1. so $L=1$

let $\lambda = \lambda_1, \lambda_2$

Hamiltonian: $1+\lambda^T f(x,u)$

$\dot{\lambda} = -\nabla_x H = 0 \Rightarrow \begin{matrix}\lambda_1 = C_1 \\ \lambda_2 = C_2\end{matrix}$ C_1 C_2 are constants

Since Hamiltonian doesn't depend on time.

We need $\nabla_u H = 0$ so $-\lambda_1 \sin(u) + \lambda_2 \cos(u) = 0 \Rightarrow u = \tan^{-1}\left(\frac{C_2}{C_1}\right) \Rightarrow$ a constant

$$\text{so}\ \left.\begin{matrix}\dot{x}_1 = s + \cos(u) \\ \dot{x}_2 = \sin(u) \\ u \text{ is a constant}\end{matrix}\right\} \begin{matrix} x_1(t) = (s+\cos(u))\,t + C_3 \\ x_2(t) = \sin(u)\,t + C_4 \end{matrix}$$

C_3, C_4 are constants

since $X_0 = (0,0) \Rightarrow C_3 = C_4 = 0$

so at time T

$$x_1(T) = (s+\cos(u))\,T = a$$
$$x_2(T) = \sin(u)\,T = b$$

u and T could be solved when a, b, s are given

and u is a constant

2) $\dot{x} = Ax + Bu$

We need $\frac{1}{2} x(t_f)^T S\, x(t_f) \leq 1$

when $\frac{1}{2} x(t_f)^T S\, x(t_f) = 1$ it fits the constraint.

assume V is in the form: $\frac{1}{2} x^T(t)\, S(t)\, x(t)$

then, $H = \frac{1}{2} u^T u + \nabla_x V^T (Ax + Bu)$

$u + \nabla_x V^T B$

$\nabla_u H = 0 \Rightarrow u^* = -B^T \nabla_x V$

according to HJB eqn:

$$-\partial_t V = \min_{u(t)} \left\{ \frac{1}{2} u^T u + \nabla_x V^T \cdot f \right\}$$

$\nabla_x V = S \cdot x$

$u = -B^T S \cdot x$

so $-\frac{1}{2} x \dot{S} x = \frac{1}{2} x^T \left(-SBB^T S + SA + A^T S \right) x$

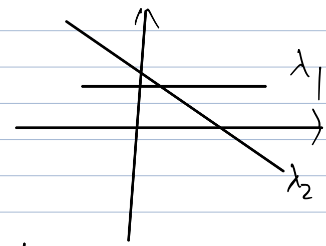
$-\dot{S} = -SBB^T S + SA + A^T S$ with $S(t_f) = S$

$\dot{S} = SBB^T S - 2SA$ after getting S, we can get the optimal u^*

3. $L = 1$, $J = \int_0^T 1\, dt = T$ $\quad \phi = 0$

$H = 1 + \lambda_1(-x_2 + u) + \lambda_2 u$

$\dot{\lambda} = -\nabla_x H = \begin{pmatrix} 0 \\ -\lambda_1 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = C_1 \\ \lambda_2 = -C_1 t + C_2 \end{cases}$



$\partial_u H = \lambda_1 + \lambda_2$

$\nabla_u^2 H = 0$

$u^* = \arg\min_u \{H\}$, so if $\lambda_1 + \lambda_2 > 0$ $\quad u = -1$

if $\lambda_1 + \lambda_2 < 0$ $\quad u = 1$

so $u^*(t) = +1$ for all $t \in [t_0, t^*]$, or

$u^*(t) = -1$ for all $t \in [t_0, t^*]$ or

$u^*(t) = +1$ for all $t \in [t_0, t_1]$ and -1 for all $t \in (t_1, t^*]$ or

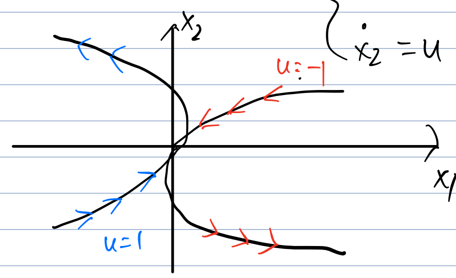
$u^*(t) = -1$ for all $t \in [t_0, t_1)$ and 1 for all $t \in (t_1, t^*]$

Since $u = \pm 1$ in all cases

we have: $\begin{cases} \dot{x}_1 = -x_2 + u \\ \dot{x}_2 = u \end{cases} \Rightarrow$

$u = 1$
$\begin{cases} x_1 = -\frac{1}{2}t^2 - (C_3 - 1)t + C_4 \\ x_2 = t + C_3 \end{cases}$

