EN530.603 Applied Optimal Control
Homework #5
October 11, 2021

Due: October 20, 2021 (before class)

Professor: Marin Kobilarov

1. (Kirk, 5-34.) Consider the system

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -ax_2(t) + u(t),$$

for $a > 0$ and $|u(t)| \leq 1$. The system must be transferred to the origin $x(t_f) = 0$ while minimizing the performance measure

$$J = \int_{t_0}^{t_f} [\gamma + |u(t)|] dt$$

The final time is free and $\gamma > 0$ is a constant.

a) Determine the adjoint equations and the control that minimizes H
b) What are the possible optimal control sequences?
c) Show that a singular interval cannot exist.
d) Determine the optimal control law.

2. (Bryson, p. 115) Consider the problem of minimizing

$$J = \|x(t_f)\|^2$$

for the system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0, \qquad t_f \text{ given}$$

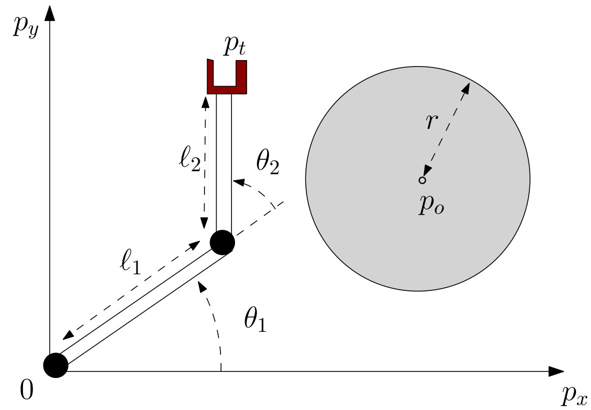
where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ with constraints

$$\|u(t)\| \leq 1.$$

Show that the optimal control for $J_{\min} > 0$ is bang-bang.

What is the analog to the figure of the time-optimal trajectories for the double-integrator problem (that we drew in class)? Either draw it by hand or simulate using Matlab by setting A and B to match the dynamics of a double integrator in 2-D.

3. Consider a two degree of freedom robotic arm operating in a workspace with a spherical obstacle. The arm base is at the origin $(0, 0)$ while the obstacle center is at position $p_o \in \mathbb{R}^2$ and its radius is r meters.



The arm configuration consists of its joint angles θ_1, θ_2 and thus the state of the arm is defined by

$$x = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2).$$

Ignoring gravity, assume that the arm is controlled using torque inputs $u = (u_1, u_2)$ so that

$$\ddot{\theta}_1 = u_1, \qquad \ddot{\theta}_2 = u_2.$$

The coordinates of the arm tip are given by

$$p_t = \begin{pmatrix} \cos(\theta_1)\ell_1 + \cos(\theta_1 + \theta_2)\ell_2 \\ \sin(\theta_1)\ell_1 + \sin(\theta_1 + \theta_2)\ell_2 \end{pmatrix}.$$

a) If the arm must move so that its *tip* does not penetrate the obstacle, give the expression for obstacle avoidance *state* inequality constraint $c(x(t), t) \leq 0$. Then derive the q-th order *state-control* inequality constraint that must be satisfied on the surface of the obstacle.

b) If the arm must move so that no part of the arm penetrates the obstacle, give the expression for obstacle avoidance *state* inequality constraint $c(x(t), t) \leq 0$. There is no need to derive the state-control inequality constraint. Note: the constraint will involve an expression defining the intersection between a line and circle.

c) Using Matlab, plot the free configuration space of the arm corresponding to the obstacle-avoidance constraint you derived in Part a). Repeat for the constraint you derived in Part b). Note: the "free configuration space" is defined as the space of joint angles for which the arm does not collide with the obstacle. Upload your code using the File Upload link on the class website; in addition attach a printout of the code and plots to your homework solutions.

4. (Kirk, 5-37) The equations of motion of a rocket in horizontal flight are given by

$$\dot{x}_1(t) = \frac{cu(t)}{x_2(t)} - \frac{D}{x_2(t)},$$
$$\dot{x}_2(t) = -u(t),$$

where $x_1(t)$ is the horizontal velocity, $x_2(t)$ is the mass of the rocket, c is the exhaust gas speed, and D is the aerodynamic drag force.

The control input $u(t)$ can be regarded as the fuel burn rate and is limited by $0 \le u(t) \le u_{\max}$. It is desired to *maximize* the range of the rocket. The initial and final values of the mass and the velocity are specified, and the terminal time is free.

a) Assume the aerodynamic drag force is constant.

    i) Determine the adjoint equations of the boundary condition relationships

    ii) Investigate the possibility of singular control intervals.

b) *(Optional, for extra credit)* Assume the aerodynamic drag force is given by

$$D(x_1(t), x_2(t)) = \alpha x_1^2(t) + \frac{\beta x_2^2(t)}{x_1^2(t)} \ge 0, \tag{1}$$

where α and β are positive constants. Again,

    i) Determine the adjoint equations of the boundary condition relationships

    ii) Investigate the possibility of singular control intervals.

Note: upload your code as a single zip file (please name it as *LastName_FirstName_HW5.zip*) to the File upload link on the class webpage; in addition attach a printout of the code and plots to your homework solutions.