

$$1) \dot{x} = ax - bu \quad x(t_0) \text{ given}, \quad J = \frac{1}{2} C [x(t_f)]^2 + \frac{1}{2} \int_{t_0}^{t_f} [u(t)]^2 dt$$

$$\text{augmented cost: } J_a = \frac{1}{2} C [x(t_f)]^2 + \int_{t_0}^{t_f} \left[\frac{1}{2} [u(t)]^2 + \lambda(t)^T (ax - bu) \right] dt$$

$$\text{let } H = \frac{1}{2} [u(t)]^2 + \lambda(t)^T [ax - bu] \quad \Phi = \frac{1}{2} C [x(t_f)]^2 = \phi$$

according to necessary conditions, we have:

$$\dot{\lambda}^T = -\partial_x H = -a\lambda^T, \text{ so } \lambda = c_1 e^{-at} \quad c_1 \text{ is a constant}$$

$$\partial_u H = 0 = u^T - b\lambda^T \text{ so } u = b\lambda \Rightarrow u = b \cdot c_1 \cdot e^{-at}$$

$$\lambda(t_f)^T = \partial_x \Phi|_{t=t_f}$$

$$\text{since } \lambda = c_1 \cdot e^{-at}, \quad u = b \cdot c_1 \cdot e^{-at}$$

$$\dot{x} = ax - b^2 c_1 e^{-at} \Rightarrow \dot{x} - ax = -b^2 c_1 e^{-at} \Rightarrow x(t) = e^{a(t-t_0)} x(t_0) - \frac{c_1 b^2 a}{2a} \left[-e^{-2at} + e^{-2at_0} \right]$$

$$\text{since } \lambda(t_f)^T = \partial_x \Phi|_{t=t_f}$$

$$\text{so } c_1 \cdot e^{-at_f} = c \cdot x(t_f) = c \cdot e^{a(t_f-t_0)} x(t_0) - \frac{c_1 b^2 a}{2a} \left[-e^{-2at_f} + e^{-2at_0} \right]$$

$$\Rightarrow c_1 \left\{ e^{-at_f} + \frac{c b^2 a}{2a} \left[-e^{-2at_f} + e^{-2at_0} \right] \right\} = c \cdot e^{a(t_f-t_0)} \cdot x(t_0)$$

$$\text{so } c_1 = c \cdot e^{a(t_f-t_0)} \cdot x(t_0) / \left\{ e^{-at_f} + \frac{c b^2 a}{2a} \left[-e^{-2at_f} + e^{-2at_0} \right] \right\}$$

$$2) \quad u = b \cdot c_1 \cdot e^{-at} = b \cdot c \cdot e^{a(t_f-t_0-t)} x(t_0) / \left\{ e^{-at_f} + \frac{c b^2 a}{2a} \left[-e^{-2at_f} + e^{-2at_0} \right] \right\}$$

$$H = u(t)^2 + \lambda(ax - bu)$$

$$\text{according to necessary condition: } \dot{\lambda} = -\lambda, \quad u - b\lambda = 0 \Rightarrow u = b\lambda$$

we need to find a "P" that

$$\dot{p} = -ap - ap + p^2 b^2 = -2ap + b^2 p^2, \text{ according to logistic model with } p(t_f) = p_f = C$$

$$\text{so } p(t) = \frac{C \cdot \left(\frac{2a}{b^2} \right)}{C + \left(\frac{2a}{b^2} - C \right) e^{2a(t-t_f)}} \\ = \frac{2ac}{b^2 C + (2a - b^2 C) e^{2a(t-t_f)}}$$

"N₀" is initial constraint corresponding to p_f, "a" = -2a

$$N(t) = \frac{\frac{N_0 M^2 e^{at}}{M - N_0}}{M + \frac{N_0 M e^{at}}{M - N_0}} = \frac{N_0 M^2 e^{at}}{M^2 - N_0 M + N_0 M e^{at}} = \frac{N_0 M e^{at}}{M - N_0 + N_0 e^{at}} = \frac{N_0 M}{N_0 + (M - N_0) e^{-at}}$$

$$\text{so } u(t) = b \cdot p(t) \cdot x(t) = \frac{2abC \cdot x(t)}{b^2 C + (2a - b^2 C) e^{2a(t-t_f)}}$$

