EN530.603 Applied Optimal Control
Lecture 5: Continuous Optimal Control Basics
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1 Continuous Systems with Terminal Constraints

Consider the cost

最终状态的cost　　过程中的cost　　此处　final time = $t_f + \delta t_f$

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt, \tag{1}$$

x_f

subject to q constraints

x_f　对于终状态

$$\psi(x(t_f), t_f) = 0$$

2种constraints

and the dynamics

对轨迹

$$\dot{x}(t) = f(x(t), u(t), t), \qquad t_0 \text{ and } x(t_0) \text{ are given.}$$

It will be useful to employ the shorthand notation $f(t) \equiv f(x(t), u(t), t)$, or $\phi(t) \equiv \phi(x(t), t)$, etc... Sometimes, f (or any other function) could also be without arguments, i.e. $f \equiv f(x(t), u(t), t)$.

Before we obtain the necessary conditions for optimality, let's see how to deal with variations of terms in the cost defined at t_f. Assume that our functional includes a term $h(x(t_f), t_f)$, i.e.

$$J = h(x(t_f), t_f) + \int_{t_0}^{t_f} \cdots \, dt,$$

where $\cdots$ represent any other terms. We can express this as

$$J = h(x(t_0), t_0) + \underbrace{\int_{t_0}^{t_f} \frac{d}{dt} h(x(t), t)}_{h(x(t_f),t_f)-h(x(t_0),t_0)} + \cdots \, dt,$$

which is equivalent to

UV　　u　　$\dot{v}$

$$J = h(x(t_0), t_0) + \int_{t_0}^{t_f} \nabla_x h(x(t), t)^T \dot{x}(t) + \partial_t h(x(t), t) + \cdots \, dt.$$

$\int u\dot{v} = uv - \int v\cdot\dot{u}$

Then using integration by parts and the definition of $\delta x_f = \delta x(t_f) + \dot{x}\delta t_f$ we have

$$\begin{aligned} \delta J =& \nabla_x h(t_f)^T \delta x_f + [\nabla_x h(t_f)^T \dot{x}(t_f) + \partial_t h(t_f) - \nabla_x h(t_f)^T \dot{x}(t_f)]\delta t_f \\ &+ \int_{t_0}^{t_f} \nabla_x [\nabla_x h(t)^T \dot{x}(t) + \partial_t h(t)] - \frac{d}{dt} \nabla_{\dot{x}} [\nabla_x h(t)^T \dot{x}(t) + \partial_t h(t)] + \delta(\cdots) \, dt \end{aligned} \tag{2}$$

After applying the derivatives under the integral, all terms there cancel and we end up with:

$$\delta J = \nabla_x h(x(t_f), t_f)^T \delta x_f + \partial_t h(x(t_f), t_f)\delta t_f + \int_{t_0}^{t_f} + \delta(\cdots) \, dt \tag{3}$$

$x = t^2$　　$t^2 + t$

直接得到　$\nabla_x h(x(t_f), t_f)^T \delta x(t_f) + \nabla_x h(x(t_f), t_f)\cdot \dot{x}\cdot\delta t_f$
$+ \partial_t h(x(t_f), t_f)\delta t_f$

and the dynamics

$$\dot{x}(t) = f(x(t), u(t), t), \qquad t_0 \text{ and } x(t_0) \text{ are given.}$$

之前是 $0 = f(x,u,t)$ 的形式

J_a: want to consider constrained problem as unconstrained problem

To obtain the necessary conditions for the original problem (??), form the augmented cost

$$J_a = \phi(t_f) + \nu^T \psi(t_f) + \int_{t_0}^{t_f} \{L(x,u,t) + \lambda^T(t)[f(x,u,t) - \dot{x}]\}\, dt.$$

