$$h = y(x) \quad \cdot P(h) = P(x) \cdot \frac{}{\partial y(x)} \quad \begin{array}{l} y = 2x \\ P(y) = \frac{P(x)}{2} \end{array}$$

EN530.603 Applied Optimal Control
Lecture 10: Optimal State Estimation
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In practical applications the state x of a system is not perfectly known but can be inferred using noisy measurements z assuming that there is a known relationship (a sensor model function) between x and z. A common technique in optimal estimation is to find x by minimizing the squared error of this relationship. This is known as the *least-squares* approach and is the basis for developing state estimation algorithms.

We are interested in the case when the state x is evolving subject to uncertainty which means that a least-squares measurement processing formulation must be combined with information about how x has changed between measurements. This is accomplished by the following methods:

- *Kalman filter (KF)*: linear discrete-time dynamics, linear measurement model, Gaussian noise

- *Kalman-Bucy filter*: same as KF but continuous-time dynamics

When the dynamics and sensor model are nonlinear and the noise is Gaussian the optimal estimate can only be approximated typically through some form of linearization. The two most common nonlinear filtering methods are:

- *Extended Kalman filter (EKF)*: linearize dynamics and sensor model and apply KF

- *Unscented Kalman Filter (UKF)*: linearization is avoided during uncertainty propagation by propagating the principle axes of the uncertainty ellipsoid through the nonlinear dynamics and then reconstructing the updated ellipsoid; measurements are still processed through linearization

Finally, when the dynamics is nonlinear and the noise is arbitrary a general solution is to approxomiate the state using a cloud of sampled states (i.e. delta functions or particles) which are propagated through the nonlinear dynamics. This leads to:

- *Particle Filter (PF)*: the random state variables approximated using weighted mixture of samples whose weights are updated using obtained measurements

Our focus would be on KF, EKF, UKF.

1 Least-squares for static estimation

Consider the estimation of a static random variable x that takes values in $\mathbb{R}^n$ using measurements z that take values in $\mathbb{R}^{k_1}$ such that
$$z = Hx + v,$$

linear situ: $Z = Hx + V$

1

目的是为了求 $\hat{x}$: the next estimation

where v is a random variable denoting the measurement error. The goal is to find the optimal estimate $\hat{x}$ which can be accomplished by defining the measurement estimate error

$$e_z = z - H\hat{x}$$

e_z

and minimizing

$$J = \frac{1}{2}e_z^T e_z = \frac{1}{2}(z - H\hat{x})^T(z - H\hat{x}).$$

The necessary condition gives

$$\nabla J = -H^T(z - H\hat{x}) = 0 \quad \Rightarrow \quad \hat{x} = (H^T H)^{-1} H^T z$$

The sufficient conditions are

$$\nabla^2 J = H^T H,$$

which must be positive definite for $\hat{x}$ to be the true estimate. This is satisfied when $k_1 \geq n$ and $\text{rank}(H) = n$.

Now assume that there was some prior information that the distribution of x has mean $\hat{x}_0$ and covariance P_0, i.e.

$$E[x] = \hat{x}_0, \qquad E[(x - \hat{x}_0)(x - \hat{x}_0)^T] = P_0,$$

and that the measurement error is zero mean and has a diagonal covariance

$$R = \begin{bmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_{k_1} \end{bmatrix}$$

or, equivalently

$$E[v] = 0, \qquad E[vv^T] = R.$$

A meaningful cost function would then be to use a scaled error e_z' where each coordinate is defined by

$$e_{z_i}' = \frac{e_{z_i}}{\sqrt{r_i}},$$

where $\sqrt{r_i}$ is the standard deviation of the i-th measurement element error. Intuitively, a small standard deviation means a bigger e_{z_i}' which means that it will carry more importance in the cost function. More generally this is formulated using the cost function

current noise   next step noise

$$J(\hat{x}) = \frac{1}{2}(\hat{x} - \hat{x}_0)^T P_0^{-1}(\hat{x} - \hat{x}_0) + \frac{1}{2}(z - H\hat{x})^T R^{-1}(z - H\hat{x})$$

e_z e_z

J 是为关于 error 的 objective function

The necessary condition $\nabla J = 0$ gives

$$P_0^{-1}(\hat{x} - \hat{x}_0) - H^T R^{-1}(z - H\hat{x}) = 0,$$

which is equivalent to

new estimate

$$\hat{x} = (H^T R^{-1} H + P_0^{-1})^{-1}(H^T R^{-1} z + P_0^{-1}\hat{x}_0). \qquad (1)$$

$$= \hat{x}_0 + \underset{2}{P} \cdot H^T R^{-1}(z - H\hat{x}_0)$$

$$= \hat{x}_0 + k(\cdot z - H\hat{x}_0)$$

$$k = P H^T R^{-1}$$

The sufficient conditions are

$$\nabla^2 J = P_0^{-1} + H^T R^{-1} H,$$

which must be positive definite for $\hat{x}$ to be the minimum. The necessary condition (1) can be expressed as

$$
\begin{aligned}
\hat{x} &= (H^T R^{-1} H + P_0^{-1})^{-1}(H^T R^{-1} z + P_0^{-1} \hat{x}_0) \\
&= (H^T R^{-1} H + P_0^{-1})^{-1}[H^T R^{-1}(z - H\hat{x}_0) + (P_0^{-1} + H^T R^{-1} H)\hat{x}_0] \\
&\triangleq \hat{x}_0 + P H^T R^{-1}(z - H\hat{x}_0) \\
&\triangleq \hat{x}_0 + K(z - H\hat{x}_0)
\end{aligned}
$$

where we used the definitions

$$P^{-1} \triangleq P_0^{-1} + H^T R^{-1} H, \tag{2}$$
$$K \triangleq P H^T R^{-1}. \tag{3}$$

