1)
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}-x_1\\ x_1x_2\\ x_2\end{bmatrix}}_{f} + \underbrace{\begin{bmatrix}1+3x_2^2\\ 1\\ 0\end{bmatrix}}_{g} u$$

$y = x_3$ $\dot{y} = x_2$ $\ddot{y} = x_1x_2 + u$ System relative degree: $r=2$

$r=2$ so we need to find η with $z_1 = x_3$ $z_2 = x_2$

the jacobian of the coordinate transformation is:

$$\partial\Phi = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ \frac{\partial\eta}{x_1} & \frac{\partial\eta}{x_2} & \frac{\partial\eta}{x_3}\end{bmatrix}$$ and require $\operatorname{rank}(\partial\Phi) = 3$, $L_g\eta = 0$

we can let $\eta = -x_1 + x_2 + x_2^3$

$$\Phi = \begin{pmatrix}x_3\\ x_2\\ -x_1+x_2+x_2^3\end{pmatrix}$$

$$\dot{\eta} = x_1 - (1+3x_2^2)u + x_1x_2 + u + 3x_2^2\cdot(x_1x_2+u)$$

when $z_1=0$ $z_2=0$ $\dot{\eta} = x_1$ $\eta = -x_1$ let $V_0 = \frac{1}{2}\eta^T\eta$, then $\dot{V}_0$ is n.d

so it is minimum phase

2)

a) $$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 w_2 w_3 \\ a_2 w_1 w_3 \\ a_3 w_1 w_2 \end{pmatrix}}_{f} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{g_1} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{g_2} u_2 \quad \text{and } y = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

differentiate y:

$$\dot{y} = \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} a_1 w_2 w_3 + u_1 \\ a_2 w_1 w_3 + u_2 \end{pmatrix}$$

let virtual input $V = \dot{y}$, then $u_1 = V_1 - a_1 w_2 w_3$

$u_2 = V_2 - a_2 w_1 w_3$

let $e = w - w_d$ then, $\dot{e} = \dot{w} = V$ $\quad \ddot{e} = \ddot{w}$

let $\dot{e} = -k_p e$ $\quad k_1, k_2 > 0$

k_p is p.d and symmetric, so $V = \begin{pmatrix} -k_1(w_1 - w_{1d}) \\ -k_2(w_2 - w_{2d}) \end{pmatrix}$ so $u_1 = -k_1(w_1 - w_{1d}) - a_1 w_2 w_3$

$u_2 = -k_2(w_2 - w_{2d}) - a_2 w_1 w_3$

relative degree is 1, 1. So we need to find η.

η need to satisfy: $\partial\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial\eta}{w_1} & \frac{\partial\eta}{w_2} & \frac{\partial\eta}{w_3} \end{pmatrix}$ $\quad$ rank$(\partial\Phi) = 3$ and $\nabla_\eta \cdot g_1 = 0$

$\nabla_\eta \cdot g_2 = 0$

we can let $\eta = w_3$ $\quad$ then. Φ: $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$\dot{\eta} = a_3 w_1 w_2$

when $w_1 = w_2 = 0$ $\quad \dot{\eta} = 0$, so the zero dynamic is stable.

so it is not minimum phase

2) b)

$y=\begin{pmatrix} w_1 \\ w_3 \end{pmatrix}$ $\dot{y}=\begin{pmatrix} a_1 w_2 w_3 + u_1 \\ a_3 w_1 w_2 \end{pmatrix}$ $\ddot{y}=\begin{pmatrix} a_1(a_2 w_1 w_3 + u_2) w_3 + a_1 w_2 a_3 w_1 w_2 + \dot{u}_1 \\ a_3(a_1 w_2 w_3 + u_1) w_2 + a_3 w_1 (a_2 w_1 w_3 + u_2) \end{pmatrix}$

$$= \begin{pmatrix} a_1 a_2 w_1 w_3^2 + a_1 a_3 w_1 w_2^2 \\ a_2 a_3 w_1^2 w_3 + a_1 a_3 w_2^2 w_3 + a_3 w_2 u_1 \end{pmatrix} + \begin{pmatrix} 1 & a_1 w_3 \\ 0 & a_3 w_1 \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ u_2 \end{pmatrix}$$

