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Thrust distribution in e^+e^- annihilation

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1 Introduction

In this thesis we are interested in a specific aspect of theoretical high energy physics. In particular, the topic of interest is that of resummantion of the Thrust event-shape distribution in electron-positron (e^+e^-) collisions.

As is typical in physics, the equations that govern the interactions are quite complicated and it is almost impossible to find exact solutions, so all functions of interest are perturbatively expanded, meaning they are expanded in a power series of a small parameter.

When the force in question is the electromagnetic interaction, one can use the fine structure constant (or electromagnetic coupling constant) $\alpha_{em} \sim \frac{1}{137}$.

When discussing particles that interact with the strong interaction, it is then natural to use the strong coupling constant α_S.

The kinematical variable we are interested in is the Thrust event-shape distribution,

which T is defined as:

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|} \stackrel{\text{def}}{=} 1 - \tau \tag{1}$$

where the sum is over all final state particles and $\vec{n}$ is a unit vector. It can be seen from this definition that the thrust is an infrared and collinear safe quantity, that is, it is insensitive to the emission of zero momentum particles and to the splitting of one particle into two collinear ones.

In fact, contribution from soft particles with $\vec{p_i} \to 0$ drop out, and collinear splitting does not change the thrust: $|(1-\lambda)\vec{p_i} \cdot \vec{n}| + |\lambda\vec{p_i} \cdot \vec{n}| = |\vec{p_i} \cdot \vec{n}|$ and $|(1-\lambda)\vec{p_i}| + |\lambda\vec{p_i}| = |\vec{p_i}|$

Formally, infrared-safe observables are the one which do not distinguish between (n+1) partons and n partons in the soft/collinear limit, *i.e*, are insensitive to what happens at long-distance.

Infra-red safe observables are important in the context of perturbative QCD, because they allow for a meaningful comparison between theory and experiment.

Thus the cross section :

$$\sigma(\tau) = \int_{1-\tau}^{1} \mathrm{d}T \, \frac{\mathrm{d}\sigma}{\mathrm{d}T} \tag{2}$$

It can be seen that a two-particle final state has fixed $T = 1$, consequently the thrust distribution receives its first non-trivial contribution from three-particle final states

The lower limit on T depends on the number of final-state particles. Neglecting masses, $T_{min} = 2/3$ for three particles, corresponding to a symmetric configuration. For four particles the minimum thrust corresponds to final-state momenta forming the vertices of a regular tetrahedron, each making an angle $\cos^{-1}(1/\sqrt(3))$ with respect to the thrust axis. Thus $T_{min} = 1/\sqrt(3) = 0.577$ in this case . For more than four particles, T_{min} approaches 1/2 from above as the number of particles increases.

At large values o f T, however, there are terms in higher order that become enhanced by powers o f $\ln(1-T)$. In this kinematical region the real expansion parameter is the large effective coupling $\alpha_s \ln^2(\tau)$ and therefore any finite-order perturbative calculation cannot give an accurate evaluation o f the cross section.

2 Resummation

2.1 Soft-gluon effects in QCD cross sections

Soft-gluon effects and the soft gluon exponentiation are reviewed in [7] and [8], here i summarize the physical motivation and main idea of the resummation of soft gluon effects in QCD.

The finite energy resolution inherent in any particle detector implies that the physical cross sections, those experimentally measured, inherently incorporate all contributions from arbitrarily soft particles produced in the final state. In other words, because we lack the ability to precisely resolve the energy of soft particles, we are unable to distinguish between their presence or absence in our calculations. Consequently, we must account for the sum over all possible final states.

This inclusiveness is essential in QCD calculations. Higher order perturbative contributions due to *virtual* gluons are infrared divergent and the divergences are exactly cancelled by radiation of undetected *real* gluons. speaking the cancellation does not necessarily take place *order by order* in perturbative theory. In particular kinematic configurations, *e.g* Thrust in the dijet limit $T \to 1$, real and virtual contributions can be highly unbalanced, because the emission of real radiation is inhibited by kinematic constrainsts, spoiling the cancellation mechanism. As a result, soft gluon contribution to QCD cross sections can still be either large or singular.

In these cases, the cancellation of infrared divergences bequeaths higher order contributions of the form:

$$C_{nm}\alpha_s^n \ln^m\left(\frac{1}{1-T}\right), \quad \text{with } m \le 2n, \tag{3}$$

that can become large, $\alpha_s \ln^2(1-x) \lesssim 1$, even if the QCD coupling α_s is in the perturbative regime
$\alpha_s \ll 1$. The logarithmically enhanced terms in eq. (3) are certainly relevant near the dijet limit $T- > 1$. In these cases, see [5],[6], the theoretical predictions can be improved by evaluating soft gluon contributions to high orders and possibly resumming to all of them in α_s.

2.2 Resummation of soft-gluon effects

The physical bases for all order summation of soft gluon contributions to QCD cross sections are the following: to QCD are dynamics and kinematics factorizations. The first factorization follows from gauge invariance and unitarity: in the soft limit multi-gluon amplitudes fulfil generalized factorization formulae given in terms of a single gluon emission probability that is universal *i.e* process independent. The second factorization regards kinematics and is strongly dependent on the observable to because computed.

If, in the appropriate soft limit, the phase space for this observable can be written in a factorized way, resummation is feasible in form of generalized exponentiation of the single gluon emission probability. Then exponentiation allows one to dfine and carry out an improved perturbative expansion that systematically resums LL, terms, NLL terms and so on.

In general the phase space is not factorizable in single particle contributions. If when factorizable, it does not occur in the space where the physical observable x is defined. Usually, it is necessary to introduce a conjugate space where the physical observable x is defined. Usually, it is necessary to introduce a conjugate space to overcome phase-space constraints.

A typical example is the energy conservation constraints that can be factorized by working in N-moment space, N being the variable conjugate to the energy x via a Mellin (or Laplace) transformation.

Large or Singular soft gluon contributions can have different origins and resummation takes different exponentiation forms depending on kinematics. This leads to varieties of Sudakov effects.

One of the effects is Sudakov suppression: Soft gluon resummation produces suppression of cross sections near the exclusive phase space boundary. Typical examples are e^+e^- event shapes distributions like Thrust T.

3 QCD running coupling

Classically, the force between two sources is then given by $F = \frac{\alpha}{r^2}$, characterized by a universal coefficient – the coupling constant α, which quantifies the force between two static bodies of unit “charge” at distance r, *i.e.*, the electric charge for QED, the color charge for QCD, the weak isospin for the weak force, or the mass for gravity. Consequently, the coupling α is defined as being proportional to the elementary charge squared, *e.g.*, $\alpha_{em} \equiv \frac{e^2}{4\pi}$ where e is the elementary electric charge, or $\alpha_s \equiv \frac{g^2}{4\pi}$ where g is the elementary gauge field coupling in QCD. In quantum field theory (QFT), $\frac{1}{r^2}$ is the coordinate-space expression for the propagator of the force carrier (gauge boson) at leading-order in perturbation theory: in momentum space, the analogous propagator is proportional to $\frac{1}{q^2}$, where q the boson 4-momentum ($Q^2 = -q^2 > 0$).

For sources interacting weakly, the one-boson exchange representation of interactions is a good first approximation. However, when interactions become strong (with "strong" to be defined below), higher orders in perturbation theory become noticeable and the $\frac{1}{r^2}$ law no longer stands. In such cases, it makes good physics sense to fold the extra r-dependence into the coupling, which thereby becomes r, or equivalently Q^2, dependent.

