1)

a) $y = x_3$ so $\dot{y} = \dot{x}_3 = -x_1 + u \Rightarrow u = x_1 + \dot{x}_3$ We can take $x_1 = a(x)$ $\dot{x}_3 = v$

then $u = a(x) + b(x)v$ $b(x) = 1$

the system has 1 relative degree, the relationship between u and v is linear

So the system is input-output linearizable

b) $y = h(x) = x_3$ We need to find $\Phi_2(x)$ and $\Phi_3(x)$ that $\begin{bmatrix} h(x) \\ \Phi_2(x) \\ \Phi_3(x) \end{bmatrix}$ is invertible

and $\frac{\partial \Phi_2}{\partial x} \cdot g = \frac{\partial \Phi_3}{\partial x} \cdot g = 0$ in this case $g(x) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}^T$

So we have: $\frac{\partial \Phi_2}{\partial x_2} + \frac{\partial \Phi_2}{\partial x_3} = 0$ $\frac{\partial \Phi_3}{\partial x_2} + \frac{\partial \Phi_3}{\partial x_3} = 0$

We can set $\Phi_2 = x_1$ $\Phi_3 = x_2 - x_3$

So the normal form is:

$$\Phi(x) = \begin{bmatrix} z_1 = x_3 \\ \eta_1 = x_1 \\ \eta_2 = x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ which is a global diffeomorphism

$$\dot{\Phi}(x) \begin{bmatrix} \dot{z}_1 = -x_1 + u \\ \dot{\eta}_1 = -x_1 + x_2 - x_3 \\ \dot{\eta}_2 = -x_1 x_3 - x_2 + x_1 \end{bmatrix}$$

c) zero dynamics:

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

A's eigenvalues are -2 and 0 so A is not Hurwitz

and the origin is stable but not asymptotically stable.

Therefore, the system is not minimum phase

2) $\dot{x}_1 = -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3, \quad \dot{x}_2 = u$

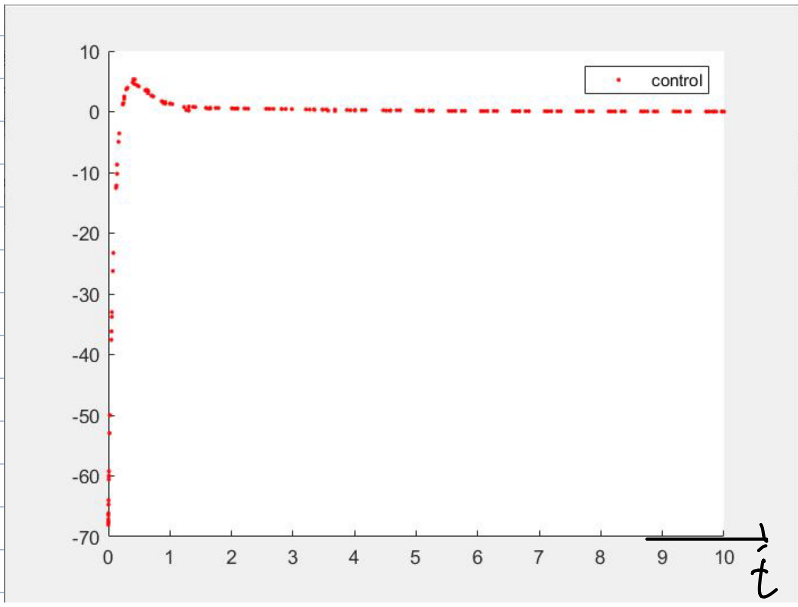
a) $\ddot{x}_1 = -\dot{x}_2 - 3x_1 \cdot \dot{x}_1 - \frac{3}{2}x_1^2 \cdot \dot{x}_1 = -u - (3x_1 + \frac{3}{2}x_1^2) \cdot \dot{x}_1$

$y = x_1$ $\quad u = -\ddot{x}_1 - (3x_1 + \frac{3}{2}x_1^2) \cdot \dot{x}_1$ $\quad V = \ddot{x}_1$ (virtual input)