t_f 不一定 fixed

ν 是只与 terminal 有关，所以与 t 无关

之前: $f(\ )=0$

Let $\Phi = \phi + \nu^T \psi$ and define the Hamiltonian H by

与 lec3 的 f 不同 这里 $f_c = \dot{x}$

$$H(x,u,\lambda,t) = L(x,u,t) + \lambda^T(t) f(x,u,t).$$

Taking variations with respect to all variables including final time t_f (and using the relationship (??)) we obtain

$$\delta J = \nabla_x h(x(t_f),t_f)^T \delta x_f + \partial_t h(x(t_f),t_f)\delta t_f + \int_{t_0}^{t_f} + \delta(\cdots)\, dt \qquad J = h(x(t_f),t_f) + \int_{t_0}^{t_f} \cdots\, dt,$$

$$\delta J_a = \left((\partial_t \Phi + L)\delta t_f + \partial_x \Phi \cdot \delta x_f\right)_{t=t_f} + \int_{t_0}^{t_f} (\partial_x H \cdot \delta x + \partial_u H \cdot \delta u - \lambda^T \delta \dot{x})\, dt.$$

$L\delta t_f \approx \int_{t_f}^{t_f+\delta t_f} L\, dt$ $\quad$ t_f 不固定

差 $+ (f-\dot{x})\cdot\delta\lambda^T$, $f-\dot{x}=0$

Integrating by parts and using the relationship

$$\delta x_f = \delta x(t_f) + \dot{x}\delta t_f,$$

$$\boxed{\int -\lambda^T \delta\dot{x}\, dt = -\lambda^T \delta x(t_f) + \int \dot{\lambda}^T \delta x\, dt}$$

we obtain

$$\delta J_a = \left([\partial_t \Phi + L + \lambda^T \dot{x}]\, \delta t_f + [\partial_x \Phi - \lambda^T]\, \delta x_f \right)_{t=t_f} + \int_{t_0}^{t_f} \left[\left(\partial_x H + \dot{\lambda}^T\right) \delta x + \partial_u H \cdot \delta u \right] dt$$

Since all variations are arbitrary and independent the necessary conditions become

$\delta J_a = 0$

$$\dot{\lambda}^T = -\partial_x H = -\lambda^T \partial_x f - \partial_x L, \tag{4}$$

$$\lambda(t_f)^T = \partial_x \Phi|_{t=t_f} = (\partial_x \phi + \nu^T \partial_x \psi)_{t=t_f}, \tag{5}$$

only when final state is free

$$\partial_u H = \lambda^T \partial_u f + \partial_u L = 0, \tag{6}$$

$$(\partial_t \Phi + L + \lambda^T \dot{x})_{t=t_f} = \left(\frac{d\Phi}{dt} + L\right)_{t=t_f} = 0, \tag{7}$$

only when t_f is free

where

推出

$$\frac{d\Phi}{dt} = \partial_t \Phi + \partial_x \Phi \cdot \dot{x}.$$

After substituting the expression for $\lambda(t_f)$ the necessary conditions are summarized according to:

optimal: $\delta J_a = 0$ 推出这些条件

$$\dot{x} = f(x,u,t) \tag{8}$$

dynamics

$$\dot{\lambda} = -\nabla_x H = -\nabla_x f \cdot \lambda - \nabla_x L, \ (4) \tag{9}$$

$$\nabla_u H = \nabla_u f \cdot \lambda + \nabla_u L = 0, \ (6) \tag{10}$$

$$\lambda(t_f) = \nabla_x \phi(x(t_f),t_f) + \nabla_x \psi(x(t_f),t_f) \cdot \nu, \ (5) \tag{11}$$

$$\left(\partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f\right)_{t=t_f} = 0, \tag{12}$$

dynamics + (5) + (7)

When the final time t_f is fixed the last relationship (??) can be dropped.