The running of the coupling is due to vacuum polarization, the vacuum is not empty, but is filled with virtual particles that are constantly created and annihilated which can interact with the propagating particles, leading to a modification of the interaction strength.

While in QED, the extra r-dependence comes only from the vacuum polarization. In QCD, α_s receives contributions from the vacuum polarization and from gluon self-interactions since the gluon has a color charge.

The two couplings have opposite trends: the QED coupling increases with energy and the theory becomes strongly coupled at high energies, whereas the opposite happens for the QCD coupling as it is large at low energies and decreases with energy. This property of being weakly coupled at high energies is known as *asymptotic freedom* and it means that perturbative calculations in QCD can only be done at high energies where α_s becomes small enough that a power expansion is meaningful.

In the framework of perturbative QCD (*pQCD*), predictions for observables are expressed in terms of the renormalized coupling $\alpha = \alpha(\mu^2)$, a function of an (unphysical) renormalization scale μ_R . The coupling satisfies the following renormalization group equation (RGE):

$$\mu^2 \frac{d\alpha}{d\mu^2} = \beta(\alpha) = -\left(b_0\alpha^2 + b_1\alpha^3 + b_2\alpha^4 + \ldots\right) \tag{4}$$

where $b_0 = \frac{11C_A - 4n_f T_R}{12\pi} = \frac{33-2n_f}{12\pi}$ is the 1-loop β-function coefficient, $b_1 = \frac{17C_A^2 - n_f T_R(10C_A+6C_F)}{24\pi^2} = \frac{153-19n_f}{24\pi^2}$ is the 2-loop coefficient, $b_2 = \frac{2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2}{128\pi^3}$ is the 3-loop coefficient. $C_A = 3$ and $C_F = \frac{4}{3}$ are the Casimir operators of the adjoint and fundamental representations of $SU(3)$, $T_R = \frac{1}{2}$ is the trace normalization, n_f is the number of active quark flavors.

It is not possible to solve eq. (4) as it is for two reasons: only the first few b_n coefficients are known (up to b_4); the exact equation becomes more and more complicated as more terms of the series are included, making it impossible to obtain an analytic solution.

In order to solve both problems, the equation is solved in the following way: at first only b_0 is included and the obtained solution is called α_{LO} , as it will only contain a term proportional to α ; then also b_1 is included and only terms up to the second order in α are kept to obtain α_{NLO} ; this same procedure is used to obtain α_{NNLO} , α_{N^3LO} , α_{N^4LO} . There will be a complication in calculating α_{NLO} and higher orders which will be explained and resolved in the following sections.

3.1 One-loop running coupling

The one-loop running coupling α_{LO} is obtained by solving the RGE with only the first term of the β-function:

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0\alpha^2 \tag{5}$$

This equation can be solved by separation of variables and imposing the boundary condition $\alpha(Q^2) = \alpha_s$:

$$\int_{\alpha(Q^2)}^{\alpha(\mu^2)} \frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0 \frac{\mathrm{d}\mu^2}{\mu^2} \tag{6}$$

and one obtains:

$$\alpha_{LO}(\mu^2) = \frac{\alpha_s}{1 + b_0 \alpha_s \log\left(\frac{\mu^2}{Q^2}\right)} \tag{7}$$

In which we can confirm the decreasing with energy trend of the running coupling that we anticipated above.

It is useful to define the variable $\lambda_\mu = b_0 \alpha_s \log\left(\frac{\mu^2}{Q^2}\right)$ so that:

$$\alpha_{LO}(\mu^2) = \frac{\alpha_s}{1 + \lambda_\mu} \tag{8}$$

3.2 Two-loop running coupling

In order to obtain the two-loop running coupling α_{NLO}, we need to solve the RGE with the first two terms of the β-function eq. (4):

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0 \alpha^2 - b_1 \alpha^3 \tag{9}$$

but this equation is not solvable in a straightforward way as the one-loop equation, we have to use the perturbative approach. We can rewrite the equation as:

$$\frac{d\alpha}{d\mu^2} = -\frac{b_0 \alpha^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha) \tag{10}$$

and expand the α term in the parenthesis as:

$$\alpha = \alpha_{LO} + \delta\alpha \tag{11}$$

where α_{LO} is the one-loop running coupling and $\delta\alpha$ contains the higher order correction, one obtains:

$$\frac{d\alpha}{d\mu^2} = -\frac{b_0\alpha^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{LO} - \frac{b_0}{b_1}\delta\alpha) \tag{12}$$

Observe that in parenthesis, by keeping 1 gave us the one-loop running coupling, by keeping $\frac{b_1}{b_0}\alpha_{LO}$ we can obtain the first order corrections and $\delta\alpha$ are needed for higher order corrections. The equation to solve is then:

$$\int_{\alpha_s}^{\alpha(\mu^2)} -\frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0\frac{\mathrm{d}\mu^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{LO}(\mu^2)) \tag{13}$$

Using *Mathematica* to solve this equation, we obtain the two-loop running coupling:

$$\alpha_{NLO}(\mu^2) = \frac{\alpha_s}{1 + \lambda_\mu + \alpha_s\frac{b_1}{b_0}\log(1 + \lambda_\mu)} \tag{14}$$

in which the expansion in powers of α_s is not explicit. One can expand in powers of α_s by keeping λ_μ fixed and only keeping terms up to $\mathcal{O}(\alpha_s^2)$ by doing so one obtains:

$$\alpha_{NLO}\big(\mu^2\big) = \alpha_{LO}\big(\mu^2\big) - \frac{b_1}{b_0}\alpha_{LO}^2\big(\mu^2\big)\log(1 + \lambda_\mu) + \mathcal{O}(\alpha_s^2) \tag{15}$$

We found the correction:

$$\delta\alpha_{NLO}\big(\mu^2\big) = -\frac{b_1}{b_0}\alpha_{LO}^2\big(\mu^2\big)\log(1 + \lambda_\mu) \tag{16}$$

By repeating the same procedure, one can obtain the three-loop running coupling α_{NNLO} and so on.

3.3 Higher order corrections

In order to calculate higher order corrections, one need to be careful of the powers of α need for the desired order, and the contributions to various orders of α_s may not be immediately apparent, but they are straightforward to compute. Expand the running coupling in powers of α_s:

$$\alpha = \alpha_{LO} + \delta\alpha_{NLO} + \delta\alpha_{NNLO} + \delta\alpha_{N^3LO} + \delta\alpha_{N^4LO} + \dots \tag{17}$$

with $\delta\alpha_{NLO} = \mathcal{O}(\alpha_s)$, $\delta\alpha_{NLO} = \mathcal{O}(\alpha_s^2)\delta\alpha_{NNLO} = \mathcal{O}(\alpha_s^3)$, $\delta\alpha_{N^3LO} = \mathcal{O}(\alpha_s^4)$, $\delta\alpha_{N^4LO} = \mathcal{O}(\alpha_s^5)$, and so on. We present these contributions in the following table:

Power	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$	$\mathcal{O}(\alpha_s^5)$
α	α_{LO}	$\delta\alpha_{NLO}$	$\delta\alpha_{NNLO}$	$\delta\alpha_{N^3LO}$	$\delta\alpha_{N^4LO}$
α^2		α_{LO}^2	$2\alpha_{LO}\delta\alpha_{NLO}$	$\delta\alpha_{NLO}^2 + 2\alpha_{LO}\delta\alpha_{NNLO}$	$2\alpha_{LO}\delta\alpha_{N^3LO} + 3\alpha_{LO}\delta\alpha_{NLO}^2$
α^3			α_{LO}^3	$3\alpha_{LO}^2\delta\alpha_{NLO}$	$3\alpha_{LO}^2\delta\alpha_{NNLO} + 3\alpha_{LO}\delta\alpha_{NLO}^2$
α^4				α_{LO}^4	$4\alpha_{LO}^3\delta\alpha_{NLO}$
α^5					α_{LO}^5

Table 1: Contributions to different powers of α_s.

