1) $\min J(x) = \int_0^1 \left\{ \frac{1}{2}\dot{x}^2(t) + 3x(t)\dot{x}(t) + 2x^2(t) + 3x(t) \right\} dt$

let $g(x(t), \dot{x}(t))dt$

With Euler-lagrange equation: $g_x(x,\dot{x}) - \frac{d}{dt} g_{\dot{x}}(x,\dot{x}) = 0 = 3\dot{x} + 4x + 3 - (\ddot{x} + 3\dot{x}) \Rightarrow \ddot{x} = 4x+3$

Since $\ddot{x} = 4x+3$

for $\ddot{x} - 4x = 0$ characteristic equation: $r^2 - 4r = 0$

then $x = a_1 \cdot e^{2t} + a_2 \cdot e^{-2t}$

for $\ddot{x} = 4x+3$. a simple Solution: $x = -\frac{3}{4}$ $\ddot{x} = 0$

so $x = a_1 \cdot e^{2t} + a_2 \cdot e^{-2t} - \frac{3}{4}$, and $x(0) = 0$ $x(1) = 4$

so $\begin{cases} a_1 + a_2 - \frac{3}{4} = 0 \\ a_1 \cdot e^2 + a_2 \cdot e^{-2} - \frac{3}{4} = 4 \end{cases} \Rightarrow \begin{cases} a_1 = \dfrac{19 - 3e^{-2}}{4(e^2 - e^{-2})} \\ a_2 = \dfrac{3e^2 - 19}{4(e^2 - e^{-2})} \end{cases}$

so $x(t) = \dfrac{19 - 3e^{-2}}{4(e^2 - e^{-2})} e^{2t} + \dfrac{3e^2 - 19}{4(e^2 - e^{-2})} e^{-2t} - \dfrac{3}{4}$

2)

$\min J(x) = \int_0^1 \left\{ \dot{x}_1^2(t) + \dot{x}_2^2(t) + 3x_1(t)\, x_2(t) \right\} dt$

then., $g(x, \dot{x}) = \dot{x}_1^2(t) + \dot{x}_2^2(t) + 3x_1(t)x_2(t)$

With Euler-lagrange equation: $g_x(x,\dot{x}) - \frac{d}{dt} g_{\dot{x}}(x,\dot{x}) = 0 \Rightarrow [3x_2 \quad 3x_1] - [2\ddot{x}_1 \quad 2\ddot{x}_2] = 0$

so $\begin{cases} 2\ddot{x}_1 - 3x_2 = 0 \\ 2\ddot{x}_2 - 3x_1 = 0 \end{cases} \Rightarrow 2x_1^{(4)} - \frac{9}{2} x_1 = 0 \Rightarrow x_1^{(4)} = \frac{9}{4} x_1 = \frac{3}{2} \ddot{x}_2$

Since $x_1^{(4)} = \frac{9}{4} x_1 \Rightarrow$ characteristic equation: $r^4 - \frac{9}{4} = 0$ $r^2 = \frac{3}{2}$

and $x_1(t) \in R$ for $t \in [t_0, t_f]$ $x_2(t) \in R$ for $t \in [t_0, t_f)$

so $x_1 = \lambda_1 \exp(\frac{\sqrt{6}}{2}t) + \lambda_2 \exp(-\frac{\sqrt{6}}{2}t) + \lambda_3 \cos\frac{\sqrt{6}}{2}t + \lambda_4 \sin\frac{\sqrt{6}}{2}t$

$x_2 = \frac{2}{3}\ddot{x}_1 = \lambda_1 \exp(\frac{\sqrt{6}}{2}t) + \lambda_2 \exp(-\frac{\sqrt{6}}{2}t) - \lambda_3 \cos\frac{\sqrt{6}}{2}t - \lambda_4 \sin\frac{\sqrt{6}}{2}t$

With $x_1(0) = 0$ $x_2(0) = 0$ $\Rightarrow \lambda_1 = -\lambda_2$ $\lambda_3 = 0$

so $x_1 = \lambda_1 \exp(\frac{\sqrt{6}}{2}t) - \lambda_1 \exp(-\frac{\sqrt{6}}{2}t) + \lambda_4 \sin\frac{\sqrt{6}}{2}t$

