1) Hamiltonian: $H = r + |u(t)| + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 = r + |u| + \lambda_1 x_2 + \lambda_2(-a x_2 + u)$

necessary condition: $\dot{\lambda} = -\nabla_x H = \begin{bmatrix} 0 \\ -\lambda_1 + a\lambda_2 \end{bmatrix}$

a) terms in H depend on u are $|u| + \lambda_2 u$, and we want to minimize it.

$$\text{so} \begin{cases} u = 1 & \text{when } \lambda_2 < -1 \\ u = -1 & \text{when } \lambda_2 > 1 \\ u = 0 & \text{when } -1 < \lambda_2 < 1 \end{cases}$$

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then, $\lambda_1 = C_1$

$\dot{\lambda}_2 = -C_1 + a\lambda_2$. so $\lambda_2 = C_2 e^{at} + \frac{C_1}{a}$, we need to find C_1 C_2.

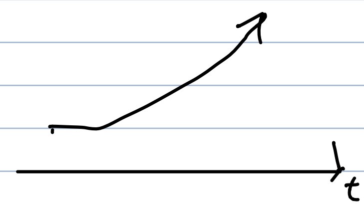
b) Since final time is free, $\left.\left(\partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f\right)\right|_{t_f} = H(t_f) = 0$

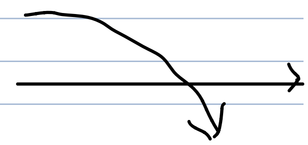
$H(t_f) = 0 \Rightarrow r + |u(t_f)| + \lambda_2(t_f) \cdot u(t_f) = 0$,

$u(t_f)$ cannot be 0 since $r > 0$

if $u(t_f) = 1$, $\lambda_2(t_f) = -(1+r)$

if $u(t_f) = -1$, $\lambda_2(t_f) = r + 1$





Since $\lambda_2 = C_2 e^{at} + \frac{C_1}{a}$, possible optimal seqences: $\{-1, 0, 1\}$ $\{1, 0, -1\}$

$\{0, 1\}$ $\{0, -1\}$

$\{1\}$ $\{-1\}$

c) at singular intervals, controls cannot be determined

$|u(t_f)| + \lambda_2(t_f) \cdot u(t_f) = [1 + \lambda_2(t_f)] u(t_f)$ when $u(t_f) > 0$

$= [-1 + \lambda_2(t_f)] u(t_f)$ when $u(t_f) < 0$

So $\lambda_2 = -1$ or 1, then, $C_2 = 0$ $C_1 = a$ or $-a$

but in b) we know that $\lambda_2(t_f) = -(1+r)$ or $r+1$. and $r > 0$

then, contradiction happens

So there is no singular internal.

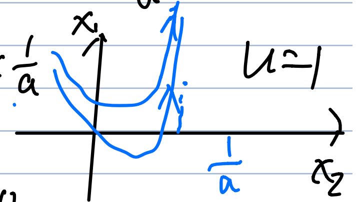
d) possible optimal seqences: $\{-1,0,1\}$ $\{1,0,-1\}$ $\{0,1\}$ $\{0,-1\}$ $\{1\}$ $\{-1\}$ and $\begin{cases} u=1 & \text{when } \lambda_2<-1 \\ u=-1 & \text{when } \lambda_2>1 \\ u=0 & \text{when } -1<\lambda_2<1 \end{cases}$

if $u=1 \Rightarrow \begin{cases} \dot{x}_1 = x_2 & x_1(t_f)=0 \\ \dot{x}_2 = -a x_2 + 1 & x_2(t_f)=0 \end{cases} \Rightarrow \begin{cases} x_2 = \frac{1}{a}\left(-e^{-a(t-t_f)}+1\right) \\ x_1 = \frac{1}{a^2} e^{-a(t-t_f)} + \frac{1}{a}(t-t_f) - \frac{1}{a^2} \end{cases}$

we need a relationship between x_1 x_2

so $x_2 = \frac{1}{a}\left(-e^{-a(t-t_f)}+1\right)$

$x_1 = -\frac{1}{a}x_2 - \frac{1}{a^2}\ln(-ax_2+1)$ and $ax_2<1 \Rightarrow x_2<\frac{1}{a}$ ①