For the three-loop running coupling, the equation to solve is:

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0\alpha^2(1 - \frac{b_1}{b_0}\alpha - \frac{b_2}{b_0}\alpha^2) \tag{18}$$

One can substitute the expansion of $\alpha = \alpha_{LO} + \delta\alpha_{NLO} + \mathcal{O}(\alpha_s^2)$ in powers of α_s and retain only terms up to $\mathcal{O}(\alpha_s^2)$ with this prescription the equation to solve is:

$$\int_{\alpha_s}^{\alpha(\mu^2)} -\frac{\mathrm{d}\alpha}{\alpha^2} = \int_{Q^2}^{\mu^2} -b_0\frac{\mathrm{d}\mu^2}{\mu^2}(1 - \frac{b_1}{b_0}\alpha_{NLO}(\mu^2) - \frac{b_2}{b_0}\alpha_{LO}^2(\mu^2)) \tag{19}$$

solve and obtain the three-loop running coupling α_{NNLO}:

$$\alpha_{NNLO}(\mu^2) = \alpha_{LO}\big(\mu^2\big) + \delta\alpha_{NLO}\big(\mu^2\big) + \delta\alpha_{NNLO}\big(\mu^2\big) \tag{20}$$

with

$$\delta\alpha_{NNLO}\big(\mu^2\big) = \frac{\alpha_{LO}^3(\mu^2)}{b_0^2}\left(b_1^2\lambda_\mu - b_0b_2\lambda_\mu + b_1^2\log^2(1+\lambda_\mu) - b_1^2\log(1+\lambda_\mu)\right) \tag{21}$$

Similarly one can obtain the four-loop running coupling α_{N^3LO} and five-loop running coupling α_{N^4LO}.

$$\alpha_{N^3LO}(\mu^2) = \alpha_{LO}\left(\mu^2\right) + \delta\alpha_{NLO}\left(\mu^2\right) + \delta\alpha_{NNLO}\left(\mu^2\right) + \delta\alpha_{N^3LO}\left(\mu^2\right) \tag{22}$$

$$\begin{aligned}\delta\alpha_{N^3LO}\left(\mu^2\right) = \frac{\alpha_{LO}^4(\mu^2)}{2b_0^3}\Big(&- \left(b_1^3 - 2b_0b_2b_1 + b_0^2b_3\right)\lambda_\mu^2 \\ &- \left(2b_0^2b_3 - 2b_0b_1b_2\right)\lambda_\mu - 2b_1^3\log^3\left(\lambda_\mu+1\right) + 5b_1^3\log^2\left(1+\lambda_\mu\right) \\ &+ \left(2b_0b_1b_2\left(2\lambda_\mu - 1\right) - 4b_1^3\lambda_\mu\right)\log\left(1+\lambda_\mu\right)\Big)\end{aligned} \tag{23}$$

$$\alpha_{N^4LO}(\mu^2) = \alpha_{LO}\left(\mu^2\right) + \delta\alpha_{NLO}\left(\mu^2\right) + \delta\alpha_{NNLO}\left(\mu^2\right) + \delta\alpha_{N^3LO}\left(\mu^2\right) + \delta\alpha_{N^4LO}\left(\mu^2\right) \tag{24}$$

$$\begin{aligned}\delta\alpha_{N^4LO} = \frac{\alpha_{LO}^5}{6b_0^4}\Big(&\left(2b_1^4 - 6b_0b_2b_1^2 + 4b_0^2b_3b_1 + 2b_0^2b_2^2 - 2b_0^3b_4\right)\lambda_\mu^3 \\ &+ \left(9b_1^4 - 24b_0b_2b_1^2 + 9b_0^2b_3b_1 + 12b_0^2b_2^2 - 6b_0^3b_4\right)\lambda_\mu^2 \\ &+ \left(6b_0^2b_1b_3 - 6b_0^3b_4\right)\lambda_\mu + 6b_1^4\log^4(1+\lambda_\mu) \\ &- 26b_1^4\log^3\left(\lambda_\mu+1\right) + 9\left(\left(2b_1^4 - 2b_0b_1^2b_2\right)\lambda_\mu + b_1^4 + 2b_0b_2b_1^2\right)\log^2(1+\lambda_\mu) \\ &+ \left(6b_1\left(b_1^3 - 2b_0b_2b_1 + b_0^2b_3\right)\lambda_\mu^2 + 6b_1\left(-3b_1^3 + b_0b_2b_1 + 2b_0^2b_3\right)\lambda_\mu \right. \\ &\left. - 6b_1b_3b_0^2\right)\log(1+\lambda_\mu)\Big)\end{aligned} \tag{25}$$

4 Calculation of the f functions

In the two-jet region the fixed-order thrust distribution is enhanced by large infrared logarithms which spoil the convergence of the perturbative series. The convergence can be restored by resumming the logarithms to all orders in the coupling constant α_s.

According to general theorems [2],[10],[11], the cross section eq. (2) has a power series expansion in $\alpha_s(Q^2)$ of the form:

$$\frac{\sigma(\tau, Q^2)}{\sigma_t} = C\big(\alpha_s(Q^2)\big)\Sigma\big(\tau, \alpha_s(Q^2)\big) + F\big(\tau, \alpha_s(Q^2)\big) \tag{26}$$

where σ_t is the total hadronic cross section and

$$C(\alpha_s) = 1 + \sum_{n=1}^{\infty} C_n \alpha_s^n \tag{27}$$

$$\Sigma(\tau, \alpha_s) = \exp\left[\sum_{n=1}^{\infty} \alpha_s^n \sum_{m=1}^{2n} G_{nm} \ln^m \tau\right] \tag{28}$$

$$F(\tau, \alpha_s) = \sum_{n=1}^{\infty} \alpha_s^n F_n(\tau) \tag{29}$$

Here C_n and G_{nm} are constants, while $F_n(\tau)$ are perturbatively computable functions that vanish at small τ. Thus at small τ (large thrust) it becomes morst important to resum the series of large logarithms in $\Sigma(\tau, \alpha_s)$. These are normally classified as *leading* logarithms when $n < m \leq 2n$, *next-to-leading* when $m = n$ and *subdominant* logarithms when $m < n$.

The term eq. (28) can be re-written as

$$\Sigma(\tau, \alpha_s) = \exp\left\{L f_1(\lambda) + \sum_{i=0}^{\infty} f_{i+1}(\lambda)\alpha_s^i\right\} \tag{30}$$

where $L = \ln N = \ln(\nu Q^2)$ and $\lambda = \alpha_s b_0 L$. We require the functions $f_i(\lambda)$ to be omogeneous,$i.e$ $f_i(0) = 0$, so that at $N^n LL$ we can write:

$$f_{n+1}(\lambda) = \sum_{k \geq n} \tilde{G}_{k,k+1-n} \alpha_s^k L^{k+1-n} \tag{31}$$

this requirement is automatically satisfied if we choose as variable $L = \ln\left(\frac{N}{N_0}\right)$ where $N_0 = e^{-\gamma_E}$, $\gamma_E = 0.5772\ldots$ being the Euler-Mascheroni constant. With the latter choice the terms proportional to γ_E and its powers disappear. The advantage of the variable N is that the total rate is directly reproduced by setting $N = 1$, while in the variable $n = N/N_0$ it is given $f_{n=1/N_0}$. These two choices differ only by terms of higher order in γ_E.