validation of answer of (1) and (2) are equal

MATLAB R2021a - academic use

```

主 页  绘图  APP  编辑器  发布  视图
+  新建  打开  保存  比较  转至  注释  插入  运行  运行并  运行并
  打印  查找  查找  断点  运行  运行并  运行并
      文件  导航  编辑  断点  运行

C:\Users\11528\OneDrive\桌面
当前文件夹  编辑器 - C:\Users\11528\OneDrive\桌面\hw4_p1_2.m  工作区

名称
AOC21
AOC-20F
CloudMus
Desktop
Jiahe Xu
lab safety
ML: reso...
NLO 2020
notes
pic
风骚律师
绝命毒师
Control Pa...
desktop.ini
essay.docx
hw4_p1_2_...
HW2.pdf
Jupyter N...

1 syms a b c u x_t0 x_t x_tf c1 tf t0 t p_t real;
2 t0 = 0;
3 c1 = c * exp(a*(tf-t0)) * x_t0 / ...
4     ( exp(-a*tf) + c*b*b*exp(a*tf)*(-exp(-2*a*tf)+exp(-2*a*t0) )/(2*a) );
5 x_t = exp(a*(t-t0))*x_t0 - c1*b*b*exp(a*t)*(-exp(-2*a*t)+exp(-2*a*t0) )/(2*a);
6 u1 = b*c1*exp(-a*t);
7
8 p_t = 2*a*c / (b*b*c + ( 2*a - b*b*c )*exp( 2*a*(t-tf) ) );
9 u2 = b * p_t * x_t;
10 simplify( u1 - u2 )
11
>> hw4_p1_2
ans =
0
fx >>

```

```

clear;
clc;

% dynamic x_dot = Ax - Bu
A = [ 0 , 1 ; 2 , -1];
B = [ 0 ; 1];

% 1/2 * x'*Q*x
Q = [ 2 , 0 ; 0 , 1 ];
R = 0.005;
Pf = zeros(4,1);
tf = 20;
x0 = [-5 , 5]';
dt = 0.001;

[t,P] = ode45( @(t,P) get_Pdot(t,P,A,B,Q,R) , tf : -dt : 0 , Pf );

len = length(t);
x = zeros(2,len);
u = zeros(len,1);
x(:,len) = [-5,5]';
for i = len:-1:2
    [xdot,u(i)] = get_xdot( t(i) , x(:,i) , A , B , P(i,:) , R );
    x(:,i-1) = x(:,i) + dt*xdot;
end

%plotting P
for i=1:4
    figure;
    plot(t,P(:,i));
    ylabel(strcat('P ',int2str(i),'(t)'));
    xlabel('time s');
end

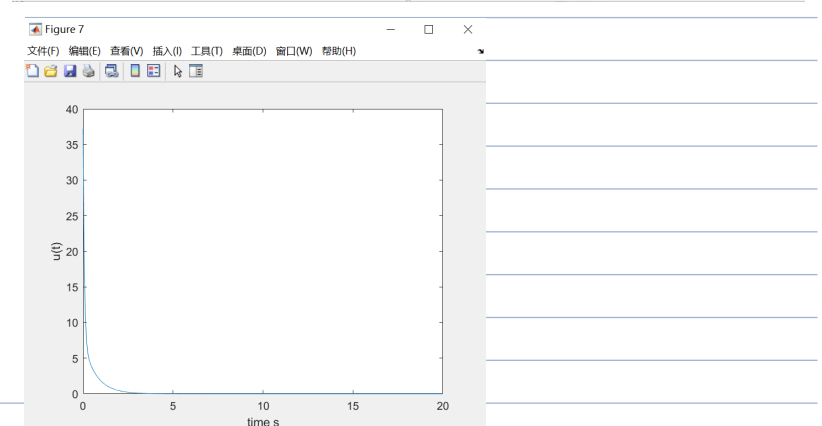
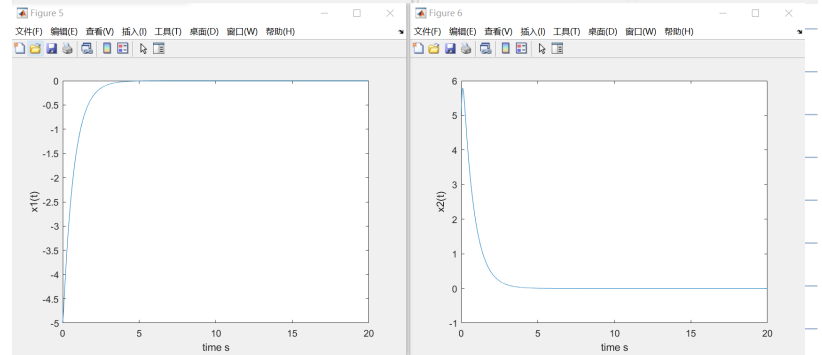
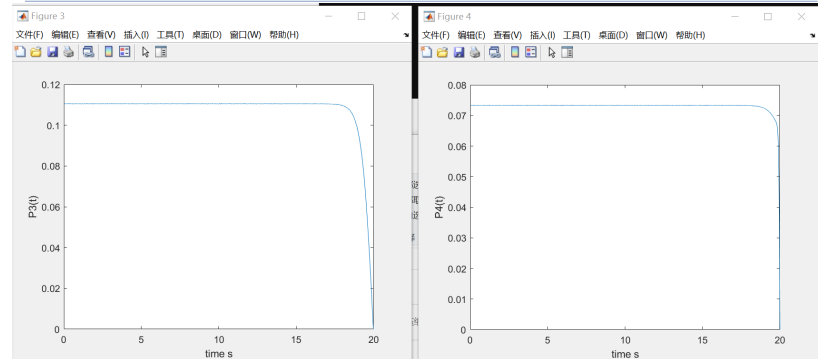
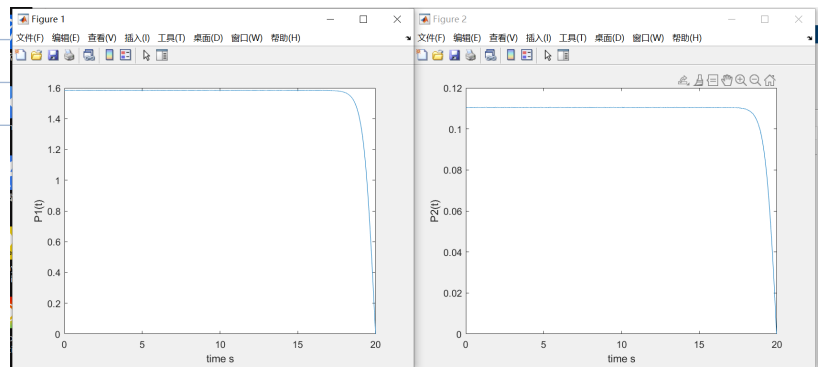
% plotting x
for i = 1:2
    figure;
    plot(t,x(i,:));
    ylabel(strcat('x ',int2str(i),'(t)'));
    xlabel('time s');
end

% plotting u
figure;
plot(t,u);
ylabel('u(t)');
xlabel('time s');

function Pdot = get_Pdot(t,P,A,B,Q,R)
    P = reshape(P,2,2);
    Pdot = -(A')*P - P*A + P*B*(B')*P*(1/R) - Q;
    Pdot = Pdot(:);
end

function [xdot, u] = get_xdot(t,x,A,B,P,R)
    Pt = reshape(P',2,2);
    u = -(1/R)*(B')*Pt*x;
    xdot = A*x + B*u;
end