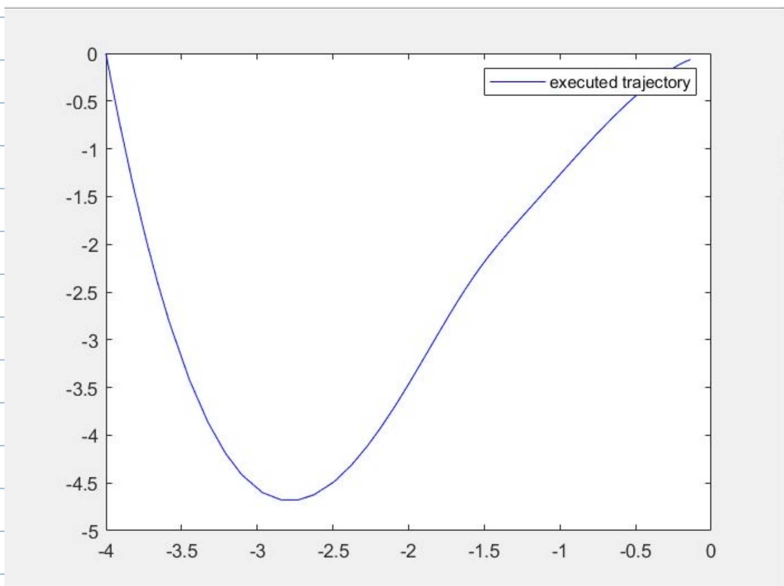
$V = \ddot{y}_d - k_d(\dot{y} - \dot{y}_d) - k_p(y - y_d)$ $\quad k_d, k_p > 0$

$z = \begin{pmatrix} y - y_d \\ \dot{y} - \dot{y}_d \end{pmatrix} \quad \dot{z} = Az \quad A = \begin{pmatrix} 0 & 1 \\ -k_p & -k_d \end{pmatrix}$

b)



plot of control



trajectory
$(-4, 0) \rightarrow (0, 0)$

3)

$$I\ddot{q}_1 + Mgl\sin q_1 + k(q_1 - q_2) = 0$$
$$J\ddot{q}_2 + k(q_2 - q_1) = u.$$

$\Downarrow$

$$\ddot{q}_1 = -I^{-1}\cdot MgL\cdot\sin q_1 + I^{-1}\cdot k(q_2 - q_1) \Rightarrow q_1^{(3)} = -I^{-1}MgL\cos q_1\cdot\dot{q}_1 + I^{-1}k(\dot{q}_2 - \dot{q}_1)$$

$$q_1^{(4)} = I^{-1}k(\ddot{q}_2 - \ddot{q}_1) + I^{-1}MgL\sin q_1\cdot\dot{q}_1^2 - I^{-1}MgL\cos q_1\cdot\ddot{q}_1$$

$$= \frac{k}{I}\left(\frac{1}{J}u - \frac{k}{J}(q_2 - q_1)\right) + I^{-1}MgL\sin q_1\cdot\dot{q}_1^2 - I^{-1}mgL\cos q_1\left(-I^{-1}\cdot MgL\cdot\sin q_1 + I^{-1}\cdot k(q_2 - q_1)\right)$$

$$-\frac{k}{IJ}u = -q_1^{(4)} - \frac{k^2}{IJ}(q_2 - q_1) + I^{-1}MgL\sin q_1\cdot\dot{q}_1^2 - I^{-1}mgL\cos q_1\left(-I^{-1}\cdot MgL\cdot\sin q_1 + I^{-1}\cdot k(q_2 - q_1)\right)$$

So

$$u = \frac{IJ}{k}\cdot q_1^{(4)} - \frac{IJ}{k}\cdot\left[-\frac{k^2}{IJ}(q_2 - q_1) + I^{-1}MgL\sin q_1\cdot\dot{q}_1^2 - I^{-1}mgL\cos q_1\left(-I^{-1}\cdot MgL\cdot\sin q_1 + I^{-1}\cdot k(q_2 - q_1)\right)\right]$$

let $x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$, this part is a function of x, denote as $\alpha(x)$

so $u = \alpha(x) + \beta(x)\cdot v$ $\quad \beta(x) = \frac{IJ}{k}$ $\quad v = q_1^{(4)}$ (virtual input)

2)

let $\dot{z} = Az$ $\quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 & -k_4 \end{pmatrix}$ $\quad v = y_d^{(4)} - k_1(y - y_d) - k_2(\dot{y} - \dot{y}_d) - k_3(\ddot{y} - \ddot{y}_d) - k_4(y^{(3)} - y_d^{(3)})$

then with $u = \alpha(x) + \beta(x)\cdot v$,

we can get u.