In the article by Catani, Turnock, Webber and Trentadue [6], it was observed that for a final state configuration corresponding to a large value of thrust , eq. (1) can be approximated by

$$\tau = 1 - T \approx \frac{k_1^2 + k_2^2}{Q^2} \tag{32}$$

where k_1^2 and k_2^2 are the invariant masses squared of two back-to-back jets and Q^2 is the energy of the center of mass. Thus the key to the evaluation of the thrust distributions is its relation to the quark jet mass distribution $J^q(Q^2, k^2) = J^q_{k^2}(Q^2)$ which denotes the probability of jet invariant mass-squared k^2 at scale Q^2, then the thrust fraction

$$R_T(\tau, \alpha_s(Q^2)) = \frac{\sigma(\tau, Q^2)}{\sigma_t} = \frac{1}{\sigma_t}\int_0^1 \frac{\mathrm{d}\sigma(\tau', Q^2)}{\mathrm{d}\tau'}\Theta(\tau - \tau')\mathrm{d}\tau' \tag{33}$$

takes the form of a convolution of two jet mass distributions $J(Q^2, k_1^2)$ and $J(Q^2, k_2^2)$

$$R_T(\tau, \alpha_s(Q^2)) \underset{\tau\ll 1}{=} \int_0^\infty \mathrm{d}k_1^2 \int_0^\infty \mathrm{d}k_2^2 J^q_{k_1^2}(Q^2) J^q_{k_2^2}(Q^2)\Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) \tag{34}$$

Introducing the Laplace transform of the jet mass distribution

$$\tilde{J}^q_\nu(Q^2) = \int_0^\infty J^q(Q^2, k^2)e^{-\nu k^2}\mathrm{d}k^2 \tag{35}$$

and using the integral representation of the Heaviside step function

$$\Theta\left(\tau - \frac{k_1^2 + k_2^2}{Q^2}\right) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{\mathrm{d}N}{N} e^{N\tau} e^{-N\frac{k_2^2+k_2^2}{Q^2}} \tag{36}$$

by substituting eq. (36) into eq. (34) and $N = \nu Q^2$:

$$\begin{aligned} R_T(\tau) &= \int_{C-i\infty}^{C+i\infty} \frac{\mathrm{d}N}{N}\frac{e^{N\tau}}{2\pi i}\left[\int_0^\infty \mathrm{d}k_1^2 e^{-\nu k_1^2} J^q_{k_1^2}(Q^2)\right]\left[\int_0^\infty \mathrm{d}k_2^2 e^{-\nu k_2^2} J^q_{k_2^2}(Q^2)\right] \\ &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{N\tau}\left[\tilde{J}^q_\nu(Q^2)\right]^2 \frac{\mathrm{d}N}{N} \end{aligned} \tag{37}$$

where C is a real positive constant to the right of all singularities of the integrand $\tilde{J}_\nu(Q^2)$ in the complex ν plane.

An integral representation for the Laplace transform $\tilde{J}_\nu(Q^2)$ is given by

$$\ln \tilde{J}^q_\nu(Q^2) = \int_0^1 \frac{\mathrm{d}u}{u}\left(e^{-u\nu Q^2} - 1\right)\left[\int_{u^2Q^2}^{uQ^2} \frac{1}{q^2} A\big(\alpha_s(q^2)\big)\mathrm{d}q^2 + \frac{1}{2} B\big(\alpha_s(uQ^2)\big)\right] \tag{38}$$

with

$$A(\alpha_s) = \sum_{n=1}^{\infty} \frac{A_n}{\pi^n}\alpha_s^n \qquad\qquad B(\alpha_s) = \sum_{n=1}^{\infty} \frac{B_n}{\pi^n}\alpha_s^n$$

Function $A(\alpha_s)$ is associated with the cusp anomalous dimension and governs the exponentiation of the leading logarithms (LL). It captures the resummation of the soft and collinear gluon emissions that dominate in the limit of large thrust values.

Function $B(\alpha_S)$ includes the next-to-leading logarithmic (NLL) corrections and accounts for subleading contributions from hard collinear emissions. It typically involves the non-cusp part of the anomalous dimensions and running of the coupling constant.

The integral as it is cannot be integrated, the u integration may be performed using the prescription in Appendix A of [9] and readapting the formula to the case of Laplace transform instead of Mellin transform.

In appendix A we show that the prescription, to evaluate the large-N Mellin moments of soft-gluon contributins at an arbitrary logarithmic accuracy, can be used for the Laplace transform as well, we can use this result to express eq. (38) in an alternative representation:

The method is a generalization of the prescription to NLL accuracy in [5]

$$e^{-u\nu Q^2} - 1 \simeq -\Theta(u - v) \quad \text{with } v = \frac{N_0}{N} \tag{39}$$

where $N_0 = e^{-\gamma_E}$, $\gamma_E = 0.5772\ldots$ being the Euler-Mascheroni constant

$$\ln \tilde{J}^q_\nu(Q^2) = -\int_{N_0/N}^{1} \frac{\mathrm{d}u}{u} \left[\int_{u^2Q^2}^{uQ^2} \frac{1}{q^2} A\big(\alpha_s(q^2)\big)\mathrm{d}q^2 + \frac{1}{2}\tilde{B}\big(\alpha_s(uQ^2)\big) \right] + \ln \tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) \tag{40}$$

Note that, due to the integration of the running coupling the integral in **??** is singular for all values of $N = \nu Q^2$. However, if we perform the integration up to a fixed logarithmic accuracy N^kLL ($i.e$ we compute the leading $\alpha_s^n \ln^{n+1} N$, next-to-leading $\alpha_s^n \ln^n N$ and so on to $\alpha_s^n \ln^{n+1-k}$ terms), we find

$$\ln \tilde{J}^q_\nu(Q^2) = \ln N f_1(\lambda) + f_2(\lambda) + \alpha_s f_3(\lambda) + \alpha_s^2 f_4(\lambda) + \alpha_s^3 f_5(\lambda) + \mathcal{O}\big(\alpha_s^n \ln^{n-4} N\big) \tag{41}$$

We observe that the N-space formula eq. (40) is finite and uniquely defined up to the very large $N = N_L = \exp\left(\frac{1}{2\alpha_s b_0}\right)$ ($\lambda = \frac{1}{2}$), thanks to the prescription above.

The region where the integral is divergent is therefore excluded from the integration if $N < N_L$.

5 Inversion of the Laplace transform

In order to find the quark jet mass distribution $J^q(Q^2, k^2)$,we have to perform the inverse Laplace transform via the Mellin's inversion formula (or the Bromwich integral) given by the line integral:

$$J^q(Q^2,k^2) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT} \mathrm{d}\nu e^{\nu k^2}\tilde{J}^q_\nu\big(Q^2\big) \tag{42}$$

where C is a real number such that C is at the right of all singularities of the integrand in the complex plane and the function $\tilde{J}^q_\nu(Q^2)$ has to be bounded on the line.

