1) at $x_0 = [0,0,0,0,0]^T$, $g_0(x_0) = 0$,
$u$ is open and its convex hull contains 0.

$$g_3 = [g_0, g_1] = \begin{pmatrix} v\cos\theta\sin\phi \\ v\sin\theta\sin\phi \\ -v\cdot\cos\phi/L \\ 0 \\ 0 \end{pmatrix} \quad g_4 = [g_0, [g_0, g_1]] = \begin{pmatrix} -v^2\sin\theta/L \\ v^2\cos\theta/L \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$g_5 = [g_0, [g_0, [g_0, g_1]]] = \begin{pmatrix} -v^3\cos\theta\cdot\sin\phi/L^2 \\ -v^3\sin\theta\sin\phi/L^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det([g_1, g_2, g_3, g_4, g_5]) = -\frac{v^6\cdot\sin(2\cdot\phi)}{2L^4} \Rightarrow \text{ so LARC of degree 4}$$

the bad brackets of degree $\leq 4$ are:

$$[g_1, [g_0, g_1]] = \begin{pmatrix} v\cos\theta\cos\phi \\ v\cdot\sin\theta\cos\phi \\ v\cdot\sin\phi/L \\ 0 \\ 0 \end{pmatrix} \overset{\Delta}{=} -v\cdot[g_0, g_2] \qquad [g_2, [g_0, g_2]] = 0$$

Hence, the system is STLC, so LA and controllable

2) $L(w) = \frac{1}{2}(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2) + \frac{1}{2} J_r (w_1+u_1)^2 + \frac{1}{2} J_r (w_2+u_2)^2$

a) $\partial_w L = \begin{pmatrix} J_1 w_1 + J_r(w_1+u_1) \\ J_2 w_2 + J_r (w_2+u_2) \\ J_3 w_3 \end{pmatrix}$

Since $\frac{d}{dt} \partial_w L = \partial_w L \times w = -\hat{w}\, \partial_w L$

so $\partial_w L = C \cdot e^{-\int_0^t \hat{w}(t)\, dt}$. Since $\partial_w L(0)=0 \Rightarrow C=0$ so $\partial_w L = 0$

$$\begin{pmatrix} J_1 w_1 + J_r(w_1+u_1) \\ J_2 w_2 + J_r(w_2+u_2) \\ J_3 w_3 \end{pmatrix} = 0 \Rightarrow w_1 = -\frac{J_r}{J_1+J_r} u_1 \quad w_2 = -\frac{J_r}{J_2+J_r} u_2$$

so $$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{J_r}{J_1+J_r} \\ 0 \\ 0 \end{pmatrix}}_{b_1} u_1 + \underbrace{\begin{pmatrix} 0 \\ -\frac{J_r}{J_2+J_r} \\ 0 \end{pmatrix}}_{b_2} u_2$$

b)

Since $R \cdot R^T = I$ and $\dot{R} = R \cdot \hat{w}$, $w = b_1 u_1 + b_2 u_2$, so $R^T \cdot \dot{R} = \hat{w}$

$\dot{R}_{11} = -\sin\beta\cos\gamma \cdot \dot{\beta} - \cos\beta \cdot \sin\gamma \cdot \dot{\gamma}$

$\dot{R}_{12} = \sin\beta \sin\gamma \cdot \dot{\beta} - \cos\beta\cos\gamma \cdot \dot{\gamma}$

$\dot{R}_{13} = \cos\beta \cdot \dot{\beta}$

$\dot{R}_{21} = (\cos\alpha \sin\beta \cos\gamma - \sin\alpha \sin\gamma) \cdot \dot{\alpha} + \sin\alpha \cos\beta \cos\gamma \cdot \dot{\beta} + (\cos\alpha \cdot \cos\gamma - \sin\alpha \sin\beta \sin\gamma)\dot{\gamma}$

$\dot{R}_{22} = (-\sin\alpha \cos\gamma - \cos\alpha \cdot \sin\beta \sin\gamma) \cdot \dot{\alpha} - \sin\alpha \cos\beta \sin\gamma \cdot \dot{\beta} - (\cos\alpha \sin\gamma + \sin\alpha \sin\beta \cos\gamma)\dot{\gamma}$

$\dot{R}_{23} = -\cos\alpha \cos\beta \cdot \dot{\alpha} + \sin\alpha \sin\beta \cdot \dot{\beta}$