let virtual input $v = \ddot{y}$

$$\begin{pmatrix} \dot{u}_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{a_1 w_3}{a_3 w_1} \\ 0 & \frac{1}{a_3 w_1} \end{pmatrix} \cdot \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} a_1 a_2 w_1 w_3^2 + a_1 a_3 w_1 w_2^2 \\ a_2 a_3 w_1^2 w_3 + a_1 a_3 w_2^2 w_3 + a_3 w_2 u_1 \end{pmatrix} \right]$$

dynamic compensator $\xi = u_1$ when $w_1 \neq 0$ controller is valid.

and $v = -k_p(y - y_d) - k_d(\dot{y} - \dot{y}_d) + \ddot{y}_d$

Since y_d is constant, $v = -k_p(y - y_d) - k_d \dot{y}$

k_p, k_d are pd and symmetric

2, c)

in a) $u_1 = -k_1(w_1 - w_{1d}) - a_1 w_2 w_3$

$u_2 = -k_2(w_2 - w_{2d}) - a_2 w_1 w_3$

in b)

$$\begin{pmatrix} \dot{u}_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{a_1 w_3}{a_3 w_1} \\ 0 & \frac{1}{a_3 w_1} \end{pmatrix} \cdot \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} a_1 a_2 w_1 w_3^2 + a_1 a_3 w_1 w_2^2 \\ a_2 a_3 w_1^2 w_3 + a_1 a_3 w_2^2 w_3 + a_3 w_2 u_1 \end{pmatrix} \right]$$

I suggest we can first follow controller in a)

after get into some near region of (w_{1d}, w_{2d}), use controller in b)

after get into some near region of (w_{1d}, w_{3d}) use controller in a)

in the process, we can shrink the boundary of region

to force the state get to (w_{1d}, w_{2d}, w_{3d})

3)

$$m\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}0\\ mg\end{bmatrix} + \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}u_1,$$

$$J_0\ddot{\theta} = u_2 - u_3,$$

$$a\ddot{\varphi} = u_3 - \ell_0 u_1 \sin\varphi - \frac{a}{J_0}(u_2 - u_3),$$

$$\begin{pmatrix}x_1^{(3)}\\ x_2^{(3)}\end{pmatrix} = \frac{1}{m}R(x_3)\cdot\begin{pmatrix}-u_1\dot{x}_3\\ \dot{u}_1\end{pmatrix} \qquad R(x) = \begin{pmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{pmatrix}$$

denote y as x_2. θ as x_3 φ as x_4

Since $J_0\ddot{x}_3 = u_2 - u_3$,

$$\begin{pmatrix}x_1^{(4)}\\ x_2^{(4)}\end{pmatrix} = \frac{R(x_3)}{m}\begin{pmatrix}-2\dot{u}_1\dot{x}_3\\ -u_1\dot{x}_3^2\end{pmatrix} + \frac{R(x_3)}{m}\begin{pmatrix}-u_1/J_0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_2-u_3\\ \ddot{u}_1\end{pmatrix}$$

let virtual input be $v = \begin{pmatrix}x_1^{(4)}\\ x_2^{(4)}\end{pmatrix}$, dynamic compensator: $\xi = \begin{pmatrix}u_1\\ \dot{u}_1\end{pmatrix}$

We can set $v = y_d^{(4)} - \sum_{i=0}^{3} k_i\left(y^{(i)} - y_d^{(i)}\right)$ k_i are p.d and symmetric

let $y = \begin{pmatrix}x_1\\ x_2\end{pmatrix}$ $y_d = \begin{pmatrix}x_{1d}\\ x_{2d}\end{pmatrix}$ We can use u_2-u_3, $\ddot{u}_1$ in controller

then,

$$\begin{bmatrix}u_2-u_3\\ \ddot{u}_1\end{bmatrix} = \begin{bmatrix}-\frac{J_0}{\xi_1} & 0\\ 0 & 1\end{bmatrix}\left[mR^T(x_3)\cdot v + \begin{pmatrix}2\xi_2\dot{x}_3\\ u_1\cdot\dot{x}_3^2\end{pmatrix}\right]$$