Instead of directly considering the expression in eq. (42), it was pointed in [5] that it is more convenient to work with the mass fraction $R^q(w)$, which gives the fraction of jets with masses less than wQ^2:

$$R^q(w) = \int_0^\infty J^q(Q^2,k^2)\Theta(wQ^2-k^2)\mathrm{d}k^2 \tag{43}$$

and using the integral representation of the Heaviside step function eq. (36)

$$\Theta(wQ^2-k^2) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{\nu(wQ^2-k^2)} \tag{44}$$

we recognize the Laplace transform of the quark jet mass distribution eq. (35)

$$\begin{aligned} R^q(w) &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{w\nu Q^2}\int_0^\infty J^q(Q^2,k^2)e^{-\nu k^2}\mathrm{d}k^2 \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{w\nu Q^2}\tilde{J}^q_\nu(Q^2) \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{w\nu Q^2}e^{\mathcal{F}(\alpha_s,\ln\left(\nu Q^2\right))} \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C'-iT}^{C'+iT}\frac{\mathrm{d}N}{N}e^{wN}e^{\mathcal{F}(\alpha_s,\ln N)} \end{aligned} \tag{45}$$

where $N=\nu Q^2$ and $\mathcal{F}$ has the logarithms expansion

$$\begin{aligned} \mathcal{F}(\alpha_s,\ln N) = f_1(b_0\alpha_s\ln N)\ln N + f_2(b_0\alpha_s\ln N) + f_3(b_0\alpha_s\ln N)\alpha_s \\ + f_4(b_0\alpha_s\ln N)\alpha_s^2 + f_5(b_0\alpha_s\ln N)\alpha_s^3 + \mathcal{O}\big(\alpha_s^4\big) \end{aligned} \tag{46}$$

Since the function $\mathcal{F}$ in the exponent varies more slowly with N than wN, we can introduce the integration variable $u = wN$ so that $\ln N = \ln u + \ln \frac{1}{w} = \ln u + L$ and Taylor expand with respect to $\ln u$ around 0, which is equivalent to expanding the original function $\mathcal{F}$ w.r.t $\ln N$ around $\ln N = \ln \frac{1}{w} \equiv L$:

$$\begin{aligned} R^q(w) &= \frac{1}{2\pi i} \int_C \frac{\mathrm{d}u}{u} e^u e^{\mathcal{F}(\alpha_s, \ln u + L)} \\ &\overset{\text{Taylor}}{=} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\mathcal{F}(\alpha_s, L) + \sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \\ &= e^{\mathcal{F}(\alpha_s, L)} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \end{aligned} \tag{47}$$

where the integral is intended as before, along the line C to the right of all singularities of the integrand, and

$$\mathcal{F}^{(n)}(\alpha_s, L) = \frac{\partial^n \mathcal{F}(\alpha_s, \ln u + L)}{\partial \ln u^n}\bigg|_{\ln u = 0} \tag{48}$$

As noticed in [5], the n-th derivative of $\mathcal{F}$ w.r.t $\ln u$ evaluated at $\ln u = 0$ is at most of logarithmic order $\alpha_s^{n+k-1} L^k$, so in order to achieve N^4LL accuracy we need to compute the first four derivatives of $\mathcal{F}$ w.r.t $\ln u$ and neglect the terms of order $\mathcal{O}(\alpha_s^4)$ that appear in the derivation. We obtain the following expressions:

$$\begin{aligned} \mathcal{F}^{(1)}(\alpha_s, L) = &\, f_1(\lambda) + \lambda f_1'(\lambda) + \alpha_s b_0 f_2'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s^3 b_0 f_4'(\lambda) \\ &+ \mathcal{O}\big(\alpha_s^n L^{n-3}\big) \end{aligned} \tag{49}$$

$$\begin{aligned} \mathcal{F}^{(2)}(\alpha_s, L) = &\, 2\alpha_s b_0 f_1'(\lambda) + \alpha_s b_0 \lambda f_1''(\lambda) + \alpha_s^2 b_0^2 f_2''(\lambda) + \alpha_s^3 b_0^2 f_3''(\lambda) \\ &+ \mathcal{O}\big(\alpha_s^n L^{n-3}\big) \end{aligned} \tag{50}$$

$$\mathcal{F}^{(3)}(\alpha_s, L) = 3\alpha_s^2 b_0^2 f_1''(\lambda) + \alpha_s^2 b_0^2 \lambda f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 f_2^{(3)}(\lambda) + \mathcal{O}\big(\alpha_s^n L^{n-3}\big) \tag{51}$$

$$\mathcal{F}^{(4)}(\alpha_s,L) = 4\alpha_s^3 b_0^3 f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 \lambda f_1^{(4)}(\lambda) + \mathcal{O}\big(\alpha_s^n L^{n-3}\big) \tag{52}$$

Here $\lambda = \alpha_s b_0 L$ and derivative w.r.t $\ln u$ and then evaluated at $\ln u = 0$, or equivalently derivative w.r.t L gives the same result.

After recasting the expansion presented in eq. (47) using the expression $\gamma(\alpha_s,L) = f_1(\lambda) + \lambda f_1'(\lambda)$ from [5], and defining $\mathcal{F}^{(1)}_{res}(\alpha_s,L) \equiv \mathcal{F}^{(1)}(\alpha_s,L) - \gamma(\alpha_s,L)$, we proceed to expand the second exponential with respect to $\ln u$ around 0, following the approach outlined in [1]. This yields the subsequent expansion:

$$\begin{aligned}
R^q(w) &= e^{\mathcal{F}(\alpha_s,L)}\int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u} e^{\mathcal{F}^{(1)}_{res}(\alpha_s,L)\ln u + \sum_{n=2}^{\infty}\frac{\mathcal{F}^{(n)}(\alpha_s,L)}{n!}\ln^n u} \\
&= \int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u}\Bigg(1 + \mathcal{F}^{(1)}_{res}\ln u + \frac{1}{2}\big(\mathcal{F}^{(2)} + (\mathcal{F}^{(1)}_{res})^2\big)\ln^2 u \\
&+ \frac{1}{6}\big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}^{(1)}_{res} + (\mathcal{F}^{(1)}_{res})^3\big)\ln^3 u \\
&+ \frac{1}{24}\big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}^{(1)}_{res} + 6\mathcal{F}^{(2)}(\mathcal{F}^{(1)}_{res})^2 + (\mathcal{F}^{(1)}_{res})^4\big)\ln^4 u \\
&+ \mathcal{O}\big(\ln^5 u\big)\Bigg)
\end{aligned} \tag{53}$$

Lastly, we utilize the following result to evaluate the integral presented in eq. (43).

$$\int_C \frac{\mathrm{d}u}{2\pi i}\ln^k u e^{u-(1-\gamma(\alpha_s,L))\ln u} = \frac{\mathrm{d}^k}{\mathrm{d}\gamma^k}\frac{1}{\Gamma(1-\gamma(\alpha_s,L))} \tag{54}$$

where Γ is the Euler Γ-function.

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Bigg[&1 + \mathcal{F}^{(1)}_{res}\psi_0(1-\gamma) + \frac{1}{2}\big(\mathcal{F}^{(2)} + (\mathcal{F}^{(1)}_{res})^2\big)\big(\psi_0^2 - \psi_1\big)(1-\gamma) \\
&+ \frac{1}{6}\big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}^{(1)}_{res} + (\mathcal{F}^{(1)}_{res})^3\big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma) \\
&+ \frac{1}{24}\big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}^{(1)}_{res} + 6\mathcal{F}^{(2)}(\mathcal{F}^{(1)}_{res})^2 + (\mathcal{F}^{(1)}_{res})^4\big) \\
&\big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Bigg]
\end{aligned}
\tag{55}
$$

where $\psi_n(z)$ are the polygamma functions, defined as:

$$
\psi_n(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\ln\Gamma(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{\Gamma'(z)}{\Gamma(z)} = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\psi_0(z) \tag{56}
$$

Substituting the expressions eqs. (49) to (52) into eq. (55) we obtain:

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Bigg[&1 + \Big(\alpha_s^3 b_0 f_4'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s b_0 f_2'(\lambda)\Big)\psi_0(1-\gamma) \\
&+ \frac{1}{2}\Big(\alpha_s^3\big(2b_0^2 f_2'(\lambda) f_3'(\lambda) + b_0^2 f_3''(\lambda)\big) + \frac{1}{2}\alpha_s^2\big(b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2\big) \\
&+ \frac{1}{2}\alpha_s(b_0\lambda f_1''(\lambda) + 2b_0 f_1'(\lambda))\Big)\big(\psi_0^2 - \psi_1\big)(1-\gamma) \\
&+ \frac{1}{6}\Big(\alpha_s^3\big(b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda) f_2''(\lambda) + 3b_0^2\lambda f_1''(\lambda) f_3'(\lambda) \\
&+ 6b_0^2 f_1'(\lambda) f_3'(\lambda)\big) + \frac{1}{6}\alpha_s^2\big(b_0^2\lambda f_1^{(3)}(\lambda) + 3b_0^2\lambda f_1''(\lambda) f_2'(\lambda) + 3b_0^2 f_1''(\lambda) \\
&+ 6b_0^2 f_1'(\lambda) f_2'(\lambda)\big)\Big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma) \\
&+ \frac{1}{24}\Big(\alpha_s^3\big(b_0^3\lambda f_1^{(4)}(\lambda) + 4b_0^3\lambda f_1^{(3)}(\lambda) f_2'(\lambda) \\
&+ 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3\lambda f_1''(\lambda) f_2''(\lambda) + 6b_0^3\lambda f_1''(\lambda) f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda) f_2'(\lambda) \\
&+ 12b_0^3 f_1'(\lambda) f_2''(\lambda) + 12b_0^3 f_1'(\lambda) f_2'(\lambda)^2\big) + \frac{1}{8}\alpha_s^2\big(b_0^2\lambda^2 f_1''(\lambda)^2 + 4b_0^2 f_1'(\lambda)^2 \\
&+ 4b_0^2\lambda f_1'(\lambda) f_1''(\lambda)\big)\Big)\big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Bigg]
\end{aligned}
\tag{57}
$$

and reorganize as a power series of α_s:

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Bigg[&1+\alpha_s b_0\Big(\psi_0(1-\gamma)f_2'(\lambda)+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\big(\lambda f_1''(\lambda)\\
&+2f_1'(\lambda)\big)\Big)+\alpha_s^2\Big(\frac{1}{8}(\psi_0^4-6\psi_1+3\psi_1^3+4\psi_0\psi_2-\psi_3)(1-\gamma)\big(b_0^2\lambda^2 f_1''(\lambda)^2\\
&+4b_0^2 f_1'(\lambda)^2+4b_0^2\lambda f_1'(\lambda)f_1''(\lambda)\big)+\frac{1}{6}(\psi_0^3-3\psi_0\psi_1+\psi_2)(1-\gamma)\big(b_0^2\lambda f_1^{(3)}(\lambda)\\
&+3b_0^2\lambda f_1''(\lambda)f_2'(\lambda)+3b_0^2 f_1''(\lambda)+6b_0^2 f_1'(\lambda)f_2'(\lambda)\big)+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\\
&\big(b_0^2 f_2''(\lambda)+b_0^2 f_2'(\lambda)^2\big)+b_0\psi_0(1-\gamma)f_3'(\lambda)\Big)+\alpha_s^3\Big(\frac{1}{24}(\psi_0^4-6\psi_1+3\psi_1^3+4\psi_0\psi_2\\
&-\psi_3)(1-\gamma)\big(b_0^3\lambda f_1^{(4)}(\lambda)+4b_0^3\lambda f_1^{(3)}(\lambda)f_2'(\lambda)+4b_0^3 f_1^{(3)}(\lambda)+6b_0^3\lambda f_1''(\lambda)f_2''(\lambda)\\
&+6b_0^3\lambda f_1''(\lambda)f_2'(\lambda)^2+12b_0^3 f_1''(\lambda)f_2'(\lambda)+12b_0^3 f_1'(\lambda)f_2''(\lambda)+12b_0^3 f_1'(\lambda)f_2'(\lambda)^2\big)\\
&+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\big(2b_0^2 f_2'(\lambda)f_3'(\lambda)+b_0^2 f_3''(\lambda)\big)+\frac{1}{6}(\psi_0^3-3\psi_0\psi_1\\
&+\psi_2)(1-\gamma)\big(b_0^3 f_2^{(3)}(\lambda)+b_0^3 f_2'(\lambda)^3+3b_0^3 f_2'(\lambda)f_2''(\lambda)+3b_0^2\lambda f_1''(\lambda)f_3'(\lambda)\\
&+6b_0^2 f_1'(\lambda)f_3'(\lambda)\big)+b_0\psi_0(1-\gamma)f_4'(\lambda)\Big)+\mathcal{O}\big(\alpha_s^4\big)\Bigg]
\end{aligned}
\tag{58}
$$

By comparing eq. (37) and eq. (45) we see that to obtain the Thrust cross section, we simply multiply by 2 the inverse Laplace transform obtained. Following the convention of [5], the final resummed expression $R_T(\tau)$ can be written as one exponential eq. (59),

$$
R_T(\tau) = \exp\big\{Lg_1(\lambda)+g_2(\lambda)+\alpha_s g_3(\lambda)+\alpha_s^2 g_4(\lambda)+\alpha_s^3 g_5(\lambda)+\mathcal{O}\big(\alpha_s^4\big)\big\} \tag{59}
$$

to do so we observe that $\frac{1}{\Gamma(1-\gamma)} = \exp\{-\ln(\Gamma(1-\gamma))\}$ corrects f_2, while the expression in parenthesis starting with 1 can be seen as the expansion of an exponential for $\alpha_s \to 0$:

$$e^{\alpha_s g_3(\lambda)+\alpha_s^2 g_4(\lambda)+\alpha_s^3 g_5(\lambda)+\mathcal{O}\left(\alpha_s^4\right)} = 1+\alpha_s g_3(\lambda)+\frac{1}{2}\alpha_s^2\big(g_3^2(\lambda)+2g_4(\lambda)\big) + \frac{1}{6}\alpha_s^3\big(g_3^3(\lambda)+6g_3(\lambda)g_4(\lambda)+6g_5(\lambda)\big)+\mathcal{O}\big(\alpha_s^4\big) \tag{60}$$

to obtain $g_3(\lambda)$, $g_4(\lambda)$ and $g_5(\lambda)$ we match eq. (58) with eq. (60) and obtain the following expressions:

$$g_1(\lambda) = 2f_1(\lambda) \tag{61}$$

$$g_2(\lambda) = 2f_2(\lambda) - \ln\Gamma(1-2f_1(\lambda)-2\lambda f_1`(\lambda)) \tag{62}$$

$$g_3(\lambda) = 2f_3(\lambda)+2\Big(\psi_0(1-\gamma)f_2'(\lambda)+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\big(\lambda f_1''(\lambda)+2f_1'(\lambda)\big)\Big) \tag{63}$$

$$g_4(\lambda) = 2f_4(\lambda)+2\Big(\frac{1}{8}(\psi_0^4-6\psi_1+3\psi_1^3+4\psi_0\psi_2-\psi_3)(1-\gamma)\big(b_0^2\lambda^2 f_1''(\lambda)^2 + 4b_0^2 f_1'(\lambda)^2+4b_0^2\lambda f_1'(\lambda)f_1''(\lambda)\big)+\frac{1}{6}(\psi_0^3-3\psi_0\psi_1+\psi_2)(1-\gamma)\big(b_0^2\lambda f_1^{(3)}(\lambda) + 3b_0^2\lambda f_1''(\lambda)f_2'(\lambda)+3b_0^2 f_1''(\lambda)+6b_0^2 f_1'(\lambda)f_2'(\lambda)\big)+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma) \big(b_0^2 f_2''(\lambda)+b_0^2 f_2'(\lambda)^2\big)+b_0\psi_0(1-\gamma)f_3'(\lambda)\Big) \tag{64}$$

