EN530.603 Applied Optimal Control
Homework #2

September 15, 2021

Due: September 22, 2021

Lecturer: Marin Kobilarov

1. Find the stationary points of the following and determine whether they are maxima, minima, or saddle points:

   (a)

$$\text{minimize } L(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) \tag{1}$$

$$\text{subject to } f(x) = x_1 + x_2 + x_3 = 0 \tag{2}$$

   (b)

$$\text{minimize } L(u) = (u_1^2 + 3u_1 - 4)(u_2^2 - u_2 + 3) \tag{3}$$

$$\text{subject to } f(u) = u_1 - 2u_2 = 0 \tag{4}$$

2. (a) Consider the optimization of a quadratic cost subject to linear constraints, i.e. minimize

$$L(x, u) = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + s^T x,$$

   subject to

$$f(x, u) = Ax + Bu + c = 0,$$

   where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; $Q \geq 0$ (positive semidefinite matrix) and $R > 0$ (positive definite matrix); $A \in \mathbb{R}^{n \times n}$, $B^{n \times m}$, $s \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$.

   - Derive the necessary and sufficient conditions for an optimal solution using the Lagrangian multiplier approach. Be careful which matrices you are allowed to invert.
   - Assume that A is full rank and compute the actual optimal solution.

   (b) Consider the optimization of a quadratic cost subject to linear constraints, i.e. minimize

$$L(y) = \frac{1}{2}y^T M y + k^T y$$

   subject to

$$f(y) = Ay + c = 0,$$

   where $y \in \mathbb{R}^n$, $M > 0$ is positive definite, $A \in \mathbb{R}^{m \times n}$ for $m < n$ is full rank, $k \in \mathbb{R}^n$, and $c \in \mathbb{R}^m$. Compute the optimal solution y^* and show that it is a global minimum.

3. Find the optimal $x^* \in \mathbb{R}^n$ and prove that is the global minimum for:

$$\min a^T x + b,$$

subject to

$$x^T x - 1 = 0,$$

where a and b are constant. How do you explain your solution geometrically (could assume $b = 0$ if that is easier)?

4. Consider the minimization of $f(x)$ for $x \in \mathbb{R}^n$. Newton's method is derived by finding the direction $d^k \in \mathbb{R}^n$ which minimizes the local quadratic approximation f^k of f at x^k defined by

$$f^k(d) = f(x^k) + \nabla f(x^k)^T d + \frac{1}{2} d^T \nabla^2 f(x^k) d.$$

In contrast, the search direction d^k in a *trust-region* Newton method is derived by solving the *constrained* optimization

$$\text{minimize} \quad f^k(d) \quad \text{subject to} \quad \|d\| \leq \gamma^k,$$

for a given $\gamma^k > 0$ called the trust-region radius. Using the Lagrangian multiplier approach prove that this optimization is equivalent to solving

$$(\nabla^2 f(x^k) + \delta^k I) d^k = -\nabla f(x^k),$$

where $\delta^k \geq 0$. How do you interpret the value δ^k. Can you propose a reasonable choice for δ^k considering the properties of $\nabla^2 f(x^k)$.