The matrix P is actually the covariance matrix of the error in the estimate $\hat{x}$ since

$$
\begin{aligned}
P &= E[(\hat{x} - x)(\hat{x} - x)^T] \\
&= E\left\{[\hat{x}_0 + K(z - H\hat{x}_0) - x][\hat{x}_0 + K(z - H\hat{x}_0) - x]^T\right\} \\
&= E\left\{[(I - KH)(\hat{x}_0 - x) + Kv][(I - KH)(\hat{x}_0 - x) + Kv]^T\right\} \\
&= (I - KH)P_0(I - KH)^T + KRK^T,
\end{aligned} \tag{4}
$$

since $\hat{x} - x$ and v are assumed independent, i.e. $E[(\hat{x} - x - v)(\hat{x} - x - v)^T] = 0$.

Notice that after multiplying (2) on the left with P and on the right P_0 we get

$$P = (I - KH)P_0$$

which is substituted in (4) to verify that

$$(I - KH)P_0(I - KH)^T + KRK^T = P - PH^T R^{-1} HP + PH^T R^{-1} HP = P.$$

These developments can be generalized for the case when new measurements appear sequentially at each time-step i. This leads to the recursive least-squares algorithm:

Given prior : mean $\hat{x}_0$ and covariance P_0
For each new measurement $z_i = H_i x + v_i$, where $v_i \sim \mathcal{N}(0, R_i)$
$$\hat{x}_i = \hat{x}_{i-1} + K_i(z_i - H_i \hat{x}_{i-1}),$$
$$P_i = (P_{i-1}^{-1} + H_i^T R_i^{-1} H_i)^{-1},$$
$$K_i = P_i H_i^T R_i^{-1}$$

K 相当于一个权重   R小. (observe 准)

An alternative update for P_i can be accomplished using the matrix inversion lemma

$$
\begin{aligned}
P_i &= (P_{i-1}^{-1} + H_i^T R_i^{-1} H_i)^{-1} \\
&= P_{i-1} - P_{i-1} H_i^T (H_i P_{i-1} H_i^T + R_i)^{-1} H_i P_{i-1}
\end{aligned}
$$

which avoids the need to compute both P_i and its inverse at each iteration.

3

$$X_i = X_{i-1} + k_i(z_i - H x_{i-1})$$

Nonlinear $Z = h(x) + v$ (handwritten)

Nonlinear least-squares. When the measurement model is nonlinear, i.e.

$$z = h(x) + v$$

we must minimize

$$J(\hat{x}) = \frac{1}{2}(\hat{x} - \hat{x}_0)^T P_0^{-1}(\hat{x} - \hat{x}_0) + \frac{1}{2}[z - h(\hat{x})]^T R^{-1}[z - h(\hat{x})]$$

h is nonlinear, use "netwon"-styk methods to solve the optimization problem (handwritten)

which can be locally approximated by first-order expansion of the nonlinear function h, i.e.

$$z \approx h(\hat{x}_0) + \partial h(\hat{x}_0)(x - \hat{x}_0) + v,$$

which can be equivalently written as

$$v = e_z$$ (handwritten)

$$dz = H dx + v,$$

where $dz = z - h(\hat{x}_0)$, $dx = x - \hat{x}_0$ and $H = \partial h(\hat{x}_0)$ so that the standard linear least-squares approach can be applied to

$$J(dx) = \frac{1}{2}dx^T P_0^{-1}dx + \frac{1}{2}[dz - Hdx]^T R^{-1}(dz - Hdx)$$

the solution to which is exactly the same as linear least squares, except that we have

$$dx = K dz,$$

or

$$\hat{x} - \hat{x}_0 = K[z - h(\hat{x}_0)].$$

The multi-stage recursive nonlinear least algorithm is then given by:

> Given prior : mean x_0 and covariance P_0
> For each new measurement $z_i = h_i(x) + v_i$, where $v_i \sim \mathcal{N}(0, R_i)$
> $$\hat{x}_i = \hat{x}_{i-1} + K_i[z_i - h(\hat{x}_{i-1})],$$
> $$P_i = P_{i-1} - P_{i-1}H_i^T(H_iP_{i-1}H_i^T + R_i)^{-1}H_iP_{i-1},$$
> $$K_i = P_iH_i^T R_i^{-1}$$

