0.1 Calculating the resummation coefficients

In this section we discuss the calculation of the resummation coefficients $f_i(\lambda)$ for the thrust distribution.

In the article by Catani, Turnock, Webber and Trentadue [1], it was observed that for a final state configuration corresponding to a large value of thrust , **??** can be approximated by

$$\tau = 1 - T \approx \frac{k_1^2 + k_2^2}{Q^2} \tag{1}$$

where k_1^2 and k_2^2 are the squared invariant masses of two back-to-back jets and Q^2 is the energy of the center of mass. Thus the key to the evaluation of the thrust distributions is its relation to the quark jet mass distribution $J^q(Q^2, k^2)$, also denoted as $J^q_{k^2}(Q^2)$, which represents the probability of finding a jet originating from quarks, with an invariant mass-squared k^2, produced in collisions with a center-of-mass energy Q^2.

Then the thrust distribution $R_T(\tau, \alpha_s(Q^2))$ **??** takes the form of a convolution of two jet mass distributions $J(Q^2, k_1^2)$ and $J(Q^2, k_2^2)$

$$R_T(\tau, \alpha_s(Q^2)) \underset{\tau \ll 1}{=} \int_0^\infty \mathrm{d}k_1^2 \int_0^\infty \mathrm{d}k_2^2 J^q_{k_1^2}(Q^2) J^q_{k_2^2}(Q^2) \Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) \tag{2}$$

Introducing the Laplace transform of the jet mass distribution:

$$\tilde{J}^q_\nu(Q^2) = \int_0^\infty J^q(Q^2, k^2) e^{-\nu k^2} \mathrm{d}k^2 \tag{3}$$

and using the integral representation of the Heaviside step function:

$$\Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) = \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} \frac{\mathrm{d}N}{N} e^{N\tau} e^{-N \frac{k_2^2 + k_2^2}{Q^2}} \tag{4}$$

by substituting eq. (4) into eq. (2) and setting $N = \nu Q^2$ we obtain:

$$\begin{aligned} R_T(\tau) &= \int_{C - i\infty}^{C + i\infty} \frac{\mathrm{d}N}{N} \frac{e^{N\tau}}{2\pi i} \left[\int_0^\infty \mathrm{d}k_1^2 e^{-\nu k_1^2} J^q_{k_1^2}(Q^2)\right] \left[\int_0^\infty \mathrm{d}k_2^2 e^{-\nu k_2^2} J^q_{k_2^2}(Q^2)\right] \\ &= \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} e^{N\tau} \left[\tilde{J}^q_\nu(Q^2)\right]^2 \frac{\mathrm{d}N}{N} \end{aligned} \tag{5}$$

where C is a real positive constant to the right of all singularities of the integrand $\tilde{J}_\nu(Q^2)$ in the complex ν plane.

An integral representation for the Laplace transform $\tilde{J}_\nu(Q^2)$ is given by [2]:

$$\ln \tilde{J}^q_\nu(Q^2) = \int_0^1 \frac{\mathrm{d}u}{u}\Big(e^{-u\nu Q^2}-1\Big)\left[\int_{u^2Q^2}^{uQ^2}\frac{1}{q^2}A\Big(\alpha_s(q^2)\Big)\mathrm{d}q^2 + \frac{1}{2}B\Big(\alpha_s(uQ^2)\Big)\right] \tag{6}$$

with

$$A(\alpha_s) = \sum_{n=1}^{\infty} A_n\Big(\frac{\alpha_s}{\pi}\Big)^n, \qquad\qquad B(\alpha_s) = \sum_{n=1}^{\infty} B_n\Big(\frac{\alpha_s}{\pi}\Big)^n.$$

Function $A(\alpha_s)$ is associated with the cusp anomalous dimension and governs the exponentiation of the leading logarithms (LL). It captures the resummation of the soft and collinear gluon emissions that dominate in the limit of large thrust values.

Function $B(\alpha_S)$ includes the next-to-leading logarithmic (NLL) corrections and accounts for subleading contributions from hard collinear emissions. It typically involves the non-cusp part of the anomalous dimensions and running of the coupling constant.

The integral as it is cannot be integrated, the u integration may be performed using the prescription discussed in Appendix A of [3] and readapting the formula to the case of Laplace transform instead of Mellin transform.

This method is a generalization of the prescription to NLL accuracy in [2]

$$e^{-u\nu Q^2} - 1 \simeq -\Theta(u-v) \quad \text{with } v = \frac{N_0}{N} \tag{7}$$

where $N_0 = e^{-\gamma_E}$.

