EN530.603 Applied Optimal Control
Homework #6

October 25, 2021

Due: November 1, 2021 (before class)

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1. Consider the minimization of

$$J = \frac{1}{2}x(1)^2 + \int_0^1 \frac{1}{2}[x(t)u(t)]^2 dt,$$

subject to the nonlinear dynamics

$$\dot{x} = xu, \qquad x(0) = 1.$$

Derive an optimal feedback control using the HJB equation. In the process, show that the HJB partial differential equation has a solution that is a quadratic function in x.

2. Consider the LQR problem with known disturbance $w(t)$ which requires the minimization of

$$J = \frac{1}{2}x(t_f)^T P_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [x(t)^T Q(t)x(t) + u(t)^T R(t)u(t)]dt,$$

subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t).$$

a) Using the HJB equation, show that a possible value function has the form

$$V(x(t), t) = \frac{1}{2}x(t)^T P(t)x(t) + b(t)^T x(t) + c(t)$$

and show that the associate optimal control is

$$u(t) = K(t)x(t) + k(t),$$

where $K(t)$ and $k(t)$ are the feedback gain matrix and feedforward term, respectively. Derive the differential equations for $\dot{P}, \dot{b}, \dot{c}$ and their boundary conditions which satisfy the HJB equation.

b) Do the discrete-time equivalent of Part a).

3. Implement in Matlab or Python the discrete-time approach from 2b, for the system with state $x_i = (p_i, v_i) \in \mathbb{R}^2$ and dynamics given by:

$$\begin{aligned} p_{i+1} &= p_i + \Delta t v_i \\ v_{i+1} &= v_i + \Delta t(-0.5v_i + 0.2p_i + u_i + 0.1), \end{aligned}$$

with cost $J = \frac{1}{2}(p_N^2 + v_N^2) + \sum_{i=0}^{N-1} \frac{1}{2}(Ru_i^2)$. Simulate the resulting trajectory for your system from the following initial conditions: $x_0 = (10, 0)$, $x_0 = (10, 5)$, $x_0 = (10, -5)$ and include the resulting plots in the (p, v) 2d plane. You can set $R = 0.04$, $N = 100$ and $\Delta t = 0.1$.

Pick several other values for R and study the change in the optimal trajectory.

Note: upload your code as a single zip file (please name it as *LastName_FirstName_HW6.zip*) to the File upload link on the class webpage; in addition attach a printout of the code and plots to your homework solutions.