only when final state is free

当 L 和 f 与 t 无关时 H(t) 是常数，即 $\dot{H}=0$

Hamiltonian conservation. Note that whenever the Hamiltonian does not depend on time (that is when f and L do not depend on time)

$$\partial_t H(x,u,\lambda,t)=0$$

then H is a conserved quantity along optimal trajectories $x^*(t), u^*(t), \lambda^*(t)$, i.e. we have that

$$\dot{H}(x,u,\lambda,t) = \partial_x H\cdot\dot{x}+\partial_u H\cdot\dot{u}+\partial_\lambda H\cdot\dot{\lambda}+\partial_t H \tag{13}$$

$$= -\underbrace{\dot{\lambda}^T f(x,u,t)}_{(4)} + \underbrace{0}_{(10)}\cdot\dot{u}+f(x,u,t)^T\dot{\lambda}+0=0 \tag{14}$$

Therefore, in this case we have $H(t)=$ const for all $t\in[t_0,t_f]$. Furthermore, in the special case when $\partial_t\phi=0$ and $\partial_t\psi=0$ the last condition (??) reduces to $H(t)=0$. $H(t_f)=0$ $t_0\sim t_f$ 只要一个点确定 H 值就确定

(12)

Minimum-time problems. For minimum-time problems we have $\phi=0$ and $L=1$ so that condition (??) reduces to

让时间最短

$$\left(\nu^T[\partial_t\psi+\nabla_x\psi^T\cdot f]+1\right)_{t=t_f}=0, \quad (12)$$

可以求出 ν

which can be used along with the constraint $\psi(x(t_f),t_f)=0$ to determine the multipliers ν and final time t_f.

都是 necessary condition.

Solution Methods

We are faced with solving the differential equations for $t=[t_0,t_f)$:

what we want to solve for $t=[t_0,t_f)$

$$\text{Euler-Lagrange (EL)}: \quad \begin{pmatrix}\dot{x}\\ \dot{\lambda}\end{pmatrix}=\begin{pmatrix}f(x,u,t)\\ -\nabla_x H\end{pmatrix} \tag{15}$$

(9)

where $u(t)$ is computed by minimizing H which corresponds to the condition 求出 $u(t)$

$$\text{Control optimization}: \quad \nabla_u H=0, \ (10)$$

不一定成立

which we assume can be solved and that $u(t)$ is then expressed as a function of $x(t)$ and $\lambda(t)$, subject to the boundary constraints 由 $x(t)$ $\lambda(t)$ 求出 $u(t)$

ψ: Si

求出 $u(t)$ 后
再根据各种条件求其它参数.

$$\text{Transversality Conditions (TC)}: \quad \begin{aligned}&\psi(x(t_f),t_f)=0 \quad \text{final terminal constraint}\\ &\lambda(t_f)=\nabla_x\phi(x(t_f),t_f)+\nabla_x\psi(x(t_f),t_f)\cdot\nu,\\ &\left(\partial_t\phi+\nu^T\partial_t\psi+L+\lambda^T f\right)_{t=t_f}=0, \quad \leftarrow \text{free end time. condition}\end{aligned} \tag{16}$$

The following solution methods are applicable based on whether EL can be integrated in closed-form and whether TC can be solved in closed form:

- general: two-point boundary value problem (BVP) works with any EL and TC, the conditions are satisfied using a numerical "collocation" procedure 用来解 $\nabla_u H$ 中的 u 再代入 EL 解 TC
- EL integrable: pick $\lambda(0)$ integrate from t_0 to t_f and solve TC as an implicit equality for the unknown $(\lambda(0),\nu)$. When final time t_f is free then solve for $(\lambda(0),\nu,t_f)$.
- EL integrable and TC solvable: closed-form solution.