$x_2 = \lambda_1 \exp(\frac{\sqrt{6}}{2}t) - \lambda_1 \exp(-\frac{\sqrt{6}}{2}t) - \lambda_4 \sin\frac{\sqrt{6}}{2}t$

a)

Since $t_f = 1$, $x_2(t_f) = 1 \Rightarrow$ $\lambda_1 \exp(\frac{\sqrt{6}}{2}) - \lambda_1 \exp(-\frac{\sqrt{6}}{2}) - \lambda_4 \sin\frac{\sqrt{6}}{2} = 1$

since x_1 has free boundary $\Rightarrow g_{\dot{x}_1}\big|_{t_f=1} = 0 \Rightarrow \dot{x}_1(1) = 0$

$\dot{x}_1(1) \Rightarrow \lambda_1 \frac{\sqrt{6}}{2} \exp(\frac{\sqrt{6}}{2}) + \lambda_1 \frac{\sqrt{6}}{2} \exp(-\frac{\sqrt{6}}{2}) + \lambda_4 \frac{\sqrt{6}}{2} \cos\frac{\sqrt{6}}{2} = 0$

sine $\dot{x}_1(1) \Rightarrow \lambda_1\frac{\sqrt6}{2}\exp(\frac{\sqrt6}{2}) + \lambda_1\frac{\sqrt6}{2}\exp(-\frac{\sqrt6}{2}) + \lambda_4\cos\frac{\sqrt6}{2} = 0$

then, $\begin{cases}\lambda_1\left(\exp(\frac{\sqrt6}{2}) - \exp(-\frac{\sqrt6}{2})\right) - \lambda_4\sin\frac{\sqrt6}{2} = 1 \\ \lambda_1\left(\exp(\frac{\sqrt6}{2}) + \exp(-\frac{\sqrt6}{2})\right) + \lambda_4\cos\frac{\sqrt6}{2} = 0\end{cases} \Rightarrow \begin{cases}\lambda_1 \approx 0.0748 \\ \lambda_4 \approx -0.8157\end{cases}$

So

$$x_1^*(t) \approx 0.0748\cdot\left(\exp(\frac{\sqrt6}{2}t) - \exp(-\frac{\sqrt6}{2}t)\right) - 0.8157\sin\frac{\sqrt6}{2}t$$

$$x_2^*(t) \approx 0.0748\cdot\left(\exp(\frac{\sqrt6}{2}t) - \exp(-\frac{\sqrt6}{2}t)\right) + 0.8157\sin\frac{\sqrt6}{2}t$$

b) Since tf is free and the final position should be on the surface

$\psi(x) = x_1(t_f) + 3x_2(t_f) + 5t - 15 = 0$

then the augmented cost: $J_a = v^T\cdot\psi + \int_{t_0}^{t_f} g(x,\dot{x},t)\,dt$

then the necessary conditions becomes:

$$\begin{cases}\nabla_{\dot{x}} g(t_f) + v\nabla_x\psi(t_f) = 0 \\ \psi(x(t_f), t_f) = 0 \\ \frac{\partial}{\partial t_f}v^T\psi + g(t_f) - \nabla_{\dot{x}}g^T\cdot\dot{x}(t_f) = 0 \\ g_x(x,\dot{x}) - \frac{d}{dt}g_{\dot{x}}(x,\dot{x}) = 0 \text{ (which already covered above)}\end{cases}$$

$\nabla_{\dot{x}}g(t_f) + v\nabla_x\psi(t_f) = 0 \Rightarrow \begin{bmatrix}2\dot{x}_1(t_f) \\ 2\dot{x}_2(t_f)\end{bmatrix} + v\cdot\begin{bmatrix}1\\3\end{bmatrix} = 0$ so $\dot{x}(t_f) = -\frac{v}{2}\cdot\begin{bmatrix}1\\3\end{bmatrix}$

since: $x_1 = \lambda_1\exp(\frac{\sqrt6}{2}t) - \lambda_1\exp(-\frac{\sqrt6}{2}t) + \lambda_4\sin\frac{\sqrt6}{2}t$