$$g_5(\lambda) = 2f_5(\lambda)+2\Big(\frac{1}{24}(\psi_0^4-6\psi_1+3\psi_1^3+4\psi_0\psi_2-\psi_3)(1-\gamma) \big(b_0^3\lambda f_1^{(4)}(\lambda)+4b_0^3\lambda f_1^{(3)}(\lambda)f_2'(\lambda)+4b_0^3 f_1^{(3)}(\lambda)+6b_0^3\lambda f_1''(\lambda)f_2''(\lambda) + 6b_0^3\lambda f_1''(\lambda)f_2'(\lambda)^2+12b_0^3 f_1''(\lambda)f_2'(\lambda)+12b_0^3 f_1'(\lambda)f_2''(\lambda)+12b_0^3 f_1'(\lambda)f_2'(\lambda)^2\big) + \frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\big(2b_0^2 f_2'(\lambda)f_3'(\lambda)+b_0^2 f_3''(\lambda)\big)+\frac{1}{6}(\psi_0^3-3\psi_0\psi_1+\psi_2)(1-\gamma) \big(b_0^3 f_2^{(3)}(\lambda)+b_0^3 f_2'(\lambda)^3+3b_0^3 f_2'(\lambda)f_2''(\lambda)+3b_0^2\lambda f_1''(\lambda)f_3'(\lambda)+6b_0^2 f_1'(\lambda)f_3'(\lambda)\big) + b_0\psi_0(1-\gamma)f_4'(\lambda)\Big) \tag{65}$$

A Laplace transform in the large N limit

Using the methodology outlined in [9], we will demonstrate that the Mellin transform prescription is also applicable to the Laplace transform. in the large moment $\nu Q^2 = N$ limit, this fact was already known in the literature [4] and we'll show it here for completeness.

We are interested in solving the following integral

$$\int_0^1 \mathrm{d}z \, \frac{e^{-N(1-z)} - 1}{1-z} F(\alpha_s, \ln(1-z)) = \int_0^1 \frac{\mathrm{d}u}{u} \big(e^{-uN} - 1\big) F(\alpha_s, \ln u) \tag{66}$$

Start by considering

$$I_n(N) = \int_0^1 \frac{\mathrm{d}u}{u} \big(e^{-uN} - 1\big) \ln^n(u) \tag{67}$$

the above integral can be evaluated as described in [3]. Using the following identity

$$\ln^n(u) = \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \epsilon}\right)^n u^{\epsilon} = \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \epsilon}\right)^n e^{\epsilon \ln u} \tag{68}$$

to replace the logarithm term in the integrand eq. (67) and straightforwardly integrate the resulting expression. We obtain

$$\begin{aligned}
I_n(N) &= \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \epsilon}\right)^n \int_0^1 \mathrm{d}u \, \big(e^{-uN} - 1\big) u^{\epsilon - 1} \\
&= \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \epsilon}\right)^n \left\{ -\frac{1}{\epsilon} + N^{-\epsilon} (\Gamma(\epsilon, 0) - \Gamma(\epsilon, N)) \right\} \\
&= \lim_{\epsilon \to 0} \lim_{N \to \infty} \left(\frac{\partial}{\partial \epsilon}\right)^n \left\{ -\frac{1}{\epsilon} + N^{-\epsilon} \Gamma(\epsilon) \right\} + e^{-N + \mathcal{O}\left((\frac{1}{N})^2\right)} \mathcal{O}\left(\frac{1}{N}\right) \\
&= \lim_{\epsilon \to 0} \lim_{N \to \infty} \left(\frac{\partial}{\partial \epsilon}\right)^n \left\{ \frac{1}{\epsilon} (N^{-\epsilon} \epsilon \Gamma(\epsilon) - 1) \right\} + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \\
&= \lim_{\epsilon \to 0} \lim_{N \to \infty} \left(\frac{\partial}{\partial \epsilon}\right)^n \left\{ \frac{1}{\epsilon} (e^{-\epsilon \ln N} \Gamma(1 + \epsilon) - 1) \right\} + \mathcal{O}\left(\frac{e^{-N}}{N}\right)
\end{aligned} \tag{69}$$

where $\Gamma(\epsilon,0)=\Gamma(\epsilon)$, $\Gamma(\epsilon,N)$ is the incomplete Gamma function and $\epsilon\Gamma(\epsilon)=\Gamma(1+\epsilon)$

$$\Gamma(\epsilon,N)=\int_N^\infty \mathrm{d}t\, t^{\epsilon-1}e^{-t} \tag{70}$$

The last equation in eq. (69) is the same as Eq. (68) obtained in [9] for the Mellin transform. Therefore, we can conclude that the Mellin transform prescription is also applicable to the Laplace transform in the large N limit.

Using the known expansion of the Gamma function for small ϵ

$$\Gamma(1+\epsilon)=\exp\left\{-\gamma_E\epsilon+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)\epsilon^n}{n}\right\} \tag{71}$$

the term in curly brackets in eq. (71) can be expanded in power of ϵ and then derive. The result for $I_n(N)$ is thus a polynomial of degree $n+1$ in the large logarithm $\ln N$:

$$\begin{aligned}I_n(N)=&\frac{(-1)^n+1}{n+1}(\ln N+\gamma_E)^{n+1}+\frac{(-1)^{n-1}}{2}n\zeta(2)(\ln N+\gamma_E)^{n-1}\\&+\sum_{k=0}^{n-2}a_{nk}(\ln N+\gamma_E)^k+\mathcal{O}\left(\frac{e^{-N}}{N}\right)\end{aligned} \tag{72}$$

This result can be generalized using the following formal identity:

$$e^{-\epsilon\ln N}\Gamma(1+\epsilon)=\Gamma(1-\frac{\partial}{\partial\ln N})e^{\epsilon\ln N} \tag{73}$$

then we can perform the n-th derivative with respect to ϵ, and obtain

$$I_n(N)=\Gamma\left(1-\frac{\partial}{\partial\ln N}\right)\frac{(-\ln N)^n+1}{n+1}+\mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{74}$$

This expression can be regarded as a replacement for eq. (69) to compute the polynomial coefficients a_{nk} in eq. (72). Moreover, by observing that

$$\frac{(-\ln N)^n + 1}{n+1} = -\int_{\frac{1}{N}}^{1} \mathrm{d}u\, \frac{\ln^n(u)}{u} \tag{75}$$

we obtain the all order generalization for of the prescriprion used in [5]:

$$e^{-uN} - 1 = -\Gamma\left(1 - \frac{\partial}{\partial \ln_N}\right)\Theta(u - \frac{1}{N}) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{76}$$

$$= -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln_N}\right)\Theta(u - \frac{N_0}{N}) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{77}$$

where

$$\tilde{\Gamma}(1-\epsilon) \equiv e^{\gamma_E \epsilon}\Gamma(1+\epsilon) = \exp\left\{\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)\epsilon^n}{n}\right\} \tag{78}$$

It is straightforward to show that the prescription can be applied to as follows:

$$\int_0^1 \frac{\mathrm{d}u}{u}\big(e^{-uN} - 1\big)F(\alpha_s, \ln u) = -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln_N}\right)\int_{\frac{N_0}{N}}^{1}\frac{\mathrm{d}u}{u}F(\alpha_s, \ln u) + \mathcal{O}\left(\frac{e^{-N}}{N}\right) \tag{79}$$

and to evaluated the ln N-contribution arising from the integration of anu soft-gluon function F that has a generic perturbative expansion of the type

$$F(\alpha_s, \ln u) = \sum_{k=1}^{\infty}\alpha_s^k \sum_{n=0}^{2k-1} F_{kn}\ln^n u \tag{80}$$

The result eq. (79) can be used to obtain eq. (40) as shown in [9].

