1 Inversion of the Laplace transform

In order to find the quark jet mass distribution $J^q(Q^2, k^2)$,we have to perform the inverse Laplace transform via the Mellin's inversion formula (or the Bromwich integral) given by the line integral:

$$J^q(Q^2, k^2) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \mathrm{d}\nu e^{\nu k^2} \tilde{J}^q_\nu\big(Q^2\big) \tag{1}$$

where C is a real number such that C is at the right of all singularities of the integrand in the complex plane and the function $\tilde{J}^q_\nu(Q^2)$ has to be bounded on the line.

Instead of directly considering the expression in eq. (1), it was pointed in [2] that it is more convenient to work with the mass fraction $R^q(w)$, which gives the fraction of jets with masses less than wQ^2:

$$R^q(w) = \int_0^\infty J^q(Q^2, k^2)\Theta(wQ^2 - k^2)\mathrm{d}k^2 \tag{2}$$

and using the integral representation of the Heaviside step function ??

$$\Theta(wQ^2 - k^2) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \frac{\mathrm{d}\nu}{\nu} e^{\nu(wQ^2 - k^2)} \tag{3}$$

we recognize the Laplace transform of the quark jet mass distribution ??

$$\begin{aligned} R^q(w) &= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \frac{\mathrm{d}\nu}{\nu} e^{w\nu Q^2} \int_0^\infty J^q(Q^2, k^2) e^{-\nu k^2} \mathrm{d}k^2 \\ &= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \frac{\mathrm{d}\nu}{\nu} e^{w\nu Q^2} \tilde{J}^q_\nu(Q^2) \\ &= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C-iT}^{C+iT} \frac{\mathrm{d}\nu}{\nu} e^{w\nu Q^2} e^{\mathcal{F}(\alpha_s, \ln\left(\nu Q^2\right))} \\ &= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{C'-iT}^{C'+iT} \frac{\mathrm{d}N}{N} e^{wN} e^{\mathcal{F}(\alpha_s, \ln N)} \end{aligned} \tag{4}$$

where $N = \nu Q^2$ and $\mathcal{F}$ has the logarithms expansion

$$\begin{aligned}\mathcal{F}(\alpha_s, \ln N) &= f_1(b_0\alpha_s \ln N)\ln N + f_2(b_0\alpha_s \ln N) + f_3(b_0\alpha_s \ln N)\alpha_s \\ &\quad + f_4(b_0\alpha_s \ln N)\alpha_s^2 + f_5(b_0\alpha_s \ln N)\alpha_s^3 + \mathcal{O}(\alpha_s^4)\end{aligned} \tag{5}$$

Since the function $\mathcal{F}$ in the exponent varies more slowly with N than wN, we can introduce the integration variable $u = wN$ so that $\ln N = \ln u + \ln \frac{1}{w} = \ln u + L$ and Taylor expand with respect to $\ln u$ around 0, which is equivalent to expanding the original function $\mathcal{F}$ w.r.t $\ln N$ around $\ln N = \ln \frac{1}{w} \equiv L$:

$$\begin{aligned}R^q(w) &= \frac{1}{2\pi i}\int_C \frac{\mathrm{d}u}{u} e^u e^{\mathcal{F}(\alpha_s, \ln u + L)} \\ &\overset{\text{Taylor}}{=} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\mathcal{F}(\alpha_s, L) + \sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u} \\ &= e^{\mathcal{F}(\alpha_s, L)} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u - \ln u} e^{\sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s, L)}{n!} \ln^n u}\end{aligned} \tag{6}$$

where the integral is intended as before, along the line C to the right of all singularities of the integrand, and

$$\mathcal{F}^{(n)}(\alpha_s, L) = \frac{\partial^n \mathcal{F}(\alpha_s, \ln u + L)}{\partial \ln u^n}\bigg|_{\ln u = 0} \tag{7}$$

