^2 - \psi_1) (1-\gamma) \right. \\
& + \frac{1}{6} \left(\alpha_s^3 (b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda) f_2''(\lambda) + 3b_0^2 \lambda f_1''(\lambda) f_3'(\lambda) \right. \\
& + 6b_0^2 f_1'(\lambda) f_3'(\lambda)) + \frac{1}{6} \alpha_s^2 (b_0^2 \lambda f_1^{(3)}(\lambda) + 3b_0^2 \lambda f_1''(\lambda) f_2'(\lambda) + 3b_0^2 f_1''(\lambda) \\
& + 6b_0^2 f_1'(\lambda) f_2'(\lambda)) \left. \left. \right) (\psi_0^3 - 3\psi_0 \psi_1 + \psi_2) (1-\gamma) \right. \\
& + \frac{1}{24} \left(\alpha_s^3 (b_0^3 \lambda f_1^{(4)}(\lambda) + 4b_0^3 \lambda f_1^{(3)}(\lambda) f_2'(\lambda) \right. \\
& + 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3 \lambda f_1''(\lambda) f_2''(\lambda) + 6b_0^3 \lambda f_1''(\lambda) f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda) f_2'(\lambda) \\
& + 12b_0^3 f_1'(\lambda) f_2''(\lambda) + 12b_0^3 f_1'(\lambda) f_2'(\lambda)^2) + \frac{1}{8} \alpha_s^2 (b_0^2 \lambda^2 f_1''(\lambda)^2 + 4b_0^2 f_1'(\lambda)^2 \\
& + 4b_0^2 \lambda f_1'(\lambda) f_1''(\lambda)) \left. \left. \right) (\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0 \psi_2 - \psi_3) (1-\gamma) + \mathcal{O}(\ln^5 u) \right] \tag{16}
\end{aligned}$$

and reorganize as a power series of α_s :

$$\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s, L)}}{\Gamma(1-\gamma)} & \left[1 + \alpha_s b_0 \left(\psi_0(1-\gamma)f_2'(\lambda) + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)(\lambda f_1''(\lambda) + 2f_1'(\lambda)) \right) \right. \\
& + \alpha_s^2 \left(\frac{1}{8}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1-\gamma)(b_0^2\lambda^2 f_1''(\lambda))^2 \right. \\
& + 4b_0^2 f_1'(\lambda)^2 + 4b_0^2 \lambda f_1'(\lambda) f_1''(\lambda) \Big) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1-\gamma)(b_0^2 \lambda f_1^{(3)}(\lambda) \\
& + 3b_0^2 \lambda f_1''(\lambda) f_2'(\lambda) + 3b_0^2 f_1''(\lambda) + 6b_0^2 f_1'(\lambda) f_2'(\lambda)) + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)(b_0^2 f_2''(\lambda) \\
& + b_0^2 f_2'(\lambda)^2) + b_0\psi_0(1-\gamma)f_3'(\lambda) \Big) + \alpha_s^3 \left(\frac{1}{24}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1-\gamma) \right. \\
& (b_0^3 \lambda f_1^{(4)}(\lambda) + 4b_0^3 \lambda f_1^{(3)}(\lambda) f_2'(\lambda) + 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3 \lambda f_1''(\lambda) f_2''(\lambda) \\
& + 6b_0^3 \lambda f_1''(\lambda) f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda) f_2'(\lambda) + 12b_0^3 f_1'(\lambda) f_2''(\lambda) + 12b_0^3 f_1'(\lambda) f_2'(\lambda)^2) \\
& + \frac{1}{2}(\psi_0^2 - \psi_1)(1-\gamma)(2b_0^2 f_2'(\lambda) f_3'(\lambda) + b_0^2 f_3''(\lambda)) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1-\gamma) \\
& (b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda) f_2''(\lambda) + 3b_0^2 \lambda f_1''(\lambda) f_3'(\lambda) + 6b_0^2 f_1'(\lambda) f_3'(\lambda)) \\
& \left. \left. + b_0\psi_0(1-\gamma)f_4'(\lambda) \right) + \mathcal{O}(\alpha_s^4) \right]
\end{aligned} \tag{17}$$

