EN530.678 Nonlinear Control and Planning in Robotics
Sample Midterm Quiz #1

March 22, 2021

Prof: Marin Kobilarov

1. Consider the system with state $x \in \mathbb{R}^2$ and dynamics given by

$$\dot{x}_1 = x_1 - x_1^3 + x_2$$
$$\dot{x}_2 = 3x_1 - x_2$$

   (a) Find all equilibria and determine their stability

   (b) Consider one of the stable equilibria. Estimate its region of attraction, e.g. by finding a Lyapunov function $V(x)$ and estimating a scalar $c > 0$ for which stability holds from all x such that $V(x) < c$.

   Hint: when the equilibrium is not at the origin, i.e. it is a point $x^* \neq 0$ there are two ways to apply our standard zero-origin stability analysis. The first approach is to "translate" the system by expressing the problem in terms of new coordinates $y = x - x^*$ for which the equilibirum is $y^* = 0$. The second is to choose a Lyapunov function based on the difference from the equilibrium, such as $V(x) \propto \|x - x^*\|^2$.

2. Consider a system with state $x \in \mathbb{R}^3$ and distribution given by the two vector fields

$$g_1 = \begin{pmatrix} 2x_3 \\ -1 \\ 0 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} -x_1 \\ -2x_2 \\ x_3 \end{pmatrix}$$

   Show that the distribution is integrable and find the integrable manifolds. What is the dimension of the integrable manifolds for the cases when: i) $x = 0$; ii) $x_3 = 0$, iii) $x_3 \neq 0$

3. Consider a planar rigid body with two forces described by the equations of motion (assuming unit mass and unit moment of inertia)

$$\ddot{x}_1 = \cos x_3 u_1 - \sin x_3 u_2,$$
$$\ddot{x}_2 = \sin x_3 u_1 + \cos x_3 u_2,$$
$$\ddot{x}_3 = -d u_2,$$

   for some constant $d > 0$.

   (a) Show that the system has flat outputs

$$y_1 = x_1 + \frac{1}{d}\cos x_3, \quad y_2 = x_2 + \frac{1}{d}\sin x_3$$

   by expressing the state and inputs as functions of $y = (y_1, y_2)$ and its derivatives.

(b) Give the details of a procedure for computing the required inputs to generate a trajectory reaching a desired final state $x_f(T)$ at a given time T.

4. Consider a satellite equipped with a body-fixed antenna pointing along a spatial direction defined by the unit vector $b(t) \in \mathbb{R}^3$. The body rotates with angular velocity $\omega(t) \in \mathbb{R}^3$ and is controlled using torque inputs $u(t) \in \mathbb{R}^3$. The dynamics is given by

$$\begin{aligned} \dot{b} &= b \times \omega, \\ I\dot{\omega} &= (I\omega) \times \omega + u. \end{aligned}$$

The goal is to align the satellite antenna (i.e. the vector $b(t)$) with a desired direction defined by the constant unit vector $b_0 \in \mathbb{R}^3$. Regarding $x \triangleq (b, \omega) \in \mathbb{R}^6$ as the system state, show that the control law

$$u = -k\omega - b \times b_0,$$

asymptotically stabilizes the system at the zero-velocity equilibrium $x_0 = (b_0, 0)$. To do this consider the Lyapunov function

$$V(x) = \frac{1}{2}\omega^T I \omega + 1 - b^T b_0.$$

Hint: in the derivation use the skew-symmetry property that $a^T(b \times a) = 0$ as well the property that $(a \times b)^T c = a^T(b \times c)$, for any $a, b, c \in \mathbb{R}^3$.