1) $\dot{x} = ax - bu$ $\quad x(t_0)$ given, $J = \frac{1}{2}c\,[x(t_f)]^2 + \frac{1}{2}\int_{t_0}^{t_f}[u(t)]^2\,dt$

augmented cost: $J_a = \frac{1}{2}c[x(t_f)]^2 + \int_{t_0}^{t_f} \frac{1}{2}[u(t)]^2 + \lambda(t)^T(ax-bu)\,dt$

let $H = \frac{1}{2}[u(t)]^2 + \lambda(t)^T[ax-bu]$ $\qquad \Phi = \frac{1}{2}c[x(t_f)]^2 = \phi$

according to necessary conditions, we have:

$\dot{\lambda}^T = -\partial_x H = -a\lambda^T$, so $\lambda = c_1 e^{-at}$ $\qquad c_1$ is a constant

$\partial_u H = 0 = u^T - b\lambda^T$ so $u = b\lambda \Rightarrow u = b\cdot c_1 \cdot e^{-at}$

$\lambda(t_f)^T = \partial_x \Phi |_{t=t_f}$

since $\lambda = c_1 \cdot e^{-at}$, $u = b\cdot c_1 \cdot e^{-at}$

$\dot{x} = ax - b^2 c_1 e^{-at} \Rightarrow \dot{x} - ax = -b^2 c_1 \cdot e^{-at} \Rightarrow x(t) = e^{a(t-t_0)}x(t_0) - \frac{c_1 b^2 e^{at}}{2a}[-e^{-2at} + e^{-2at_0}]$

since $\lambda(t_f)^T = \partial_x \Phi |_{t=t_f}$

so $c_1 \cdot e^{-at_f} = c\cdot x(t_f) = c\cdot e^{a(t_f-t_0)}x(t_0) - \frac{c\cdot c_1 b^2 e^{at_f}}{2a}[-e^{-2at_f} + e^{-2at_0}]$

$\Rightarrow c_1\left\{e^{-at_f} + \frac{cb^2 e^{at_f}}{2a}[-e^{-2at_f} + e^{-2at_0}]\right\} = c\cdot e^{a(t_f-t_0)}\cdot x(t_0)$

so $c_1 = c\cdot e^{a(t_f-t_0)}\cdot x(t_0) \Big/ \left\{e^{-at_f} + \frac{b^2 c\, e^{at_f}}{2a}[-e^{-2at_f} + e^{-2at_0}]\right\}$

$u = b\cdot c_1 \cdot e^{-at} = b\cdot c\cdot e^{a(t_f-t_0-t)}\cdot x(t_0) \Big/ \left\{e^{-at_f} + \frac{cb^2 e^{at_f}}{2a}[-e^{-2at_f} + e^{-2at_0}]\right\}$

2)

$H = u(t)^2 + \lambda(ax-bu)$

according to necessary condition: $\dot{\lambda} = -\lambda$, $\quad u - b\lambda = 0 \Rightarrow u = b\lambda$

we need to find a "P" that

$\dot{p} = -ap - ap + p^2b^2 = -2ap + b^2p^2$, according to logistic model with $P(t_f) = P_f = C$

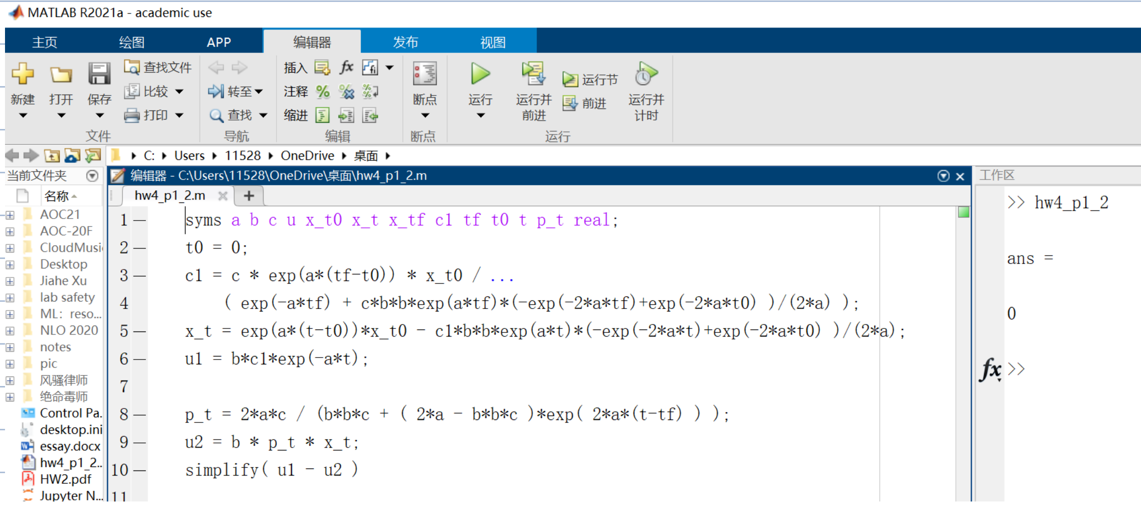
so $P(t) = \dfrac{C\cdot\left(\frac{2a}{b^2}\right)}{C + \left(\frac{2a}{b^2} - C\right)e^{2a(t-t_f)}}$ $\qquad$ "N_0" is initial constraint corresponding to P_f, "a" $= -2a$