↑ we can track $x(t)$ $y(t)$

then let $\ddot{x}_4 = \ddot{x}_{4d} - k_4(x_4 - x_{4d}) - k_5(\dot{x}_4 - \dot{x}_{4d})$, $k_4, k_5 > 0$

so $u_3 = a\ddot{x}_4 + L_0 u_1 \sin x_4 + \frac{a}{J_0}(u_2 - u_3)$ get u_1, u_2-u_3 from there

3) b)

$\theta(t) + \varphi(t) = x_3 + x_4$ let $x_5 = x_3 + x_4$

$$\ddot{x}_5 = \ddot{x}_{5d} - k_6(x_5 - x_{5d}) - k_7(\dot{x}_5 - \dot{x}_{5d}), \quad k_6, k_7 > 0$$

$$\ddot{x}_5 = \ddot{x}_3 + \ddot{x}_4 = \frac{u_3}{a} - \frac{L_0}{a}u_1\cdot\sin x_4 - \frac{1}{J_0}(u_2 - u_3) + \frac{1}{J_0}(u_2 - u_3)$$

$$= \frac{u_3}{a} - \frac{L_0}{a}u_1\sin x_4$$

$$u_3 = a\ddot{x}_5 + L_0 u_1 \sin x_4$$

reuse $u_2 - u_3$, and $\ddot{u}_1$ get before.

4) $y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

a)

$$\ddot{y} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{bmatrix} -\sin x_3 \cdot u_2 x_4^2 + \beta\cos x_3 \cdot u_2 \cdot x_4^2 + \cos x_3 u_1 + \beta \sin x_3 u_1 \\ \cos x_3 \cdot u_2 x_4^2 + \beta \sin x_3 \cdot u_2 \cdot x_4^2 + \sin x_3 u_1 - \beta\cos x_3 u_1 \end{bmatrix}$$

Virtual input v

$$= \underbrace{\begin{bmatrix} \cos x_3 + \beta\sin x_3 & -\sin x_3 \cdot x_4^2 + \beta\cos x_3 x_4^2 \\ \sin x_3 - \beta\cos x_3 & \cos x_3 x_4^2 + \beta\cos x_3 x_4^2 \end{bmatrix}}_{\text{denote this as } A} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

let error dynamic: $e = y - y_d$

let $\begin{pmatrix} \dot{e} \\ \ddot{e} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -k_p & -k_d \end{pmatrix} \begin{pmatrix} e \\ \dot{e} \end{pmatrix}$ k_p, k_d and p.d and symmetric

then, $v = \ddot{y}_d - k_p(y - y_d) - k_d(\dot{y} - \dot{y}_d)$

and then $u = A^{-1} \cdot v$

4b) $\delta = \begin{pmatrix} (\beta - \hat{\beta}) \sin x_3 \cdot x_4 \\ (\hat{\beta} - \beta) \cos x_3 \cdot x_4 \end{pmatrix} = (\beta - \hat{\beta}, \hat{\beta} - \beta) \cdot \begin{pmatrix} \sin x_3 \\ \cos x_3 \end{pmatrix} \cdot x_4$

$$(\beta - \hat{\beta}, \hat{\beta} - \beta) \cdot \begin{pmatrix} \sin x_3 \\ \cos x_3 \end{pmatrix} \leq \|(\beta - \hat{\beta}, \hat{\beta} - \beta)\| \cdot \|(\sin x_3, \cos x_3)\|$$

$$= \left|\sqrt{2}(\beta - \hat{\beta})\right|$$

So. $\|\delta\| \leq \underbrace{\left|\sqrt{2}(\beta - \hat{\beta})\right|}_{a_1} \cdot |x_4|$

5) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix}}_{g_0} + \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{g_1} u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{g_2} u_2$$

$$[g_1, g_2] = -\begin{bmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{bmatrix} = g_3$$

$\det([g_1 \; g_2 \; g_3]) = 1$ LARC of degree 2

there is no bad brackets of degree ≤ 2

so LARC is satisfied, the system is STLC