1. $J = \underbrace{\frac{1}{2} x(1)^2}_{\uparrow \phi} + \int_0^1 \underbrace{\frac{1}{2}(xu)^2}_{\uparrow L} dt.$ with $f = xu$

$$u^* = \operatorname{argmin}\left\{ \frac{1}{2} x^2 u^2 + \nabla_x V^T (xu) \right\}$$

from necessary condition: $x^2 u + \nabla_x V^T x = 0$ so $u = -\frac{1}{x} \nabla_x V$

according to HJB eqn: $-\partial_t V = -\frac{1}{2}(\nabla_x V)^2$

assume V to be of the form $V(t,x) = \frac{1}{2} P(t) x^2$

then, $\frac{1}{2} \dot{P} x^2 = \frac{1}{2} P^2 x^2 \Rightarrow \dot{P} = P^2$ with $P(t_f) = 1$

So $P(t) = \frac{-1}{t-2} \Rightarrow V = \frac{-1}{2(t-2)} x^2$, $u = \frac{1}{t-2}$

↑ HJB eqn has a sol that is quadratic func in X

2)

a) $J = \underbrace{\frac{1}{2} x(t_f)^T P_f x(t_f)}_{\uparrow \phi} + \int_{t_0}^{t_f} \underbrace{\frac{1}{2}\left[x(t)^T Q(t) x(t) + u(t)^T R(t) u(t)\right]}_{\uparrow L} dt$ $\quad \underbrace{\dot{x}(t) = A(t)x(t) + B(t)X(t) + W(t)}_{\uparrow f}$

$u = \operatorname{argmin}\{L + \nabla_x V^T f\}$ $\quad u^* = -R^{-1} B^T \nabla_x V$

HJB becomes: $-\partial_t V(t,x) = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \nabla_x V^T (Ax + Bu + w)$

$$= \frac{1}{2} x^T Q x - \frac{1}{2} \nabla_x V^T B R^{-1} B^T \nabla_x V + \nabla_x V^T (Ax + w)$$

assume V has the form of: $V = \frac{1}{2} x^T P x + b^T x + c$

then $\left.\begin{aligned} \partial_t V &= \frac{1}{2} x^T \dot{P} x + \dot{b}^T x + \dot{c} \\ \nabla_x V &= Px + b \end{aligned}\right\}$ insert into HJB eqn

so $-\left(\frac{1}{2} \dot{P} x^2 + \dot{b}^T x + \dot{c}\right) = \frac{1}{2} x^T Q x - \frac{1}{2}(Px+b)^T B R^{-1} B^T (Px+b) + (Px+b)^T (Ax+w)$

$0 = \frac{1}{2} x^T \dot{P} x + \dot{b}^T x + \dot{c} + \frac{1}{2} x^T Q x - \frac{1}{2} x^T P^T B R^{-1} B^T P x - b^T B R^{-1} B^T P x - \frac{1}{2} b^T B R^{-1} B^T b + x^T P^T A x + b^T A x + x^T P^T w + b^T w$

So $\frac{1}{2} x^T \left(\dot{P} + Q - P^T B R^{-1} B^T P + 2 P^T A\right) x + \left(\dot{b}^T - b^T B R^{-1} B^T P + b^T A + w^T P\right) x + \left(\dot{c} - \frac{1}{2} b^T B R^{-1} B^T b + b^T w\right)$

with $P(t_f) = P_f$ $\quad b(t_f) = 0$ $\quad c(t_f) = 0$

Since x could be non-zero

so $\dot{P} = -Q + P^T B R^{-1} B^T P - (P^T A + A^T P)$ with $P(t_f) = P_f$

$\dot{b} = (P^T B R^{-1} B^T - A^T) b - P^T W$ with $b(t_f) = 0$

$\dot{c} = \frac{1}{2} b^T B R^{-1} B^T b - b^T W$ with $c(t_f) = 0$

after we solved P, b, c.