2 Propagation of Uncertainty

Consider the case when the state $x(t)$ is changing. For example, consider the simple discrete-time dynamics

$$x_k = x_{k-1} + w_{k-1},$$

where $x_0 = 0$ and $w_k \sim N(0, q^2)$. There are many possible evolutions $x_0, x_1, x_2, \ldots$. If we study the mean and covariance of x_k we would expect that it would grow with k since trajectories will diffuse in all directions as shown below

4
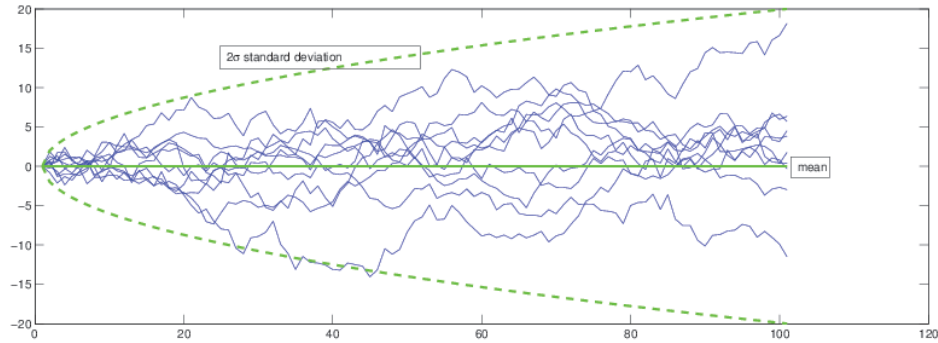
Note that if we know the first two moments, we can deduce (based on a standard statistical test) that 95% of random states x_k will fall within $1.96\sigma_k$ where σ_k is the evolved standard devition of x_k at k (i.e. the square of the second central moment).

2.1 Continuous-time noise

Consider the restricted class of systems

$$\dot{x}(t) = f(x(t), u(t)) + L(t)w(t), \tag{5}$$

where *L(t) is a given matrix* and where the *noise $w(t)$ evolves* in continuous time. We make the standard assumption that it is uncorrelated in time, i.e.

assume : noise is uncorrelated in time

$$E[w(t)w(\tau)^T] = Q'_c(t)\delta(t - \tau), \tag{6}$$

where $Q'_c(t)$ is the continuous covariance (also called spectral density), and $\delta(t - \tau)$ is the *Dirac delta* function defined by

$$\delta(t - \tau) = \left\{ \begin{array}{ll} 0, & t \neq \tau \\ \infty, & t = \tau \end{array} \right.$$

which satisfies the property

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

need to deal with this later in integration.

Practically speaking, δ is an impulse of infinitely high power but infinitely small duration which ultimately integrates to 1 over the time-domain (think about multiplying the power $1/\Delta t$ by the duration Δt in the limit as $\Delta t \to 0$).

2.2 Interpreting the noise through Brownian motion

The above relationship is actually based on the heuristic notion that $w(t)$ corresponds to the time-derivative of a fundamental process denoted by $\beta(t)$ according to

So $\dot{x}(t) = f(x(t), u(t)) + L(t) S(t) \dot{\beta}(t)$

$$w(t) = S(t)\dot{\beta}(t),$$

where S satisfies $SS^T = Q'_c$. We say "heuristic" because for stochastic processes even the notion of differentiability and smoothness is not strict. The stochastic process $\beta(t)$ is called a *Wiener process* or a *Brownian motion* if

i. $\beta(0) = 0$ *Start from 0*

ii. each sample path is continuous *continuous*

iii. $\beta(t) \sim \mathcal{N}(0, t)$, i.e. it is a zero-mean Gaussian with variance t *0 mean with an expanding covariance*

iv. for all $0 < t_1 < t_2 < t_3 < \dots$ the random variables *uncorrelated on different time interval*

$$\beta(t_1), \beta(t_2) - \beta(t_1), \beta(t_3) - \beta(t_2), \dots$$

are uncorrelated in time.

properties

In the multidimensional case, we have $\beta(t) = [\beta_1(t), \beta_2(t), \dots]$ and it is assumed that these are uncorrelated so that $\beta(t)$ is $\mathcal{N}(0, tI)$, where I is the identity matrix.

The dynamics (5) can be written as the *Itô stochastic differential equation*

So $\dot{x}(t) = f(x(t), u(t)) + L(t) S(t) \dot{\beta}(t)$

$$dx(t) = f(x(t), u(t))dt + L(t)S(t)d\beta(t),$$

$d\beta(t)$: change in Brownian motion

which is obtained by multiplying (5) by dt. This form suggests that we can regard the change in the state dx as a change in time dt and change in a Brownian motion $d\beta$.