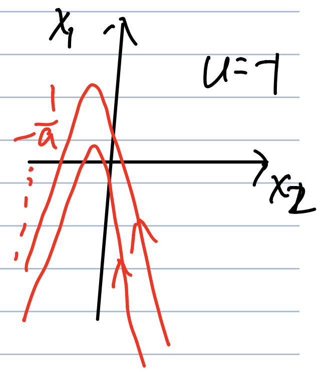
if $u=-1 \Rightarrow \begin{cases} \dot{x}_1 = x_2 & x_1(t_f)=0 \\ \dot{x}_2 = -a x_2 - 1 & x_2(t_f)=0 \end{cases} \Rightarrow \begin{cases} x_2 = \frac{1}{a}\left(e^{-a(t-t_f)}-1\right) = \dot{x}_1 \\ x_1 = \frac{-1}{a^2} e^{-a(t-t_f)} - \frac{1}{a}(t-t_f) + \frac{1}{a^2} \end{cases}$

we need a relationship between x_1 x_2

$x_2 = \frac{1}{a}\left(e^{-a(t-t_f)}-1\right)$

$x_1 = -\frac{1}{a}x_2 + \frac{1}{a^2}\ln(ax_2+1)$ and $ax_2>-1 \Rightarrow x_2>-\frac{1}{a}$

②

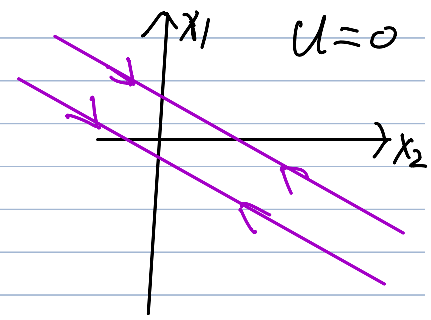


if $u=0 \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a x_2 \end{cases} \Rightarrow \begin{cases} x_2 = e^{-at} x_2(0) \\ x_1 = \frac{-1}{a} e^{-at} x_2(0) + \frac{1}{a} x_2(0) + x_1(0) \end{cases}$

we need a relationship between x_1 x_2

$x_2 = e^{-at} x_2(0)$

$x_1 = -\frac{1}{a}x_2 + \frac{1}{a}x_2(0) + x_1(0)$ ③



so when $(x_1(0), x_2(0))$ is not on ① or ②

just let the system follow ③ (no control), until it hit ① or ②

and then follows it the control seq: $\{0,1\}$ or $\{0,-1\}$

when $(x_1(0), x_2(0))$ is on ①, it may go to the origin, or intersect with ③ and then hit ② and follow ②

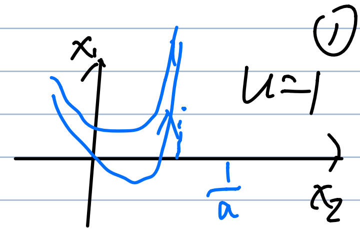
the control seq would be $\{1\}$ or $\{1,0,-1\}$

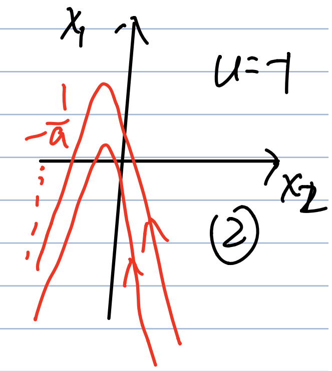
Similar things when $(x_1(0), x_2(0))$ is on ②, the control seq would be $\{-1\}$ or $\{-1,0,1\}$