so $u^* = -R^{-1} B^T V_x = -R^{-1} B^T (Px + b) = -R^{-1} B^T P x - R^{-1} B^T b$

so $K(t) = -R^{-1} B^T P$ $\quad k(t) = -R^{-1} B^T b$

b) we seperate t_f into many small pieces $1 \cdots N$. any P_i should be symmetric

$$L_i = \frac{1}{2}(x^T Q_i x + u^T R_i u) \qquad x_{i+1} = A_i x + B_i u + W_i$$

according to Bellman eqn: $V_i = \min_u \{L_i + V_{i+1}\}$ with $V_N(x) = \phi(x, t_N) = \frac{1}{2} x^T P_f x$

$$u_i^* = \arg\min_u \left\{ \frac{1}{2}(x^T Q_i x + u^T R_i u) + \frac{1}{2}(A_i x + B_i u + W_i)^T P_{i+1} (A_i x + B_i u + W_i) + b_{i+1}^T (A_i x + B_i u + W_i) + C_{i+1} \right\}$$

according necessary condition: $R_i u + B_i^T P_{i+1}(A_i x + B_i u + W_i) + B_i^T b_{i+1} = 0$

so $u_i^* = -(R_i + B_i^T P_{i+1} B_i)^{-1} (B_i^T P_{i+1}(A_i x + W_i) + B_i^T b_{i+1})$

so $V_i = \frac{1}{2}(x^T Q_i x + u^{*T} R_i u^*) + \frac{1}{2}(A_i x + B_i u^* + W_i)^T P_{i+1}(A_i x + B_i u^* + W_i) + b_{i+1}^T (A_i x + B_i u^* + W_i) + C_{i+1}$

for simplicity: we take $(R_i + B_i^T P_{i+1} B_i) = S$

$$V_i = \frac{1}{2} x^T (Q_i + A_i^T P_{i+1} A_i) x + \frac{1}{2} u^{*T} S u^* + \left[(A_i x + W_i)^T P_{i+1} B_i + b_{i+1}^T B_i \right] u^* + (W_i^T P_{i+1} A_i + b_{i+1}^T A_i) x$$
$$+ \frac{1}{2} W_i^T P_{i+1} W_i + b_{i+1}^T W_i + C_{i+1}$$

$$\frac{1}{2} u^{*T} S u^* = \frac{1}{2} (B_i^T P_{i+1} A_i x + B_i^T P_{i+1} W_i + B_i^T b_{i+1})^T S^{-1} (B_i^T P_{i+1} A_i x + B_i^T P_{i+1} W_i + B_i^T b_{i+1})$$

$$= \frac{1}{2} \left\{ x^T A_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i x + W_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} W_i + b_{i+1}^T B_i S^{-1} B_i^T b_{i+1} \right.$$
$$\left. + 2 W_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i x + 2 b_{i+1}^T B_i S^{-1} B_i^T P_{i+1} W_i + 2 b_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i x \right\}$$

$$\left[(A_i x + w_i)^T P_{i+1} B_i + b_{i+1}^T B_i\right] u^* = \left[(A_i x + w_i)^T P_{i+1} B_i + b_{i+1}^T B_i\right]\cdot(-S^{-1})\cdot(B_i^T P_{i+1} A_i x + B_i^T P_{i+1} w_i + B_i^T b_{i+1})$$

$$= -(x^T A_i^T P_{i+1} B_i + w_i^T P_{i+1} B_i + b_{i+1}^T B_i) S^{-1} (B_i^T P_{i+1} A_i x + B_i^T P_{i+1} w_i + B_i^T b_{i+1})$$

$$= -\{ x^T A_i^T P_{i+1} B_i S^{-1} B_i^T P_{i+1} A_i x + x^T A_i^T P_{i+1} B_i S^{-1} B_i^T P_{i+1} w_i + x^T A_i^T P_{i+1} B_i S^{-1} B_i^T b_{i+1}$$

$$+ w_i^T P_{i+1} B_i S^{-1} B_i^T P_{i+1} A_i x + w_i^T P_{i+1} B_i S^{-1} B_i^T P_{i+1} w_i + w_i^T P_{i+1} B_i S^{-1} B_i^T b_{i+1}$$

$$+ b_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i x + b_{i+1}^T B_i S^{-1} B_i^T P_{i+1} w_i + b_{i+1}^T B_i S^{-1} B_i^T b_{i+1} \}$$