$$N(t) = \frac{\frac{N_0M^2e^{at}}{M-N_0}}{M + \frac{N_0Me^{at}}{M-N_0}} = \frac{N_0M^2e^{at}}{M^2 - N_0M + N_0Me^{at}} = \frac{N_0Me^{at}}{M - N_0 + N_0e^{at}} = \frac{N_0M}{N_0 + (M-N_0)e^{-at}}.$$

$= \dfrac{2ac}{b^2c + (2a - b^2c)e^{2a(t-t_f)}}$

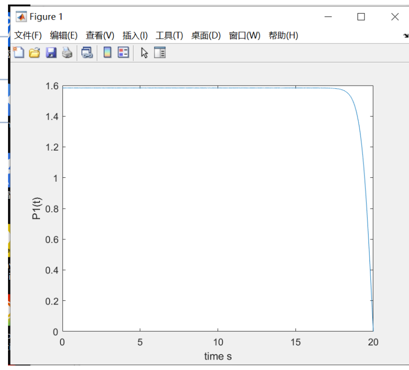
So $u(t) = b\cdot P(t)\cdot x(t) = \dfrac{2abc\cdot x(t)}{b^2c + (2a - b^2c)e^{2a(t-t_f)}}$

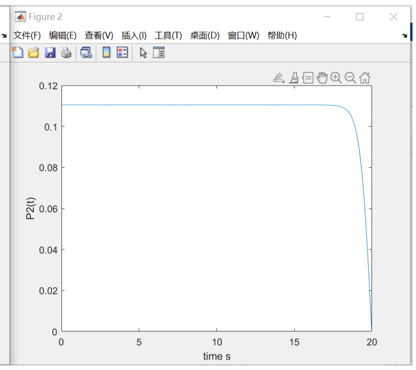
Validation of answer of (1) and (2) are equal

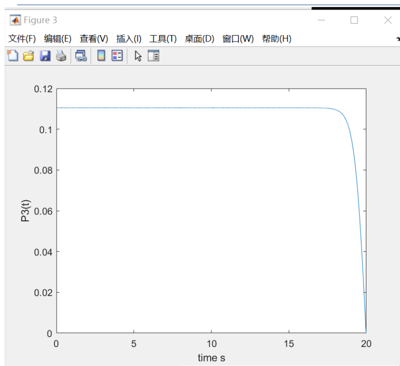


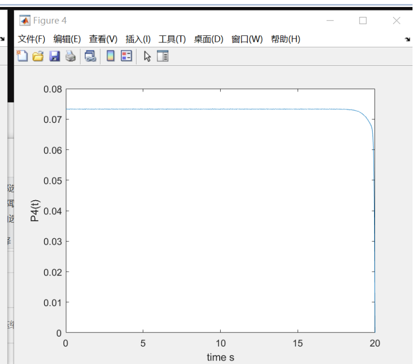
3

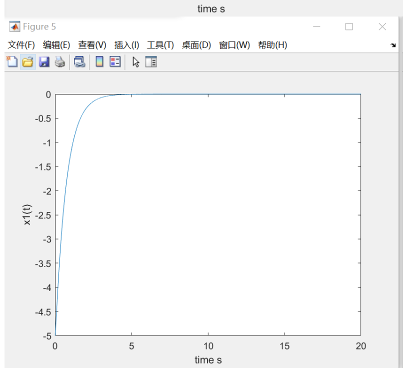
```
clear;
clc;

% dynamic x_dot = Ax - Bu
A = [ 0 , 1 ; 2 , -1];
B = [ 0 ; 1];

% 1/2 * x'*Q*x
Q = [ 2 , 0 ; 0 , 1 ];
R = 0.005;
Pf = zeros(4,1);
tf = 20;
x0 = [-5 , 5]';
dt = 0.001;

[t,P] = ode45( @(t,P)get_Pdot(t,P,A,B,Q,R) , tf : -dt : 0 , Pf );

len = length(t);
x = zeros(2,len);
u = zeros(len,1);
x(:,len) = [-5,5]';
for i = len:-1:2
    [xdot,u(i)] = get_xdot( t(i) , x(:,i) , A , B , P(i,:) , R );
    x(:,i-1) = x(:,i) + dt*xdot;
end

%plotting P
for i=1:4
    figure;
    plot(t,P(:,i));
    ylabel(strcat('P ',int2str(i),'(t)'));
    xlabel('time s');
end
% plotting x
for i = 1:2
    figure;
    plot(t,x(i,:));
    ylabel(strcat('x ',int2str(i),'(t)'));
    xlabel('time s');
end
% plotting u
figure;
plot(t,u);
ylabel('u(t)');
xlabel('time s');


function Pdot = get_Pdot(t,P,A,B,Q,R)
    P = reshape(P,2,2);
    Pdot = -(A')*P - P*A + P*B*(B')*P*(1/R) - Q;
    Pdot = Pdot(:);
end

function [xdot, u] = get_xdot(t,x,A,B,P,R)
    Pt = reshape(P',2,2);
    u = -(1/R)*(B')*Pt*x;
    xdot = A*x + B*u;
end
```