$x_1 = -\frac{1}{a}x_2 - \frac{1}{a^2}\ln(-ax_2+1)$ ①

$x_1 = -\frac{1}{a}x_2 + \frac{1}{a^2}\ln(ax_2+1)$ ②

$x_1 = -\frac{1}{a}x_2 + \frac{1}{a}x_2(0) + x_1(0)$ ③





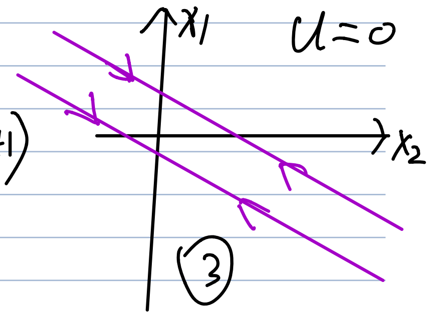
When $x_2(t) > -\frac{1}{a}$ and $x_1(t) = -\frac{1}{a}x_2(t) + \frac{1}{a^2}\ln(ax_2(t)+1)$

then, control will be -1 … "A"

when $x_2(t) < \frac{1}{a}$ and $x_1(t) = -\frac{1}{a}x_2(t) + \frac{1}{a^2}\ln(ax_2(t)+1)$

then, control will be 1 "B"

otherwise the control will be 0 "C"



when the system change from A to C status on intersections, it should follow the control rule of the latter one.

2) minimize: $J = \|x(t_f)\|^2$

$L=0$, so. $H = \lambda^T \cdot f = \lambda^T(Ax+Bu)$

terminal cost: $\phi = x(t_f)^T x(t_f)$, since no terminal constraint, then, $\Phi = \phi$

from necessary condition:

$$\dot{\lambda} = -\nabla_x H = -A^T\lambda$$

$$\partial_u H = \lambda^T B$$

$$\lambda(t_f) = \nabla_x \phi = 2x(t_f)$$

term in H depend on u is $\lambda^T B u$, and we want to minimize it

so $u = \begin{cases} 1 & \text{if } \lambda^T B < 0 \\ -1 & \text{if } \lambda^T B > 0 \\ \text{singular}, & \text{if } \lambda^T B = 0 \end{cases}$ $\Rightarrow$ bang bang control

for double integrator problem, we have $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\Rightarrow \dot{\lambda} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix}$

$u^*(t) = \begin{cases} -1 & \text{if } \lambda_2 > 0 \\ 1 & \text{if } \lambda_2 < 0 \\ \text{singular} & \text{if } \lambda_2 = 0 \end{cases}$ when $u = \pm 1$ $x_1 = \pm\frac{1}{2}t^2 + C_3 t + C_4$

$x_2 = \pm t + C_3$

so $\begin{cases} \dot{\lambda}_1 = 0 \\ \dot{\lambda}_2 = -\lambda_1 \end{cases} \Rightarrow \begin{cases} \lambda_1(t) = C_1 \\ \lambda_2(t) = -C_1 t + C_2 \end{cases}$

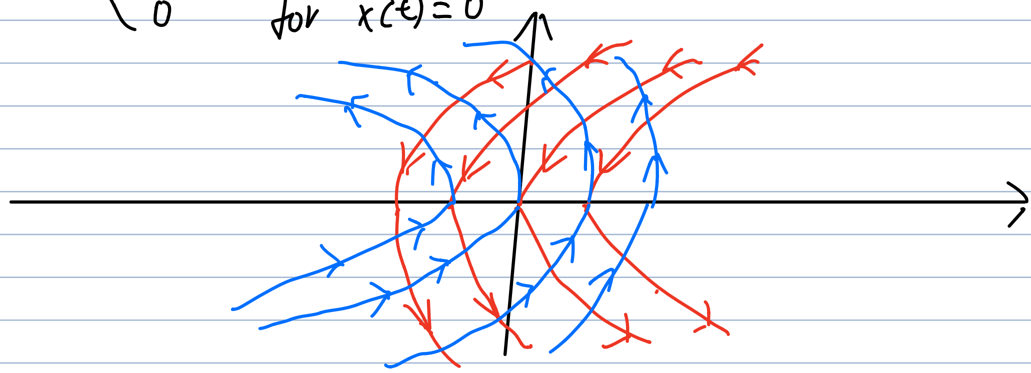
$\lambda_2(t)$ is a line, when $\lambda_2 = 0$, we cannot control the system (u can be arbitrary)

