

Problem 1

a) When $\dot{x} = \vec{0}$, $\dot{x}_3 = 0$, then $x_3 = 1$ or 0

If $x_3 = 0$ then $\dot{x}_1 = 1$. So x_3 can only be 1.

$x_3 = 1$, then $\dot{x}_1 = -x_2 + 1$, so $x_2 = 1 \Rightarrow \dot{x}_2 = x_1 - 1 \Rightarrow x_1 = 1$

so the system only has one equilibrium point $(1, 1, 1)$

b) when $x = (1, 1, 1)$

$$A = Df \Big|_{x=(1,1,1)} = \begin{bmatrix} 0 & -x_2 & -x_2 \\ x_3 & -1 & x_1 \\ 0 & 0 & 2x_3 - 3x_3^2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The eigenvalues are: $-1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Since $\operatorname{Re} \lambda_i(A) < 0$, then the equilibrium point is asymptotically stable

When $x_3 = 0$, $\dot{x}_1 = 1$, $\dot{x}_2 = -x_2$. Since $\dot{x}_1 = 1$, which is a positive constant so x_1 will keep increasing. Therefore, the equilibrium is not globally asymptotically stable



2) With $U_1 = U_2 = U_3 = 0$ $W = 0$

a) Let $V(W) = \frac{1}{2} (J_1 W_1^2 + J_2 W_2^2 + J_3 W_3^2)$? $V(W)$ is p.d.

$$\dot{V} = \left[(J_2 - J_3) + (J_3 - J_1) + (J_1 - J_2) \right] W_1 W_2 W_3 = 0$$

So, the origin is stable

b) Since \dot{V} is always zero, it's not ? asymptotically stable

c) Let $V(W) = \frac{1}{2} (J_1 W_1^2 + J_2 W_2^2 + J_3 W_3^2)$ $V(W)$ is p.d.
with $U_i = -k_i W_i$

$$\dot{V}(W) = -k_1 W_1^2 - k_2 W_2^2 - k_3 W_3^2, \text{ so } \dot{V} \text{ is negative definite}$$

Hence, the origin of the system is globally asymptotically stable.



3) with $u=0$

a) $V = \frac{1}{2} \dot{q}^T M(q) \dot{q} + P(q)$, since $M(q)$ is pd. and $P(q)$ is pd.
then V is pd.

$$\dot{V} = \dot{q}^T M(q) \ddot{q} + \frac{1}{2} \dot{q}^T M(q) \dot{q} + g(q)^T \cdot \dot{q}$$

$$= \frac{1}{2} \dot{q}^T M(q) \dot{q} - \dot{q}^T (C(q, \dot{q}) \dot{q} + D \dot{q} + g(q)) + g(q)^T \cdot \dot{q}$$

$$= \frac{1}{2} \dot{q}^T (M(q) - 2C) \dot{q} - \dot{q}^T D \dot{q} - \dot{q}^T g(q) + g(q)^T \cdot \dot{q}$$

Since $M-2C$ is skew-symmetric, then $\dot{q}^T (M(q) - 2C) \dot{q} = 0$

so $\dot{V} = -\dot{q}^T D \dot{q}$. and D is p.s.d.

Hence, the origin is stable

b) With $u = -k_d \dot{q}$, use V in (a) V is pd.

then, we have $\dot{V} = -\dot{q}^T (k_d I + D) \dot{q}$

$$= -\dot{q}^T k_d \dot{q} - \dot{q}^T D \dot{q}, \text{ so } \dot{V} \leq 0$$

when $\dot{V} = 0$, then $\dot{q} = 0$, so $\ddot{q} = 0$.

according to dynamics $g(q) = 0$, so $q = 0$

Since the \dot{V} is 0 only when $q = 0$

by LaSalle's principle, the origin is asymptotically stable

3c) With $u = g(q) - k_p(q - q^*) - k_d \dot{q}$

We have: $M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + k_p(q - q^*) - k_d \dot{q} = 0$

When $q = q^*$, $\dot{q} = 0$. the dynamic system is 0

so $(q^*, 0)$ is an equilibrium point.

let $e = q - q^*$, then $\dot{e} = \dot{q}$ (q^* is constant)

let $V = \frac{1}{2} \dot{e}^T M(q) \dot{e} + \frac{1}{2} e^T K_p e$, which is p.d

$$\begin{aligned}\dot{V} &= \dot{q}^T M(q) \ddot{q} + \frac{1}{2} \dot{q}^T M(q) \dot{q} + e^T K_p \dot{q} \\ &= \frac{1}{2} \dot{q}^T M(q) \dot{q} - \dot{q}^T C(q) \dot{q} - \dot{q}^T K_p e - \dot{q}^T K_d e - \dot{q}^T D \dot{q} + e^T K_p \dot{q} \\ &= \frac{1}{2} \dot{q}^T (M - 2C) \dot{q} - \dot{q}^T K_p e - \dot{q}^T (K_d + D) \dot{q} + \dot{q}^T K_p e \\ &= -\dot{q}^T (K_d + D) \dot{q} = -\dot{e}^T (K_d + D) \dot{e} \quad \text{so } \dot{V} \leq 0\end{aligned}$$

When $\dot{V} = 0$, $\dot{e} = 0 \Rightarrow \dot{q} = 0$ so $\ddot{q} = 0$

from the dynamics: $k_p(q - q^*) = 0$ so $q = q^*$

Since $\dot{V} = 0$ only when $\dot{q} = 0$ and $q = q^*$

according to LaSalle's principle, the point $(q^*, 0)$ is an asymptotically stable equilibrium point.