$\dot{R}_{31} = (\cos\alpha \cdot \sin\gamma + \sin\alpha \sin\beta \cos\gamma) \cdot \dot{\alpha} - \cos\alpha \cos\beta \cos\gamma \cdot \dot{\beta} + (\sin\alpha \cos\gamma + \cos\alpha \sin\beta \sin\gamma) \cdot \dot{\gamma}$

$\dot{R}_{32} = (\cos\alpha \cos\gamma - \sin\alpha \sin\beta \sin\gamma) \cdot \dot{\alpha} + \cos\alpha \cos\beta \sin\gamma \cdot \dot{\beta} + (\cos\alpha \sin\beta \cos\gamma - \sin\alpha \sin\gamma) \cdot \dot{\gamma}$

$\dot{R}_{33} = -\sin\alpha \cos\beta \cdot \dot{\alpha} - \cos\alpha \sin\beta \cdot \dot{\beta}$

$$R^T \cdot \dot{R} = \begin{pmatrix} 0 & -\dot{\gamma} - \dot{\alpha}\cdot\sin\beta & \dot{\beta}\cdot\cos\gamma - \dot{\alpha}\cos\beta\sin\gamma \\ \dot{\gamma} + \dot{\alpha}\cdot\sin\beta & 0 & -\dot{\beta}\sin\gamma - \dot{\alpha}\cos\beta\cos\gamma \\ -\dot{\beta}\cdot\cos\gamma + \dot{\alpha}\cos\beta\sin\gamma & \dot{\beta}\sin\gamma + \dot{\alpha}\cos\beta\cos\gamma & 0 \end{pmatrix}$$

$$\text{so} \begin{pmatrix} \dot{\beta}\sin\gamma + \dot{\alpha}\cos\beta\cos\gamma \\ \dot{\beta}\cdot\cos\gamma - \dot{\alpha}\cos\beta\sin\gamma \\ \dot{\gamma} + \dot{\alpha}\cdot\sin\beta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\beta\cos\gamma & \sin\gamma & 0 \\ -\cos\beta\sin\gamma & \cdot\cos\gamma & 0 \\ \sin\beta & 0 & 1 \end{pmatrix}}_{\substack{\parallel \\ T}} \cdot \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = b_1 u_1 + b_2 u_2$$

$$T^{-1} = \begin{pmatrix} \frac{\cos\gamma}{\cos\beta} & -\frac{\sin\gamma}{\cos\beta} & 0 \\ \sin\gamma & \cos\gamma & 0 \\ -\frac{\sin\beta\cos\gamma}{\cos\beta} & \frac{\sin\beta\sin\gamma}{\cos\beta} & 1 \end{pmatrix}, \text{ then } \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = -\underbrace{\frac{J_r}{J_1+J_r}\begin{pmatrix} \frac{\cos\gamma}{\cos\beta} \\ \sin\gamma \\ -\frac{\sin\beta\cos\gamma}{\cos\beta} \end{pmatrix}}_{g_1} u_1 - \underbrace{\frac{J_r}{J_2+J_r}\begin{pmatrix} -\frac{\sin\gamma}{\cos\beta} \\ \cos\gamma \\ \frac{\sin\beta\sin\gamma}{\cos\beta} \end{pmatrix}}_{g_2} u_2$$

$$[g_1, g_2] = \frac{J_r^2}{(J_1+J_r)(J_2+J_r)} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdots g_3$$

$$\det([g_1, g_2, g_3]) = \frac{J_r^4}{\cos\beta \cdot (J_1+J_r)^2 (J_2+J_r)^2}$$

so $[g_1\ g_2\ g_3]$ is invertible when $\cos\beta \neq 0$,

which means when $\cos\beta \neq 0$ the driftless system is LA $\Leftrightarrow$ controllable $\Leftrightarrow$ STLC

3) equation of motion:

$$m\ddot{x} = (u_1+u_2)\sin\theta$$

$$m\ddot{y} = (u_1+u_2)\cos\theta - mg$$

$$J\ddot{\theta} = (u_2-u_1)r$$

let $q = \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

then: $\theta = \arcsin\left( \dfrac{m\ddot{q}_1}{\sqrt{(m\ddot{q}_1)^2 + (m\ddot{q}_2 + mg)^2}} \right)$ except when $\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$

and we can get $\ddot{\theta}$ except when $\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$

$$u_2 = \frac{1}{2}\left( \frac{J\ddot{\theta}}{r} + \sqrt{(m\ddot{q}_1)^2 + (m\ddot{q}_2 + mg)^2} \right)$$