In **??** we show that the prescription to evaluate the large-N Mellin moments of soft-gluon contributins at an arbitrary logarithmic accuracy, can be used for the Laplace transform as well, then we can use this result to express **??** in an alternative representation:

$$\ln \tilde{J}^q_\nu(Q^2) = -\int_{N_0/N}^1 \frac{\mathrm{d}u}{u}\left[\int_{u^2Q^2}^{uQ^2}\frac{1}{q^2}A\Big(\alpha_s(q^2)\Big)\mathrm{d}q^2 + \frac{1}{2}\tilde{B}\Big(\alpha_s(uQ^2)\Big)\right] + \ln\tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right). \tag{8}$$

Note that, due to the integration of the running coupling the integral in **??** is singular for all values of $N = \nu Q^2$. However, if we perform the integration up to a fixed logarithmic accuracy $N^k LL$ (*i.e* we compute the leading $\alpha_s^n \ln^{n+1} N$, next-to-leading $\alpha_s^n \ln^n N$ and so on to $\alpha_s^n \ln^{n+1-k}$ terms), we find the form factor:

$$\ln \tilde{J}^q_\nu(Q^2) = \ln N f_1(\lambda) + f_2(\lambda) + \alpha_s f_3(\lambda) + \alpha_s^2 f_4(\lambda) + \alpha_s^3 f_5(\lambda) + \mathcal{O}\left(\alpha_s^n \ln^{n-4} N\right) + \ln \tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) \tag{9}$$

From now on we'll always use eq. (8), so i'll drop the ˜ notation for B.

We observe that the N-space formula eq. (8) is finite and uniquely defined up to the very large $N = N_L = \exp\left(\frac{1}{2\alpha_s b_0}\right)$ ($\lambda = \frac{1}{2}$), thanks to the prescription above.

0.1.1 resummation functions

To calculate explicit expressions for the first few f_i terms, we first write explicitly the internal integral of **??**, for now let's forget the $\ln \tilde{C}$ term, it can be absorbed into the definition of A and B:

The q^2 integration also becomes simple if we use the renormalization group equation **??** to change the integration variable to as

$$\begin{aligned}\frac{\mathrm{d}q^2}{q^2} &= -\frac{\mathrm{d}\alpha_s}{b_0\alpha_s^2}\left(1 - \frac{b_1}{b_0}\alpha_s + \frac{\left(b_1^2 - b_2 b_0\right)}{b_0^2}\alpha_s^2 + \frac{\left(-b_3 b_0^2 + 2b_2 b_1 b_0 - b_1^3\right)}{b_0^3}\alpha_s^3 \right.\\ &\left. + \frac{\left(-b_4 b_0^3 + 2b_3 b_1 b_0^2 + b_2^2 b_0^2 - 3b_2 b_1^2 b_0 + b_1^4\right)}{b_0^4}\alpha_s^4 + \mathcal{O}\left(\alpha_s^5\right)\right) \\ &= -\frac{\mathrm{d}\alpha_s}{b_0\alpha_s^2}(N_0 + N_1\alpha_s + N_2\alpha_s^2 + N_3\alpha_s^3 + N_4\alpha_s^4 + \mathcal{O}\left(\alpha_s^5\right))\end{aligned} \tag{10}$$

where for convenience's sake we have defined:

$$\begin{aligned}&N_0 = 1 \qquad N_1 = \frac{b_1}{b_0} \qquad N_2 = \frac{\left(b_1^2 - b_2 b_0\right)}{b_0^2} \\ &N_3 = \frac{\left(-b_3 b_0^2 + 2b_2 b_1 b_0 - b_1^3\right)}{b_0^3} \\ &N_4 = \frac{\left(-b_4 b_0^3 + 2b_3 b_1 b_0^2 + b_2^2 b_0^2 - 3b_2 b_1^2 b_0 + b_1^4\right)}{b_0^4}\end{aligned} \tag{11}$$

Subsequently, the integral in eq. (8) can be expressed as:

$$\int_{\alpha_s(u^2Q^2)}^{\alpha_s(uQ^2)} \frac{\mathrm{d}\alpha_s}{b_0\alpha_s^2}(N_0 + N_1\alpha_s + N_2\alpha_s^2 + N_3\alpha_s^3 + N_4\alpha_s^4)\sum_{n=1}^{\infty} A_n\Big(\frac{\alpha_s}{\pi}\Big)^n$$
$$= \int_{\alpha_s(u^2Q^2)}^{\alpha_s(uQ^2)} \frac{\mathrm{d}\alpha_s}{b_0}\Bigg(\frac{A_1N_0}{\pi\alpha_s} + \frac{\pi A_1N_1 + A_2}{\pi^2} + \frac{\Big(\pi A_2N_1 + \pi^2 A_1N_2 + A_3\Big)}{\pi^3}\alpha_s$$
$$+ \frac{\Big(\pi A_3N_1 + \pi^2 A_2N_2 + \pi^3 A_1N_3 + A_4\Big)}{\pi^4}\alpha_s^2 \tag{12}$$
$$+ \frac{\Big(\pi A_4N_1 + \pi^2 A_3N_2 + \pi^3 A_2N_3 + \pi^4 A_1N_4 + A_5\Big)}{\pi^5}\alpha_s^3 + \mathcal{O}\Big(\alpha_s^4\Big)\Bigg)$$

and keeping only terms up to N_0,A_1 and α_s^0 we get contributions to f_1, keeping terms up to N_1,A_2 and α_s^1 yields contributions to f_2 and so on. That's because after integration, we evaluate the integrand at $\alpha_s(u^2Q^2)$ and $\alpha_s(uQ^2)$, where α_s is α_{LO} for f_1, α_{NLO} for f_2 and so on.