As noticed in [2], the n-th derivative of $\mathcal{F}$ w.r.t $\ln u$ evaluated at $\ln u = 0$ is at most of logarithmic order $\alpha_s^{n+k-1} L^k$, so in order to achieve N^4LL accuracy we need to compute the first four derivatives of $\mathcal{F}$ w.r.t $\ln u$ and neglect the terms of order $\mathcal{O}(\alpha_s^4)$ that appear in the derivation. We obtain the following expressions:

$$\begin{aligned}\mathcal{F}^{(1)}(\alpha_s, L) &= f_1(\lambda) + \lambda f_1'(\lambda) + \alpha_s b_0 f_2'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s^3 b_0 f_4'(\lambda) \\ &\quad + \mathcal{O}(\alpha_s^n L^{n-3})\end{aligned} \tag{8}$$

$$
\begin{aligned}
\mathcal{F}^{(2)}(\alpha_s, L) = & \; 2\alpha_s b_0 f_1'(\lambda) + \alpha_s b_0 \lambda f_1''(\lambda) + \alpha_s^2 b_0^2 f_2''(\lambda) + \alpha_s^3 b_0^2 f_3''(\lambda) \\
& + \mathcal{O}\big(\alpha_s^n L^{n-3}\big)
\end{aligned} \tag{9}
$$

$$
\mathcal{F}^{(3)}(\alpha_s, L) = 3\alpha_s^2 b_0^2 f_1''(\lambda) + \alpha_s^2 b_0^2 \lambda f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 f_2^{(3)}(\lambda) + \mathcal{O}\big(\alpha_s^n L^{n-3}\big) \tag{10}
$$

$$
\mathcal{F}^{(4)}(\alpha_s, L) = 4\alpha_s^3 b_0^3 f_1^{(3)}(\lambda) + \alpha_s^3 b_0^3 \lambda f_1^{(4)}(\lambda) + \mathcal{O}\big(\alpha_s^n L^{n-3}\big) \tag{11}
$$

Here $\lambda = \alpha_s b_0 L$ and derivative w.r.t $\ln u$ and then evaluated at $\ln u = 0$, or equivalently derivative w.r.t L gives the same result.

After recasting the expansion presented in eq. (6) using the expression $\gamma(\alpha_s, L) = f_1(\lambda) + \lambda f_1'(\lambda)$ from [2], and defining $\mathcal{F}^{(1)}_{res}(\alpha_s, L) \equiv \mathcal{F}^{(1)}(\alpha_s, L) - \gamma(\alpha_s, L)$, we proceed to expand the second exponential with respect to $\ln u$ around 0, following the approach outlined in [1]. This yields the subsequent expansion:

$$
\begin{aligned}
R^q(w) = & \; e^{\mathcal{F}(\alpha_s, L)} \int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u} e^{\mathcal{F}^{(1)}_{res}(\alpha_s,L)\ln u + \sum_{n=2}^{\infty} \frac{\mathcal{F}^{(n)}(\alpha_s,L)}{n!} \ln^n u} \\
= & \int_C \frac{\mathrm{d}u}{2\pi i} e^{u-(1-\gamma(\alpha_s,L))\ln u} \bigg(1 + \mathcal{F}^{(1)}_{res} \ln u + \frac{1}{2}\Big(\mathcal{F}^{(2)} + (\mathcal{F}^{(1)}_{res})^2\Big) \ln^2 u \\
& + \frac{1}{6}\Big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}^{(1)}_{res} + (\mathcal{F}^{(1)}_{res})^3\Big) \ln^3 u \\
& + \frac{1}{24}\Big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}^{(1)}_{res} + 6\mathcal{F}^{(2)}(\mathcal{F}^{(1)}_{res})^2 + (\mathcal{F}^{(1)}_{res})^4\Big) \ln^4 u \\
& + \mathcal{O}\big(\ln^5 u\big) \bigg)
\end{aligned} \tag{12}
$$

Lastly, we utilize the following result to evaluate the integral presented in eq. (2).

$$
\int_C \frac{\mathrm{d}u}{2\pi i} \ln^k u e^{u-(1-\gamma(\alpha_s,L))\ln u} = \frac{\mathrm{d}^k}{\mathrm{d}\gamma^k} \frac{1}{\Gamma(1-\gamma(\alpha_s, L))} \tag{13}
$$

where Γ is the Euler Γ-function.

