1 QCD running coupling

Classically, the force between two sources is then given by $F = \frac{\alpha}{r^2}$, characterized by a universal coefficient – the coupling constant α, which quantifies the force between two static bodies of unit "charge" at distance r, *i.e.*, the electric charge for QED, the color charge for QCD, the weak isospin for the weak force, or the mass for gravity. Consequently, the coupling α is defined as being proportional to the elementary charge squared, *e.g.*, $\alpha_{em} \equiv \frac{e^2}{4\pi}$ where e is the elementary electric charge, or $\alpha_s \equiv \frac{g^2}{4\pi}$ where g is the elementary gauge field coupling in QCD. In quantum field theory (QFT), $\frac{1}{r^2}$ is the coordinate-space expression for the propagator of the force carrier (gauge boson) at leading-order in perturbation theory: in momentum space, the analogous propagator is proportional to $\frac{1}{q^2}$, where q the boson 4-momentum ($Q^2 = -q^2 > 0$).

For sources interacting weakly, the one-boson exchange representation of interactions is a good first approximation. However, when interactions become strong (with "strong" to be defined below), higher orders in perturbation theory become noticeable and the $\frac{1}{r^2}$ law no longer stands. In such cases, it makes good physics sense to fold the extra r-dependence into the coupling, which thereby becomes r, or equivalently Q^2, dependent.

The running of the coupling is due to vacuum polarization, the vacuum is not empty, but is filled with virtual particles that are constantly created and annihilated which can interact with the propagating particles, leading to a modification of the interaction strength.

While in QED, the extra r-dependence comes only from the vacuum polarization. In QCD, α_s receives contributions from the vacuum polarization and from gluon self-interactions since the gluon has a color charge.

The two couplings have opposite trends: the QED coupling increases with energy and the theory becomes strongly coupled at high energies, whereas the opposite happens for the QCD coupling as it is large at low energies and decreases with energy. This property of being weakly coupled at high energies is known as *asymptotic freedom* and it means that perturbative calculations in QCD can only be done at high energies where α_s becomes small enough that a power expansion is meaningful.

In the framework of perturbative QCD (*pQCD*), predictions for observables are expressed in terms of the renormalized coupling $\alpha = \alpha(\mu^2)$, a function of an (unphysical) renormalization scale μ_R . The coupling satisfies the following renormalization group equation (RGE):

$$\mu^2 \frac{d\alpha}{d\mu^2} = \beta(\alpha) = -\left(b_0\alpha^2 + b_1\alpha^3 + b_2\alpha^4 + \ldots\right) \tag{1}$$

where $b_0 = \frac{11C_A - 4n_f T_R}{12\pi} = \frac{33-2n_f}{12\pi}$ is the 1-loop β-function coefficient, $b_1 = \frac{17C_A^2 - n_f T_R(10C_A + 6C_F)}{24\pi^2} = \frac{153-19n_f}{24\pi^2}$ is the 2-loop coefficient, $b_2 = \frac{2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2}{128\pi^3}$ is the 3-loop coefficient. $C_A = 3$ and $C_F = \frac{4}{3}$ are the Casimir operators of the adjoint and fundamental representations of $SU(3)$, $T_R = \frac{1}{2}$ is the trace normalization, n_f is the number of active quark flavors.

It is not possible to solve eq. (1) as it is for two reasons: only the first few b_n coefficients are known (up to b_4); the exact equation becomes more and more complicated as more terms of the series are included, making it impossible to obtain an analytic solution.

In order to solve both problems, the equation is solved in the following way: at first only b_0 is included and the obtained solution is called α_{LO} , as it will only contain a term proportional to α ; then also b_1 is included and only terms up to the second order in α are kept to obtain α_{NLO} ; this same procedure is used to obtain α_{NNLO} , α_{N^3LO} , α_{N^4LO} . There will be a complication in calculating α_{NLO} and higher orders which will be explained and resolved in the following sections.

1.1 One-loop running coupling

The one-loop running coupling α_{LO} is obtained by solving the RGE with only the first term of the β-function:

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0\alpha^2 \tag{2}$$

This equation can be solved by separation of variables and imposing the boundary condition $\alpha(Q^2) = \alpha_s$:

$$\int_{\alpha(Q^2)}^{\alpha(\mu^2)} \frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0 \frac{\mathrm{d}\mu^2}{\mu^2} \tag{3}$$

and one obtains:

$$\alpha_{LO}(\mu^2) = \frac{\alpha_s}{1 + b_0\alpha_s \log\left(\frac{\mu^2}{Q^2}\right)} \tag{4}$$

In which we can confirm the decreasing with energy trend of the running coupling that we anticipated above.

