IEOR 4404 Simulation

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Contents

Chapter 1

Probability Review

1.1 Basics

1.1.1 Definitions

1. **Sample Space**: the set of all possible outcomes.
2. **Probabilities**: assignments of the likelihoods on these possible outcomes.
3. **Event**: the set of outcomes in the sample space.

1.1.2 Counting

1. There are k containers. If container i has n_i objects, and all objects are distinct. I pick one object from each container (ordering matters, with replacement). The total number of possible outcomes is $n_1 \times n_2 \times \dots n_k$.
2. A permutation of k distinct objects is an ordering of these objects (ordering matters, no replacement). The total number of permutations of k distinct objects out of n ones is $P_{n,k} = \frac{n!}{(n-k)!}$.
3. A combination of k objects out of n distinct objects is a collection of any k objects out of the n ones (ordering does not matter). The total number of combinations of k distinct objects out of n ones is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

1.1.3 Algebra of Events

Define A, B are two events (sets of outcomes in the sample space).

1. $A \cup B$ is the event that A **OR** B occurs.
2. $A \cap B$ is the event that both A **AND** B occurs ($A \cap B$ can be empty, written as $A \cap B = \emptyset$, named as **mutually exclusive**).
3. A^c is the complement of A.
4. $E \subseteq F$: E is a subset of F

1.1.4 Axioms of Probability

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$ if $A_1, A_2, \dots$ are mutually exclusive events in the sample space

We can derive the following properties from the Axioms of Probability:

1. $P(A^c) = 1 - P(A)$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
3. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

1.1.5 Conditional Probability

For two events A and B (with $P(B) > 0$), the conditional probability $P(A|B) := \frac{P(A \cap B)}{P(B)}$.

1. $P(A|B) := \frac{P(A \cap B)}{P(B)}$
2. $P(A \cap B) = P(A|B) \times P(B)$
3. $P(A \cap B \cap C) = P(C|A \cap B)P(B|A)P(A)$

1.1.6 Independence

Two events A and B are independent if $P(A \cap B) = P(A) \times P(B)$. Thus, an equivalent definition of independence between A and B is $P(A) = P(A|B)$, and also $P(B) = P(B|A)$.

1. $P(A \cap B) = P(A) \times P(B)$
2. $P(A) = P(A|B)$
3. $P(B) = P(B|A)$
4. The events $A_1, A_2, \dots, A_n$ are independent if for every possible subcollection of these events, the probability of all of them happening together equals the product of their individual probabilities: $P(A_{1'} \cap A_{2'} \cap \dots \cap A_{r'}) = P(A_{1'})\, P(A_{2'}) \, \dots \, P(A_{r'})$ where $A_{1'}, A_{2'}, \cdots, A_{r'}$ is any subset of the original events.

1.1.7 Law of Total Probability

1. Let $B_1, B_2, \ldots, B_n$ be a collection of **mutually exclusive** events, so that whenever $i \neq j$, we have $B_i \cap B_j = \emptyset$. Also, $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$ is the **whole sample space**. Then, $B_1, B_2, \ldots, B_n$ are said to be a **partition** of Ω.
2. Then for any event A:
$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$$
$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_n)P(B_n)$$
3. Special case when $n = 2$:
$$P(A) = P(A \cap B) + P(A \cap B^c)$$
$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c)$$

1.1.8 Exercises

Conditional Probabilities

Example 1. *A bin contains 5 defective, 10 partially defective and 25 acceptable transistors. A transistor is picked at random and put to use. If it does not immediately fail, what is the probability that it is acceptable?*

$P(A) = \frac{25}{40}, \quad P(P) = \frac{10}{40}, \quad P(D) = \frac{5}{40}$
$N = A \cup P, \quad P(N) = P(A) + P(P) = \frac{25+10}{40} = \frac{35}{40}$
$P(A \mid N) = \frac{P(A)}{P(N)} = \frac{25/40}{35/40} = \frac{25}{35} = \frac{5}{7}$

Example 2. *A family has two children. Given that there is at least one girl in the family, what's the probability that there are two girls?*

Example 3. *A family has two children. Given that the older child is a girl, what's the probability that there are two girls?*

Example 4. *You know that with 30% probability that your company will open a new branch in New York. You are also 60% certain that if the branch is opened you will become its new manager. What is the probability that you will be a new branch manager?*

Independence

Example 5. *Two fair dice are thrown. Let A be the event that the number shown by the first die is even, and B the event that the sum of the dice is odd. Are A and B independent?*

1.2 Random Variables

1.2.1 Definitions (Discrete Random Variables)

1. A random variable is a function that assigns a numerical value to each outcome in the sample space.
   To specify a random variable, identify its **range of values** and **probability** assigned to each value.
2. **Probability Mass Function (PMF)**: $p(a) = P(X = a)$ of a discrete random variable X for each possible value a of X.
3. **Cumulative Distribution Function (CDF)**: $F(x) = P(X \leq x) = \sum_{a \leq x} P(X = a) = \sum_{a \leq x} p(a)$ for X.
   Knowing pmf is equivalent to knowing cdf, which is equivalent to specifying X completely.