EN530.603 Applied Optimal Control
Final Exam
December 17, 2021

Note: you may use the specified course textbooks and lecture notes, and your own notes. Show all your work for full credit. Please turn in your solutions by 2pm using the File upload link.

1. A boat moves with constant unit velocity in a stream moving at constant velocity s. The problem is to find the steering angle $u(t)$, $0 \le t \le T$, which minimizes the time T required for the boat to move between point $(0,0)$ to a given point (a,b). The equations of motion are

$$\dot{x}_1(t) = s + \cos u(t), \qquad \dot{x}_2(t) = \sin u(t),$$

where $x_1(t)$ and $x_2(t)$ are the positions of the boat parallel and perpendicular to the stream velocity, respectively. Show that the optimal solution is to steer at a constant angle. *(Problem credit: D. Bertsekas)*

2. Consider the system $\dot{x} = Ax + Bu$, where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The goal is to optimally control the system so that its final state $x(t_f)$ at a given time t_f lies within an ellipsoidal region defined by

$$\frac{1}{2}x(t_f)^T S x(t_f) - 1 \le 0$$

while minimizing the cost

$$J = \int_{t_0}^{t_f} \frac{1}{2}\|u(t)\|^2 dt.$$

Derive the optimal control $u(t)$ for the problem. Can the optimal control be expressed as a linear function of the state? Assume that the initial state $x(t_0)$ is given and is outside of the ellipsoid.

3. Consider the minimum-time control problem for the system with dynamics:

$$\begin{aligned}\dot{x}_1 &= -x_2 + u,\\ \dot{x}_2 &= u,\end{aligned}$$

subject to $|u| \le 1$. Derive the switching control law that drives the system to the origin $x = (0,0)$. Express the control by obtaining the equations of the switching curves, e.g. in the form of a switching function defined in terms of the variables x_1 and x_2.

4. A model suitable for tracking some airborne or underwater objects is described by the state $x = (p, v, a)$ where $p, v, a \in \mathbb{R}^3$ denote the position, velocity, and acceleration, respectively. The dynamics is given by

$$\begin{aligned}\dot{p}(t) &= v(t)\\ \dot{v}(t) &= a(t)\\ \dot{a}(t) &= da(t) + kv(t) + w(t)\end{aligned}$$

where d and k are constants and $w(t)$ is a Gaussian with variance $E[w(t)w(\tau)] = Q_c'\delta(t-\tau)$. The system can be expressed in the standard form $\dot{x} = Fx + Lw$ (there are no controls). Your task is to obtain (by using the best possible approximation, if an approximation is required):

- the discrete-time analog $x_k = \Phi_{k-1}x_{k-1} + \Lambda_{k-1}w_{k-1}$ for some given time-step Δt
- the covariance Q_k of the noise term $\Lambda_k w_k$, for any k.
- an estimator/filter for given bearing and velocity measurements given by

$$z = \begin{bmatrix} \frac{p}{\|p\|} + \eta_p \\ v + \eta_v \end{bmatrix}$$

  where $\eta_p \sim \mathcal{N}(0, R_p)$ and $\eta_v \sim \mathcal{N}(0, R_v)$. Note that the "bearing" measurement is encoded as a unit vector from the origin to the object (for instance from a radar sensor placed at the origin).

5. A young investor has earned in the stock market a large amount of money and plans to spend it to maximize his enjoyment through the rest of his life without working. He estimates that he will live exactly T more years and that his capital $x(t)$ should be reduced to zero at time T, i.e. $x(T) = 0$. Also he models the evolution of his capital by the differential equation

$$\dot{x}(t) = \alpha x(t) - u(t),$$

where $x(0) = x_0$ is his initial capital, $\alpha > 0$ is a given interest rate, and $u(t) \geq 0$ is his rate of expenditure. The total enjoyment he will obtain is given by

$$\int_0^T e^{-\beta t}\sqrt{u(t)}dt,$$

where β is some positive scalar which serves to discount future enjoyment. Find the optimal $\{u(t) \mid t \in [0, T]\}$. *(Problem credit: D. Bertsekas)*

6. **Optional for extra-credit:** *(Stochastic Control with perfect measurements).* Consider the scalar system

$$\dot{x}(t) = x(t) + u(t) + a + w(t),$$

for some constant $a > 0$ and zero-mean noise with unit variance $E[w(t)w(\tau)] = \delta(t-\tau)$, i.e. $q_c' = 1$. The cost function is

$$J = \frac{1}{2}x(1)^2 + \int_0^1 \frac{1}{2}u(t)^2 dt.$$

   a) Compute the optimal feedback control minimizing the expected value of J assuming perfect state information. In the process, show that a value function of the form

$$V(x,t) = \frac{1}{2}s(t)x^2 + b(t)x + v(t)$$

   satisfies the stochastic Hamilton-Jacobi-Bellman principle for this problem.

   b) Compute the minimum value of the expected cost function.