B Equivalence between resummation formulae

Here i adapt the result in [9] to show the equivalence between the resummation formulae in **??** and eq. (40) in the case of Thrust resummation.

It is straightforward to show that equation (90) in [9] becomes:

$$\int_{N_0/N}^{1} \frac{\mathrm{d}u}{u} \frac{1}{2}\Big(\tilde{B}(\alpha_s(uQ^2)) - B(\alpha_s(uQ^2))\Big) - \log \tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) =$$
$$\Gamma_2\left(\frac{\partial}{\partial \log N}\right)\left\{A(\alpha_s(\frac{N_0}{N}Q^2)) - \frac{1}{2}\frac{\partial}{\partial \log N}B(\alpha_s(\frac{N_0}{N}Q^2)) - 2A(\alpha_s(\frac{N_0^2}{N^2}Q^2))\right\} \tag{81}$$

Observe that using the renormalization group equation eq. (4) and chain rule we can write the following relation:

$$\begin{aligned}
\frac{\partial}{\partial \log N}B(\alpha_s(\frac{k}{N})) &= \frac{\partial B(\alpha_s)}{\partial \alpha_s}\frac{\partial \alpha_s(\frac{k}{N})}{\partial \frac{k}{N}}\frac{\partial \frac{k}{N}}{\partial \log N} = -\frac{\partial \alpha_s(\frac{k}{N})}{\partial \log \frac{k}{N}}\frac{\partial B(\alpha_s)}{\partial \alpha_s} \\
&= -\beta(\alpha_s)\alpha_s \frac{\partial B(\alpha_s)}{\partial \alpha_s} \\
\frac{\partial}{\partial \log N}A(\alpha_s(\frac{k}{N^2})) &= -2\beta(\alpha_s)\alpha_s \frac{\partial A(\alpha_s)}{\partial \alpha_s}
\end{aligned} \tag{82}$$

define the differential operator $\partial(\alpha_s)$ as:

$$\partial_{\alpha_s} \equiv -\beta(\alpha_s)\alpha_s \frac{\partial}{\partial \alpha_s} \tag{83}$$

Substituting the above relations in the previous equation, we obtain the equivalent of equation (92) in [9]:

$$\int_{N_0/N}^{1} \frac{\mathrm{d}u}{u} \frac{1}{2}\Big(\tilde{B}(\alpha_s(uQ^2)) - B(\alpha_s(uQ^2))\Big) - \log \tilde{C}\left(\alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) =$$
$$\Gamma_2(\partial_{\alpha_s})\left\{A(\alpha_s(\frac{N_0}{N}Q^2)) - \frac{1}{2}\partial_{\alpha_s}B(\alpha_s(\frac{N_0}{N}Q^2))\right\} - 2\Gamma(2\partial_{\alpha_s})A(\alpha_s(\frac{N_0^2}{N^2}Q^2)) \tag{84}$$

Now by setting $N = N_0$ or applying $\frac{\partial}{\partial \log N}$ one obtains respectively the functions $\tilde{C}$ and $\tilde{B}$ as functions of A and B:

$$\begin{aligned}
\tilde{C}(\alpha_s) = \exp&\left\{-\Gamma_2(\partial_{\alpha_s})\left[A(\alpha_s) - \frac{1}{2}B(\alpha_s)\right]\right\} \\
&- 2\Gamma(2\partial_{\alpha_s})A(\alpha_s(\frac{N_0^2}{N^2}Q^2))
\end{aligned} \tag{85}$$

$$\frac{\tilde{B}(\alpha_s)}{2} = \frac{B(\alpha_s)}{2} + \partial_{\alpha_s}\left\{\Gamma_2(\partial_{\alpha_s})\left[A(\alpha_s) - \frac{1}{2}\partial_{\alpha_s}B(\alpha_s)\right]\right\}$$
$$- 4\partial_{\alpha_s}\left\{\Gamma_s(2\partial_{\alpha_s})A(\alpha_s(\frac{N_0^2}{N^2}Q^2))\right\} \tag{86}$$

by inserting the expansion

$$\Gamma_2(\epsilon) = -\frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3)\epsilon - \frac{9}{16}\zeta(4)\epsilon^2 - \left(\frac{1}{6}\zeta(2)\zeta(3) + \frac{1}{5}\zeta(5)\right)\epsilon^3$$
$$- \left(\frac{1}{18}\zeta(3)^2 - \frac{61}{128}\zeta(6)\right)\epsilon^4 + \mathcal{O}(\epsilon^5) \tag{87}$$

in eq. (86) and eq. (85), we can obtain the coefficients $\tilde{B}$ and $\tilde{C}$ in terms of the coefficients A and B up to N^4LL accuracy:

$$\tilde{B}(\alpha_s(uQ^2)) = B(\alpha_s(uQ^2)) \tag{88}$$

$$\log\tilde{C}\left(\alpha_s(mu^2), \frac{\mu^2}{Q^2}\right) = \frac{A_1}{\pi}(-\zeta(2) - 1)\alpha_s + \left(\frac{-2A_2\zeta(2) - 2A_2 + \pi b_0 B_1}{2\pi^2}\right.$$
$$\left. + \frac{A_1 b_0\left(3\zeta(2)\log\left(\frac{\mu^2}{Q^2}\right) + 3\log\left(\frac{\mu^2}{Q^2}\right) - 4\zeta(3)\right)}{3\pi}\right)\alpha_s^2$$
$$+ \left(\frac{A_1}{3\pi}\left(-27b_0^2\zeta(4) - 3b_0^2\zeta(2)\log^2\left(\frac{\mu^2}{Q^2}\right) - 3b_0^2\log^2\left(\frac{\mu^2}{Q^2}\right)\right.\right.$$
$$+ 8b_0^2\zeta(3)\log\left(\frac{\mu^2}{Q^2}\right) + 3b_1\zeta(2)\log\left(\frac{\mu^2}{Q^2}\right) + 3b_1\log\left(\frac{\mu^2}{Q^2}\right)$$
$$\left. - 4b_1\zeta(3)\right) - \frac{1}{6\pi^3}\left(-12\pi A_2 b_0\zeta(2)\log\left(\frac{\mu^2}{Q^2}\right)\right.$$
$$- 12\pi A_2 b_0\log\left(\frac{\mu^2}{Q^2}\right) + 16\pi A_2 b_0\zeta(3) + 6A_3\zeta(2) + 6A_3$$
$$\left.\left. + 6\pi^2 b_0^2 B_1\log\left(\frac{\mu^2}{Q^2}\right) - 6\pi b_0 B_2 - 3\pi^2 b_1 B_1\right)\right)\alpha_s^3 + \mathcal{O}(\alpha_s^4) \tag{89}$$

We note that $\tilde{B}$ corrects the B terms so it has to be expanded up to α_s^4 to achieve N^4LL accuracy while $\ln \tilde{C}$ corrects the f_i functions so they have to be expanded up to α_s^3 to achieve N^4LL accuracy, these corrections are necessary only for NNLL accuracy and beyond, consistent with the results in [5].

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