

1 Laplace and Mellin transform in the large N limit

Using the methodology outlined in [4], we will demonstrate that the Mellin transform prescription is also applicable to the Laplace transform. in the large moment $\nu Q^2 = N$ limit, this fact was already known in the literature [2] and we'll show it here for completeness.

We are interested in solving the following integral

$$\int_0^1 dz \frac{e^{-N(1-z)} - 1}{1-z} F(\alpha_s, \ln(1-z)) = \int_0^1 \frac{du}{u} (e^{-uN} - 1) F(\alpha_s, \ln u) \quad (1)$$

Start by considering

$$I_n(N) = \int_0^1 \frac{du}{u} (e^{-uN} - 1) \ln^n(u) \quad (2)$$

the above integral can be evaluated as described in [1]. Using the following identity

$$\ln^n(u) = \lim_{\epsilon \rightarrow 0} \left(\frac{\partial}{\partial \epsilon} \right)^n u^\epsilon \quad (3)$$

to replace the logarithm term in the integrand eq. (2) and straightforwardly integrate the resulting expression. We obtain

$$\begin{aligned}
I_n(N) &= \lim_{\epsilon \rightarrow 0} \left(\frac{\partial}{\partial \epsilon} \right)^n \int_0^1 du (e^{-uN} - 1) u^{\epsilon-1} \\
&= \lim_{\epsilon \rightarrow 0} \left(\frac{\partial}{\partial \epsilon} \right)^n \left\{ -\frac{1}{\epsilon} + N^{-\epsilon} (\Gamma(\epsilon, 0) - \Gamma(\epsilon, N)) \right\} \\
&= \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left(\frac{\partial}{\partial \epsilon} \right)^n \left\{ -\frac{1}{\epsilon} + N^{-\epsilon} \Gamma(\epsilon) \right\} + e^{-N+\mathcal{O}((\frac{1}{N})^2)} \mathcal{O}\left(\frac{1}{N}\right) \\
&= \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left(\frac{\partial}{\partial \epsilon} \right)^n \left\{ \frac{1}{\epsilon} (N^{-\epsilon} \Gamma(\epsilon) - 1) \right\} + e^{-N+\mathcal{O}((\frac{1}{N})^2)} \mathcal{O}\left(\frac{1}{N}\right) \\
&= \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left(\frac{\partial}{\partial \epsilon} \right)^n \left\{ \frac{1}{\epsilon} (e^{-\epsilon \ln N} \Gamma(1 + \epsilon) - 1) \right\} + e^{-N+\mathcal{O}((\frac{1}{N})^2)} \mathcal{O}\left(\frac{1}{N}\right)
\end{aligned} \tag{4}$$

where $\Gamma(\epsilon, 0) = \Gamma(\epsilon)$, $\Gamma(\epsilon, N)$ is the incomplete Gamma function and $\epsilon \Gamma(\epsilon) = \Gamma(1 + \epsilon)$

$$\Gamma(\epsilon, N) = \int_N^\infty dt t^{\epsilon-1} e^{-t} \tag{5}$$

The last equation in eq. (4) is the same as Eq. (68) obtained in [4] for the Mellin transform. Therefore, we can conclude that the Mellin transform prescription is also applicable to the Laplace transform in the large N limit.

Using the known expansion of the Gamma function for small ϵ

$$\Gamma(1 + \epsilon) = \exp \left\{ -\gamma_E \epsilon + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) \epsilon^n}{n} \right\} \tag{6}$$

the term in curly brackets in eq. (6) can be expanded in power of ϵ and then derive. The result for $I_n(N)$ is thus a polynomial of degree $n + 1$ in the large logarithm $\ln N$:

$$\begin{aligned}
I_n(N) &= \frac{(-1)^n + 1}{n + 1} (\ln N + \gamma_E)^{n+1} + \frac{(-1)^{n-1}}{2} n \zeta(2) (\ln N + \gamma_E)^{n-1} \\
&\quad + \sum_{k=0}^{n-2} a_{nk} (\ln N + \gamma_E)^k + \mathcal{O}\left(\frac{1}{N}\right)
\end{aligned} \tag{7}$$

This result can be generalized using the following formal identity:

$$e^{-\epsilon \ln N} \Gamma(1 + \epsilon) = \Gamma\left(1 - \frac{\partial}{\partial \ln N}\right) e^{\epsilon \ln N} \quad (8)$$

then we can perform the n -th derivative with respect to ϵ , and obtain

$$I_n(N) = \Gamma\left(1 - \frac{\partial}{\partial \ln N}\right) \frac{(-\ln N)^n + 1}{n + 1} + \mathcal{O}\left(\frac{1}{N}\right) \quad (9)$$

This expression can be regarded as a replacement for eq. (4) to compute the polynomial coefficients a_{nk} in eq. (7). Moreover, by observing that

$$\frac{(-\ln N)^n + 1}{n + 1} = - \int_{\frac{1}{N}}^1 du \frac{\ln^n(u)}{u} \quad (10)$$

we obtain the all order generalization for of the prescription used in [3]:

$$e^{-uN} - 1 = -\Gamma\left(1 - \frac{\partial}{\partial \ln_N}\right) \Theta\left(u - \frac{1}{N}\right) + \mathcal{O}\left(\frac{1}{N}\right) \quad (11)$$

$$= -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln_N}\right) \Theta\left(u - \frac{N_0}{N}\right) + \mathcal{O}\left(\frac{1}{N}\right) \quad (12)$$

where

$$\tilde{\Gamma}(1 - \epsilon) \equiv e^{\gamma_E \epsilon} \Gamma(1 + \epsilon) = \exp\left\{\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) \epsilon^n}{n}\right\} \quad (13)$$

It is straightforward to show that the prescription can be applied to as follows:

$$\int_0^1 \frac{du}{u} (e^{-uN} - 1) F(\alpha_s, \ln u) = -\tilde{\Gamma}\left(1 - \frac{\partial}{\partial \ln_N}\right) \int_{\frac{N_0}{N}}^1 \frac{du}{u} F(\alpha_s, \ln u) + \mathcal{O}\left(\frac{1}{N}\right) \quad (14)$$

and to evaluated the $\ln N$ -contribution arising from the integration of anu soft-gluon function F that has a generic perturbative expansion of the type

$$F(\alpha_s, \ln u) = \sum_{k=1}^{\infty} \alpha_s^k \sum_{n=0}^{2k-1} F_{kn} \ln^n u \quad (15)$$

The result eq. (14) can be used to obtain ?? as shown in [4].

