Problem 1 M: the ellipsoidal shell in $\mathbb{R}^3$ given $x^2+y^2+4z^2=1$

chart 1: for $U = M \setminus (0,0,\frac{1}{2})$

let φ be a stereographic projection, with $(0,0,\frac{1}{2})$ as its north pole.

$$\varphi([x,y,z]^T) = \left[\frac{x}{1-2z}, \frac{y}{1-2z}\right]^T = [A,B]^T$$

$$\varphi^{-1}([A\ B]^T) = \left[\frac{2A}{1+A^2+B^2}, \frac{2B}{1+A^2+B^2}, \frac{-1+A^2+B^2}{(1+A^2+B^2)2}\right]^T = [x,y,z]^T$$

chart 2: for $U' = M \setminus (0,0,-\frac{1}{2})$

let φ' be a stereographic projection, with $(0,0,-\frac{1}{2})$ as its north pole.

$$\varphi'([x,y,z]^T) = \left[\frac{x}{1+2z}, \frac{y}{1+2z}\right]^T = [A',B']^T$$

$$\varphi'^{-1}([A'\ B']^T) = \left[\frac{2A}{1+A^2+B^2}, \frac{2B}{1+A^2+B^2}, \frac{-1+A^2+B^2}{(1+A^2+B^2)2}\right]^T = [x,y,z]^T$$

So the 2 charts cover the ellipsoidal shell's full space

and $\varphi' \circ \varphi^{-1}([A\ B]^T) = [A', B']^T$

$\varphi \circ (\varphi')^{-1}([A'\ B']^T) = [A, B]^T$

which are smooth and compatible, so M is a manifold

Problem 2

a) for every point (x,y,z) on the sphere, the gradient is $(2x, 2y, 2z)$

since $[2x, 2y, 2z]\begin{bmatrix}0\\-z\\y\end{bmatrix} = 0$, and $[2x, 2y, 2z]\begin{pmatrix}y\\-x\\0\end{pmatrix} = 0$

then, vectors g_1 and g_2 are perpendicular to the gradient at (x,y,z)

Therefore, g_1 and g_2 can be defined as vector fields

b)

let $q = (x, y, z) \in \mathbb{R}^3 \quad x^2+y^2+z^2=1$

$$[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 = \begin{bmatrix}0 & 1 & 0\\-1 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{pmatrix}0\\-z\\y\end{pmatrix} - \begin{pmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{pmatrix}\begin{bmatrix}y\\-x\\0\end{bmatrix}$$

$$= \begin{bmatrix}-z\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\-x\end{bmatrix} = \begin{pmatrix}-z\\0\\x\end{pmatrix}$$

Problem 3

a) since $6ax + 2by + 10cz = 0$

then, vectors (a,b,c) are perpendicular to vector $(6x, 2y, 10z)$

so for vectors (a,b,c) we can have:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underbrace{\begin{pmatrix} y \\ -3x \\ 0 \end{pmatrix}}_{g_1} u_1 + \underbrace{\begin{pmatrix} 0 \\ -5z \\ y \end{pmatrix}}_{g_2} u_2$$

(q is (x,y,z))

$$[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -5 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} y \\ -3x \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -5z \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -3x \end{pmatrix} - \begin{pmatrix} -5z \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5z \\ 0 \\ -3x \end{pmatrix}$$

$$\begin{pmatrix} 5z \\ 0 \\ -3x \end{pmatrix} = \left(\begin{pmatrix} y \\ -3x \\ 0 \end{pmatrix} \cdot 5z + \begin{pmatrix} 0 \\ -5z \\ y \end{pmatrix} \cdot (-3x)\right) \cdot \frac{1}{y}$$

so the distribution is involutive., which means it is integrable

b) Since vectors (a,b,c) can form a plane, and those vectors are always perpendicular to vector $(6x, 2y, 10z)$

it is obvious that the gradient at each point (x,y,z) is $(6x, 2y, 10z)$

so the manifold is $3x^2 + y^2 + 5z^2 = c$ $(c > 0)$

Problem 4

$$\dot{q} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{g_1} u_1 + \underbrace{\begin{pmatrix} q_2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{g_2} u_2 + \underbrace{\begin{pmatrix} q_3 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_{g_3} u_3$$

$$[g_2, g_3] = \frac{\partial g_3}{\partial q} g_2 - \frac{\partial g_2}{\partial q} g_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_2 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_3 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

obviously, $[1\ 0\ 0\ 0]^T$ doesn't belong to span(g_1, g_2, g_3)

so the dynamic system is nonholonomic

Problem 5

Since $(0, 1, \rho\sin q_5, \rho\cos q_3, \cos q_5)^T \cdot \dot{q} = 0$

then $(0, 1, \rho\sin q_5, \rho\cos q_3, \cos q_5)^T$ could be taken as the gradient at point q

so we have
$$\dot{q} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{g_1} u_1 + \underbrace{\begin{pmatrix} 0 \\ -\rho \\ \sin q_5 \\ 0 \\ \rho\cos q_5 \end{pmatrix}}_{g_2} u_2 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ -\cos q_3 \\ \sin q_5 \\ 0 \end{pmatrix}}_{g_3} u_3$$

(g_1, g_2, g_3 are perpendicular to the 'gradient'

$$[g_2, g_3] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos q_5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ -\rho \\ \sin q_5 \\ 0 \\ \rho\cos q_5 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos q_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\rho\sin q_5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\cos q_3 \\ \sin q_5 \\ 0 \end{pmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \sin q_3 \sin q_5 \\ \rho\cos q_5 \cos q_5 \\ 0 \end{bmatrix}$$ which doesn't belong to $\mathrm{span}(g_1, g_2, g_3)$

so, it is not involutive, which means the constraint is nonholonomic