1 Inversion of the Laplace transform

In order to find the quark jet mass distribution $J^q(Q^2,k^2)$,we have to perform the inverse Laplace transform via the Mellin's inversion formula (or the Bromwich integral) given by the line integral:

$$J^q(Q^2,k^2) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT} \mathrm{d}\nu e^{\nu k^2}\tilde{J}^q_\nu\big(Q^2\big) \tag{1}$$

where C is a real number such that C is at the right of all singularities of the integrand in the complex plane and the function $\tilde{J}^q_\nu(Q^2)$ has to be bounded on the line.

Instead of directly considering the expression in eq. (1), it was pointed in [2] that it is more convenient to work with the mass fraction $R^q(w)$, which gives the fraction of jets with masses less than wQ^2:

$$R^q(w) = \int_0^\infty J^q(Q^2,k^2)\Theta(wQ^2-k^2)\mathrm{d}k^2 \tag{2}$$

and using the integral representation of the Heaviside step function **??**

$$\Theta(wQ^2-k^2) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{\nu(wQ^2-k^2)} \tag{3}$$

we recognize the Laplace transform of the quark jet mass distribution **??**

$$\begin{aligned}
R^q(w) &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{w\nu Q^2}\int_0^\infty J^q(Q^2,k^2)e^{-\nu k^2}\mathrm{d}k^2 \\
&= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{w\nu Q^2}\tilde{J}^q_\nu(Q^2) \\
&= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C-iT}^{C+iT}\frac{\mathrm{d}\nu}{\nu}e^{w\nu Q^2}e^{\mathcal{F}(\alpha_s,\ln\left(\nu Q^2\right))} \\
&= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{C'-iT}^{C'+iT}\frac{\mathrm{d}N}{N}e^{wN}e^{\mathcal{F}(\alpha_s,\ln N)}
\end{aligned} \tag{4}$$

where $N = \nu Q^2$ and $\mathcal{F}$ has the logarithms expansion

$$
\begin{aligned}
\mathcal{F}(\alpha_s, \ln N) = f_1(b_0\alpha_s \ln N)\ln N + f_2(b_0\alpha_s \ln N) + f_3(b_0\alpha_s \ln N)\alpha_s \\
+ f_4(b_0\alpha_s \ln N)\alpha_s^2 + f_5(b_0\alpha_s \ln N)\alpha_s^3 + \mathcal{O}\big(\alpha_s^4\big)
\end{aligned}
\tag{5}
$$

Since the function $\mathcal{F}$ in the exponent varies more slowly with N than wN, we can introduce the integration variable $u = wN$ so that $\ln N = \ln u + \ln \frac{1}{w} = \ln u + L$ and Taylor expand with respect to $\ln u$ around 0, which is equivalent to expanding the original function $\mathcal{F}$ w.r.t $\ln N$ around $\ln N = \ln \frac{1}{w} \equiv L$:

$$
\begin{aligned}
R^q(w) &= \frac{1}{2\pi i}\int_C \frac{\mathrm{d}u}{u} e^u e^{\mathcal{F}(\alpha_s, \ln u + L)} \\
&\overset{\text{Taylor}}{=} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\mathcal{F}(\alpha_s, L) + \sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \\
&= e^{\mathcal{F}(\alpha_s, L)} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u}
\end{aligned}
\tag{6}
$$

where the integral is intended as before, along the line C to the right of all singularities of the integrand, and

$$
\mathcal{F}^{(n)}(\alpha_s, L) = \left.\frac{\partial^n \mathcal{F}(\alpha_s, \ln u + L)}{\partial \ln u^n}\right|_{\ln u = 0}
\tag{7}
$$

As noticed in [2], the n-th derivative of $\mathcal{F}$ w.r.t $\ln u$ evaluated at $\ln u = 0$ is given by the n-th derivative $\mathcal{F}^{(n)}(\alpha_s, L)$ is at most of logarithmic order $\alpha_s^{n+k-1} L^k$, so in order to achieve N^4LL accuracy we need to compute the first four derivatives of $\mathcal{F}$ w.r.t $\ln u$ and neglect the terms of order $\mathcal{O}\big(\alpha_s^4\big)$ that appear in the derivation. We obtain the following expansions:

$$
\begin{aligned}
\mathcal{F}^{(1)}(\alpha_s, L) = f_1(\lambda) + \lambda f_1'(\lambda) + \alpha_s b_0 f_2'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s^3 b_0 f_4'(\lambda) \\
+ \mathcal{O}\big(\alpha_s^n L^{n-3}\big)
\end{aligned}
\tag{8}
$$

$$\mathcal{F}^{(2)}(\alpha_s, L) = 2\alpha_s b_0 f_1'(\lambda) + \alpha_s b_0 \lambda f_1''(\lambda) + \alpha_s^2 b_0^2 f_2''(\lambda) + \alpha_s^3 b_0^2 f_3''(\lambda) + \mathcal{O}\big(\alpha_s^n L^{n-3}\big) \tag{9}$$

$$\mathcal{F}^{(3)}(\alpha_s, L) = 3\alpha_s^2 b_0^2 f_1''(\lambda) + \alpha_s^2 b_0^2 \lambda f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 f_2^{(3)}(\lambda) + \mathcal{O}\big(\alpha_s^n L^{n-3}\big) \tag{10}$$

$$\mathcal{F}^{(4)}(\alpha_s, L) = 4\alpha_s^3 b_0^3 f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 \lambda f_1^{(4)}(\lambda) + \mathcal{O}\big(\alpha_s^n L^{n-3}\big) \tag{11}$$

Here $\lambda = \alpha_s b_0 L$ and derivative w.r.t $\ln u$ and then evaluated at $\ln u = 0$, or equivalently derivative w.r.t L gives the same result.