$u = -1$
$\begin{cases} x_1 = \frac{1}{2}t^2 - (C_3 + 1)t + C_4 \\ x_2 = -t + C_3 \end{cases}$



when $u = 1$ $\quad x_1 = -\frac{1}{2}x_2^2 + x_2 + C_5$

$u = -1$ $\quad x_1 = \frac{1}{2}x_2^2 + x_2 + C_6$

switching function: $S(x(t)) = x_1 - \frac{1}{2}x_2|x_2| - x_2$

$u^* = \begin{cases} +1 & \text{for } x(t) \text{ that } S(x(t)) > 0 \\ -1 & \text{for } x(t) \text{ that } S(x(t)) < 0 \\ +1 & \text{for } S(x) = 0 \quad x_2 < 0 \\ -1 & \text{for } S(x) = 0 \quad x_2 > 0 \\ 0 & \text{for } x = (0,0) \end{cases}$

4 $x = \begin{bmatrix} p \\ v \\ a \end{bmatrix}$ $\quad \dot{x} = Fx + Lw = \underbrace{\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & k\cdot I & d\cdot I \end{bmatrix}}_{F} \begin{bmatrix} p \\ v \\ a \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{L} w$

$F \in \mathbb{R}^{9\times 9} \quad L \in \mathbb{R}^{9\times 1}$

We need x_k in the form: $x_k = \Phi_{k-1} \cdot x_{k-1} + \Lambda_{k-1} w_{k-1}$

$\Phi_{k-1} x(t_{k-1}) = x(t_k) + \int_{t_{k-1}}^{t_k} F(\tau) x(\tau) \, d\tau$ $\quad$ I here are 3x3 identity matrix

Since F is constant. $\quad \Phi_{k-1} \approx I_{9\times 9} + \Delta t F(t_{k-1}) = \begin{bmatrix} I & \Delta t \cdot I & 0 \\ 0 & I & \Delta t \cdot I \\ 0 & k \cdot \Delta t \cdot I & I + d \cdot \Delta t \cdot I \end{bmatrix}$

Since $\frac{x_k - x_{k-1}}{\Delta t} = F \cdot x_{k-1} + L w_{k-1}$, $\quad$ so $\Lambda_{k-1} = \Delta t \cdot L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Delta t \\ \Delta t \\ \Delta t \end{bmatrix}$

$$X_k = \begin{bmatrix} I & \Delta t \cdot I & 0 \\ 0 & I & \Delta t \cdot I \\ 0 & k \cdot \Delta t \cdot I & I + d \cdot \Delta t \cdot I \end{bmatrix} X_{k-1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Delta t \\ \Delta t \\ \Delta t \end{bmatrix} W_{k-1}$$

$$P_k = \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + Q_{k-1}$$

$$Q_{k-1} \approx \Lambda_{k-1} Q_c' \Lambda_{k-1}^T = Q_c' \cdot \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \end{bmatrix} \cdot \Delta t^2$$

$$z = \begin{bmatrix} \frac{p}{\|p\|} \\ v \end{bmatrix} + \begin{bmatrix} n_p \\ n_v \end{bmatrix} \Rightarrow h = \begin{bmatrix} \frac{p}{\|p\|} \\ v \end{bmatrix} \quad H_k = \partial_x h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

z is nonlinear, so use EKF.

so; prediction:

$$\hat{x}_{k|k-1} = \Phi_{k-1} \cdot \hat{X}_{k-1|k-1}$$

$$P_{k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^{T} + Q_{k-1}$$

correction: $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k [z_k - h(\hat{x}_{k|k-1})]$

$$P_{k|k} = P_{k|k-1} - P_{k|k-1} H_k^{T} (H_k P_{k|k-1} H_k^{T} + R_k)^{-1} H_k P_{k|k-1}$$

$$K_k = P_k H_k^{T} R_k^{-1}$$

5 $\dot{x} = ax - u$ and $x(T)=0$, $x(0) = x_0$, $u(t) \geq 0$

Since the enjoyment is $\int_0^T e^{-\beta t}\sqrt{u(t)}\,dt$

the cost function is $\int_0^T - e^{-\beta t}\sqrt{u(t)}\,dt$, and we want to minimize it

$L = -e^{-\beta t}\sqrt{u}$

Hamiltonion: $-e^{-\beta t}\sqrt{u} + \lambda \cdot (ax-u)$

So: $\dot{\lambda} = -\nabla_x H = -\lambda a$ so $\lambda = C_1 e^{-at}$ C_1 is a constant

there have constraints on u.

so $u^* = \underset{u}{\operatorname{argmin}}\{H\} = \underset{u\geq 0}{\operatorname{argmin}}\left\{-e^{-\beta t}\sqrt{u} + \lambda(ax^* - u)\right\}$

and we have: $\nabla_u H = 0 \Rightarrow -e^{-\beta t}\cdot\frac{1}{2}u^{-\frac{1}{2}} - \lambda = 0$

So $u = \frac{1}{4\lambda^2} e^{-2\beta t}$

and $\nabla_u^2 H = \frac{1}{4}\cdot e^{-\beta t} u^{-\frac{3}{2}}$

fit the constraint

When $u = \frac{1}{4\lambda^2}e^{-2\beta t}$, $u \geq 0$, $\nabla_u^2 H > 0$, so $u^* = \frac{1}{4\lambda^2} e^{-2\beta t}$

and $\lambda = C_1 e^{-at} \Rightarrow u^* = \frac{1}{4C_1^2} e^{(2a-2\beta)t}$

So $\dot{x} = ax - \frac{1}{4C_1^2} e^{(2a-2\beta)t}$ which is a linear ODE

We need to find a Homogeneous solution, and a particular solution

Homogenerous solution: $x = C_2 e^{at}$ C_2 is a constant.