```



4) error dynamics:

a)

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_d(t) = \begin{bmatrix} V \cdot \cos \theta \\ V \cdot \sin \theta \\ V \cdot \tan \delta \\ u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u \cos \theta - 1 \\ u \sin \theta - 2 \\ u \tan \delta \\ u_1 \\ u_2 \end{bmatrix}$$

b) we want $\dot{e} \approx Ae + Bs$

$$A(e) \triangleq \partial_x f(x_d(t), u_d(t))$$

$$B(e) \triangleq \partial_u f(x_d(t), u_d(t))$$

So $A = \begin{bmatrix} 0 & 0 & -V_d \sin \theta_d & \cos \theta_d & 0 \\ 0 & 0 & V_d \cos \theta_d & \sin \theta_d & 0 \\ 0 & 0 & 0 & \tan \theta_d & V_d / \cos^2 \theta_d \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ in this case,
 $V_d = \sqrt{5}$
 $\theta_d = \arctan 2$
 $\delta_d = 0$

So $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

c)

(d)

```
hw4_p4.m x hw4_p3.m +
1 clc;
2 clear;
3
4 % x y theta v phi
5 x0 = [0,0,0,0,0]';
6 % simulate dynamics for 5 seconds
7 [ts, xs] = ode45(@point_ode, [0 5], x0, []);
8
9 % legend for trajectory
10 % uncomment the following line if you want to plot trajectory
11 % legend('real path', 'desired path')
12
13 % legend for u
14 % uncomment the following line if you want to plot u
15 legend('u1', 'u2')
16 xlabel('time s');
17
18 % x y theta v phi
19 function dx = point_ode(t, x)
20
21 theta = x(3);
22 v = x(4);
23 phi = x(5);
24
25 xd = [t, 2*t, atan2(2,1), sqrt(5), 0]';
26
27 % uncomment the following line if you want to plot trajectory
28 % plot(x(1), x(2), 'b');
29 % hold on;
30 % plot(xd(1), xd(2), 'r');
31 % hold on;
32
33 ud = [0, 0]';
34
35 thetad = xd(3);
36 v_d = xd(4);
37 phid = xd(5);
38 A = [0, 0, -v_d * sin(thetad), cos(thetad), 0;
39 0, 0, v_d * cos(thetad), sin(thetad), 0;
40 0, 0, 0, tan(phid), v_d / (cos(phid)^2);
41 0, 0, 0, 0, 0;
42 0, 0, 0, 0, 0];
43
44 B = [0 0; 0 0; 0 0; 1 0; 0 1];
45
46 Q = diag([5, 5, 0.01, 0.1, 0.1]);
47 R = diag([0.5, 0.1]);
48 [K, P] = lqr(A, B, Q, R);
49 u = -K*(x - xd) + u_d;
50
51 % uncomment the following line if you want to plot u
52 plot(t, u(1), 'b');
53 hold on;
54 plot(t, u(2), 'r');
55 hold on;
56
57 dx = [v * cos(theta); v * sin(theta); v * tan(phi); u(1); u(2)];
58 end
```