so when $u = \pm 1$, we have $x_1 = \pm t^2 + C_3 t + C_4$

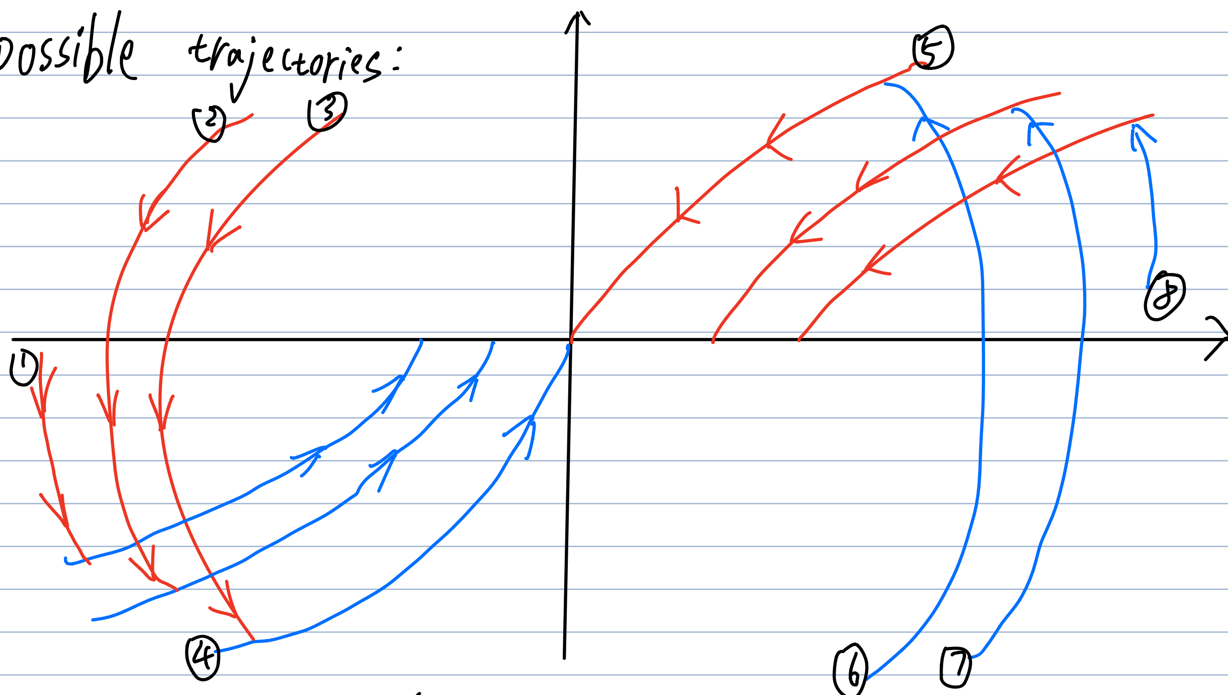
$x_2 = \pm t + C_3$

let $x_1(t) = -\frac{1}{2} x_2(t) \left| x_2(t) \right|$

$$u^*(t) = \begin{cases} +1 & \text{for } x(t) \text{ that } S(x(t)) > 0 \\ -1 & \text{for } x(t) \text{ that } S(x(t)) < 0 \\ +1 & \text{for } x(t) \text{ that } S(x(t)) = 0 \text{ and } x_2 > 0 \\ -1 & \text{for } x(t) \text{ that } S(x(t)) = 0 \text{ and } x_2 < 0 \\ 0 & \text{for } x(t) = 0 \end{cases}$$



possible trajectories:



When t_f is long enough it will follow curves like 3, 4, 5, 6

Otherwise it look like curves 1 or 2 or 7 8

3

a) $P_t = \begin{bmatrix} L_1 \cdot \cos\theta_1 + L_2 \cdot \cos(\theta_1+\theta_2) \\ L_1 \cdot \sin\theta_1 + L_2 \cdot \sin(\theta_1+\theta_2) \end{bmatrix}$ to avoid obstacle: $\|P_t - P_0\|_2^2 \geq r^2$

so $C(t) = r^2 - (P_t-P_0)^T(P_t-P_0) \leq 0$

$$\dot{C}(t) = -2(P_t-P_0)^T \cdot \dot{P}_t = -2(P_t-P_0)^T \begin{bmatrix} -L_1 \cdot \sin\theta_1 \cdot \dot\theta_1 - L_2 \cdot \sin(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2) \\ L_1 \cdot \cos\theta_1 \cdot \dot\theta_1 + L_2 \cos(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2) \end{bmatrix}$$