Heuristically, since $\beta(t)$ is $\mathcal{N}(0, tI)$ we can think of $d\beta(t)$ as $\mathcal{N}(0, dtI)$, or equivelently

where S satisfies $SS^T = Q'_c$.

$$E[d\beta(t)d\beta(\tau)^T] = \begin{cases} dtI, & \text{for } t = \tau \\ 0, & \text{for } t \neq \tau \end{cases}.$$

$dt \cdot I$

确定的 所以可以 直接拿出 $E[\ ]$

Note that using the relationship $w(t) = \frac{1}{dt}S(t)d\beta(t)$ we have

$\dot{\beta} = \frac{d\beta(t)}{dt}$ *$\delta_{ij} - $个 dt*

$$E[w(t)w(\tau)^T] = \frac{1}{dt^2}E[S(t)d\beta(t)d\beta(\tau)^T S(t)^T] = \frac{1}{dt^2}S(t)E[d\beta(t)d\beta(\tau)^T]S(\tau)^T = \frac{1}{dt}Q'_c(t)\delta_{jk},$$

discrete version.

where δ_{jk} is the Kronecker delta (i.e. it is 1 when $j = k$ and 0 otherwise) in view of defining $t = t_0 + jdt$, $\tau = t_0 + kdt$. As $dt \to 0$ the above corresponds to the definition (6).

$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\delta_{jk} = \delta(t-\tau)$

2.3 Linear Systems

For linear systems with Gaussian noise we can precisely determine the resulting distribution at each time-step k in terms of its mean and covariance. For nonlinear systems one can approximate this evolution.

Consider the system with state $x \in \mathbb{R}^n$ evolving according to

$$x_k = \Phi_{k-1}x_{k-1} + \Gamma_{k-1}u_{k-1} + \Lambda_{k-1}w_{k-1},$$

where $E[w_k] = 0$, $E[w_k w_k^T] = Q'_k$ and $E[w_k w_{k-\ell}^T] = 0$ for all integers $\ell \neq k$. The last condition means that the noise is uncorrelated in time. The initial conditions are

又知 x_0 的 mean 和 variance

$$E[x_0] = \hat{x}_0, \qquad E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] = P_0,$$

Assume $x_0 \sim \mathcal{N}(\hat{x}_0, P_0)$

while the controls are known and the state, noise, and controls are uncorrelated. Assume that x has a Gaussian density $x_0 \sim \mathcal{N}(\hat{x}_0, P_0)$, i.e.

$$p(x_0) = \frac{1}{(2\pi)^{\frac{n}{2}}|P_0|^{\frac{1}{2}}}e^{-\frac{1}{2}(x_0 - \hat{x}_0)^T P_o^{-1}(x_0 - \hat{x}_0)}$$

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Our goal is to propagate the mean and covariance. We have

$$E[x_k] = E[\Phi_{k-1}x_{k-1} + \Gamma_{k-1}u_{k-1} + \Lambda_{k-1}w_{k-1}]$$
$$= \Phi_{k-1}E[x_{k-1}] + \Gamma_{k-1}E[u_{k-1}] + \Lambda_{k-1}E[w_{k-1}]$$

which is equivalent to

control is given

$$\hat{x}_k = \Phi_{k-1}\hat{x}_{k-1} + \Gamma_{k-1}u_{k-1}.$$

First note that　　和k 号的 deviation

$$x_k - \hat{x}_k = \Phi_{k-1}x_{k-1} + \Gamma_{k-1}u_{k-1} + \Lambda_{k-1}w_{k-1} - \Phi_{k-1}\hat{x}_{k-1} - \Gamma_{k-1}u_{k-1}$$
$$= \boxed{\Phi_{k-1}(x_{k-1} - \hat{x}_{k-1}) + \Lambda_{k-1}w_{k-1}.}$$

和k-1 号的 deviation

The covariance is then

$$P_k = E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] =$$
$$= E\left\{[\Phi_{k-1}(x_{k-1} - \hat{x}_{k-1}) + \Lambda_{k-1}w_{k-1}][\Phi_{k-1}(x_{k-1} - \hat{x}_{k-1}) + \Lambda_{k-1}w_{k-1}]^T\right\}$$
$$= \Phi_{k-1}E[(x_{k-1} - \hat{x}_{k-1})(x_{k-1} - \hat{x}_{k-1})^T]\Phi_{k-1}^T + \Phi_{k-1}E[(x_{k-1} - \hat{x}_{k-1})w_{k-1}^T]\Lambda_{k-1}^T$$
$$+ \Lambda_{k-1}E[w_{k-1}(x_{k-1} - \hat{x}_{k-1})^T]\Phi_{k-1}^T + \Lambda_{k-1}E[w_{k-1}w_{k-1}^T]\Lambda_{k-1}^T$$
$$= \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + \Lambda_{k-1}Q'_{k-1}\Lambda_{k-1}^T$$

where we used the assumption the state error and noise and uncorrelated, i.e. that

$$E[(x_{k-1} - \hat{x}_{k-1})w_{k-1}^T] = E[w_{k-1}(x_{k-1} - \hat{x}_{k-1})^T] = 0.$$

Let

$$Q_k \triangleq \Lambda_{k-1}Q'_{k-1}\Lambda_{k-1}^T,$$

The resulting update $(\hat{x}_{k-1}, P_{k-1}) \to (\hat{x}_k, P_k)$ are called a *Gauss-Markov sequence*, given by

$$\hat{x}_k = \Phi_{k-1}\hat{x}_{k-1} + \Gamma_{k-1}u_{k-1},$$
$$P_k = \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + Q_{k-1}.$$