And there's an easy way to see why it's like this, in fact it's also possible to do the q^2 integration directly, using **??????????** from **??** and keeping in mind **??**.

$$\int_{u^2Q^2}^{uQ^2} \frac{\mathrm{d}q^2}{q^2}\sum_{n=1}^{\infty}\frac{A_n}{\pi^n}\Big(\alpha_{\text{LO}}(q^2) + \delta\alpha_{\text{NLO}}(q^2) + \delta\alpha_{\text{NNLO}}(q^2) + \delta\alpha_{\text{N}^3\text{LO}}(q^2) + \delta\alpha_{\text{N}^4\text{LO}}(q^2) + \dots\Big)^n \tag{13}$$

now we can see that if we consider terms up to α_s^1 only A_1 contributes and this gives f_1, if we consider terms up to α_s^2 we see A_2 starts to contribute together with $A_1\delta\alpha_{\text{NLO}}$ and this gives f_2 , for f_3 we need to consider terms up to α_s^3 and so on.

For the B-term it's similar. However, since the B-term is "already integrated" in q^2, it contributes one order lower in α_s compared to the A-term. Specifically, B_1 starts to contribute from f_2, B_2 from f_3 and so on.

Now armed with this knowledge we can calculate eq. (8) and find:

$$\ln \tilde{J}_\nu^q(Q^2) = \ln N f_1(\lambda) + f_2(\lambda) + \alpha_s f_3(\lambda) + \alpha_s^2 f_4(\lambda) + \alpha_s^3 f_5(\lambda) + \mathcal{O}\Big(\alpha_s^n \ln^{n-4} N\Big) \tag{14}$$

where $\lambda = \alpha_s b_0 \ln N$, $N = \nu Q^2$, $\alpha_s = \alpha_s(Q^2)$ and

$$f_1(\lambda) = -\frac{A_1}{2\pi b_0\lambda}[(1-2\lambda)\ln(1-2\lambda) - 2(1-\lambda)\ln(1-\lambda)] \tag{15}$$

$$f_2(\lambda) = -\frac{A_2}{2\pi^2 b_0^2}[2\ln(1-\lambda) - \ln(1-2\lambda)] + \frac{B_1\ln(1-\lambda)}{2\pi b_0} + \frac{\gamma_E A_1}{\pi b_0}[\ln(1-2\lambda) - \ln(1-\lambda)]$$

$$- \frac{A_1 b_1}{2\pi b_0^3} \left[-\ln^2(1-\lambda) + \frac{1}{2}\ln^2(1-2\lambda) - 2\ln(1-\lambda) + \ln(1-2\lambda) \right] \tag{16}$$

$$\begin{aligned}
f_3(\lambda) = &-\frac{A_3}{2\pi^3 b_0^2(\lambda-1)(2\lambda-1)}\lambda^2 + \frac{B_2}{2\pi^2 b_0(\lambda-1)}\lambda + \frac{B_2\lambda}{2\pi^2 b_0(\lambda-1)} \\
&+ \frac{b_1 A_2}{2\pi^2 b_0^3(\lambda-1)(2\lambda-1)} \left[3\lambda^2 + (1-\lambda)\ln(1-2\lambda) - 2(1-2\lambda)\ln(1-\lambda)\right] \\
&- \frac{\gamma_E A_2 \lambda}{\pi^2 b_0(\lambda-1)(2\lambda-1)} - \frac{B_1 b_1}{2\pi b_0^2(\lambda-1)}[\lambda + \ln(1-\lambda)] + \frac{\gamma_E B_1 \lambda}{2\pi(\lambda-1)} + \frac{\gamma_E^2 A_1 \lambda(2\lambda-3)}{2\pi(\lambda-1)(2\lambda-1)} \\
&+ \frac{b_1 \gamma_E A_1}{\pi b_0^2(\lambda-1)(2\lambda-1)}[-\lambda + (1-2\lambda)\ln(1-\lambda) - (1-\lambda)\ln(1-2\lambda)] \qquad (17) \\
&- \frac{b_1^2 A_1}{2\pi b_0^4(\lambda-1)(2\lambda-1)}\left[\left(\lambda^2 + (2\lambda-1)\ln(1-\lambda)(2\lambda + \ln(1-\lambda))\right)\right. \\
&\left. + \frac{1}{2}\left((1-\lambda)\ln^2(1-2\lambda)\right) - 2(\lambda-1)\lambda\ln(1-2\lambda)\right] \\
&- \frac{b_0 b_2 A_1}{2\pi b_0^4(\lambda-1)(2\lambda-1)}\left[\lambda^2 + (\lambda-1)(2\lambda-1)(2\ln(1-\lambda) + \ln(1-2\lambda))\right]
\end{aligned}$$

