EN530.603 Applied Optimal Control
Midterm #1
October 19, 2020

1. (10 pts) Find the optimum of the function

$$J(x_1, x_2) = x_1^2 + 4x_2^2 + x_1x_2 + x_2,$$

subject to

$$x_1 - 3x_2 - 10 \geq 0.$$

2. (10 pts) Consider a linear system with state $x \in \mathbb{R}^n$, controls $u \in \mathbb{R}^m$, dynamics

$$\dot{x} = Ax + Bu + c,$$

for given constant terms $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, c \in \mathbb{R}^n$ and cost function

$$J = \int_{t_0}^{t_f} \left[q^T x(t) + \frac{1}{2} u(t)^T R u(t) \right] dt,$$

where $q \in \mathbb{R}^n$ and $R \in \mathbb{R}^{m\times m}$ is symmetric positive definite. The final state $x(t_f)$ is free and the final time t_f is fixed.

Find the optimal control law, i.e. express u as a function of the state x.

In the process, show that there is a matrix $P(t)$ and vector $s(t)$ such that choosing the multipliers according to $\lambda(t) = P(t)x(t) + s(t)$ satisfies the optimality conditions. Derive the *generalized* Riccati ODE for $P(t)$ and $s(t)$, and specify any required boundary conditions.

3. (10 pts) Consider the triple integrator system with state $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and dynamics

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u,$$

where u is the control input constrained by

$$|u| \leq 1.$$

Describe the procedure for computing the minimum time control law taking the system from a starting state $x(t_0) = x_0$ to a final fixed state $x(t_f) = 0$. The cost function is $J = \int_{t_0}^{t_f} 1dt$. In the process, determine whether there are any singular intervals, find the number of switching times that are expected to occur along the optimal solution, and give a procedure to compute the exact switching times.

Note: the solution might involve finding the roots of set of nonlinear algebraic equations. You are only required to specify what these equations are and not to solve them analytically.

4. (Extra credit: 10pts) Let $x(t)$ denote a stock to be reinvested or sold for revenue, with dynamics given by

$$\dot{x} = xu,$$

where $0 \leq u(t) \leq 1$ denotes the fraction of stock to be reinvested. The initial value of the stock is $x(0) = x_0 > 0$. Compute the optimal strategy over a given time horizon $[0, T]$ for a given T to *maximize* the total sales expressed as:

$$\int_0^T (1-u)x \; dt.$$