So $V_i = \frac{1}{2} x^T (Q_i + A_i^T P_{i+1} A_i) x + \frac{1}{2} u^{*T} S u^* + \left[(A_i x + w_i)^T P_{i+1} B_i + b_{i+1}^T B_i\right] u^* + (w_i^T P_{i+1} A_i + b_{i+1}^T A_i) x$

$$+ \frac{1}{2} w_i^T P_{i+1} w_i + b_{i+1}^T w_i + c_{i+1}$$

$$= \frac{1}{2} x^T (Q_i + A_i^T P_{i+1} A_i) x + \cancel{\frac{1}{2} x^T A_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i x} + \cancel{\frac{1}{2} w_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} w_i} + \cancel{\frac{1}{2} b_{i+1}^T B_i S^{-1} B_i^T b_{i+1}}$$

$$+ \cancel{w_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i x} + \cancel{b_{i+1}^T B_i S^{-1} B_i^T P_{i+1} w_i} + \cancel{b_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i x} - \frac{1}{2} x^T A_i^T P_{i+1} B_i S^{-1} B_i^T P_{i+1} A_i x$$

$$- x^T A_i^T P_{i+1} B_i S^{-1} B_i^T P_{i+1} w_i - \cancel{x^T A_i^T P_{i+1} B_i S^{-1} B_i^T b_{i+1}} - \cancel{w_i^T P_{i+1} B_i S^{-1} B_i^T P_{i+1} A_i x}$$

$$- w_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} w_i \frac{1}{2} - \cancel{w_i^T P_{i+1} B_i S^{-1} B_i^T b_{i+1}} - b_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i x$$

$$- b_{i+1}^T B_i S^{-1} B_i^T P_{i+1} w_i - b_{i+1}^T B_i S^{-1} B_i^T b_{i+1} \cdot \frac{1}{2}$$

$$+ (w_i^T P_{i+1} A_i + b_{i+1}^T A_i) x + \frac{1}{2} w_i^T P_{i+1} w_i + b_{i+1}^T w_i + c_{i+1}$$

$$= \frac{1}{2} x^T \left(Q_i + A_i^T P_{i+1} A_i - A_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i\right) x$$

$$+ \left(w_i^T P_{i+1} A_i + b_{i+1}^T A_i - b_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i - w_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1}^T A_i\right) x$$

$$+ \frac{1}{2} w_i^T P_{i+1} w_i + b_{i+1}^T w_i + c_{i+1} - \frac{1}{2} w_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} w_i - \frac{1}{2} b_{i+1}^T B_i S^{-1} B_i^T b_{i+1}$$

$$- w_i^T P_{i+1} B_i S^{-1} B_i^T b_{i+1}$$

$$= \frac{1}{2} x^T \left(Q_i + A_i^T P_{i+1} A_i - A_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i\right) x$$

$$+ (w_i^T P_{i+1} + b_{i+1}^T)(A_i - B_i S^{-1} B_i^T P_{i+1} A_i) x$$

$$- \frac{1}{2} (b_{i+1}^T B_i + w_i^T P_{i+1}^T B_i) S^{-1} (B_i^T P_{i+1} w_i + B_i^T b_{i+1}) + \frac{1}{2} w_i^T P_{i+1} w_i + b_{i+1}^T w_i + c_{i+1}$$

Since V_i should in the form of: $\frac{1}{2}x^T P_i x + b_i x + C_i$

So P_i. b_i C_i should be:

$$P_i = Q_i + A_i^T P_{i+1} A_i - A_i^T P_{i+1}^T B_i S^{-1} B_i^T P_{i+1} A_i$$

$$b_i = (w_i^T P_{i+1} + b_{i+1}^T)(A_i - B_i S^{-1} B_i^T P_{i+1} A_i)$$

$$C_i = \frac{1}{2} w_i^T P_{i+1} w_i + b_{i+1}^T w_i + C_{i+1}$$

With. $(R_i + B_i^T P_{i+1} B_i) = S$, $P_N = P_f$ $b_N = 0$ $C_N = 0$

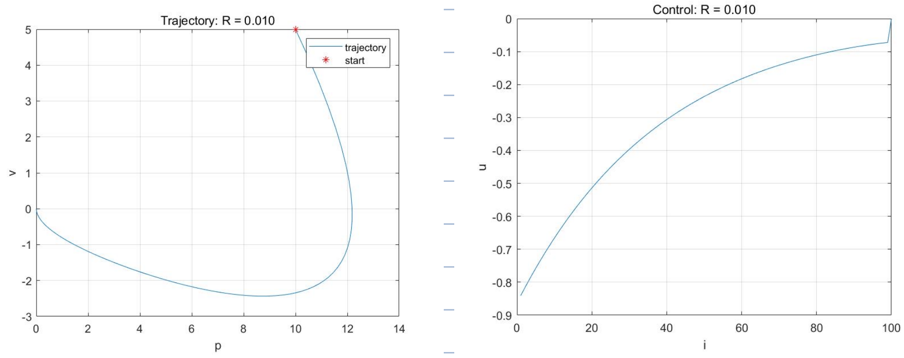
and $u_i^* = -(R_i + B_i^T P_{i+1} B_i)^{-1} (B_i^T P_{i+1}(A_i x + w_i) + B_i^T b_{i+1})$