$$\begin{aligned}
f_4(\lambda) = &-\frac{A_4\lambda^2\left(2\lambda^2 - 6\lambda + 3\right)}{6\pi^4 b_0^2(\lambda-1)^2(2\lambda-1)^2} + \frac{B_3(\lambda-2)\lambda}{4\pi^3 b_0(\lambda-1)^2} + \frac{b_1 A_3}{12\pi^3 b_0^3(\lambda-1)^2(2\lambda-1)^2}\left[15\lambda^2\right. \\
&\left. + 10(\lambda-3)\lambda^3 + 3(\lambda-1)^2\ln(1-2\lambda) - 6(1-2\lambda)^2\ln(1-\lambda)\right] \\
&+ \frac{\gamma_E A_3 \lambda(3\lambda-2)}{2\pi^3 b_0(\lambda-1)^2(2\lambda-1)^2} + \frac{\gamma_E B_2(\lambda-2)\lambda}{2\pi^2(\lambda-1)^2} - \frac{b_1 B_2\left[\lambda^2 - 2\lambda - 2\ln(1-\lambda)\right]}{4\pi^2 b_0^2(\lambda-1)^2} \\
&+ \frac{\gamma_E^2 A_2 \lambda\left(4\lambda^3 - 12\lambda^2 + 15\lambda - 6\right)}{2\pi^2(\lambda-1)^2(2\lambda-1)^2} + \frac{b_1\gamma_E A_2}{2\pi^2 b_0^2(\lambda-1)^2(2\lambda-1)^2}\left[\lambda(2-3\lambda)\right. \\
&\left. + 2(\lambda-1)^2\ln(1-2\lambda) - 2(1-2\lambda)^2\ln(1-\lambda)\right] + \frac{b_2 A_2 \lambda^3(4\lambda-3)}{3\pi^2 b_0^3(\lambda-1)^2(2\lambda-1)^2} \\
&- \frac{b_1^2 A_2}{12\pi^2 b_0^4(\lambda-1)^2(2\lambda-1)^2}\left[\lambda^2\left(22\lambda^2 - 30\lambda + 9\right) + 3(\lambda-1)^2\ln^2(1-2\lambda)\right. \\
&\left. + 3(\lambda-1)^2\ln(1-2\lambda) - 6(1-2\lambda)^2\ln(1-\lambda)(\ln(1-\lambda)+1)\right] \qquad (18) \\
&+ \frac{b_0\gamma_E^2 B_1(\lambda-2)\lambda}{4\pi(\lambda-1)^2} + \frac{b_1\gamma_E B_1\ln(1-\lambda)}{2\pi b_0(\lambda-1)^2} + \frac{b_2 B_1 \lambda^2}{4\pi b_0^2(\lambda-1)^2} \\
&+ \frac{b_1^2 B_1(\lambda - \ln(1-\lambda))(\lambda + \ln(1-\lambda))}{4\pi b_0^3(\lambda-1)^2} + \frac{b_0\gamma_E^3 A_1 \lambda\left(12\lambda^3 - 36\lambda^2 + 39\lambda - 14\right)}{6\pi(\lambda-1)^2(2\lambda-1)^2} \\
&+ \frac{b_1\gamma_E^2 A_1}{2\pi b_0(\lambda-1)^2(2\lambda-1)^2}\left[2(\lambda-1)^2\ln(1-2\lambda) - (1-2\lambda)^2\ln(1-\lambda)\right]
\end{aligned}$$

$$- \frac{b_1^2 \gamma_E A_1}{2\pi b_0^3 A_1 (\lambda-1)^2 (2\lambda-1)^2} \left[(4\lambda-3)\lambda^2 - (1-2\lambda)^2 \ln^2(1-\lambda) + (\lambda-1)^2 \ln^2(1-2\lambda) \right]$$
$$- \frac{b_1^3 A_1}{12\pi b_0^5 (\lambda-1)^2 (2\lambda-1)^2} \left[4(3-4\lambda)\lambda^3 + 2(1-2\lambda)^2 \ln(1-\lambda) \left(\ln^2(1-\lambda) - 3\lambda^2 \right) \right.$$
$$\left. + 12(\lambda-1)^2 \lambda^2 \ln(1-2\lambda) - (\lambda-1)^2 \ln^3(1-2\lambda) \right] + \frac{b_2 \gamma_E A_1 \lambda^2 (4\lambda-3)}{2\pi b_0^2 (\lambda-1)^2 (2\lambda-1)^2}$$
$$\frac{b_1 b_2 A_1}{12\pi b_0^4 (\lambda-1)^2 (2\lambda-1)^2} \left[\lambda^2 (2\lambda(3-7\lambda)+3) + 3\left(8\lambda^2 - 4\lambda + 1\right)(\lambda-1)^2 \ln(1-2\lambda) \right.$$
$$\left. - 6(1-2\lambda)^2 (2(\lambda-1)\lambda+1) \ln(1-\lambda) \right] + - \frac{b_3 A_1}{12\pi b_0^3 (\lambda-1)^2 (2\lambda-1)^2} \left[(2(\lambda-3)\lambda+3)\lambda^2 \right.$$
$$\left. - 6\left(2\lambda^2 - 3\lambda + 1\right)^2 \ln(1-\lambda) + 3\left(2\lambda^2 - 3\lambda + 1\right)^2 \ln(1-2\lambda) \right]$$

