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Assignment 1

1、 combine the three points' column vectors to a matrix :

$$A = \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 1 & \frac{1}{3} & -1 \\ 1 & \frac{4}{3} & 2 \end{pmatrix}$$

if the points are collinear then A will lose a rank.

then rank(A) = 2, det(A) = 0.

$$\begin{aligned} \det A &= \tfrac{1}{2}\times\tfrac{1}{3}\times 2 + 1\times 1\times(-1) + 2\times 1\times\tfrac{4}{3} \\ &\quad - \left(\tfrac{1}{2}\times(-1)\times\tfrac{4}{3} + 1\times 1\times 2 + 2\times 1\times\tfrac{1}{3}\right) \\ &= \tfrac{1}{3} - 1 + \tfrac{8}{3} - \left(-\tfrac{2}{3} + 2 + \tfrac{2}{3}\right) \\ &= 3 - 1 - 2 \\ &= 0. \end{aligned}$$

Therefore these points are collinear.

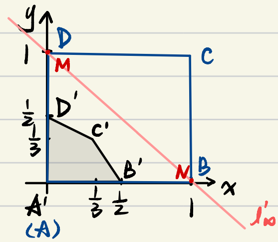
2. (a) suppose $A(0,0), B(1,0), C(1,1), D(0,1)$

$$\tilde{A}' = A\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{B}' = H\tilde{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\tilde{C}' = H\tilde{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\tilde{D}' = H\tilde{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$



$\therefore$ $A' = (0,0)^T$, $B' = (\frac{1}{2}, 0)^T$, $C' = (\frac{1}{3}, \frac{1}{3})^T$, $D' = (0, \frac{1}{2})^T$

(b) $l_{AB}: x=0 \Rightarrow l_{AB} = (1,0,0)^T$ suppose the two vanishing points are M and N

$l_{CD}: x=1 \Rightarrow l_{CD} = (1,0,-1)^T$

$$l_{AB} \times l_{CD} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow M$$

$l_{AD}: y=0 \Rightarrow l_{AD} = (0,1,0)^T$

$l_{BC}: y=1 \Rightarrow l_{BC} = (0,1,-1)^T$

$$l_{AD} \times l_{BC} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow N$$

$$\therefore \quad \tilde{M}' = AM = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\tilde{N}' = HN = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$\therefore$ $M' = (0,1)^T$, $N' = (1,0)^T$

$$l'_\infty = l_{M'N'} = \tilde{M}' \times \tilde{N}' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow x+y-1=0$$

l'_∞, M', N' are shown in the figure in red.

(c) suppose $H_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_1 & v_2 & v \end{pmatrix}$, together with $l_\infty' = (1,1,-1)^T$

we have $l_\infty'^T = (0,0,1) H_p^{-1}$

$$(1,1,-1) = (0,0,1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{v_1}{v} & -\frac{v_2}{v} & \frac{1}{v} \end{pmatrix} = \left(-\frac{v_1}{v}, -\frac{v_2}{v}, \frac{1}{v}\right).$$

$$\therefore \begin{cases} \frac{1}{v} = -1 \\ -\frac{v_1}{v} = 1 \\ -\frac{v_2}{v} = 1 \end{cases} \Rightarrow \begin{cases} v = -1 \\ v_1 = 1 \\ v_2 = 1 \end{cases}$$

$$\therefore H_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

(d) $$H_p \tilde{A}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$H_p \tilde{B}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$H_p \tilde{C}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$H_p \tilde{D}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

so the four points are $(0,0)^T$, $(-1,0)^T$, $(-1,-1)^T$, $(0,-1)^T$

3. a) suppose $p_1 = (1,1)^T$, $p_2 = (2,1)^T$, $p_3 = (0,1)^T$

$p_1' = (-1,1)^T$, $p_2' = (4,1)^T$, $p_3' = (1,1)^T$

$\because$ $p_i' = Hp_i$, suppose $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$

$$\therefore \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} h_{11}+h_{12} \\ h_{21}+h_{22} \end{pmatrix} \Rightarrow \frac{h_{11}+h_{12}}{h_{21}+h_{22}} = \frac{-1}{1}$$

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2h_{11}+h_{12} \\ 2h_{21}+h_{22} \end{pmatrix} \Rightarrow \frac{2h_{11}+h_{12}}{2h_{21}+h_{22}} = \frac{4}{1}$$

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix} \Rightarrow \frac{h_{12}}{h_{22}} = \frac{1}{1}$$

$\therefore$ $h_{12} = h_{22}$

$$h_{11} + h_{12} = -h_{21} - h_{12} \Rightarrow h_{11} + h_{21} = -2h_{12}$$

$$2h_{11} + h_{12} = 8h_{21} + 4h_{12} \Rightarrow 2h_{11} - 8h_{21} = 3h_{12}$$

$$\Rightarrow 2h_{11} + 2h_{21} = -4h_{12}$$

$$2h_{11} - 8h_{21} = 3h_{12}$$

$$10h_{21} = -7h_{12}$$

$$h_{21} = -\frac{7}{10}h_{12}$$

$$h_{11} = -2h_{12} + \frac{7}{10}h_{12} = -\frac{13}{10}h_{12}$$

$$\therefore H = \begin{pmatrix} -\frac{13}{10} & 1 \\ -\frac{7}{10} & 1 \end{pmatrix} h_{12}$$

$\because$ H is homogeneous

$$\therefore H = \begin{pmatrix} -13 & 10 \\ -7 & 10 \end{pmatrix}$$

b) $$p' = Hp = \begin{pmatrix} -13 & 10 \\ -7 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -13 \\ -7 \end{pmatrix}$$

4. from the cross ratio definition we have

$$\mathrm{Cross}(p_1, p_2, p_3, p_4) = \frac{\overline{p_1p_3}\ \overline{p_2p_4}}{\overline{p_2p_3}\ \overline{p_1p_4}}$$

p_4 is the infinite point $\Rightarrow p_{42} = 0$.

suppose $p_{12} = p_{22} = p_{32} = 1$

$$\overline{p_1p_3} = \det\begin{pmatrix} p_{11} & p_{31} \\ p_{12} & p_{32} \end{pmatrix} = p_{11}p_{32} - p_{12}p_{31} = p_{11} - p_{31}$$

$$\overline{p_2p_4} = \det\begin{pmatrix} p_{21} & p_{41} \\ p_{22} & p_{42} \end{pmatrix} = p_{21}p_{42} - p_{22}p_{41} = -p_{41}$$

$$\overline{p_2p_3} = \det\begin{pmatrix} p_{21} & p_{31} \\ p_{22} & p_{32} \end{pmatrix} = p_{21}p_{32} - p_{22}p_{31} = p_{21} - p_{31}$$

$$\overline{p_1p_4} = \det\begin{pmatrix} p_{11} & p_{41} \\ p_{12} & p_{42} \end{pmatrix} = p_{11}p_{42} - p_{12}p_{41} = -p_{41}$$

$$\therefore \mathrm{Cross}(p_1, p_2, p_3, p_4) = \frac{\overline{p_1p_3}\ \overline{p_2p_4}}{\overline{p_2p_3}\ \overline{p_1p_4}}$$

$$= \frac{(p_{11} - p_{31})(-p_{41})}{(p_{21} - p_{31})(-p_{41})}$$

$$= \frac{p_{11} - p_{31}}{p_{21} - p_{31}}$$

$$= \frac{\overline{p_1p_3}}{\overline{p_2p_3}}$$

$\ast$ substitute from textbook version

$p_1 \Rightarrow p_3$

$p_2 \Rightarrow p_1$

$p_3 \Rightarrow p_2$