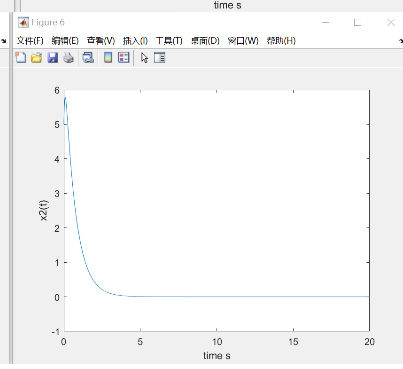


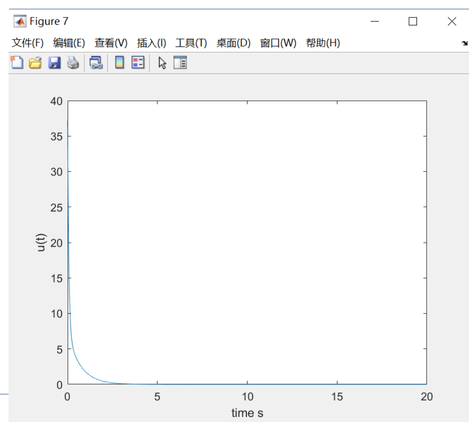












4) error dynamics:

a) $\dot{e}(t) = \dot{x}(t) - \dot{x}_d(t) = \begin{pmatrix} v\cdot\cos\theta \\ v\cdot\sin\theta \\ v\cdot\tan\delta \\ u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v\cos\theta - 1 \\ v\cdot\cos\theta - 2 \\ v\tan\delta \\ u_1 \\ u_2 \end{pmatrix}$

b) We want $\dot{e} \approx Ae + Bs$

$A(t) \triangleq \partial_x f(x_d(t), u_d(t)) \qquad B(t) \triangleq \partial_u f(x_d(t), u_d(t))$

So $A = \begin{pmatrix} 0 & 0 & -V_d\sin\theta_d & \cos\theta_d & 0 \\ 0 & 0 & V_d\cos\theta_d & \sin\theta_d & 0 \\ 0 & 0 & 0 & \tan\delta_d & V_d/\cos^2\delta_d \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

in this case,
$V_d = \sqrt{5}$
$\theta_d = \arctan 2$
$\delta_d = 0$

So $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

c)

hw4_p4.m   hw4_p3.m

```
clc;
clear;

% x y theta v phi
x0 = [0,0,0,0,0]';
% simulate dynamics for 5 seconds
[ts, xs] = ode45(@point_ode, [0 5], x0,[]);

% legend for trajectory
%uncomment the following line if you want to plot trajectory
%legend('real path','desired path')

% legend for u
%uncomment the following line if you want to plot u
legend('u1','u2')
xlabel('time s');

% x y theta v phi
function dx = point_ode(t, x)

theta = x(3);
v = x(4);
phi = x(5);

xd = [ t , 2*t , atan2(2,1),sqrt(5),0]';

%uncomment the following line if you want to plot trajectory
% plot(x(1),x(2),'*b');
% hold on;
% plot(xd(1),xd(2),'*r');
% hold on;

ud = [0,0]';

thetad = xd(3);
vd = xd(4);
phid = xd(5);
A = [0, 0, -vd* sin(thetad), cos(thetad), 0;
 0, 0, vd* cos(thetad), sin(thetad), 0;
 0, 0, 0, tan(phid), vd/(cos(phid)^2);
 0, 0, 0, 0, 0;
 0, 0, 0, 0, 0];

B = [ 0 0; 0 0;0 0; 1 0; 0 1];

Q = diag([5, 5, 0.01, 0.1, 0.1]);
R = diag([0.5, 0.1]);
[K, P] = lqr(A, B, Q, R);
u = -K*(x-xd) + ud;

%uncomment the following line if you want to plot u
plot(t,u(1),'*b');
hold on;
plot(t,u(2),'*r');
hold on;

dx = [ v*cos(theta); v*sin(theta); v*tan(phi); u(1); u(2);];
end
```

(d)