A need to be hurtwiz, we need the real part of all eigenvalues < 0

$\det(A - \lambda I) = \lambda^4 + k_4\lambda^3 + k_3\cdot\lambda^2 + k_2\cdot\lambda + k_1$ which is a quartic equation of one unknown

for a quartic equation of one unknown: $Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$

Ferrari's method: (from wiki)

extra point

$$\alpha = -\frac{3B^2}{8A^2} + \frac{C}{A} \qquad \beta = \frac{B^3}{8A^3} - \frac{BC}{2A^2} + \frac{D}{A}$$

$$\gamma = -\frac{3B^4}{256A^4} + \frac{CB^2}{16A^3} - \frac{BD}{4A^2} + \frac{E}{A}$$

if $\beta = 0$ $\quad x = -\frac{B}{4A} \pm_s \sqrt{\frac{-\alpha \pm_t \sqrt{\alpha^2 - 4\gamma}}{2}}$

Otherwise.

$$P = -\frac{\alpha^2}{12} - \gamma \qquad Q = -\frac{\alpha^3}{108} + \frac{\alpha\gamma}{3} - \frac{\beta^2}{8}$$

$$R = -\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}} \qquad U = \sqrt[3]{R}$$

$$y = -\frac{5}{6}\alpha + \begin{cases} U = 0 & \rightarrow -\sqrt[3]{Q} \\ U \neq 0 & \rightarrow U - \frac{P}{3U} \end{cases} \qquad W = \sqrt{\alpha + 2y}$$

$$x = -\frac{B}{4A} \pm_s \frac{W \pm_t \sqrt{-\left(3\alpha + 2y \pm_s \frac{2\beta}{W}\right)}}{2}$$

$\pm_s$ should have same sign.

so in order to make all the real part of eigenvalues < 0

(in our case, $A = 1$) $\qquad$ assume $k_1, k_2, k_3, k_4 > 0$

so $\quad -\frac{B}{4} \pm W$ is smaller than 1.

$$\text{so } -\frac{B}{4} + \sqrt{-\frac{3B^2}{8} + C - \frac{5}{3} \cdot \left(-\frac{3}{8}B^2 + C\right)} < 0$$

$$\frac{B^2}{4} - \frac{2}{3}C < \frac{B^2}{16}$$

$$C > \frac{9}{32}B^2$$

So $k_4 > 0$ and $k_3 > \frac{9}{32} k_4$

4.

a) let $\frac{\tan u_1}{L} = u_1'$ since $\tan u_1 \in (-\infty, \infty)$ so every u_1 can map to a unique u_1'

let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $u' = \begin{pmatrix} u_1' \\ u_2 \end{pmatrix}$ $u_1 = \text{atan}(u_1' \cdot L)$

So once we have u' we can have u.

let $x = x_1$ $y = x_2$ $\theta = x_3$ $v = x_4$

then
$$\dot{x}_1 = x_4 \cdot \cos x_3 \qquad \ddot{x}_1 = \dot{x}_4 \cos x_3 + x_4 \cdot (-\sin x_3) \dot{x}_3$$
$$\dot{x}_2 = x_4 \cdot \sin x_3 \qquad \ddot{x}_2 = \dot{x}_4 \sin x_3 + x_4 \cdot \cos x_3 \cdot \dot{x}_3$$
$$\dot{x}_3 = u_1' \cdot x_4 \quad \Rightarrow \quad \ddot{y} = \begin{bmatrix} -x_4^2 \sin x_3 & \cos x_3 \\ x_4^2 \cos x_3 & \sin x_3 \end{bmatrix} \cdot \begin{pmatrix} u_1' \\ u_2 \end{pmatrix}$$
$$\dot{x}_4 = u_2$$
$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad u' = \begin{bmatrix} -\frac{1}{x_4^2} \sin x_3 & \frac{1}{x_4^2} \cos x_3 \\ \cos x_3 & \sin x_3 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix}}_{V \text{ (virtual input)}}$$

$$V = \ddot{y}_d - k_d(\dot{y} - \dot{y}_d) - k_p(y - y_d) \qquad k_d, k_p > 0$$

then we can get u', then u

b)