EL: always applies: what happens in trajectory
optimal control cond: same as EL
TC: what happens for final state

Example 1. Minimum Control Effort Landing Consider a second order system with state $x = (p, v) \in \mathbb{R}^4$ where $p \in \mathbb{R}^2$ is the position and $v \in \mathbb{R}^2$ is the velocity. The system has a *double integrator dynamics* given by

$$\begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix},$$

u:改变加速度,fuel....

where $u \in \mathbb{R}^2$ is the acceleration control input. The system starts with known initial state $x_0 = (p_0, v_0)$ and must "land" with a prescribed velocity v_f somewhere on a unit circle centered at the origin, i.e. the final configuration must satisfy $\psi(x(t_f)) = 0$, where

$$\psi(x) = p^T p - 1.$$

降落在一个圆上

$V(t_f) = V_f = 0$ $\quad \psi(x) = \begin{bmatrix} p^T p - 1 \\ V \end{bmatrix}$

$\delta V(t_f) = 0$

$V(t_f) = 0$

The objective function is the control effort given by

$$L(x, u) = \frac{1}{2}\|u\|^2$$

We start with the Hamiltonian, and the multipliers $\lambda = (\lambda_p, \lambda_v)$

$$H = \frac{1}{2}u^T u + \underbrace{\lambda_p^T v + \lambda_v^T u}_{\lambda \cdot \dot{f}},$$

$$\dot{x} = f(x, u, t)$$
$$\dot{\lambda} = -\nabla_x H = -\nabla_x f \cdot \lambda - \nabla_x L,$$
$$\nabla_u H = \nabla_u f \cdot \lambda + \nabla_u L = 0,$$
$$\lambda(t_f) = \nabla_x \phi(x(t_f), t_f) + \nabla_x \psi(x(t_f), t_f) \cdot \nu,$$
$$\left(\partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f\right)_{t=t_f} = 0,$$

We have

λ_p is constant

$$\dot{\lambda} = -\nabla_x H \quad \Rightarrow \quad \dot{\lambda}_p = 0, \quad \dot{\lambda}_v = -\lambda_p$$
$$\nabla_u H = 0 \quad \Rightarrow \quad u = -\lambda_v,$$

$\dot{u} = \lambda_p$

求出 u.

from which we get

$$\ddot{u} = -\ddot{\lambda}_v = \dot{\lambda}_p = 0,$$

$\Rightarrow \dot{u}$ is constant $\quad$ u is linear $u = a_1 t + a_2$

which means that the path $p(t)$ is a cubic spline that can be written according to

V is quadratic $\quad$ P is cubic.

$$p(t_0 + t) = c_3 t^3 + c_2 t^2 + v_0 t + p_0, \tag{17}$$

while the velocity is

又 $V = \dot{p}$ $\qquad \psi(x(t_f)) = p^T p - 1 = 0$

$$v(t_0 + t) = 3c_3 t^2 + 2c_2 t + v_0. \tag{18}$$

Now from

min time, 无 terminal cost

$$\lambda_p(t_f) = \nabla_p \psi(x(t_f))\nu = 2p(t_f)\nu.$$

(16)

Note that above since the velocity is not present in the terminal constraint ψ, then there is no additional condition on $\lambda_v(t_f)$.

Now considering that $\lambda_p(t_f) = \dot{u}(t_f) = 6c_3$ the above is equivalent to

C_0 $\quad C_1$ $\quad$ 都是已知 (由条件和初始值决定)

$$6c_3 = 2p(t_f)\nu.$$

1D 2D 2D

Finally, assuming t_f is given we can solve for ν, c_2, c_3 (5 unknowns) the implicit equations (5 equations):

$$6c_3 - 2p(t_f)\nu = 0, \tag{19}$$

2eq $\quad$ 2D

$$p(t_f)^T p(t_f) - 1 = 0, \tag{20}$$

1eq $\quad$ 1D

$$v(t_f) - v_f = 0, \tag{21}$$

2eq $\quad$ 2D

option 1:

$\delta x(t_f)$ is free.