$x_2 = \lambda_1\exp(\frac{\sqrt6}{2}t) - \lambda_1\exp(-\frac{\sqrt6}{2}t) - \lambda_4\sin\frac{\sqrt6}{2}t \Rightarrow$

$\dot{x}_1 = \frac{\sqrt6}{2}\lambda_1\exp(\frac{\sqrt6}{2}t) + \frac{\sqrt6}{2}\lambda_1\exp(-\frac{\sqrt6}{2}t) + \frac{\sqrt6}{2}\lambda_4\cos\frac{\sqrt6}{2}t$

$\dot{x}_2 = \frac{\sqrt6}{2}\lambda_1\exp(\frac{\sqrt6}{2}t) + \frac{\sqrt6}{2}\lambda_1\exp(-\frac{\sqrt6}{2}t) - \frac{\sqrt6}{2}\lambda_4\cos\frac{\sqrt6}{2}t$

So

$$\begin{bmatrix}\lambda_1\\\lambda_4\end{bmatrix} = \frac{v}{2}\cdot\frac{2}{\sqrt6}\cdot\begin{bmatrix}\frac{-4}{\exp(\frac{\sqrt6}{2}t_f) + \exp(-\frac{\sqrt6}{2}t_f)} \\ \frac{1}{\cos(\frac{\sqrt6}{2}t_f)}\end{bmatrix} \Rightarrow x(t_f) = \frac{v}{2}\cdot\frac{2}{\sqrt6}\begin{bmatrix}\frac{-2\{\exp(\frac{\sqrt6}{2}t_f) - \exp(-\frac{\sqrt6}{2}t_f)\}}{\exp(\frac{\sqrt6}{2}t_f) + \exp(-\frac{\sqrt6}{2}t_f)} + \tan(\frac{\sqrt6}{2}t_f) \\ \frac{-2\{\exp(\frac{\sqrt6}{2}t_f) - \exp(-\frac{\sqrt6}{2}t_f)\}}{\exp(\frac{\sqrt6}{2}t_f) + \exp(-\frac{\sqrt6}{2}t_f)} - \tan(\frac{\sqrt6}{2}t_f)\end{bmatrix}$$

since $\frac{\partial}{\partial t_f}v\psi + g(t_f) - \nabla_{\dot{x}}g^T\cdot\dot{x}(t_f) = 0$

then $5v + \dot{x}_1^2(t_f) + \dot{x}_2^2(t_f) + 3x_1(t_f)x_2(t_f) - 2\dot{x}_1^2(t_f) - 2\dot{x}_2^2(t_f) = 0$

so $-\frac{5}{2}v^2 + 5v + 3x_1(t_f)x_2(t_f) = 0$

$$= -\frac{5}{2}v^2 + 5v + \frac{v^2}{2}\left(4\left\{\frac{\exp(\frac{\sqrt6}{2}t_f) - \exp(-\frac{\sqrt6}{2}t_f)}{\exp(\frac{\sqrt6}{2}t_f) + \exp(-\frac{\sqrt6}{2}t_f)}\right\}^2 - \left(\tan(\frac{\sqrt6}{2}t_f)\right)^2\right) = 0$$

Since $V > 0$, $\frac{V}{2}\left(4\left\{\frac{\exp(\frac{\sqrt{6}}{2}tf)-\exp(-\frac{\sqrt{6}}{2}tf)}{\exp(\frac{\sqrt{6}}{2}tf)+\exp(-\frac{\sqrt{6}}{2}tf)}\right\}^2-\left(\tan(\frac{\sqrt{6}}{2}tf)\right)^2-5\right)+5=0$

since $x_1(tf)+3x_2(tf)+5tf=15$

$$\frac{V}{2}\cdot\frac{2}{\sqrt{6}}\left(\frac{-8\left\{\exp(\frac{\sqrt{6}}{2}tf)-\exp(-\frac{\sqrt{6}}{2}tf)\right\}}{\exp(\frac{\sqrt{6}}{2}tf)+\exp(-\frac{\sqrt{6}}{2}tf)}-2\tan(\frac{\sqrt{6}}{2}tf)\right)+5tf=15$$