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Big[1 &+ \mathcal{F}^{(1)}_{res}\psi_0(1-\gamma) + \frac{1}{2}\Big(\mathcal{F}^{(2)} + (\mathcal{F}^{(1)}_{res})^2\Big)\big(\psi_0^2 - \psi_1\big)(1-\gamma) \\
&+ \frac{1}{6}\Big(\mathcal{F}^{(3)} + 3\mathcal{F}^{(2)}\mathcal{F}^{(1)}_{res} + (\mathcal{F}^{(1)}_{res})^3\Big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma) \\
&+ \frac{1}{24}\Big(\mathcal{F}^{(4)} + 3(\mathcal{F}^{(2)})^2 + 4\mathcal{F}^{(3)}\mathcal{F}^{(1)}_{res} + 6\mathcal{F}^{(2)}(\mathcal{F}^{(1)}_{res})^2 + (\mathcal{F}^{(1)}_{res})^4\Big) \\
&\big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Big]
\end{aligned}
\tag{14}
$$

where $\psi_n(z)$ are the polygamma functions, defined as:

$$
\psi_n(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\ln\Gamma(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{\Gamma'(z)}{\Gamma(z)} = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\psi_0(z) \tag{15}
$$

Substituting the expressions eqs. (8) to (11) into eq. (14) we obtain:

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Big[1 &+ \Big(\alpha_s^3 b_0 f_4'(\lambda) + \alpha_s^2 b_0 f_3'(\lambda) + \alpha_s b_0 f_2'(\lambda)\Big)\psi_0(1-\gamma) \\
&+ \frac{1}{2}\Big(\alpha_s^3\big(2b_0^2 f_2'(\lambda)f_3'(\lambda) + b_0^2 f_3''(\lambda)\big) + \frac{1}{2}\alpha_s^2\big(b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2\big) \\
&+ \frac{1}{2}\alpha_s\big(b_0\lambda f_1''(\lambda) + 2b_0 f_1'(\lambda)\big)\Big)\big(\psi_0^2 - \psi_1\big)(1-\gamma) \\
&+ \frac{1}{6}\Big(\alpha_s^3\big(b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda)f_2''(\lambda) + 3b_0^2\lambda f_1''(\lambda)f_3'(\lambda) \\
&+ 6b_0^2 f_1'(\lambda)f_3'(\lambda)\big) + \frac{1}{6}\alpha_s^2\big(b_0^2\lambda f_1^{(3)}(\lambda) + 3b_0^2\lambda f_1''(\lambda)f_2'(\lambda) + 3b_0^2 f_1''(\lambda) \\
&+ 6b_0^2 f_1'(\lambda)f_2'(\lambda)\big)\Big)\big(\psi_0^3 - 3\psi_0\psi_1 + \psi_2\big)(1-\gamma) \\
&+ \frac{1}{24}\Big(\alpha_s^3\big(b_0^3\lambda f_1^{(4)}(\lambda) + 4b_0^3\lambda f_1^{(3)}(\lambda)f_2'(\lambda) \\
&+ 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3\lambda f_1''(\lambda)f_2''(\lambda) + 6b_0^3\lambda f_1''(\lambda)f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda)f_2'(\lambda) \\
&+ 12b_0^3 f_1'(\lambda)f_2''(\lambda) + 12b_0^3 f_1'(\lambda)f_2'(\lambda)^2\big) + \frac{1}{8}\alpha_s^2\big(b_0^2\lambda^2 f_1''(\lambda)^2 + 4b_0^2 f_1'(\lambda)^2 \\
&+ 4b_0^2\lambda f_1'(\lambda)f_1''(\lambda)\big)\Big)\big(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3\big)(1-\gamma) + \mathcal{O}\big(\ln^5 u\big)\Big]
\end{aligned}
\tag{16}
$$