$$f_5(\lambda) = - \frac{A_5 \lambda^2 \left(4\lambda^4 - 18\lambda^3 + 33\lambda^2 - 24\lambda + 6\right)}{12\pi^5 b_0^2 (\lambda-1)^3 (2\lambda-1)^3} + \frac{B_4 \lambda \left(\lambda^2 - 3\lambda + 3\right)}{6\pi^4 b_0 (\lambda-1)^3} \tag{19}$$
$$- \frac{\gamma_E A_4 \lambda \left(7\lambda^2 - 9\lambda + 3\right)}{3\pi^4 b_0 (\lambda-1)^3 (2\lambda-1)^3} + \frac{b_1 A_4}{36\pi^4 b_0^3 (\lambda-1)^3 (2\lambda-1)^3} \left[7\lambda^2 (\lambda(\lambda(2\lambda(2\lambda-9)+33)-24) \right.$$
$$\left. + 6) - 6(\lambda-1)^3 \ln(1-2\lambda) + 12(2\lambda-1)^3 \ln(1-\lambda) \right] + \frac{\gamma_E B_3 \lambda \left(\lambda^2 - 3\lambda + 3\right)}{2\pi^3 (\lambda-1)^3} +$$
$$\frac{b_1 B_3 \left(\lambda^3 - 3\lambda^2 + 3\lambda + 3\ln(1-\lambda)\right)}{6\pi^3 b_0^2 (\lambda-1)^3} + \frac{\gamma_E^2 A_3 \lambda \left(8\lambda^5 - 36\lambda^4 + 66\lambda^3 - 69\lambda^2 + 39\lambda - 9\right)}{2\pi^3 (\lambda-1)^3 (2\lambda-1)^3}$$
$$- \frac{b_1 \gamma_E A_3}{3\pi^3 b_0^2 (\lambda-1)^3 (2\lambda-1)^3} \left[\lambda((9-7\lambda)\lambda - 3) + 3(\lambda-1)^3 \ln(1-2\lambda) \right.$$
$$\left. - 3(2\lambda-1)^3 \ln(1-\lambda) \right] + \frac{b_2 A_3 \lambda^3 \left(4\lambda^3 - 18\lambda^2 + 19\lambda - 6\right)}{4\pi^3 b_0^3 (\lambda-1)^3 (2\lambda-1)^3} -$$
$$\frac{b_1^2 A_3}{36\pi^3 b_0^4 (\lambda-1)^3 (2\lambda-1)^3} \left[\lambda^3 (\lambda(26\lambda(2\lambda-9)+303)-150) - 9(\lambda-1)^3 \ln^2(1-2\lambda) \right.$$
$$\left. + 24\lambda^2 - 6(\lambda-1)^3 \ln(1-2\lambda) + 6(2\lambda-1)^3 \ln(1-\lambda)(3\ln(1-\lambda)+2) \right]$$
$$+ \frac{b_0 \gamma_E^2 B_2 \lambda \left(\lambda^2 - 3\lambda + 3\right)}{2\pi^2 (\lambda-1)^3} - \frac{b_1 \gamma_E B_2 \ln(1-\lambda)}{\pi^2 b_0 (\lambda-1)^3} + \frac{b_1^2 B_2 \left(\lambda^3 - 3\lambda^2 + 3\ln^2(1-\lambda)\right)}{6\pi^2 b_0^3 (\lambda-1)^3}$$
$$- \frac{b_2 B_2 (\lambda-3)\lambda^2}{6\pi^2 b_0^2 (\lambda-1)^3} + \frac{b_0 \gamma_E^3 A_2 \lambda \left(24\lambda^5 - 108\lambda^4 + 198\lambda^3 - 193\lambda^2 + 99\lambda - 21\right)}{3\pi^2 (\lambda-1)^3 (2\lambda-1)^3}$$
$$+ \frac{b_1^2 \gamma_E A_2}{3\pi^2 b_0^3 (\lambda-1)^3 (2\lambda-1)^3} \left[\lambda^2 (\lambda(18\lambda-25)+9) + 3(\lambda-1)^3 \ln^2(1-2\lambda) \right.$$
$$\left. - 3(2\lambda-1)^3 \ln^2(1-\lambda) \right] - \frac{b_2 \gamma_E A_2 \lambda^2 \left(18\lambda^2 - 25\lambda + 9\right)}{3\pi^2 b_0^2 (\lambda-1)^3 (2\lambda-1)^3}$$