numerical solutions:

hw2.m

```matlab
clc;
clear;

syms tf v
a = sqrt(6)/2;

F = @(x) [ x(2) * ( 4*( (exp(a*x(1))-exp(-a*x(1)))/(exp(a*x(1))+exp(-a*x(1))) )^2 -(tan(a*x(1)))^2 -5) + 10 ;
    x(2)/(2*a)*(-8* (exp(a*x(1))-exp(-a*x(1)))/(exp(a*x(1))+exp(-a*x(1)))-2*tan(a*x(1)) )+5*x(1) - 15];
all_ans = [];
num_ans = 0;

for i=0:0.1:10
    for j=-10:0.1:10
        x0 = [i;j];
        x = fsolve(F,x0);
        x= x';
        res = F(x);
        if( res'*res < 1e-3)
            found = 0;
            for k=1:num_ans
                if( ( x - all_ans(k,:))' * (x-all_ans(k,:)) < 1e-3 )
                    found = 1;
                end
            end
            if(found == 0)
                all_ans = [all_ans;x];
                num_ans = num_ans+1;
            end
        end
    end
end
size(all_ans)
all_ans
```

```
value of the function toler

<stopping criteria details>

No solution found.

fsolve stopped because the
but the vector of function
value of the function toler

<stopping criteria details>

No solution found.

fsolve stopped because the
but the vector of function
value of the function toler

<stopping criteria details>

ans =

     5     2


all_ans =

    7.2788    7.6158
    4.3404    3.2202
    3.6665    0.4840
    5.9142    3.2871
    8.1885    6.7745

fx >>
```

tf   V.

3) Since $J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$ $\quad \left.\frac{dJ(x^*+\varepsilon\eta)}{d\varepsilon}\right|_{\varepsilon=0} = 0$

$$\frac{dg(x^*+\varepsilon\eta, \dot{x}^*+\varepsilon\dot{\eta})}{d\varepsilon} = \frac{\partial g}{\partial(x^*+\varepsilon\eta)} \cdot \eta + \frac{\partial g}{\partial(\dot{x}^*+\varepsilon\dot{\eta})} \cdot \dot{\eta}$$

Since $\left.\frac{dJ(x^*+\varepsilon\eta)}{d\varepsilon}\right|_{\varepsilon=0} = 0$

then, $\int_{t_0}^{t_f} \left.\frac{dg(x^*+\varepsilon\eta, \dot{x}^*+\varepsilon\dot{\eta})}{d\varepsilon}\right|_{\varepsilon=0} dt = 0$

$$= \int_{t_0}^{t_f} \frac{\partial g}{\partial x^*} \cdot \eta + \frac{\partial g}{\partial \dot{x}} \cdot \dot{\eta} \; dt = 0$$

$$\int_{t_0}^{t_f} \frac{\partial g}{\partial \dot{x}^*} \cdot \dot{\eta}\, dt = \left.\frac{\partial g}{\partial \dot{x}^*}\eta(t)\right|_{t_0}^{t_f} - \int_{t_0}^{t_f} d\left(\frac{\partial g}{\partial \dot{x}^*}\right) \cdot \eta(t)\, dt$$

Since $\eta(t_0) = \eta(t_f) = 0$ ↑ this term is 0

$$\text{So} \int_{t_0}^{t_f} \frac{\partial g}{\partial x^*} \cdot \eta + \frac{\partial g}{\partial \dot{x}} \cdot \dot{\eta}\; dt = 0 = \int_{t_0}^{t_f} \left(\frac{\partial g}{\partial x^*} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}^*}\right)\right) \cdot \eta(t)\, dt$$

Since $\eta(t)$ is arbitrary, $\frac{\partial g}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)$ needs to be 0 for $t \in (t_0, t_f)$ when x is optimal

which is EL necessary condition