Chapter 1

Resummation

1.1 Soft-gluon effects in QCD cross sections

Soft-gluon effects and the soft-gluon exponentiation are reviewed in [4] and [5], here i summarize the physical motivation and main idea of the resummation of soft gluon effects in QCD.

The finite energy resolution inherent in any particle detector implies that the physical cross sections, those experimentally measured, inherently incorporate all contributions from arbitrarily soft particles produced in the final state. In other words, because we lack the ability to precisely resolve the energy of soft particles, we are unable to distinguish between their presence or absence in our calculations fig. 1. Consequently, we must account for the sum over all possible final states.

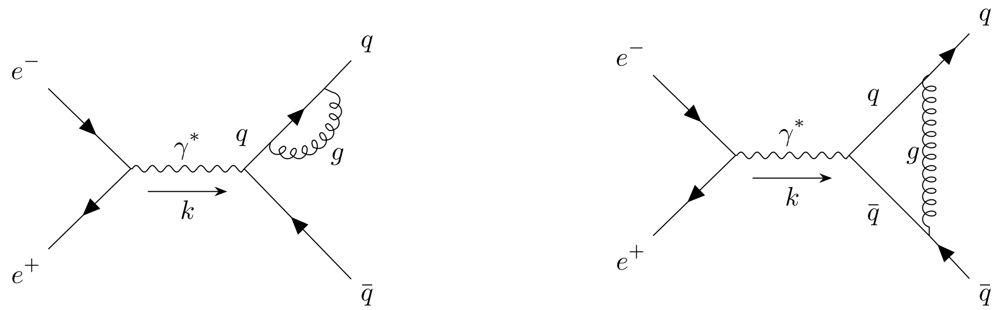

Figure 1: Next-to-leading order $\mathcal{O}\left(\alpha_s^2\right)$ diagrams for $e^+e^- \to q\bar{q}g$ process whose final state is identical to the Tree level process **??** $e^+e^- \to q\bar{q}$

This inclusiveness is essential in QCD calculations. Higher order perturbative contributions due to *virtual* gluons are infrared divergent and the divergences are exactly cancelled by radiation of undetected *real* gluons. In particular kinematic configurations, *e.g* Thrust in the dijet limit $T \to 1$, real and virtual contributions can be highly unbalanced, because the emission of real radiation is inhibited by kinematic constrainsts, spoiling the cancellation mechanism. As a result, soft gluon contribution to QCD cross sections can still be either large or singular.

In these cases, the cancellation of infrared divergences bequeaths higher order contributions of the form:

$$G_{nm}\alpha_s^n \ln^m \frac{1}{\tau}, \quad \text{with } m \leq 2n, \tag{1}$$

that can become large, $\alpha_s \ln^2 \frac{1}{\tau} \gtrsim 1$, even if the QCD coupling α_s is in the perturbative regime $\alpha_s \ll 1$. These logarithmically enhanced terms in eq. (1) are certainly relevant near the dijet limit $\tau \to 0$. In these cases, the theoretical predictions can be improved by evaluating soft gluon contributions to high orders and possibly resumming to all of them in α_s [3, 8].

Resummation of large logarithms in event shape distributions was described by CTTW [2].

The physical basis for all-order resummation of soft-gluon contributions to QCD cross sections are dynamics and kinmatics factorizations. The first factorization follows from gauge invariance and unitarity while the second factorization is strongly depends on the observable to be computed.

In the appropriate soft limit, if the phase-space for this observable can be written in a factorized way, then resummation is feasible in the form of a generalized exponentiation [9]. However even when phase-space factorization is achievable, it does not always occur in the space where the physical observable x is defined. Usually, it is necessary to introduce a conjugate space to overcome phase-space constraints. A typical example is the energy-conservation constraint that can be factorized by working in N-moment space, N being the variable conjugate to the energy x via a Mellin (or Laplace) transformation.

1.2 CTTW convention

According to general theorems [1],[6],[7], the cumulant cross section **??** $R(\tau)$ has a power series expansion in $\alpha_s(Q^2)$ of the form:

$$R(\tau) = C\Big(\alpha_s\Big(Q^2\Big)\Big)\Sigma\Big(\tau, \alpha_s\Big(Q^2\Big)\Big) + F\Big(\tau, \alpha_s\Big(Q^2\Big)\Big) \tag{2}$$

where

$$C(\alpha_s) = 1 + \sum_{n=1}^{\infty} C_n \bar{\alpha_s}^n \tag{3}$$

$$\Sigma(\tau, \alpha_s) = \exp\left[\sum_{n=1}^{\infty} \bar{\alpha_s}^n \sum_{m=1}^{2n} G_{nm} \ln^m \tau\right] \tag{4}$$

$$D(\tau, \alpha_s) = \sum_{n=1}^{\infty} \bar{\alpha_s}^n D_n(\tau) \tag{5}$$

Here C_n and G_{nm} are constants and $\bar{\alpha_s} = \frac{\alpha_s}{2\pi}$, while $D_n(\tau)$ are perturbatively computable functions that vanish at small τ.

Thus at small τ (large thrust) it becomes morst important to resum the series of large logarithms in $\Sigma(\tau, \alpha_s)$. These are normally classified as *leading* logarithms when $n < m \leq 2n$, *next-to-leading* when $m = n$ and *subdominant* logarithms when $m < n$.

The cumulant cross section $R(\tau)$ can be written in general as:

$$R(\tau) = \left(1 + \sum_{n=1}^{\infty} C_n \bar{\alpha_s}^n\right) \exp\Big[L g_1(\lambda) + g_2(\lambda) + g_3(\lambda)\alpha_s + g_4(\lambda)\alpha_s^2 + g_5(\lambda)\alpha_s^3 + \mathcal{O}\Big(\alpha_s^4\Big)\Big] \tag{6}$$

where $L = \ln\frac{1}{\tau}$ and $\lambda = \alpha_s b_0 L$. The function g_1 encodes all the leading logarithms, the function g_2 resums all next-to-leading logarithms and so on.