The sequence is called *Markov* when the current state x_k only depends on the previous state x_{k-1}. This can also be expressed using conditional probabilities according to

$$p(x_k) = \int p(x_k|x_{k-1})p(x_{k-1})dx_{k-1} = \int p(x_k|x_{k-1})p(x_{k-1}|x_{k-2})p(x_{k-2})dx_{k-1,k-2} = \cdots = \int p(x_0)\Pi_{i=1}^k p(x_i|x_{i-1})d$$

2.4 From continuous to discrete evolution.

In practice, the discrete dynamics must be constructed from the actual continuous system dynamics generally given by

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) + L(t)w(t), \quad 一般是这个形式$$

We have

$$x(t_k) = x(t_{k-1}) + \int_{t_{k-1}}^{t_k} [F(\tau)x(\tau) + G(\tau)u(\tau) + L(\tau)w(\tau)]d\tau$$
$$= \Phi(t_k, t_{k-1})x(t_{k-1}) + \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)[G(\tau)u(\tau) + L(\tau)w(\tau)]d\tau,$$

$$\dot{x} = \int_{t_0} Ax + B$$

come from solution of linear ODE

where the *state transition matrix* $\Phi(t_k, t_{k-1})$ is such that

from solution of linear ODE

$$\Phi(t_k, t_{k-1})x(t_{k-1}) = x(t_k) + \int_{t_{k-1}}^{t_k} [F(\tau)x(\tau)]d\tau.$$

For instance, for constant $F(t) = F$ we simply have $\Phi(t_k, t_{k-1}) = e^{(t_k - t_{k-1})F}$. In general, for small $\Delta t = t_k - t_{k-1}$ the approximation

$$\Phi(t_k, t_{k-1}) \approx I + \Delta t F(t_{k-1})$$

could be employed in practice. The mean evolves according to

$$\hat{x}_k = \Phi_{k-1}\hat{x}_{k-1} + \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)G(\tau)u(\tau)d\tau$$

where $\Phi_{k-1} \triangleq \Phi(t_k, t_{k-1})$. Note that if we assume that the control is constant during the sampling interval then we can write

$$\hat{x}_k = \Phi_{k-1}\hat{x}_{k-1} + \Gamma_{k-1}u_{k-1},$$

where

$$\Gamma_{k-1} \triangleq \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)G(\tau)d\tau.$$

The covariance evolves according to

$$P_k = \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + E\left\{\left[\int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)L(\tau)w(\tau)d\tau\right]\left[\int_{t_{k-1}}^{t_k} \Phi(t_k, \alpha)L(\alpha)w(\alpha)d\alpha\right]^T\right\}$$

$$= \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)L(\tau)\underbrace{E[w(\tau)w(\alpha)^T]}_{Q_c'\delta(\tau-\alpha)}L(\alpha)^T\Phi^T(t_k, \alpha)d\tau d\alpha \quad \text{only whe } \tau = \alpha \text{ 时为0, 所以去掉了} \\ \text{一个} \int$$

$$= \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)L(\tau)Q_c'L(\tau)^T\Phi^T(t_k, \tau)d\tau$$

$$\triangleq \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + Q_{k-1}, \qquad Q_{k-1}$$

where Q_{k-1} is defined through the last relationship above. Note that for small $\Delta t = t_k - t_{k-1}$ the simple first-order approximation

$$Q_{k-1} \approx L(t_{k-1})Q_c'(t_{k-1})L^T(t_{k-1})\Delta t$$

could be employed in practice.

Example 1. Consider the double integrator with state consisting of the position p and velocity v, i.e. $x = (p, v)$ and dynamics given by

$$\dot{p} = v$$
$$\dot{v} = u + w,$$

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with $x_0 \sim \mathcal{N}(\hat{x}_0, P_0)$ and $w \sim \mathcal{N}(0, q_c)$. We have

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q_c(t) = q_c$$

Assume that we have a sampling rate of Δt during which the control u_k is constant. Then

$$\Phi_k = e^{\Delta t F} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \qquad \text{\textcolor{red}{$I + F \cdot \Delta t$}}$$

and the mean evolves according to

$$\hat{x}_k = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \hat{x}_{k-1} + \begin{bmatrix} \Delta t & \Delta t^2/2 \\ 1 & \Delta t \end{bmatrix} G u_{k-1}$$

or equivalently

$$\begin{bmatrix} p_k \\ v_k \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ v_{k-1} \end{bmatrix} + \begin{bmatrix} \Delta t^2/2 \\ \Delta t \end{bmatrix} u_{k-1}.$$

To compute the covariance we need to find

$$Q_{k-1} = \int_{t_{k-1}}^{t_k} \begin{bmatrix} \tau - t_{k-1} \\ 1 \end{bmatrix} q_c \begin{bmatrix} \tau - t_{k-1} \\ 1 \end{bmatrix}^T d\tau = q_c \int_{t_{k-1}}^{t_k} \begin{bmatrix} (\tau - t_{k-1})^2 & (\tau - t_{k-1}) \\ (\tau - t_{k-1}) & 1 \end{bmatrix} d\tau = q_c \begin{bmatrix} \frac{\Delta t^3}{3} & \frac{\Delta t^2}{2} \\ \frac{\Delta t^2}{2} & \Delta t \end{bmatrix}.$$