After recasting the expansion presented in eq. (6) using the expression $\gamma(\alpha_s, L) = f_1(\lambda) + \lambda f_1'(\lambda)$ from [2], and defining $\mathcal{F}^{(1)}_{res}(\alpha_s, L) \equiv \mathcal{F}^{(1)}(\alpha_s, L) - \gamma(\alpha_s, L)$, we proceed to expand the second exponential with respect to $\ln u$ around 0, following the approach outlined in [1]. This yields the subsequent expansion:

$$\begin{aligned}
R^q(w) &= e^{\mathcal{F}(\alpha_s,L)} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u} e^{\mathcal{F}^{(1)}_{res}(\alpha_s,L)\ln u + \sum_{n=2}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s,L)}{n!} \ln^n u} \\
&= \int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u} \bigg(1 + \mathcal{F}^{(1)}_{res} \ln u + \frac{1}{2}\Big(\mathcal{F}^{(2)} + (\mathcal{F}^{(1)}_{res})^2\Big) \ln^2 u \\
&+ \frac{1}{6}\Big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}^{(1)}_{res} + (\mathcal{F}^{(1)}_{res})^3\Big) \ln^3 u \\
&+ \frac{1}{24}\Big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}^{(1)}_{res} + 6\mathcal{F}^{(2)}(\mathcal{F}^{(1)}_{res})^2 + (\mathcal{F}^{(1)}_{res})^4\Big) \ln^4 u \\
&+ \mathcal{O}\big(\ln^5 u\big)\bigg)
\end{aligned} \tag{12}$$

Lastly, we utilize the following result to evaluate the integral presented in eq. (2).

$$\int_C \frac{\mathrm{d}u}{2\pi i} \ln^k u e^{u-(1-\gamma(\alpha_s,L))\ln u} = \frac{\mathrm{d}^k}{\mathrm{d}\gamma^k} \frac{1}{\Gamma(1-\gamma(\alpha_s, L))} \tag{13}$$

where Γ is the Euler Γ-function.

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Big[& 1 + \mathcal{F}^{(1)}_{res}\psi_0(1-\gamma) + \frac{1}{2}\Big(\mathcal{F}^{(2)} + (\mathcal{F}^{(1)}_{res})^2\Big)\big(\psi_0^2 - \psi_1\big)(1-\gamma) \\
& + \frac{1}{6}\Big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}^{(1)}_{res} + (\mathcal{F}^{(1)}_{res})^3\Big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma) \\
& + \frac{1}{24}\Big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}^{(1)}_{res} + 6\mathcal{F}^{(2)}(\mathcal{F}^{(1)}_{res})^2 + (\mathcal{F}^{(1)}_{res})^4\Big) \\
& \big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Big]
\end{aligned}
\tag{14}
$$

where $\psi_n(z)$ are the polygamma functions, defined as:

$$
\psi_n(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\ln\Gamma(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{\Gamma'(z)}{\Gamma(z)} = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\psi_0(z)
\tag{15}
$$

Substituting the expressions eqs. (8) to (11) into eq. (14) we obtain:

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Big[& 1 + \Big(\alpha_s^3 b_0 f_4'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s b_0 f_2'(\lambda)\Big)\psi_0(1-\gamma) \\
& + \frac{1}{2}\Big(\alpha_s^3\big(2b_0^2 f_2'(\lambda) f_3'(\lambda) + b_0^2 f_3''(\lambda)\big) + \frac{1}{2}\alpha_s^2\big(b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2\big) \\
& + \frac{1}{2}\alpha_s\big(b_0\lambda f_1''(\lambda) + 2b_0 f_1'(\lambda)\big)\Big)\big(\psi_0^2 - \psi_1\big)(1-\gamma) \\
& + \frac{1}{6}\Big(\alpha_s^3\big(b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda) f_2''(\lambda) + 3b_0^2\lambda f_1''(\lambda) f_3'(\lambda) \\
& + 6b_0^2 f_1'(\lambda) f_3'(\lambda)\big) + \frac{1}{6}\alpha_s^2\big(b_0^2\lambda f_1^{(3)}(\lambda) + 3b_0^2\lambda f_1''(\lambda) f_2'(\lambda) + 3b_0^2 f_1''(\lambda) \\
& + 6b_0^2 f_1'(\lambda) f_2'(\lambda)\big)\Big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma) \\
& + \frac{1}{24}\Big(\alpha_s^3\big(b_0^3\lambda f_1^{(4)}(\lambda) + 4b_0^3\lambda f_1^{(3)}(\lambda) f_2'(\lambda) \\
& + 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3\lambda f_1''(\lambda) f_2''(\lambda) + 6b_0^3\lambda f_1''(\lambda) f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda) f_2'(\lambda) \\
& + 12b_0^3 f_1'(\lambda) f_2''(\lambda) + 12b_0^3 f_1'(\lambda) f_2'(\lambda)^2\big) + \frac{1}{8}\alpha_s^2\big(b_0^2\lambda^2 f_1''(\lambda)^2 + 4b_0^2 f_1'(\lambda)^2 \\
& + 4b_0^2\lambda f_1'(\lambda) f_1''(\lambda)\big)\Big)\big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Big]
\end{aligned}
\tag{16}
$$