$$
\begin{aligned}
&+\frac{b_1^3 A_2}{36\pi^2 b_0^5(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(2\lambda(50\lambda-171)+339)-114)+6)\\
&-6(\lambda-1)^3\ln^3(1-2\lambda)+12(2\lambda-1)^3\ln(1-\lambda)\left(-3\lambda+\ln^2(1-\lambda)+1\right)\\
&+6(6\lambda-1)(\lambda-1)^3\ln(1-2\lambda)\Big]+\frac{b_3A_2\lambda^3\left(20\lambda^3-54\lambda^2+45\lambda-12\right)}{12\pi^2 b_0^3(\lambda-1)^3(2\lambda-1)^3}\\
&-\frac{b_1\gamma_E^2 A_2}{\pi^2 b_0(\lambda-1)^3(2\lambda-1)^3}\Big[2(\lambda-1)^3\ln(1-2\lambda)-(2\lambda-1)^3\ln(1-\lambda)\Big]\\
&-\frac{b_2 b_1 A_2}{18\pi^2 b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(4\lambda(20\lambda-63)+237)-75)+3)\\
&+3(6\lambda-1)(\lambda-1)^3\ln(1-2\lambda)-6(2\lambda-1)^3(3\lambda-1)\ln(1-\lambda)\Big]\\
&+\frac{b_0^2\gamma_E^3 B_1\lambda\left(\lambda^2-3\lambda+3\right)}{6\pi(\lambda-1)^3}-\frac{b_1^2\gamma_E B_1\left(\lambda-\ln^2(1-\lambda)+\ln(1-\lambda)\right)}{2\pi b_0^2(\lambda-1)^3}-\\
&\frac{b_1^3 B_1}{12\pi b_0^4(\lambda-1)^3}(\lambda+\ln(1-\lambda))\left(2\lambda^2-3\lambda+2\ln^2(1-\lambda)-2\lambda\ln(1-\lambda)-3\ln(1-\lambda)\right)\\
&+\frac{b_2\gamma_E B_1\lambda}{2\pi b_0(\lambda-1)^3}-\frac{b_3 B_1\lambda^2(2\lambda-3)}{12\pi b_0^2(\lambda-1)^3}+\frac{b_1\gamma_E^2 B_1\left(\lambda^3-3\lambda^2+3\lambda-2\ln(1-\lambda)\right)}{4\pi(\lambda-1)^3}\\
&+\frac{b_2 b_1 B_1\lambda\left(2\lambda^2-3\lambda-3\ln(1-\lambda)\right)}{6\pi b_0^3(\lambda-1)^3}+\frac{b_0^2\gamma_E^4 A_1}{12\pi(\lambda-1)^3(2\lambda-1)^3}\lambda(2\lambda-3)\Big[28\lambda^4\\
&-84\lambda^3+105\lambda^2-63\lambda+15\Big]+\frac{b_1^3\gamma_E A_1}{6\pi b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(4\lambda(3(\lambda-3)\lambda+8)-9)\\
&-2(\lambda-1)^3\ln^3(1-2\lambda)+3(\lambda-1)^3\ln^2(1-2\lambda)+12\lambda(\lambda-1)^3\ln(1-2\lambda)\\
&+(2\lambda-1)^3\ln(1-\lambda)(\ln(1-\lambda)(2\ln(1-\lambda)-3)-6\lambda)\Big]\\
&+\frac{b_1^4 A_1}{24\pi b_0^6(\lambda-1)^3(2\lambda-1)^3}\Big[8(\lambda-1)^3\lambda^2(4\lambda-3)\ln(1-2\lambda)\\
&-2(2\lambda-1)^3\ln(1-\lambda)\left(2(2\lambda-3)\lambda^2+\ln(1-\lambda)((\ln(1-\lambda)-2)\ln(1-\lambda)-6\lambda)\right)\\
&+2\lambda^3\left(\lambda\left(-28\lambda^2+54\lambda-33\right)+6\right)+(\lambda-1)^3\ln^4(1-2\lambda)-2(\lambda-1)^3\ln^3(1-2\lambda)-\\
&12(\lambda-1)^3\lambda\ln^2(1-2\lambda)\Big]-\frac{b_2\gamma_E^2 A_1\lambda\left(4\lambda^3-6\lambda+3\right)}{2\pi b_0(\lambda-1)^3(2\lambda-1)^3}+\frac{b_3\gamma_E\lambda^2\left(12\lambda^3-36\lambda^2+32\lambda-9\right)}{6\pi b_0^2(\lambda-1)^3(2\lambda-1)^3}\\
&+\frac{b_2^2 A_1}{36\pi b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(6-\lambda(\lambda(2\lambda(10\lambda+9)-69)+42))\\
&-12\left(2\lambda^2-3\lambda+1\right)^3\ln(1-\lambda)+6\left(2\lambda^2-3\lambda+1\right)^3\ln(1-2\lambda)\Big]\\
&+\frac{b_1^2\gamma_E^2 A_1}{2\pi b_0^2(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda\left(4\lambda^3-6\lambda+3\right)+2(\lambda-1)^3\ln^2(1-2\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&- 2(\lambda-1)^3\ln(1-2\lambda) - (2\lambda-1)^3(\ln(1-\lambda)-1)\ln(1-\lambda)\Big] \\
&- \frac{b_2 b_1^2 A_1}{36\pi b_0^5(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(2(153-82\lambda)\lambda-165)+12)+6) \\
&- 18\lambda(\lambda-1)^3\ln^2(1-2\lambda) + 6\left(6(1-2\lambda)^2\lambda-1\right)(\lambda-1)^3\ln(1-2\lambda) \\
&- 6(2\lambda-1)^3\ln(1-\lambda)\left(6\lambda(\lambda-1)^2 - 3\lambda\ln(1-\lambda)-2\right)\Big] \\
&+ \frac{b_1\gamma_E^3 A_1}{6\pi(\lambda-1)^3(2\lambda-1)^3}\Big[24\lambda^6 - 108\lambda^5 + 198\lambda^4 - 193\lambda^3 + 16\lambda^3\ln(1-\lambda) \\
&- 8\lambda^3\ln(1-2\lambda) + 99\lambda^2 - 24\lambda^2\ln(1-\lambda) + 24\lambda^2\ln(1-2\lambda) - 21\lambda \\
&+ 12\lambda\ln(1-\lambda) - 24\lambda\ln(1-2\lambda) - 2\ln(1-\lambda) + 8\ln(1-2\lambda)\Big] \\
&- \frac{b_2 b_1\gamma_E A_1}{3\pi b_0^3(\lambda-1)^3(2\lambda-1)^3}\lambda\Big[\lambda(4\lambda(3(\lambda-3)\lambda+8)-9) + 6(\lambda-1)^3\ln(1-2\lambda) \\
&- 3(2\lambda-1)^3\ln(1-\lambda)\Big] + \frac{b_3 b_1 A_1}{18\pi b_0^4(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^3(\lambda(4(18-7\lambda)\lambda-51)+6) \\
&+ 3\lambda^2 + 3(2\lambda(\lambda(8\lambda-9)+3)-1)(\lambda-1)^3\ln(1-2\lambda) \\
&- 3(2\lambda-1)^3(\lambda(\lambda(4\lambda-9)+6)-2)\ln(1-\lambda)\Big] \\
&- \frac{b_4 A_1}{36\pi b_0^3(\lambda-1)^3(2\lambda-1)^3}\Big[\lambda^2(\lambda(\lambda(2\lambda(2\lambda-9)+33)-24)+6) \\
&- 12\left(2\lambda^2-3\lambda+1\right)^3\ln(1-\lambda) + 6\left(2\lambda^2-3\lambda+1\right)^3\ln(1-2\lambda)\Big]
\end{aligned}
$$