$$\psi(x) = \begin{bmatrix} p^T p - 1 \\ V - V_f \end{bmatrix} = 0$$

option 2: $\delta p(t_f)$ is free $\qquad \psi(x) = p^T p - 1 = 0$

$\delta v(t_f) = 0, \; V = V_f$

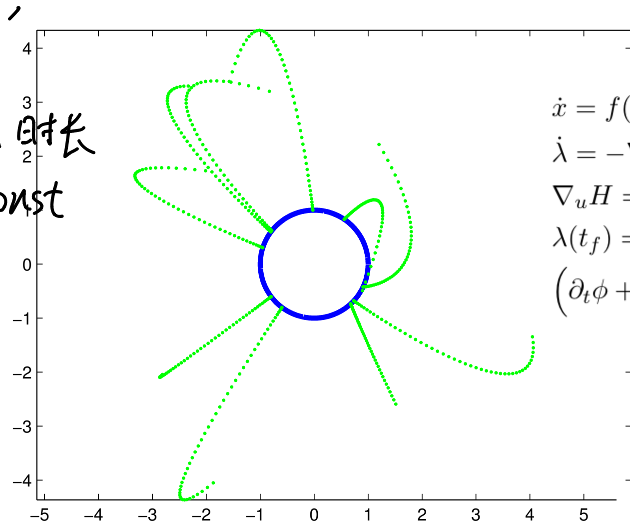
where $p(t_f)$ and $v(t_f)$ are given by (??) and (??). Note that it is necessary that $\nu \neq 0$ to ensure that the constraint is satisfied.

Examples of the resulting trajectories from randomly initialized states are given. In all examples we have $v_f = (0, 0)$

知道 control law

不要初始值,反点时长

若想减小时间 可以让 $L = \frac{1}{2}\|u^2\| + b$ const



$$
\begin{aligned}
\dot{x} &= f(x, u, t) \\
\dot{\lambda} &= -\nabla_x H = -\nabla_x f \cdot \lambda - \nabla_x L, \\
\nabla_u H &= \nabla_u f \cdot \lambda + \nabla_u L = 0, \\
\lambda(t_f) &= \nabla_x \phi(x(t_f), t_f) + \nabla_x \psi(x(t_f), t_f) \cdot \nu, \\
&\left(\partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f\right)_{t=t_f} = 0,
\end{aligned}
$$

$L = 1$

$\phi = 0$ 无ψ

$$\left(\nu^T[\partial_t \psi + \nabla_x \psi^T \cdot f] + 1\right)_{t=t_f}$$

帆船问题

Example 2. Example: Zermelo's problem (Bryson §2.7) Consider a ship with dynamics

u, v 是已知函数

$U = \theta$: control, 要求的表达式

$$\dot{x} = V \cos\theta + u(x, y) \tag{22}$$

$$\dot{y} = V \sin\theta + v(x, y), \tag{23}$$

where (x, y) is the position, V is a constant velocity, θ is the heading angle input and u and v denote velocity due to currents. The goal is to travel between points A and B in minimum time.

The Hamiltonian is

$$H = \underbrace{\lambda_x(V\cos\theta + u) + \lambda_y(V\sin\theta + v)}_{\lambda \cdot f} + \underbrace{1}_{\sim L \text{ w.r.t. time}}.$$

The Euler-Lagrange equations are

根据EL求λ

$$\dot{\lambda}_x = -\partial_x H = -\lambda_x \partial_x u - \lambda_y \partial_x v \tag{24}$$

$$\dot{\lambda}_y = -\partial_y H = -\lambda_x \partial_y u - \lambda_y \partial_y v \tag{25}$$

optimal control cond:

$$0 = \partial_\theta H = V(-\lambda_x \sin\theta + \lambda_y \cos\theta) \Rightarrow \tan\theta = \frac{\lambda_y}{\lambda_x} \tag{26}$$

Since this is a minimum-time problem we have $H = 0$ and from (??) that

L. f 与t无关

第4条没有 x_f fixed

$$\lambda_x = \frac{-\cos\theta}{V + u\cos\theta + v\sin\theta}, \quad \lambda_y = \frac{-\sin\theta}{V + u\cos\theta + v\sin\theta}$$

$\phi = 0$

ψ与t无关. 第5条: $\partial_t \psi = 0$ $H(t_f) = 0$ → $H = 0$

This leads to

$$\dot{\theta} = \sin^2\theta \partial_x v + \sin\theta\cos\theta(\partial_x u - \partial_y v) - \cos^2\theta \partial_y u$$

Now, in order to reach B one has to select the start angle θ_A and the final time t_f.