The coefficients \tilde{B} and \tilde{C} are related to the coefficients A and B in the following way:

$$\tilde{B}(\alpha_s) = B(\alpha_s) + 4\partial_\alpha \Gamma_2(\partial_\alpha) \left[A(\alpha_s) - \frac{1}{4} B(\alpha_s) \right] \quad (16)$$

$$\tilde{C}(\alpha_s) = \exp \left\{ -4\Gamma_2(\partial_\alpha) \left[A(\alpha_s) - \frac{1}{4} B(\alpha_s) \right] \right\} \quad (17)$$

where

$$\Gamma_2(\partial_\alpha) = \frac{1}{\epsilon^2} \left[1 - e^{-\gamma_E \epsilon \Gamma(1-\epsilon)} \right] = \frac{1}{\epsilon^2} \left\{ 1 - \exp \left[\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} \epsilon^n \right] \right\} \quad (18)$$

$$\partial_\alpha \equiv -2\beta(\alpha_s) \alpha_s \frac{\partial}{\partial \alpha_s} \quad (19)$$

$\beta(\alpha_s)$ is the QCD beta function ??

by inserting the expansion

$$\begin{aligned} \Gamma_2(\epsilon) = & \frac{1}{2} \zeta(2) - \frac{1}{3} \zeta(3) \epsilon - \frac{9}{16} \zeta(4) \epsilon^2 - \left(\frac{1}{6} \zeta(2) \zeta(3) + \frac{1}{5} \zeta(5) \right) \epsilon^3 \\ & - \left(\frac{1}{18} \zeta(3)^2 - \frac{61}{128} \zeta(6) \right) \epsilon^4 + \mathcal{O}(\epsilon^5) \end{aligned} \quad (20)$$

in eq. (16) and eq. (17), we can obtain the coefficients \tilde{B} and \tilde{C} in terms of the coefficients A and B up to N^4LL accuracy:

$$\begin{aligned}
\tilde{B}(\alpha_s) - B(\alpha_s) = & -\frac{4A_1b_0}{\pi}(\zeta(2))\alpha_s^2 \\
& + \left(-\frac{8A_2b_0}{\pi^2}\zeta(2) + \frac{4b_0^2B_1}{\pi}\zeta(2) - \frac{4A_1}{3\pi}(8b_0^2\zeta(3) + 3b_1\zeta(2)) \right) \alpha_s^3 \\
& + \left(-\frac{12A_3b_0}{\pi^3}\zeta(2) + \frac{12b_0^2B_2}{\pi^2}\zeta(2) - \frac{8A_2}{\pi^2}(4b_0^2\zeta(3) + b_1\zeta(2)) \right. \\
& + \frac{B_1}{3\pi}(48b_0^3\zeta(3) + 30b_0b_1\zeta(2)) - \frac{4A_1}{(3\pi)}(20b_0b_1\zeta(3) + 3b_2\zeta(2) \\
& \left. + 81b_0^3\zeta(4)) \right) \alpha_s^4 + \mathcal{O}(\alpha_s^5)
\end{aligned} \tag{21}$$

$$\begin{aligned}
\ln \tilde{C}(\alpha_s) = & \frac{2A_1\zeta(2)}{\pi}\alpha_s \\
& + \left(\frac{2A_2}{\pi^2}\zeta(2) - \frac{b_0B_1}{\pi}\zeta(2) + \frac{A_1}{3\pi}(8b_0\zeta(3) - 6b_0\ln\left[\frac{\mu^2}{Q^2}\right]\zeta(2)) \right) \alpha_s^2 \\
& + \left(\frac{2A_3}{\pi^3}\zeta(2) - \frac{2b_0B_2}{\pi^2}\zeta(2) + \frac{A_2}{3\pi^2}(16b_0\zeta(3) - 12b_0\ln\left[\frac{\mu^2}{Q^2}\right]\zeta(2)) \right. \\
& + \frac{B_1}{3\pi}(-8b_0^2\zeta(3) - 3b_1\zeta(2) + 6b_0^2\ln\left[\frac{\mu^2}{Q^2}\right]\zeta(2)) + \frac{A_1}{3\pi}(8b_1\zeta(3) \\
& + 6b_0^2\ln^2\left[\frac{\mu^2}{Q^2}\right]\zeta(2) - 2\ln\left[\frac{\mu^2}{Q^2}\right](8b_0^2\zeta(3) + 3b_1\zeta(2)) + 54b_0^2\zeta(4)) \left. \right) \alpha_s^3 \\
& + \mathcal{O}(\alpha_s^4)
\end{aligned} \tag{22}$$

We note that \tilde{B} corrects the B terms so it has to be expanded up to α_s^4 to achieve N^4LL accuracy while $\ln \tilde{C}$ corrects the f_i functions so they have to be expanded up to α_s^3 to achieve N^4LL accuracy, these corrections are necessary only for NNLL accuracy and beyond, consistent with the results in [3].