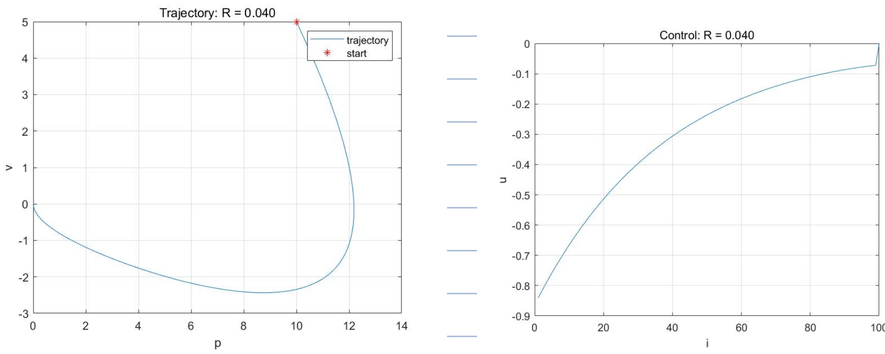
Since we want $u(t) = K(t)\, x(t) + k(t)$

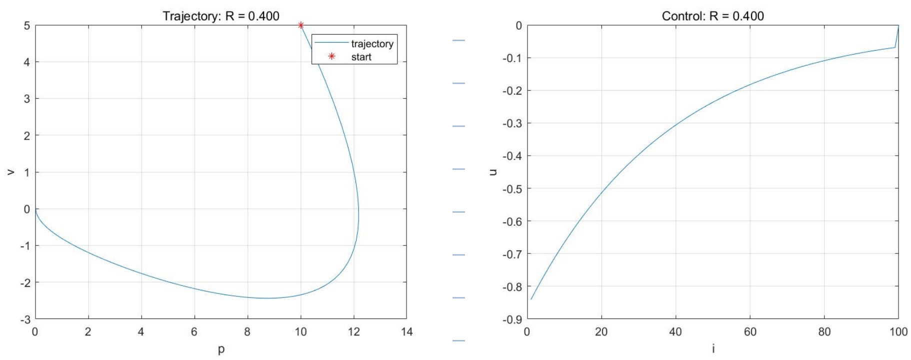
So $K(t) = -(R_i + B_i^T P_{i+1} B_i)^{-1} B_i^T P_{i+1} A_i$