It is useful to define the variable $\lambda_\mu = b_0\alpha_s \log\left(\frac{\mu^2}{Q^2}\right)$ so that:

$$\alpha_{LO}(\mu^2) = \frac{\alpha_s}{1 + \lambda_\mu} \tag{5}$$

1.2 Two-loop running coupling

In order to obtain the two-loop running coupling α_{NLO}, we need to solve the RGE with the first two terms of the β-function eq. (1):

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0\alpha^2 - b_1\alpha^3 \tag{6}$$

but this equation is not solvable in a straightforward way as the one-loop equation, we have to use the perturbative approach. We can rewrite the equation as:

$$\frac{d\alpha}{d\mu^2} = -\frac{b_0\alpha^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha) \tag{7}$$

and expand the α term in the parenthesis as:

$$\alpha = \alpha_{LO} + \delta\alpha \tag{8}$$

where α_{LO} is the one-loop running coupling and $\delta\alpha$ contains the higher order correction, one obtains:

$$\frac{d\alpha}{d\mu^2} = -\frac{b_0\alpha^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{LO} - \frac{b_0}{b_1}\delta\alpha) \tag{9}$$

Observe that in parenthesis, by keeping 1 gave us the one-loop running coupling, by keeping $\frac{b_1}{b_0}\alpha_{LO}$ we can obtain the first order corrections and $\delta\alpha$ are needed for higher order corrections. The equation to solve is then:

$$\int_{\alpha_s}^{\alpha(\mu^2)} -\frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0\frac{\mathrm{d}\mu^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{LO}(\mu^2)) \tag{10}$$

Using *Mathematica* to solve this equation, we obtain the two-loop running coupling:

$$\alpha_{NLO}(\mu^2) = \frac{\alpha_s}{1 + \lambda_\mu + \alpha_s\frac{b_1}{b_0}\log\left(1 + \lambda_\mu\right)} \tag{11}$$

in which the expansion in powers of α_s is not explicit. One can expand in powers of α_s by keeping λ_μ fixed and only keeping terms up to $\mathcal{O}\left(\alpha_s^2\right)$ by doing so one obtains:

$$\alpha_{NLO}\left(\mu^2\right) = \alpha_{LO}\left(\mu^2\right) - \frac{b_1}{b_0}\alpha_{LO}^2\left(\mu^2\right)\log\left(1 + \lambda_\mu\right) + \mathcal{O}(\alpha_s^2) \tag{12}$$

We found the correction:

$$\delta\alpha_{NLO}\big(\mu^2\big) = -\frac{b_1}{b_0}\alpha_{LO}^2\big(\mu^2\big)\log\big(1+\lambda_\mu\big) \tag{13}$$

By repeating the same procedure, one can obtain the three-loop running coupling α_{NNLO} and so on.

1.3 Higher order corrections

In order to calculate higher order corrections, one need to be careful of the powers of α need for the desired order, and the contributions to various orders of α_s may not be immediately apparent, but they are straightforward to compute. Expand the running coupling in powers of α_s:

$$\alpha = \alpha_{LO} + \delta\alpha_{NLO} + \delta\alpha_{NNLO} + \delta\alpha_{N^3LO} + \delta\alpha_{N^4LO} + \dots \tag{14}$$

with $\delta\alpha_{NLO} = \mathcal{O}(\alpha_s)$, $\delta\alpha_{NLO} = \mathcal{O}\big(\alpha_s^2\big)\delta\alpha_{NNLO} = \mathcal{O}(\alpha_s^3)$, $\delta\alpha_{N^3LO} = \mathcal{O}(\alpha_s^4)$, $\delta\alpha_{N^4LO} = \mathcal{O}(\alpha_s^5)$, and so on. We present these contributions in the following table:

Power	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$	$\mathcal{O}(\alpha_s^5)$
α	α_{LO}	$\delta\alpha_{NLO}$	$\delta\alpha_{NNLO}$	$\delta\alpha_{N^3LO}$	$\delta\alpha_{N^4LO}$
α^2		α_{LO}^2	$2\alpha_{LO}\delta\alpha_{NLO}$	$\delta\alpha_{NLO}^2 + 2\alpha_{LO}\delta\alpha_{NNLO}$	$2\alpha_{LO}\delta\alpha_{N^3LO} + 3\alpha_{LO}\delta\alpha_{NLO}^2$
α^3			α_{LO}^3	$3\alpha_{LO}^2\delta\alpha_{NLO}$	$3\alpha_{LO}^2\delta\alpha_{NNLO} + 3\alpha_{LO}\delta\alpha_{NLO}^2$
α^4				α_{LO}^4	$4\alpha_{LO}^3\delta\alpha_{NLO}$
α^5					α_{LO}^5

Table 1: Contributions to different powers of α_s.