and reorganize as a power series of α_s:

$$
\begin{aligned}
R^q(w) = \frac{e^{\mathcal{F}(\alpha_s,L)}}{\Gamma(1-\gamma)}\Big[1 &+ \alpha_s b_0\Big(\psi_0(1-\gamma)f_2'(\lambda) + \frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\big(\lambda f_1''(\lambda) \\
&+ 2f_1'(\lambda)\big)\Big) + \alpha_s^2\Big(\frac{1}{8}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 - \psi_3)(1-\gamma)\big(b_0^2\lambda^2 f_1''(\lambda)^2 \\
&+ 4b_0^2 f_1'(\lambda)^2 + 4b_0^2\lambda f_1'(\lambda)f_1''(\lambda)\big) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 + \psi_2)(1-\gamma)\big(b_0^2\lambda f_1^{(3)}(\lambda) \\
&+ 3b_0^2\lambda f_1''(\lambda)f_2'(\lambda) + 3b_0^2 f_1''(\lambda) + 6b_0^2 f_1'(\lambda)f_2'(\lambda)\big) + \frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma) \\
&\big(b_0^2 f_2''(\lambda) + b_0^2 f_2'(\lambda)^2\big) + b_0\psi_0(1-\gamma)f_3'(\lambda)\Big) + \alpha_s^3\Big(\frac{1}{24}(\psi_0^4 - 6\psi_1 + 3\psi_1^3 + 4\psi_0\psi_2 \\
&- \psi_3)(1-\gamma)\big(b_0^3\lambda f_1^{(4)}(\lambda) + 4b_0^3\lambda f_1^{(3)}(\lambda)f_2'(\lambda) + 4b_0^3 f_1^{(3)}(\lambda) + 6b_0^3\lambda f_1''(\lambda)f_2''(\lambda) \\
&+ 6b_0^3\lambda f_1''(\lambda)f_2'(\lambda)^2 + 12b_0^3 f_1''(\lambda)f_2'(\lambda) + 12b_0^3 f_1'(\lambda)f_2''(\lambda) + 12b_0^3 f_1'(\lambda)f_2'(\lambda)^2\big) \\
&+ \frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\big(2b_0^2 f_2'(\lambda)f_3'(\lambda) + b_0^2 f_3''(\lambda)\big) + \frac{1}{6}(\psi_0^3 - 3\psi_0\psi_1 \\
&+ \psi_2)(1-\gamma)\big(b_0^3 f_2^{(3)}(\lambda) + b_0^3 f_2'(\lambda)^3 + 3b_0^3 f_2'(\lambda)f_2''(\lambda) + 3b_0^2\lambda f_1''(\lambda)f_3'(\lambda) \\
&+ 6b_0^2 f_1'(\lambda)f_3'(\lambda)\big) + b_0\psi_0(1-\gamma)f_4'(\lambda)\Big) + \mathcal{O}\big(\alpha_s^4\big)\Big]
\end{aligned}
\tag{17}
$$

By comparing **??** and eq. (4) we see that to obtain the Thrust cross section, we simply multiply by 2 the inverse Laplace transform obtained. Following the convention of [2], the final resummed expression $R_T(\tau)$ can be written as one exponential eq. (18),

$$
R_T(\tau) = \exp\big\{Lg_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \alpha_s^3 g_5(\lambda) + \mathcal{O}\big(\alpha_s^4\big)\big\} \tag{18}
$$

to do so we observe that $\frac{1}{\Gamma(1-\gamma)} = \exp\{-\ln(\Gamma(1-\gamma))\}$ corrects f_2, while the expression in parenthesis starting with 1 can be seen as the expansion of an exponential for $\alpha_s \to 0$:

$$e^{\alpha_s g_3(\lambda)+\alpha_s^2 g_4(\lambda)+\alpha_s^3 g_5(\lambda)+\mathcal{O}\left(\alpha_s^4\right)} = 1+\alpha_s g_3(\lambda)+\frac{1}{2}\alpha_s^2\big(g_3^2(\lambda)+2g_4(\lambda)\big) + \frac{1}{6}\alpha_s^3\big(g_3^3(\lambda)+6g_3(\lambda)g_4(\lambda)+6g_5(\lambda)\big)+\mathcal{O}\big(\alpha_s^4\big) \tag{19}$$

to obtain $g_3(\lambda)$, $g_4(\lambda)$ and $g_5(\lambda)$ we match eq. (17) with eq. (19) and obtain the following expressions:

$$g_1(\lambda) = 2f_1(\lambda) \tag{20}$$

$$g_2(\lambda) = 2f_2(\lambda) - \ln\Gamma(1-2f_1(\lambda)-2\lambda f_1‘(\lambda)) \tag{21}$$

$$g_3(\lambda) = 2f_3(\lambda) + 2\Big(\psi_0(1-\gamma)f_2'(\lambda)+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\big(\lambda f_1''(\lambda)+2f_1'(\lambda)\big)\Big) \tag{22}$$

$$\begin{aligned} g_4(\lambda) = 2f_4(\lambda) + 2\Big(&\frac{1}{8}(\psi_0^4-6\psi_1+3\psi_1^3+4\psi_0\psi_2-\psi_3)(1-\gamma)\big(b_0^2\lambda^2 f_1''(\lambda)^2 \\ &+4b_0^2 f_1'(\lambda)^2+4b_0^2\lambda f_1'(\lambda)f_1''(\lambda)\big)+\frac{1}{6}(\psi_0^3-3\psi_0\psi_1+\psi_2)(1-\gamma)\big(b_0^2\lambda f_1^{(3)}(\lambda) \\ &+3b_0^2\lambda f_1''(\lambda)f_2'(\lambda)+3b_0^2 f_1''(\lambda)+6b_0^2 f_1'(\lambda)f_2'(\lambda)\big)+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma) \\ &\big(b_0^2 f_2''(\lambda)+b_0^2 f_2'(\lambda)^2\big)+b_0\psi_0(1-\gamma)f_3'(\lambda)\Big) \end{aligned} \tag{23}$$

$$\begin{aligned} g_5(\lambda) = 2f_5(\lambda) + 2\Big(&\frac{1}{24}(\psi_0^4-6\psi_1+3\psi_1^3+4\psi_0\psi_2-\psi_3)(1-\gamma) \\ &\big(b_0^3\lambda f_1^{(4)}(\lambda)+4b_0^3\lambda f_1^{(3)}(\lambda)f_2'(\lambda)+4b_0^3 f_1^{(3)}(\lambda)+6b_0^3\lambda f_1''(\lambda)f_2''(\lambda) \\ &+6b_0^3\lambda f_1''(\lambda)f_2'(\lambda)^2+12b_0^3 f_1''(\lambda)f_2'(\lambda)+12b_0^3 f_1'(\lambda)f_2''(\lambda)+12b_0^3 f_1'(\lambda)f_2'(\lambda)^2\big) \\ &+\frac{1}{2}(\psi_0^2-\psi_1)(1-\gamma)\big(2b_0^2 f_2'(\lambda)f_3'(\lambda)+b_0^2 f_3''(\lambda)\big)+\frac{1}{6}(\psi_0^3-3\psi_0\psi_1+\psi_2)(1-\gamma) \\ &\big(b_0^3 f_2^{(3)}(\lambda)+b_0^3 f_2'(\lambda)^3+3b_0^3 f_2'(\lambda)f_2''(\lambda)+3b_0^2\lambda f_1''(\lambda)f_3'(\lambda)+6b_0^2 f_1'(\lambda)f_3'(\lambda)\big) \\ &+b_0\psi_0(1-\gamma)f_4'(\lambda)\Big) \end{aligned} \tag{24}$$