$$k(t) = -(R_i + B_i^T P_{i+1} B_i)^{-1} B_i^T (P_{i+1} w_i + b_{i+1})$$

Q3. $x_0 = (10, 5)$

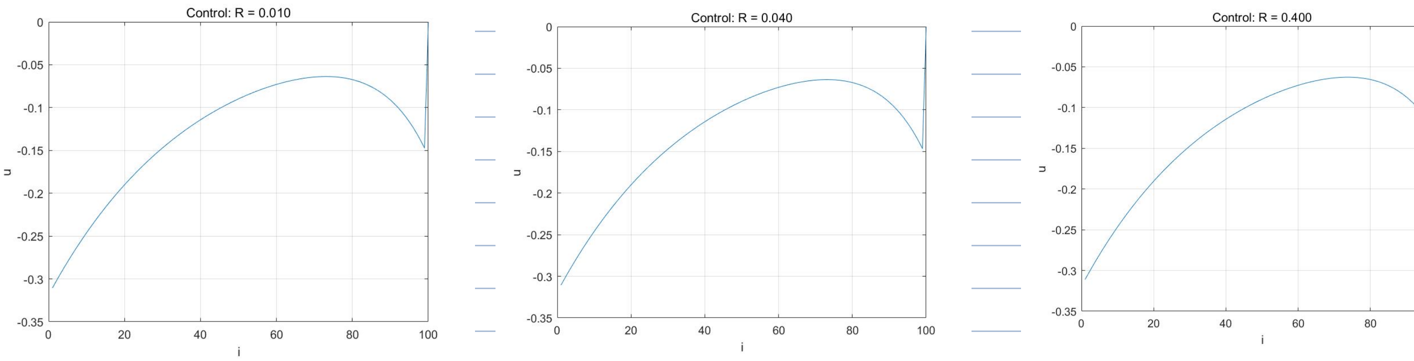


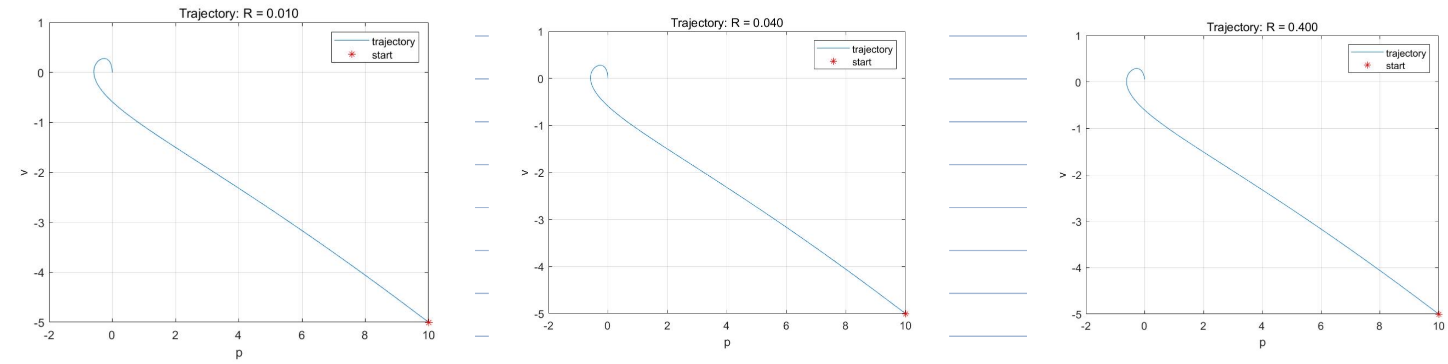





```
clc;
clear;
N = 100;
x0 = [10; 0];
% x0 = [10; 5];
% x0 = [10; -5];
Pf = eye(2);
R = 0.01;
dt = 0.1;
% x_dot = Ax + Bu + w
A = [ 1          , dt ;
      0.2 * dt , 1 - 0.5*dt ];

B = [0 ; 1];
w = [0 ; 0.1];

% Value function params
P = cell(N);
P{N} = Pf;

b = cell(N);
b{N} = [0;0];

c = cell(N);
c{N} = 0;

% trajectory
x = zeros(2,N);
x(:,1) = x0;

% control law
K = cell(N);
k = cell(N);
u = zeros(1,N);

% get P_i, b_i, c_i, K_i, k_i
for i = N-1:-1:1
    % every variable has the same notatoin within Q2
    % A B w R are constants here
    S = ( R + B' * P{i + 1} * B);
    inv_S = inv( R + B' * P{i + 1} * B);

    K_i = - inv_S * B' * P{i + 1} * A;
    k_i = - inv_S * B' * (P{i + 1} * w + b{i + 1});
    % determine P_i, b_i, c_i
    % there is no Q_i in Q3
    P_i = A'*( P{i+1} - P{i+1}' * B * inv_S * B' * P{i+1}  )* A;
    b_i = (( w' * P{i+1} + b{i+1}' )*( A - B*inv_S*(B')*P{i+1}*A ))';
    M = (B'*( P{i+1}*w + b{i+1}));

    c_i = 1/2 * w'*P{i+1}*w - 1/2*(M')*inv_S*M +(b{i+1}')*w + c{i+1};

    K{i} = K_i;
    k{i} = k_i;

    P{i} = P_i;
    b{i} = b_i;
    c{i} = c_i;

end

for i = 1:N-1
    % get x_i to integrate to x_i+1
    x_i = x(:,i);  % i-th column

    % get the control law
    K_i = K{i};
    k_i = k{i};
    u_i = K_i * x_i + k_i;
    % get x_i+1
    x_next = A * x_i + B * u_i + w;

    x(:,i+1) = x_next;
    u(i) = u_i;
end

% Plotting results
f_traj = figure(1);
plot(x(1,:), x(2,:), 'DisplayName', 'trajectory');
title(sprintf('Trajectory: R = %.3f', R));
xlabel('p'); ylabel('v');
legend();
hold on;
plot(x0(1), x0(2), 'r*', 'DisplayName', 'start');
grid on;

f_ctrl = figure(2);
plot(u);
title(sprintf('Control: R = %.3f', R));
xlabel('i'); ylabel('u');
grid on;
```

when $X_0 = (10,-5)$





when $X_0 = (10,0)$