The last equation gives a better prediction of the thrust distribution in the two-jet region, but fails to describe the multijet region $_{max}$, where non-singular pieces of the fixed-order prediction become important. To achieve a reliable description of the observable over a broader kinematical range the two expressions **??** and eq. (2) can be matched, taking care of double counting of logarithms appearing in both expressions.

Expanding eq. (6) in powers of α_s we have:

$$
\begin{aligned}
R(\tau) &= 1 + R^{(1)}(\tau)\bar{\alpha_s} + R^{(2)}(\tau)\bar{\alpha_s}^2 + R^{(3)}(\tau)\bar{\alpha_s}^3 + \dots \\
&= 1 + (C_1 + G_{12}\log^2(\tau) + G_{11}\log(\tau))\bar{\alpha_s} \\
&+ \Bigg(C_2 + \log^2(\tau)\Big(C_1 G_{12} + \frac{G_{11}^2}{2} + G_{22}\Big) + \log(\tau)\Big(C_1 G_{11} + G_{21}\Big) \\
&+ \frac{1}{2}G_{12}^2\log^4(\tau) + (G_{11}G_{12} + G_{23})\log^3(\tau)\Bigg)\bar{\alpha_s}^2 \\
&+ \Bigg(C_3 + \log^4(\tau)\left(\frac{1}{2}C_1 G_{12}^2 + \frac{1}{2}G_{12}G_{11}^2 + G_{23}G_{11} + G_{12}G_{22} + G_{34}\right) \\
&+ \log^3(\tau)\left(C_1 G_{12}G_{11} + C_1 G_{23} + \frac{G_{11}^3}{6} + G_{22}G_{11} + G_{12}G_{21} + G_{33}\right) \\
&+ \log^2(\tau)\left(\frac{1}{2}C_1 G_{11}^2 + C_2 G_{12} + C_1 G_{22} + G_{21}G_{11} + G_{32}\right) \\
&+ \log(\tau)\left(C_2 G_{11} + C_1 G_{21} + G_{31}\right) + \frac{1}{6}G_{12}^3\log^6(\tau) \\
&+ \left(\frac{1}{2}G_{11}G_{12}^2 + G_{23}G_{12}\right)\log^5(\tau)\Bigg)\bar{\alpha_s}^3 + \dots
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
R(\tau) &= C(\alpha_s)\exp\left[\frac{\alpha_s}{4\pi}\left(c_{12}L^2 + c_{11}L\right)\right] \\
&+ \left(\frac{\alpha_s}{4\pi}\right)^2\left[c_{23}L^3 + c_{22}L^2 + c_{21}L\right] \\
&+ \left(\frac{\alpha_s}{4\pi}\right)^3\left[c_{34}L^4 + c_{33}L^3 + c_{32}L^2 + c_{31}L\right] + \cdots \\
&+ D(\alpha_s),
\end{aligned}
$$

LL $\quad$ + NLL $\quad$ + NNLL $\quad$ + N^{3}LL $\quad$ $+\cdots$

we'll see in the next chapter **??** that the exponent function in Laplace space has the form:

$$
\delta F\left(\alpha_s, \ln N\right) = L f_1(\lambda) + f_2(\lambda) + f_3(\lambda)\alpha_s + f_4(\lambda)\alpha_s^2 + f_5(\lambda)\alpha_s^3 + \mathcal{O}\Big(\alpha_s^4\Big) \tag{8}
$$

where $L = \ln N = \ln\Big(\nu Q^2\Big)$ and $\lambda = \alpha_s b_0 L$. We require the functions $f_i(\lambda)$ to be omogeneous,*i.e* $f_i(0) = 0$, so that at $N^n LL$ we can write:

$$
f_{n+1}(\lambda) = \sum_{k\geq n}\tilde{G}_{k,k+1-n}\alpha_s^k L^{k+1-n} \tag{9}
$$

this requirement is automatically satisfied if we choose as variable $L = \ln\Big(\frac{N}{N_0}\Big)$ where $N_0 = e^{-\gamma_E}$, $\gamma_E = 0.5772\dots$ being the Euler-Mascheroni constant. With the latter choice the terms proportional to γ_E and its powers disappear. The advantage of the variable N is that the total rate is directly reproduced by setting $N = 1$, while in the variable $n = N/N_0$ it is given $f_{n=1/N_0}$. These two choices differ only by terms of higher order in γ_E.

For the case of the thrust distribution, it was shown in [2] that this problem can be recast in the form of an integral equation **??** in the Laplace space, whose solution gives directly the exponent function.

$$\ln \tilde{J}^q_\nu(Q^2) = \int_0^1 \frac{\mathrm{d}u}{u}\left(e^{-u\nu Q^2} - 1\right)\left[\int_{u^2Q^2}^{uQ^2} \frac{1}{q^2} A\Big(\alpha_s(q^2)\Big)\mathrm{d}q^2 + \frac{1}{2}B\Big(\alpha_s(uQ^2)\Big)\right] \tag{10}$$

The exponent function in Laplace space has the form:

$$\begin{aligned}\mathcal{F}(\alpha_s, \ln N) = {} & f_1(b_0\alpha_s \ln N)\ln N + f_2(b_0\alpha_s \ln N) + f_3(b_0\alpha_s \ln N)\alpha_s + f_4(b_0\alpha_s \ln N)\alpha_s^2 \\ & + f_5(b_0\alpha_s \ln N)\alpha_s^3 + \mathcal{O}\Big(\alpha_s^4\Big)\end{aligned} \tag{11}$$