4) from equation of motion: $\ddot{q} = M^{-1} \cdot B(q) \cdot u - m^{-1} D \cdot \dot{q}$

a) with $q^* = (0, 0)$, then, $e = q - q^* = q$

the error dynamic: $\frac{d}{dt} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{bmatrix} 0 & I \\ -k_p & -k_d \end{bmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$

let $k_d = m^{-1} D$, since $m^{-1} D$ is p.d and symmetric. //

then we need let $-k_p \cdot q = M^{-1} B(q) u$. denote as A

$$M^{-1} B(q) = \begin{bmatrix} \frac{\cos \theta}{m} & \frac{\cos \theta}{m} & -\frac{\sin \theta}{m} \\ \frac{\sin \theta}{m} & \frac{\sin \theta}{m} & \frac{\cos \theta}{m} \\ -\frac{r}{J} & \frac{r}{J} & 0 \end{bmatrix} \quad B = \begin{bmatrix} \cos \theta & \cos \theta & -\sin \theta \\ \sin \theta & \sin \theta & \cos \theta \\ -r & r & 0 \end{bmatrix}$$

$$\text{let } u = \begin{bmatrix} \frac{\cos \theta}{2} & \frac{\sin \theta}{2} & -\frac{1}{2r} \\ -\frac{\cos \theta}{2} & \frac{\sin \theta}{2} & \frac{1}{2r} \\ -\sin \theta & \cos \theta & 0 \end{bmatrix} \cdot q$$

then $M^{-1} B(q) \cdot u = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{J} \end{bmatrix} \cdot (-q)$

this matrix is p.d and symmetric

Since $x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$, $\dot{x} = Ax$

obviously. $q^*(0, 0)$ is an equilibrium point

and A is Hurwitz (theorem 7), so the system can stabilize at the origin $q^*(0, 0)$

Hence the controller is exponentially stable (theorem 7)

\hookrightarrow denote $B(q)u$ as $k \cdot q$ denote the combination of D and steering force as matrix k_d (they are related to \dot{q})

let $V = \frac{1}{2} (q^T k q + \dot{q}^T M \dot{q})$ V is pd. $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$

$$\begin{aligned} \dot{V} &= \dot{q}^T k q + \dot{q}^T M \ddot{q} \\ &= -\dot{q}^T k q + \dot{q}^T (-k q - k_d \cdot \dot{q}) = -\dot{q}^T (k_d) \cdot \dot{q} \end{aligned}$$

when the robot doesn't apply the 'steering' force

$k_d = D$, which is p.d and symmetric so $\dot{V} \leq 0$

When steering force applies : k_d becomes:

since k_d is on diagonal
and $\frac{k_0}{d(q)} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is skew-symmetric

$$X^T S X = 0 \text{ for } \forall X \text{ if } S \text{ is skew-symmetric}$$

$$\begin{bmatrix} dx & -\frac{k_0}{d(q)} dy & 0 \\ \frac{k_0}{d(q)} dy & dy & 0 \\ 0 & 0 & d\theta \end{bmatrix}$$

so for $\forall x \in \mathbb{R}^3$ $x^T k_d x'$ will remain the same
so $\dot{V} \leq 0$

from a) we know that $B(q) \cdot u = -q$

then the motion function is: $M \cdot \ddot{q} + D \cdot \dot{q} + q = 0$

when $\dot{q} = 0, \ddot{q} = 0 \Rightarrow q = 0$

according to LaSalle's principle, the origin is asymptotically stable
since V is radially unbounded, the system is globally AS stable

5 let $V(x) = \frac{1}{2}x^T x$, then $\frac{1}{2}\|x\|^2 \leq V(x) \leq \frac{1}{2}\|x\|^2 \dots \textcircled{1}$

$$\dot{V} = x^T \dot{x} = -\alpha [x^T x + x^T S(x)x + (x^T x)^2]$$

since $S(x)$ is skew-symmetric, then $x^T S(x)x = 0$

$$\begin{aligned} \dot{V} &= -\alpha [I + x^T x] \cdot x^T x \\ &= -\alpha x^T x - \alpha x^T x \cdot x^T x \end{aligned}$$

since $x^T x \geq 0$, then $\dot{V} \leq -\alpha x^T x \dots \textcircled{2}$

$$\left\| \frac{\partial V}{\partial x} \right\| = \|x\| \leq \|x\| \dots \textcircled{3}$$

from \textcircled{1} : let $d_1 = d_2 = \frac{1}{2}$



\textcircled{2} : let $d_3 = \alpha$

\textcircled{3} : let $d_4 = 1$

Using theorem 6: exponential stability theorem
the origin is globally exponentially stable.