$$u_1 = \frac{1}{2}\left( \sqrt{(m\ddot{q}_1)^2 + (m\ddot{q}_2 + mg)^2} - \frac{J\ddot{\theta}}{r} \right)$$

thus we have $q = \begin{bmatrix} x \\ y \end{bmatrix}$, $\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \varphi = (q, \ddot{q})$, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha(\ddot{q}, q^{(4)})$

so the system is differentially flat

4) we denote the point between the rear wheels as $\underline{q}$

$$\underline{q} = \begin{pmatrix} q_1 = x - d\cdot\cos\theta_1 \\ q_2 \quad y - d\cdot\sin\theta_1 \end{pmatrix} \text{ then } \dot{\underline{q}} = \begin{pmatrix} \dot x + d\cdot\sin\theta_1\cdot\dot\theta_1 \\ \dot y - d\cdot\cos\theta_1\cdot\dot\theta_1 \end{pmatrix} = \begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix}$$

$$\ddot{\underline{q}} = \begin{pmatrix} \ddot x + d(\sin\theta_1\cdot\ddot\theta_1 + \cos\theta_1\cdot\dot\theta_1^2) \\ \ddot y - d(\cos\theta_1\cdot\ddot\theta_1 - \sin\theta_1\cdot\dot\theta_1^2) \end{pmatrix} = \begin{pmatrix} \ddot q_1 \\ \ddot q_2 \end{pmatrix}$$

$$\dot q_1 = \dot x + d\cdot\sin\theta_1\cdot\dot\theta_1 = \cos\theta\, u_1 + \sin\theta_1\cdot\sin(\theta-\theta_1)\cdot u_1 = \cos(\theta-\theta_1)\cos\theta_1\, u_1$$

$$\dot q_2 = \dot y - d\cdot\cos\theta_1\cdot\dot\theta_1 = \sin\theta\, u_1 - \cos\theta_1\cdot\sin(\theta-\theta_1)\cdot u_1 = \sin\theta_1\cos(\theta-\theta_1)\, u_1$$

So $\theta_1 = \mathrm{atan}(\dot q_1/\dot q_2)$, and we can get $\dot\theta_1$ and $\ddot\theta_1$ when $\dot q_2 \neq 0$

$x = q_1 + d\cdot\cos\theta_1$ $\qquad$ $y = q_2 + d\cdot\sin\theta_1$

$\dot x = \dot q_1 - d\cdot\sin\theta_1\cdot\dot\theta_1$ $\qquad$ $\dot y = \dot q_2 + d\cdot\sin\theta_1\,\dot\theta_1$

$\ddot x = \ddot q_1 - d(\sin\theta_1\cdot\ddot\theta_1 + \cos\theta_1\cdot\dot\theta_1^2)$ $\qquad$ $\ddot y = \ddot q_2 + d\cdot(\cos\theta_1\cdot\ddot\theta_1 - \sin\theta_1\cdot\dot\theta_1^2)$

$u_1 = \sqrt{\dot x^2 + \dot y^2}$ $\quad$ $\dot u_1 = \frac{1}{2}\frac{1}{\sqrt{\dot x^2+\dot y^2}}\cdot(2\dot x\ddot x + 2\dot y\ddot y)$ when $\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$

$\dot x = \cos\theta\cdot u_1 \Rightarrow \theta = \arccos\left(\frac{\dot x}{u_1}\right) \Rightarrow \dot\theta = -\frac{1}{\sqrt{1-\left(\frac{\dot x}{u_1}\right)^2}}\cdot\frac{\ddot x u_1 - \dot x\dot u_1}{u_1^2}$

when $\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$

$\dot\theta = \frac{\tan\phi}{L}\cdot u_1 \Rightarrow \tan\phi = \frac{\dot\theta\cdot L}{u_1}$

so $\phi = \mathrm{atan}\left(\frac{\dot\theta\cdot L}{u_1}\right) \Rightarrow u_2 = \dot\phi = \frac{1}{1+\left(\frac{\dot\theta\cdot L}{u_1}\right)^2}\cdot\frac{L(\ddot\theta u_1 - \dot\theta\dot u_1)}{u_1^2}$

so $(x, y, \theta)$, could be represented by $\varphi(q, \dot q)$

$(u_1, u_2)$ could be represent by $\alpha(q, \dot q, \ddot q, q^{(3)})$

Hence the system is differentially flat