By comparing ?? and eq. (4) we see that to obtain the Thrust cross section, we simply multiply by 2 the inverse Laplace transform obtained. Following the convention of [2], the final resummed expression $R_T(\tau)$ can be written as one exponential eq. (18), to do so we observe that $\frac{1}{\Gamma(1-\gamma)} = \exp\{-\ln(\Gamma(1-\gamma))\}$ corrects f_2 , while the expression in parenthesis starting with 1 can be seen as the expansion of an exponential for $\alpha_s \rightarrow 0$, then the terms proportional to α_s corrects f_3 , the terms proportional to α_s^2 corrects f_4 and so on.

$$R_T(\tau) = \exp\{Lg_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \alpha_s^3 g_5(\lambda) + \mathcal{O}(\alpha_s^4)\} \tag{18}$$

where

$$g_1(\lambda) = 2f_1(\lambda) \quad (19)$$

$$g_2(\lambda) = 2f_2(\lambda) - \ln \Gamma(1 - 2f_1(\lambda) - 2\lambda f_1'(\lambda)) \quad (20)$$

$$g_3(\lambda) = 2f_3(\lambda) + 2\left(\psi_0(1 - \gamma)f_2'(\lambda) + \frac{1}{2}(\psi_0^2 - \psi_1)(1 - \gamma)(\lambda f_1''(\lambda) + 2f_1'(\lambda))\right) \quad (21)$$

$$\begin{aligned} g_4(\lambda) = 2f_4(\lambda) + 2\left(\frac{1}{8}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1 - \gamma)(b_0^2\lambda^2 f_1''(\lambda)^2 \right. \\ \left. + 4b_0^2 f_1'(\lambda)^2 + 4b_0^2 \lambda f_1'(\lambda) f_1''(\lambda)) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1 - \gamma)(b_0^2 \lambda f_1^{(3)}(\lambda) \right. \\ \left. + 3b_0^2 \lambda f_1''(\lambda) f_2'(\lambda) + 3b_0^2 f_1''(\lambda) + 6b_0^2 f_1'(\lambda) f_2'(\lambda)) + \frac{1}{2}(\psi_0^2 - \psi_1)(1 - \gamma) \right. \\ \left. (b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2) + b_0\psi_0(1 - \gamma)f_3'(\lambda)\right) \end{aligned} \quad (22)$$

$$\begin{aligned} g_5(\lambda) = 2f_5(\lambda) + 2\left(\frac{1}{24}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1 - \gamma) \right. \\ \left. (b_0^3 \lambda f_1^{(4)}(\lambda) + 4b_0^3 \lambda f_1^{(3)}(\lambda) f_2'(\lambda) + 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3 \lambda f_1''(\lambda) f_2''(\lambda) \right. \\ \left. + 6b_0^3 \lambda f_1''(\lambda) f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda) f_2'(\lambda) + 12b_0^3 f_1'(\lambda) f_2''(\lambda) + 12b_0^3 f_1'(\lambda) f_2'(\lambda)^2) \right. \\ \left. + \frac{1}{2}(\psi_0^2 - \psi_1)(1 - \gamma)(2b_0^2 f_2'(\lambda) f_3'(\lambda) + b_0^2 f_3''(\lambda)) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1 - \gamma) \right. \\ \left. (b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda) f_2''(\lambda) + 3b_0^2 \lambda f_1''(\lambda) f_3'(\lambda) + 6b_0^2 f_1'(\lambda) f_3'(\lambda)) \right. \\ \left. + b_0\psi_0(1 - \gamma)f_4'(\lambda)\right) \end{aligned} \quad (23)$$

References

- [1] Ugo Aglietti and Giulia Ricciardi. “Approximate NNLO threshold resummation in heavy flavor decays”. In: *Phys. Rev. D* 66 (2002), p. 074003. DOI: [10.1103/PhysRevD.66.074003](https://doi.org/10.1103/PhysRevD.66.074003). arXiv: [hep-ph/0204125](https://arxiv.org/abs/hep-ph/0204125).

- [2] S. Catani et al. “Resummation of large logarithms in e^+e^- event shape distributions”. In: *Nuclear Physics B* 407.1 (1993), pp. 3–42. ISSN: 0550-3213. DOI: [https://doi.org/10.1016/0550-3213\(93\)90271-P](https://doi.org/10.1016/0550-3213(93)90271-P). URL: <https://www.sciencedirect.com/science/article/pii/055032139390271P>.