$$\ddot{c}(t) = -2(P_t-P_0)^T \cdot \ddot{P}_t - 2\dot{P}_t^T \dot{P}_t$$

$$= -2(P_t-P_0)^T \begin{bmatrix} -L_1 \sin\theta_1 \cdot \ddot\theta_1 - L_1\cos\theta_1 \cdot \dot\theta_1^2 - L_2\sin(\theta_1+\theta_2)(\ddot\theta_1+\ddot\theta_2) - L_2\cos(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2)^2 \\ L_1\cos\theta_1 \cdot \ddot\theta_1 - L_1 \sin\theta_1 \cdot \dot\theta_1^2 + L_2\cos(\theta_1+\theta_2)(\ddot\theta_1+\ddot\theta_2) - L_2\sin(\dot\theta_1+\dot\theta_2)^2 \end{bmatrix}$$

and we can see controls now

So we can have constraints:

$$c(x(t), t) \leq 0$$

$$\Downarrow$$

$$\dot{c} = -2(P_t-P_0)^T \begin{bmatrix} -L_1 \cdot \sin\theta_1 \cdot \dot\theta_1 - L_2 \cdot \sin(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2) \\ L_1 \cdot \cos\theta_1 \cdot \dot\theta_1 + L_2 \cos(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2) \end{bmatrix} \leq 0$$

$$\Downarrow$$

$$\ddot{c} = -2(P_t-P_0)^T \begin{bmatrix} -L_1 \sin\theta_1 \cdot \ddot\theta_1 - L_1\cos\theta_1 \cdot \dot\theta_1^2 - L_2\sin(\theta_1+\theta_2)(\ddot\theta_1+\ddot\theta_2) - L_2\cos(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2)^2 \\ L_1\cos\theta_1 \cdot \ddot\theta_1 - L_1 \sin\theta_1 \cdot \dot\theta_1^2 + L_2\cos(\theta_1+\theta_2)(\ddot\theta_1+\ddot\theta_2) - L_2\sin(\dot\theta_1+\dot\theta_2)^2 \end{bmatrix} \leq 0$$

$$\Rightarrow (P_0-P_t)^T \begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$+ (P_0-P_t)^T \begin{bmatrix} -L_1\cos\theta_1 \cdot \dot\theta_1^2 - L_2\cos(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2)^2 \\ -L_1\sin\theta_1 \cdot \dot\theta_1^2 - L_2\sin(\dot\theta_1+\dot\theta_2)^2 \end{bmatrix} \leq 0$$

b) denote one end of the link as A_i (start) and the other as B_i (end)

the direction of link i : $\dfrac{\vec{B_i}-\vec{A_i}}{\|A_i-B_i\|} = \vec{d_i}$

the projection of $\overrightarrow{A_iP_o}$ on $\overrightarrow{A_iB_i}$ is $(P_o-A_i)^T\vec{d_i} = S_i$

$S_i = \min\{\max\{0,S_i\}, \|A_i-B_i\|\}$ so S_i will be between A_iB_i

$\vec{A_i} + S_i \cdot \vec{d_i}$ is the closest point between A_iB_i to P_o

let $\vec{A_i} + S_i \cdot \vec{d_i} = \vec{C_i}$

$\|C_i - P_o\| \geq r$ when there is no collision

for link 1 :

$$A_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} L_1\cos\theta_1 \\ L_1\sin\theta_1 \end{bmatrix} \qquad d_1 = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix}$$

$$S_1 = \min\left\{\max\{0, \vec{P_o}^T d_1\}, L_1\right\} \qquad C_1 = A_1 + S_1 \cdot d_1 = S_1 \cdot d_1$$