Particular solution: if $x(t) = C_3 e^{(2a-2\beta)}$

then, $C_3 (2a-2\beta) e^{(2a-2\beta)t} = a C_3 e^{(2a-2\beta)t} - \frac{1}{4C_1^2} e^{(2a-2\beta)t}$

so $C_3 = -\frac{1}{4C_1^2 (a-2\beta)}$ if $(a \neq 2\beta)$

So the general solution is:

$$x(t) = C_2 e^{at} - \frac{1}{4C_1^2 (a-2\beta)} e^{(2a-2\beta)t}$$

Since $x(0) = x_0$ $x(T) = 0$

$$\left.\begin{aligned} & C_2 - \frac{1}{4C_1^2 (a-2\beta)} = x_0 \\ & C_2 e^{aT} - \frac{1}{4C_1^2 (a-2\beta)} e^{(2a-2\beta)T} = 0 \end{aligned}\right\}$$

When $(a \neq 2\beta)$

$$\frac{1}{4C_1^2} = \frac{-x_0 (a-2\beta)}{1-e^{(a-2\beta)T}}$$

$$C_2 = \frac{-x_0 e^{(a-2\beta)T}}{1-e^{(a-2\beta)T}}$$

if $a = 2\beta$

$$\dot{x} = ax - \frac{1}{4C_1^2} e^{at}$$

for $\dot{x} = -\frac{1}{4C_1^2} e^{at}$ x should in the form $x = C_4 t e^{at}$

$$\dot{x} = C_4 e^{at} + aC_4 t e^{at} = aC_4 t e^{at} - \frac{1}{4C_1^2} e^{at}$$

So $C_4 = -\frac{1}{4C_1^2}$, so $x = C_2 e^{at} - \frac{1}{4C_1^2} t e^{at}$,

and $x(0) = x_0$ $x(T) = 0 \Rightarrow \frac{1}{4C_1^2} = \frac{x_0}{T}$, $C_2 = x_0$

if $a = 2\beta$

So when $a \neq 2\beta$

$$u^*(t) = \frac{x_0 (2\beta - a)}{1 - e^{(a-2\beta)T}} e^{(2a-2\beta)t}$$

when $a = 2\beta$

$$u^*(t) = \frac{x_0}{T} e^{at}$$

6 assume the value function is in the form:

a) $V(x,t) = \frac{1}{2} s(t) x^2 + b(t)x + v(t)$

the dynamic: $\dot{x} = x + u + a + w$

the HJB equation becomes:

$$-\partial_t V(x,t) = \min_{u(t)} \left\{ \frac{1}{2}u^2 + \nabla_x V(x+a+u) + \frac{1}{2} tr\left[\nabla_x^2 V \cdot W\right] \right\} \qquad W = E[ww^T] = 1$$

so $-\left(\frac{1}{2}\dot{s}x^2 + \dot{b}x + \dot{v}\right) = \min_{u} \left\{ \frac{1}{2}u^2 + (sx+b)(x+a+u) + \frac{1}{2}s \right\}$

$\Rightarrow$ u^* should satiesfy $u^* + sx + b = 0$ so $u^* = -sx - b$

so $\min_{u} \left\{ \frac{1}{2}u^2 + (sx+b)(x+a+u) + \frac{1}{2}s \right\}$

$$= \frac{1}{2}(sx+b)^2 + (sx+b)(x - sx + a - b) + \frac{1}{2}s$$

$$= \cancel{\frac{1}{2}s^2x^2} + \cancel{sxb} + \frac{1}{2}b^2 + \cancel{sx^2} + \cancel{bx} - \cancel{s^2x^2} - \cancel{bsx} + \cancel{asx} + ab - \cancel{bsx} - b^2 + \frac{1}{2}s$$

$$= \left(-\frac{1}{2}s^2x^2 + sx^2\right) + (bx + asx - bsx) + b(a-b) + \frac{1}{2}s + \frac{b^2}{2}$$

$$= \left(-\frac{1}{2}s^2 + s\right)x^2 + (b + as - bs)x + ab - \frac{b^2}{2} + \frac{1}{2}s$$

so $\dot{s} = s^2 - 2s$ $\qquad s(t) = \dfrac{2}{e^{c_1+2t}+1}$ $\qquad c_1$ is a constant

$\dot{b} = -b - as + bs \Rightarrow$

$\dot{v} = -ab + \frac{b^2}{2} - \frac{1}{2}s$

b) $E\left[\frac{1}{2}X(1)^2 + \int_0^1 \frac{1}{2}u^2(t)\, dt\right]$