For the three-loop running coupling, the equation to solve is:

$$\mu^2\frac{d\alpha}{d\mu^2} = -b_0\alpha^2(1 - \frac{b_1}{b_0}\alpha - \frac{b_2}{b_0}\alpha^2) \tag{15}$$

One can substitute the expansion of $\alpha = \alpha_{LO} + \delta\alpha_{NLO} + \mathcal{O}(\alpha_s^2)$ in powers of α_s and retain only terms up to $\mathcal{O}\left(\alpha_s^2\right)$ with this prescription the equation to solve is:

$$\int_{\alpha_s}^{\alpha(\mu^2)} -\frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0 \frac{\mathrm{d}\mu^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{NLO}(\mu^2) - \frac{b_2}{b_0}\alpha_{LO}^2(\mu^2)) \tag{16}$$

solve and obtain the three-loop running coupling α_{NNLO}:

$$\alpha_{NNLO}(\mu^2) = \alpha_{LO}\left(\mu^2\right) + \delta\alpha_{NLO}\left(\mu^2\right) + \delta\alpha_{NNLO}\left(\mu^2\right) \tag{17}$$

with

$$\delta\alpha_{NNLO}\left(\mu^2\right) = \frac{\alpha_{LO}^3\left(\mu^2\right)}{b_0^2}\left(b_1^2\lambda_\mu - b_0 b_2 \lambda_\mu + b_1^2 \log^2(1+\lambda_\mu) - b_1^2 \log\left(1+\lambda_\mu\right)\right) \tag{18}$$

Similarly one can obtain the four-loop running coupling α_{N^3LO} and five-loop running coupling α_{N^4LO}.

$$\alpha_{N^3LO}(\mu^2) = \alpha_{LO}\left(\mu^2\right) + \delta\alpha_{NLO}\left(\mu^2\right) + \delta\alpha_{NNLO}\left(\mu^2\right) + \delta\alpha_{N^3LO}\left(\mu^2\right) \tag{19}$$

$$\begin{aligned}\delta\alpha_{N^3LO}\left(\mu^2\right) = \frac{\alpha_{LO}^4\left(\mu^2\right)}{2b_0^3}\Big(&-\left(b_1^3 - 2b_0 b_2 b_1 + b_0^2 b_3\right)\lambda_\mu^2 \\ &-\left(2b_0^2 b_3 - 2b_0 b_1 b_2\right)\lambda_\mu - 2b_1^3 \log^3\left(\lambda_\mu + 1\right) + 5b_1^3 \log^2\left(1+\lambda_\mu\right) \\ &+\left(2b_0 b_1 b_2\left(2\lambda_\mu - 1\right) - 4b_1^3\lambda_\mu\right)\log\left(1+\lambda_\mu\right)\Big)\end{aligned} \tag{20}$$

$$\alpha_{N^4LO}(\mu^2) = \alpha_{LO}\left(\mu^2\right) + \delta\alpha_{NLO}\left(\mu^2\right) + \delta\alpha_{NNLO}\left(\mu^2\right) + \delta\alpha_{N^3LO}\left(\mu^2\right) + \delta\alpha_{N^4LO}\left(\mu^2\right) \tag{21}$$

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\begin{aligned}
\delta\alpha_{N^4LO} = \frac{\alpha_{LO}^5}{6b_0^4}\Big(&\left(2b_1^4 - 6b_0b_2b_1^2 + 4b_0^2b_3b_1 + 2b_0^2b_2^2 - 2b_0^3b_4\right)\lambda_\mu^3 \\
&+ \left(9b_1^4 - 24b_0b_2b_1^2 + 9b_0^2b_3b_1 + 12b_0^2b_2^2 - 6b_0^3b_4\right)\lambda_\mu^2 \\
&+ \left(6b_0^2b_1b_3 - 6b_0^3b_4\right)\lambda_\mu + 6b_1^4\log^4(1+\lambda_\mu) \\
&- 26b_1^4\log^3\left(\lambda_\mu + 1\right) + 9\left(\left(2b_1^4 - 2b_0b_1^2b_2\right)\lambda_\mu + b_1^4 + 2b_0b_2b_1^2\right)\log^2(1+\lambda_\mu) \\
&+ \left(6b_1\left(b_1^3 - 2b_0b_2b_1 + b_0^2b_3\right)\lambda_\mu^2 + 6b_1\left(-3b_1^3 + b_0b_2b_1 + 2b_0^2b_3\right)\lambda_\mu \right. \\
&\left. - 6b_1b_3b_0^2\right)\log\big(1+\lambda_\mu\big)\Big)
\end{aligned}
\tag{22}
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