In eqs. (17) to (19) we have removed some constant terms in order to make them homogeneous, *i.e.* $f_i(0) = 0$, those constant can be reabsorbed in the C-term **??**.

The functions $f_1(\lambda)$, $f_2(\lambda)$ and $f_3(\lambda)$ were already known in the literature. Our results agree with those in [2], [4], [5] [1]. In [4], the resummation coefficients $f_i(\lambda)$ were derived from a similar integral but instead of Laplace space they're problem factorizes in Mellin space, since in the large N limit the Mellin and Laplace transforms are equivalent **??**, we obtain their same functional form and thus compare our results. In [5], they applied Soft-Collinear Effective Theory (SCET) theory, in which contributions from Hard (H), Collinear (or Jet, J) and Soft (S) functions are treated separately at their respective scales [6]: the hard scale of the e^-e^+ collision $\mu_H \sim Q$, the invariant mass of the two back-to-back jets eq. (1) $\mu_J^2 \sim k_1^2 + k_2^2 \sim \tau Q^2$ and the soft scale $\mu_S \sim \frac{\mu_J^2}{\mu_H} \sim \tau Q$. These functions are evaluated at their respective scales and then evolved to a common scale μ using renormalization group equations **??**. The cross section is then factorized into a convolution of these functions and the resummation coefficients are obtained by performing the Laplace transform of the convolution. The results of [5] are in agreement with our results;

[1] We assume there is a misprint in $f_3(\lambda)$ in [5], the last term proportional $A_1\beta_1^2$ in parethesis should have a plus sign instead of the minus sign.

in fact their $f_1(\lambda)$ and $f_2(\lambda)$ are the same [2] as the one originally obtained in [2], however $f_3(\lambda)$ has two additional terms, one proportional to $c_s^{(1)}$ and the other to $c_j^{(1)}$ which we absorbed in the coefficients of A_3 and B_2.