3 Linear-optimal Estimation

\textcolor{red}{Δt is small} \qquad \textcolor{red}{$\uparrow$ domain term}

We can now combine measurement updates and uncertainty propagation to optimally estimate the state of the dynamic random variable $x(t)$. We consider a discrete linear, time-varying (LTV) model given by:

$$x_k = \Phi_{k-1} x_{k-1} + \Gamma_{k-1} x_{k-1} + \Lambda_{k-1} w_{k-1}, \tag{7}$$
$$z_k = H_k x_k + n_k. \tag{8}$$

Estimating the state x_k optimally is performed by iterating between uncertainty propagation and measurement updates. The mean and covariance after propagation from step $k-1$ to step k are denoted by $(x_{k|k-1}, P_{k|k-1})$. The mean and covariance after a measurement update at step k are denoted by $(x_{k|k}, P_{k|k})$. This procedure is called the *Kalman filter* and is given by

Prediction: \quad \textcolor{red}{$state\ transition$} $\textcolor{red}{\downarrow}$

$$\hat{x}_{k|k-1} = \Phi_{k-1} \hat{x}_{k-1|k-1} + \Gamma_{k-1} u_{k-1}$$
$$P_{k|k-1} = \Phi_{k-1} P_{k-1|k-1} \Phi_{k-1}^T + Q_{k-1},$$

\textcolor{red}{$measurement\ at\ time\ k$}

Correction: $\textcolor{red}{\downarrow}$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(z_k - H_k \hat{x}_{k|k-1}),$$
$$P_{k|k} = P_{k|k-1} - P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1} H_k P_{k|k-1},$$
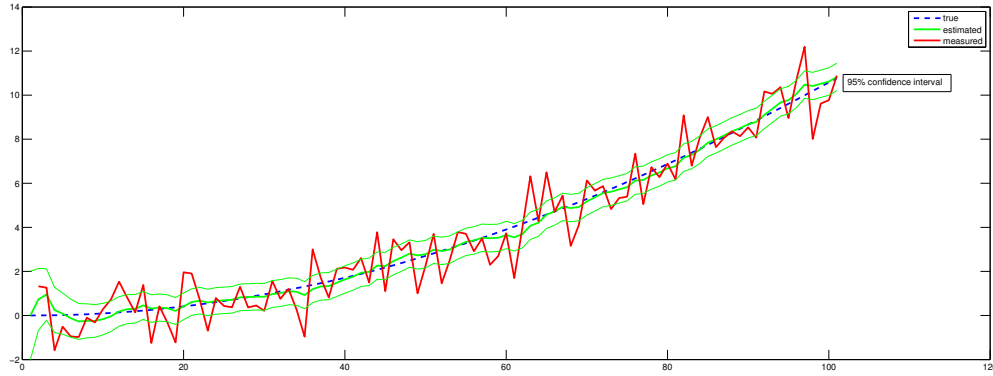$$K_k = P_{k|k} H_k^T R_k^{-1}$$

There are other (equivalent) forms of the filter depending on whether the gain matrix K or the covariance matrix P is computed first. It is common to use the correction:

$$
\begin{aligned}
\text{Correction:} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(z_k - H_k\hat{x}_{k|k-1}), \\
K_k &= P_{k|k-1}H_k^T(H_k P_{k|k-1}H_k^T + R_k)^{-1} \\
P_{k|k} &= (I - K_k H_k)P_{k|k-1},
\end{aligned}
$$

Example 2. Consider the double-integrator example implemented in `int_test.m`:

Observability. A system is said to be *observable* if the full state can be reconstructed by knowing the input u and processing measurements z. For linear systems with constant matrices F and H there is a simple observability tests, i.e. by checking that the matrix

$$
\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}
$$

has rank n. For example, in the double integrator estimation when we can *measure position* we have $H = [1 \ 0]$ and the observability matrix is

$$
\begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

which has rank 2. If instead, we were *measuring velocity* we have $H = [0 \ 1]$ and the observability matrix is

$$
\begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
$$

with rank 1. Intuitively, if we only observe velocity then it is not possible to correctly reconstruct the position variable which will drift with time.

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4 Nonlinear Estimation

Now consider a nonlinear discrete-time model with additive noise given by

$$x_k = f(x_{k-1}, u_{k-1}) + w_{k-1}, \tag{9}$$
$$z_k = h_k(x_k) + v_k. \tag{10}$$

where $w_k \sim \mathcal{N}(0, Q_k)$ and $E[w_k w_{k-\ell}^T] = 0$ for all integers $\ell \neq k$. Similarly we assume that $v_k \sim \mathcal{N}(0, R_k)$ and $E[v_k v_{k-\ell}^T] = 0$. The last condition means that the noise is uncorrelated in time. The initial conditions are

$$E[x_0] = \hat{x}_0, \qquad E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] = P_0.$$

We will be concerned with estimation schemes which propagate and maintain only the first two moments of the distribution of x – the mean $\hat{x}_k$ and the covariance P_k. We are again faced with the two problems: how to process measurements and how to update the distribution between times-steps. As we will see, measurements can be processed through the already established nonlinear least-squares procedure. Uncertainty propagation for nonlinear systems is much more complicated. The simplest approach is to perform *linearized error propagation* which essentially amounts to linearizing the dynamics and applying a Kalman filter procedure (termed *Extended Kalman Filter* (EKF)). Another, more accurate approach is to regard the distribution as an hyper-ellipsoid and propagate its center and axes through the nonlinear dynamics and then reconstruct another ellipsoid to represent the updated distribution (this procedure is the *Unscented Kalman Filter* (UKF)).