we need $\|C_1 - P_o\| \geq r$

for link 2

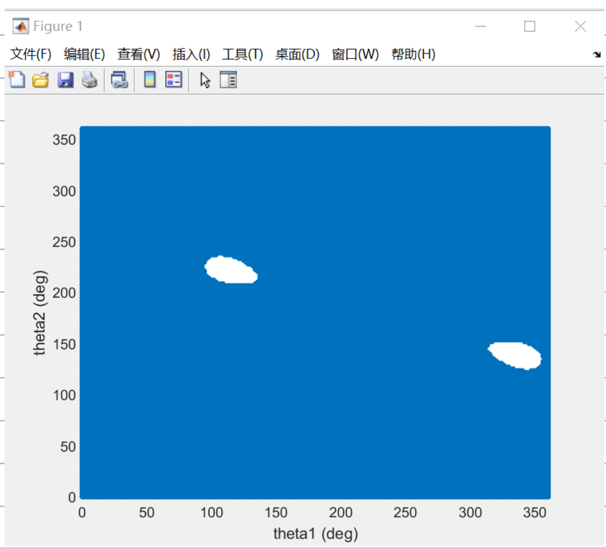
$$A_2 = \begin{bmatrix} L_1\cos\theta_1 \\ L_1\sin\theta_1 \end{bmatrix} \qquad B_2 = \begin{bmatrix} L_1\cos\theta_1 + L_2\cdot\cos(\theta_1+\theta_2) \\ L_1\sin\theta_1 + L_2\cos(\theta_1+\theta_2) \end{bmatrix} \qquad d_2 = \begin{bmatrix} \cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) \end{bmatrix}$$

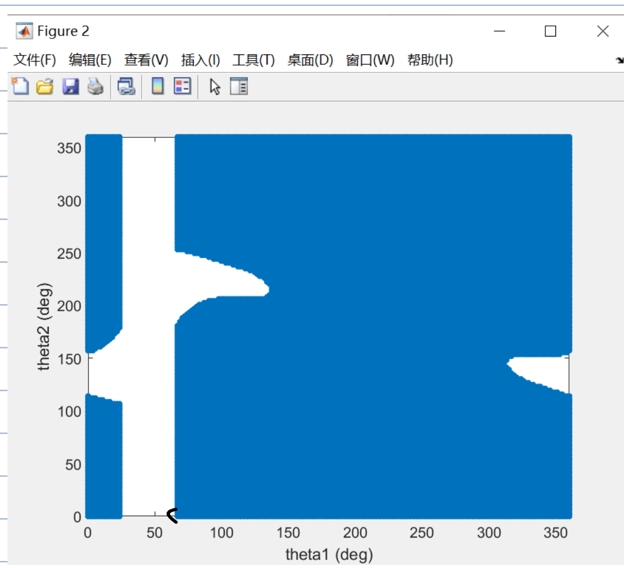
$$S_2 = \min\left\{\max\{0, (P_o-A_2)^T d_2\}, L_2\right\} \qquad C_2 = A_2 + S_2 \cdot d_2$$

we need $\|C_2 - P_o\| \geq r$

c)

```matlab
clc;
clear;

l1 = 2;
l2 = 2;
po = 1/2*[2;2];
r = 1/2;

steps = 200;
theta1s = linspace(0, 2*pi, steps);
theta2s = linspace(0, 2*pi, steps);
valid_tips = [];

for i = 1:steps
    for j = 1:steps
        th1 = theta1s(i);
        th2 = theta2s(j);
        tip = [cos(th1)*l1+cos(th1+th2)*l2-po(1);
               sin(th1)*l1+sin(th1+th2)*l2-po(2)];
        if (r - norm(tip) <= 0 )
            valid_tips = [ valid_tips , [th1;th2]];
        end
    end
end
%valid_tips= valid_tips';
figure;
title('problem3 (a)');
valid_tips = rad2deg(valid_tips);
plot(valid_tips(1,:), valid_tips(2,:), '.', 'MarkerSize', 14);
xlabel('theta1 (deg)'); ylabel('theta2 (deg)');
xlim([0, 360]); ylim([0, 360]);
valid_tips = [];
valid_map = zeros(steps,steps);

for i = 1:steps
    for j = 1:steps
        th1 = theta1s(i);
        th2 = theta2s(j);

        d1 = [cos(th1);sin(th1)];
        s1 = min(max(0,po'*d1),l1);
        c1 = s1*d1;

        A2 = l1*d1;
        B2 = [cos(th1)*l1+cos(th1+th2)*l2;
              sin(th1)*l1+sin(th1+th2)*l2];
        d2 = [cos(th1+th2);sin(th1+th2)];
        s2 = (po-A2)'*d2;
        s2 = min(max(0,s2),l2);
        c2 = A2+s2*d2;
        if ( norm(c1-po) >= r && norm(c2-po) >= r)
            valid_tips = [ valid_tips , [th1;th2]];
        end
    end
end
%valid_tips= valid_tips';
figure;
title('problem3 (b)');
valid_tips = rad2deg(valid_tips);
plot(valid_tips(1,:), valid_tips(2,:), '.', 'MarkerSize', 14);
xlabel('theta1 (deg)'); ylabel('theta2 (deg)');
xlim([0, 360]); ylim([0, 360]);
```