The functions $f_4(\lambda)$ and $f_5(\lambda)$, as far as we know they never appeared in the literature and are one the main results of the present work.

All the relevant constant can be found in **??**. The coefficients A_1, A_2 and B_1 were already known in [2] and were obtained from the one and two loop splitting functions. A_3 and B_2 were obtained by comparing out $f_3(\lambda)$ eq. (17) with the corresponding expression in Soft Collinear Effective Theory (SCET) [5] (Eq(4.17)) by absorbing the jet $c_J^{(1)}$ and soft $c_S^{(1)}$ terms into their A_3 and B_2 terms. We also compared our G_{21} (in which B_2 appear) and G_{32} (in which both A_3 and B_2 appear) with the exact expression of G_{21} and G_{32} in [6] and found perfect agreement.

B_3 was extracted by comparing our G_{31} coefficient in **??** with the exact expression of the G_{31} coefficient in [5] A_4, A_5 and B_4 were derived from [7] by incorporating the hard form factor of [8] as appropriate.

The same procedure can be applied to obtain A_4 and B_3 by calculating the expression of $f_4(\lambda)$ in SCET and absorbing the jet $c_J^{(1)}, c_J^{(2)}$ and soft $c_S^{(1)}, c_S^{(2)}$ terms given in [5] into their A_4 and B_3 terms. While A_5 and B_4 (and also the lower order coefficients) can be derived from [7] by incorporating the hard form factor of [8] as appropriate.

To obtain the Laplace trasform of the thrust distribution **??**, we perform the Laplace transform of the convolution eq. (2). By applying the convolution theorem, we find that it is twice the integral given by **??** that we have just calculated. Therefore, we multiply by 2 the $f_i(\lambda)$ we just obtained eqs. (15) to (19).

[2] The results differ by a factor 2 because we are considering the jet mass distribution, while the thrust distributions is the convolution of two jet mass distributions eq. (2).

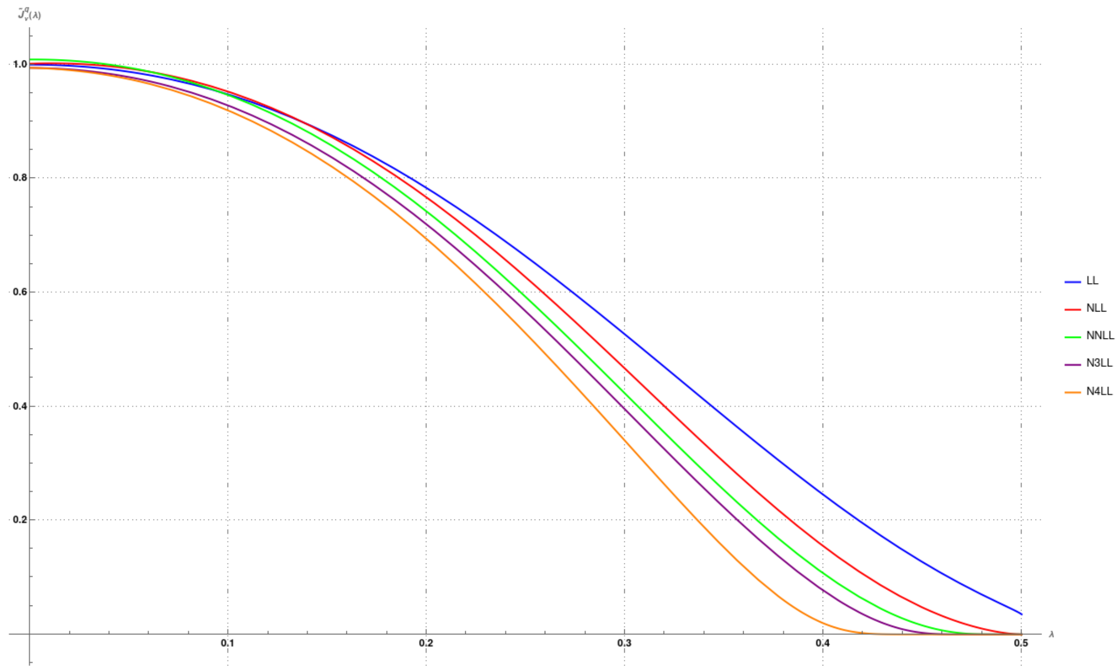

Figure 1: Plot of J^q_ν function defined in eq. (9) at different logarithmin orders.