Note. The EKF, UKF, and related filters are also applicable to the more general case

$$x_k = f(x_{k-1}, u_{k-1}, w_{k-1}),$$
$$z_k = h_k(x_k, v_k),$$

i.e. where non-additive noise terms are present.

4.1 Extended Kalman Filtering

The EKF is constructed using the dynamics and measurement Jacobians defined by

$$F_k = \partial_x f(\hat{x}_{k|k}, u_k), \qquad H_k = \partial_x h(\hat{x}_{k|k-1}).$$

The algorithm is given by

Prediction:
$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_{k-1})$$
$$P_{k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^T + Q_{k-1},$$

Correction:
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k[z_k - h(\hat{x}_{k|k-1})],$$
$$P_{k|k} = P_{k|k-1} - P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1} H_k P_{k|k-1},$$
$$K_k = P_k H_k^T R_k^{-1}$$

There are other (equivalent) forms of the filter depending on whether the gain matrix K or the covariance matrix P is computed first. It is common to use the correction:

Correction:
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k[z_k - h(\hat{x}_{k|k-1})],$$
$$K_k = P_{k|k-1}H_k^T(H_kP_{k|k-1}H_k^T + R_k)^{-1}$$
$$P_{k|k} = (I - K_kH_k)P_{k|k-1},$$

Derivation. The mean in the EKF is updated simply using the dynamics since the noise is zero mean. The covariance prediction is derived approximately through linearization. First note that

$$x_{k|k-1} - \hat{x}_{k|k-1} = f(x_{k-1|k-1}, u_{k-1}) + w_{k-1} - f(\hat{x}_{k-1|k-1}, u_{k-1})$$
$$\approx F_{k-1} \cdot (x_{k-1|k-1} - \hat{x}_{k-1|k-1}) + w_{k-1}$$

Then we have

$$P_{k|k-1} = E[(x_{k|k-1} - \hat{x}_{k|k-1})(x_{k|k-1} - \hat{x}_{k|k-1})^T]$$
$$\approx E\left\{[F_{k-1} \cdot (x_{k-1|k-1} - \hat{x}_{k-1|k-1}) + w_{k-1}][F_{k-1} \cdot (x_{k-1|k-1} - \hat{x}_{k-1|k-1}) + w_{k-1}]^T\right\}$$
$$= F_{k-1}P_{k-1|k-1}F_{k-1}^T + Q_{k-1},$$

where we used the assumption the state error and noise and uncorrelated, i.e. $E[(x_{k-1|k-1} - \hat{x}_{k-1|k-1})w_{k-1}^T] = E[w_{k-1}(x_{k-1|k-1} - \hat{x}_{k-1|k-1})^T] = 0$.

The EKF correction step is simply an application of nonlinear least-squares are already discussed.

Example 3. Wheeled robot with a range and bearing sensor. Consider a wheeled robot with state (x, y, θ) where $p = (x, y)$ is the position and θ is its orientation. The robot is controlled using forward and angular velocities denoted by the inputs $u_k = (v_k, \omega_k)$. The dynamics is given by

$$\begin{pmatrix} x_k \\ y_k \\ \theta_k \end{pmatrix} = \begin{pmatrix} x_{k-1} + \Delta t \cos(\theta_{k-1})v_{k-1} \\ y_{k-1} + \Delta t \sin(\theta_{k-1})v_{k-1} \\ \theta_{k-1} + \Delta t\omega_{k-1} \end{pmatrix} + w_{k-1}, \tag{11}$$

where $w_{k-1} \sim \mathcal{N}(0, Q_{k-1})$ is the process noise. There is a beacon in the environment at a fixed position $p' = (x', y') \in \mathbb{R}^2$ which the robot can sense through measurements $z = (\phi, r)$, where ϕ is the relative angle between the robot and the beacon and r is the range to the beacon. The sensor model is