4) Since we want to maximize the range of the rocket.

we hope to maximize $\int_{t_0}^{t_f} x_1(t)\,dt$ $\quad x_1(t)$: horizontal velocity

so we want minimize $\int_{t_0}^{t_f} -x_1(t)\,dt$.

so $H = -x_1(t) + \lambda_1\left(\frac{c\,u(t)}{x_2(t)} - \frac{D}{x_2(t)}\right) - \lambda_2 u(t)$ $\quad$ with $\begin{cases} \dot{x}_1(t) = \frac{c\,u(t)}{x_2(t)} - \frac{D}{x_2(t)} \\ \dot{x}_2(t) = -u(t) \end{cases}$

since t_f is free, $\left(\partial_t\phi + \nu^T\partial_t\psi + L + \lambda^T f\right)_{t_f} = H(t_f) = 0$

a) i) since H is not explicity depend on time, so $H(t) = 0$

adjoint eq: so $\dot{\lambda} = -\nabla_x H = -\begin{bmatrix} -1 \\ -\frac{(cu-D)\lambda_1}{x_2^2} \end{bmatrix} = \begin{bmatrix} 1 \\ (cu-D)\frac{\lambda_1}{x_2^2} \end{bmatrix} = \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix}$

$\lambda(t_f) = 0$

terms related to u in H: $\frac{\lambda_1 c}{x_2(t)}u(t) - \lambda_2 u(t) = \left(\frac{\lambda_1 c}{x_2} - \lambda_2\right)u$

so $u = \begin{cases} u_{max} & \text{when } \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) < 0 \\ 0 & \text{when } \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) > 0 \\ \text{singular} & \text{when } \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) = 0 \end{cases}$

a) ii) $\nabla_u H = \frac{\lambda_1 c}{x_2} - \lambda_2$

when the control is singular: $\nabla_u H = 0$ $\quad$ so $\lambda_2 = \frac{\lambda_1 c}{x_2}$

to get the possibility of singular control intervals,
we need to know how long $\nabla_u H$ will be 0.

so $\frac{d(\nabla_u H)}{dt} = \frac{\dot{\lambda}_1\cdot c\cdot x_2 - \lambda_1 c\cdot\dot{x}_2}{x_2^2} - \dot{\lambda}_2$

we have: $\begin{bmatrix} 1 \\ (cu-D)\frac{\lambda_1}{x_2^2} \end{bmatrix} = \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix}$ $\quad$ and $\begin{cases} \dot{x}_1(t) = \frac{c\,u(t)}{x_2(t)} - \frac{D}{x_2(t)} \\ \dot{x}_2(t) = -u(t) \end{cases}$

so $\frac{d}{dt}(\nabla_u H) = \frac{c x_2}{x_2^2} + \frac{\lambda_1 \cdot c \cdot u}{x_2^2} - (u-D)\frac{\lambda_1}{x_2^2} = \frac{c x_2 + \lambda_1 D}{x_2^2}$

in order to keep $\nabla_u H = 0$, $\frac{d}{dt}(\nabla_u H)$ needs to be 0 so $\lambda_1 = \frac{-c}{D} x_2$

but still, we haven't see u, so $\lambda_2 = -\frac{c^2}{D}$

so keep differentiate $\frac{d \nabla_u H}{dt}$.

then, $\frac{d^2}{dt^2}(\nabla_u H) = \frac{cu}{x_2^2} + \frac{D \cdot x_2^2 + 2x_2 u \cdot \lambda_1 D}{x_2^4}$ needs to be 0

so $\frac{cu}{x_2^2} + \frac{D}{x_2^2} + \frac{2u \cdot \lambda_1 D}{x_2^3} = 0$

$u(c x_2 + 2\lambda_1 D) = -D x_2$ so $u = \frac{-D x_2}{c x_2 + 2\lambda_1 D}$

since $\lambda_1 = \frac{-c}{D} x_2$ $u = \frac{D x_2}{c x_2}$

but u need to be singular, so $x_2 = 0$

so when $\lambda_1 = \frac{-c}{D} x_2$, and $x_2 = 0$ happens, the system will be in singular interval for some time.

