EN530.678 Nonlinear Control and Planning in Robotics
Midterm #1

March 29, 2021

Prof: Marin Kobilarov

Note: You must complete the test during 3:00-4:30 pm. Upload your solutions as a single pdf using `https://forms.gle/Z2AYx3FRNJHtXTR47`. You have 10 extra minutes to assemble and upload your solutions. No uploads past 4:40pm will be accepted. You can use any of the lecture notes or previous homeworks used in the class. You must NOT share this pdf with anyone else.

1. Show that the origin $(0, 0)$ is an asymptotically stable solution of

$$\dot{x}_1 = -x_1^3 + 2x_2^3,$$
$$\dot{x}_2 = -2x_1x_2^2.$$

2. Controllability

   (a) The configuration of a hopping robot during flight is $q = (\psi, \ell, \theta)$, where ψ is the leg angle, $\ell \geq 0$ is the leg extension, and θ is the body orientation. The equations of motion are

$$\dot{\psi} = u_1,$$
$$\dot{\ell} = u_2,$$
$$\dot{\theta} = -\frac{m(\ell+1)^2}{I + m(\ell+1)^2}u_1,$$

   where m and I are the constant mass and moment of inertia, and u_1 and u_2 are the control inputs. Show that the system is STLC, and hence nonholonomic.

   (b) Consider the system with state $x = (x_1, x_2, x_3)$ and dynamics given by

$$\dot{x} = \begin{pmatrix} -1 \\ 0 \\ 3x_1^2 + x_2^2 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ -2x_1x_2 \end{pmatrix} u_2$$

   Show that the system is holonomic, i.e. that it has an integrable distribution, and derive the integrable manifolds in the form $h(x) = c$ for some constant c that depends on where the system starts. If the system starts at the origin, can it reach point $x_f = (1, 1, -2)$? *Note*: recall that if you have two non-zero linearly independent vectors $a, b \in \mathbb{R}^3$ you can find a third vector $c \in \mathbb{R}^3$ such that $c^T a = 0$ and $c^T b = 0$ simply by setting

$$c = a \times b = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

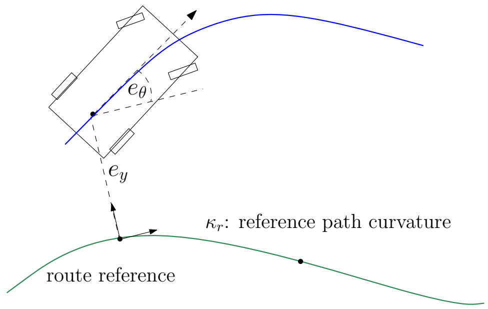

Figure 1: Vehicle description in route-relative coordinates.

3. Consider a car model expressed in a route frame (Figure 1) with state $x = (e_y, e_\theta, v, \kappa)$ defined by its *lateral offset* e_y, *relative angle* e_θ, forward velocity v, curvature κ and control inputs $u = (u_1, u_2)$ defining the path acceleration u_1 and curature-rate u_2. The equations of motion are:

$$\dot{e}_y = v \sin e_\theta, \tag{1}$$

$$\dot{e}_\theta = v \left(\kappa - \frac{\kappa_r}{1 - \kappa_r e_y} \cos e_\theta \right), \tag{2}$$

$$\dot{v} = u_1, \tag{3}$$

$$\dot{\kappa} = u_2 \tag{4}$$

where $\kappa_r(t)$ denotes the reference road curvature.

   (a) Show that the system is differentially flat with respect to outputs $y = (e_y, v)$, i.e. express the state and controls as functions of $y, \dot{y}, \ddot{y}, \ldots$.

   (b) Describe a procedure (at a high level) for computing the required inputs to generate a trajectory between two given states x_0 and x_f with time duration T.

4. Consider a satellite around a circular orbit with a camera aligned with its body-fixed vertical axis. The direction of the camera with respect to a spatial frame F is given by the unit vector $b(t) \in \mathbb{R}^3$. The body rotates with angular velocity $\omega(t) \in \mathbb{R}^3$ and is controlled using torque inputs $u(t) \in \mathbb{R}^3$. The dynamics is given by

$$\dot{b} = b \times \omega,$$
$$I\dot{\omega} = (I\omega) \times \omega + u + 3\omega_c^2 b \times (Ib),$$

where the last term denotes gravity gradient torque resulting from the orbital angular rate $\omega_c > 0$ and $I = diag([I_1, I_2, I_3])$ is a diagonal inertia matrix. The goal is to align the satellite antenna (i.e. the vector $b(t)$) with a desired direction defined by the constant unit vector $b_0 \in \mathbb{R}^3$ in frame F.

   Design a control law to accomplish this assuming that the initial conditions (at time t_0) are such that $b(t_0)^T b_0 > -1$ and $\omega(t_0) = 0$. Prove that it is locally asymptotically stable.

   Hint: Two unit vectors $x, y \in \mathbb{R}^n$ are equal if and only if $x^T y = 1$. In addition, in the derivation use the skew-symmetry property that $a^T(b \times a) = 0$ as well the property that $(a \times b)^T c = a^T(b \times c)$, for any $a, b, c \in \mathbb{R}^3$.