$$z_k = h([x_k, y_k, \theta_k]) + v_k,$$

where

$$h([x, y, \theta]) = \begin{pmatrix} \arctan \frac{y'-y}{x'-x} - \theta \\ \|p' - p\| \end{pmatrix}$$
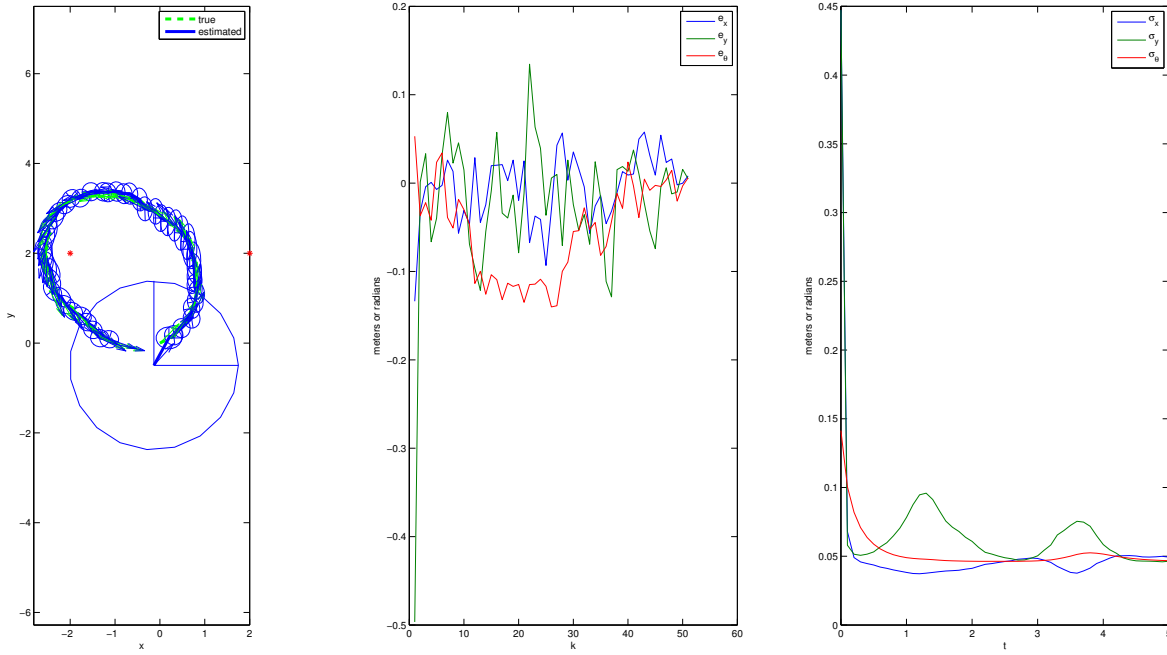
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Figure 1: With two beacons: observable

and where $v_k \sim \mathcal{N}(0, R_k)$ is the sensor noise. The EKF can be implemented using the Jacobians

$$F = \begin{bmatrix} 1 & 0 & -\Delta t v \sin\theta \\ 0 & 1 & \Delta t v \cos\theta \\ 0 & 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} \frac{q_y - p_y}{\|q-p\|^2} & -\frac{q_x - p_x}{\|q-p\|^2} & -1 \\ \frac{p_x - q_x}{\|q-p\|^2} & \frac{p_y - q_y}{\|q-p\|^2} & 0 \end{bmatrix}.$$

An EKF implementation of this model is provided in file `uni_br_ekf.m`

4.2   Unscented Kalman Filtering

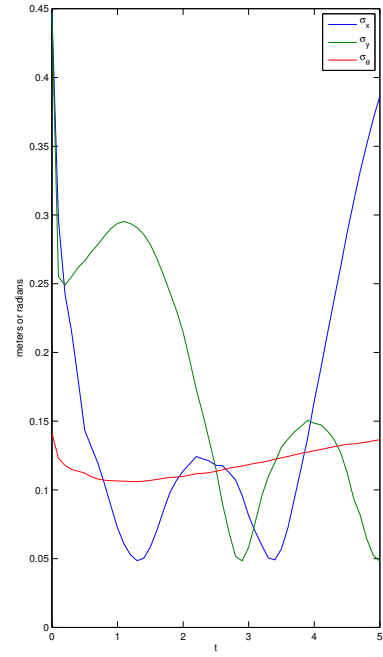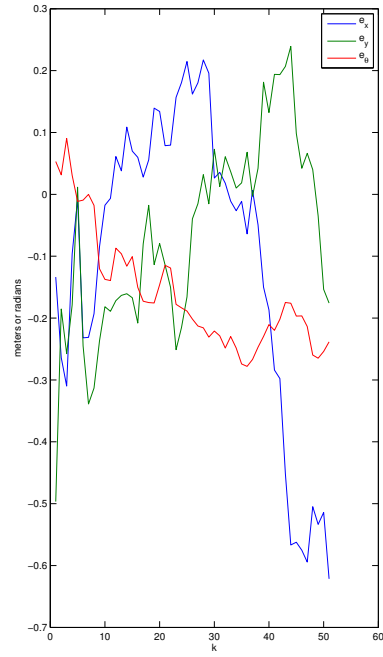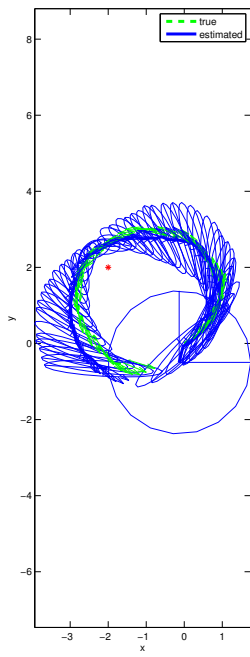Read *The Unscented Kalman Filter for Nonlinear Estimation* by Eric A. Wan and Rudolph van der Merwe[**?**].

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Figure 2: With one beacon: unobservable

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