只能pick θ_A, t_f

integrate $\dot{\theta}$ to get $x(t_f)$ $y(t_f)$ until close enough to (0,0) goal

移多使

Special case!!!

Special Case. For the special case when

$$u = -V(y/h), \quad v = 0$$

风只在 x 轴方向上

consider starting at (x_0, y_0) with the goal to reach the origin $(0,0)$. We have

$$H 是 0 \Rightarrow \quad \dot{\lambda}_x = 0 \;\Rightarrow\; \lambda_x = \text{const}$$

and therefore

$$\lambda_x = \frac{-\cos\theta}{V - V(y/h)\cos\theta} = \frac{-\cos\theta_f}{V} = -\text{const} \quad \Rightarrow \quad \cos\theta = \frac{\cos\theta_f}{1 + (y/h)\cos\theta_f}$$

In the above, it turned out that it is convenient to work in terms of θ_f rather than t_f. The solution can be obtained analytically as

根据 x, y 可求出 θ (方向)

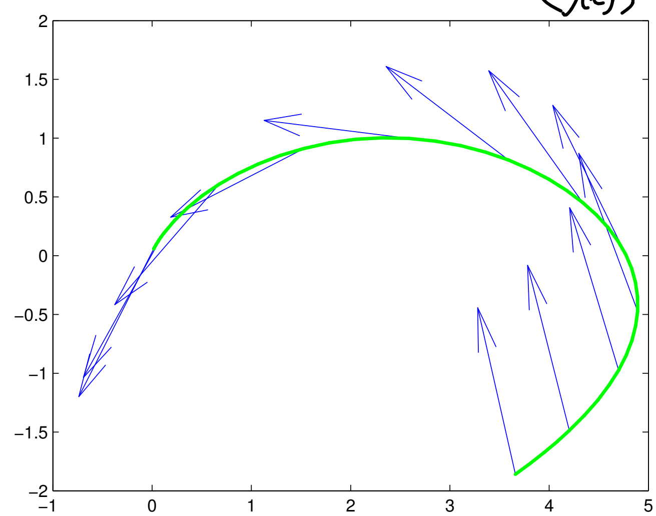
$$x = \frac{h}{2}\left[\sec\theta_f(\tan\theta_f - \tan\theta) - \tan\theta(\sec\theta_f - \sec\theta) + \log\frac{\tan\theta_f + \sec\theta_f}{\tan\theta + \sec\theta}\right], \tag{27}$$

$$y = h(\sec\theta - \sec\theta_f), \tag{28}$$

from which one can compute the initial angle θ and final angles θ_f to achieve given final position (x, y).

The computed path with initial conditions given by $x_0 = 3.66$ and $y_0 = -1.86$ with $h = 1$, $V = .3$ are given below

solve $\begin{cases} x(t) \\ y(t) \end{cases} = 0$ in terms of θ_0 and θ_f



Example 3. Minimum Control Effort Landing with Optimal Time Consider the minimum control effort landing §?? with free final time t_f and a cost function given by

$$L(x, u) = b + \frac{1}{2}\|u\|^2,$$

for some constant $b > 0$ which controls the balance between penalizing total time and total control effort.

We need to add the third transversality condition from (**??**)

$$\partial_t \phi(t_f) + H(t_f) = 0,$$

which in our case is

$$b - \frac{1}{2}\|u(t_f)\|^2 + \dot{u}(t_f)^T v(t_f) = 0$$

This can be solved along with the other five conditions to obtain the unknowns c_2, c_3, ν, t_f. Plots of computed trajectories with varying b are given below.