b)

we want minimize $\int_{t_0}^{t_f} -x_1(t)\, dt$.

so $H = -x_1(t) + \lambda_1 \left(\frac{c u(t)}{x_2(t)} - \frac{D}{x_2(t)}\right) - \lambda_2 u(t)$ with $\begin{cases} \dot{x}_1(t) = \frac{c u(t)}{x_2(t)} - \frac{D}{x_2(t)} \\ \dot{x}_2(t) = -u(t) \end{cases}$

since t_f is free, $\left(\partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f\right)_{t_f} = H(t_f) = 0$

$D(x_1(t), x_2(t)) = a x_1^2(t) + \beta \frac{x_2^2(t)}{x_1^2(t)}$ so $H = -x_1 + \frac{cu}{x_2}\lambda_1 - \frac{a x_1^2}{x_2}\lambda_1 - \frac{\beta x_2}{x_1^2}\lambda_1 - \lambda_2 u$

so $\dot{\lambda} = -\nabla_x H = -\begin{bmatrix} -1 - \frac{2a x_1}{x_2}\lambda_1 + \frac{2\beta x_2}{x_1^3}\lambda_1 \\ -\frac{cu\lambda_1}{x_2^2} + \frac{a x_1^2 \lambda_1}{x_2^2} - \frac{\beta \lambda_1}{x_1^2} \end{bmatrix} = \begin{bmatrix} 1 + \frac{2a x_1 \lambda_1}{x_2} - \frac{2\beta x_2}{x_1^3}\lambda_1 \\ \frac{cu\lambda_1}{x_2^2} - \frac{a x_1^2 \lambda_1}{x_2^2} + \frac{\beta \lambda_1}{x_1^2} \end{bmatrix} = \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix}$

adjoint eq: $\Rightarrow$

terms related to u: $\frac{\lambda_1 c}{x_2}u - \lambda_2 u = \left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) u$

so $u = \begin{cases} u_m & \text{when } \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) < 0 \\ 0 & \text{when } \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) > 0 \\ \text{singular} & \text{when } \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) = 0 \end{cases}$

$$\nabla_u H = \frac{\lambda_1 c}{x_2} - \lambda_2$$

when the control is singular: $\nabla_u H = 0$ so $\lambda_2 = \frac{\lambda_1 c}{x_2}$

$$\frac{d}{dt}(\nabla_u H) = \frac{\dot{\lambda}_1 c \cdot x_2 - \lambda_1 c \dot{x}_2}{x_2^2} - \dot{\lambda}_2$$

$$= \frac{\left(1 + \frac{2a x_1 \lambda_1}{x_2} - \frac{2\beta x_2}{x_1^3}\lambda_1\right) c \cdot x_2 + \lambda_1 \cdot c \cdot u}{x_2^2} - \left(\frac{cu\lambda_1}{x_2} - \frac{a x_1^2 \lambda_1}{x_2^2} + \frac{\beta\lambda_1}{x_1^2}\right)$$

$$= \frac{c}{x_2} + \frac{2a x_1 \lambda_1 c}{x_2^2} - \frac{2\beta\lambda_1 c}{x_2^4} + \cancel{\frac{\lambda_1 c \cdot u}{x_2^2}} - \cancel{\frac{cu\lambda_1}{x_2}} + \frac{a x_1^2 \lambda_1}{x_2^2} - \frac{\beta\lambda_1}{x_1^2}$$

if $\frac{d}{dt}(\nabla_u H) = 0$

$$\frac{cu\lambda_1(1-x_2)}{x_2^2} = -\frac{c x_2^3 + 2a x_1 x_2^2 \lambda_1 c - 2\beta\lambda_1 c + a x_1^2 x_2^2 \lambda_1}{x_2^4} - \frac{\beta\lambda_1}{x_1^2}$$