1. a) $y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\dot{y} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix}$

$x_3 = \operatorname{atan2}(\dot{y}_2, \dot{y}_1)$ when $u_1 \neq 0$

$u_1 = \sqrt{\dot{y}_1 + \dot{y}_2}$

$u_2 = \operatorname{atan}\left(\frac{\dot{x}_3 \cdot L}{u_1}\right) = \operatorname{atan}\left(\frac{\ddot{y}_2 \dot{y}_1 - \dot{y}_2 \ddot{y}_1}{(\dot{y}_1^2 + \dot{y}_2^2)^{\frac{3}{2}}} \cdot L\right)$ when $u_1 \neq 0$

b) for $X_0 \to X_m$ $T = 10$

let the velocity at X_0 and X_m be 1, $y(t) = A\lambda(t)$ $\lambda(t) = [t^3 \ t^2 \ t \ 1]^T$

then $Y = [y_0, \dot{y}_0, y_m, \dot{y}_m] = \begin{bmatrix} 0 & 1 & 5 & 1 \\ 5 & 0 & 2.5 & 0 \end{bmatrix}$

so $A = \begin{bmatrix} 0.01 & -0.15 & 1 & 0 \\ 0.005 & -0.075 & 0 & 5 \end{bmatrix}$

$y(t) = \begin{pmatrix} 0.01t^3 - 0.15t^2 + t \\ 0.005t^3 - 0.075t^2 + 5 \end{pmatrix}$ for $X_0 \to X_m$.

for $X_m \to X_f$. $T = 10$

let the velocity be -1 $y(t) = A\lambda(t)$ $\lambda(t) = [t^3 \ t^2 \ t \ 1]^T$

then $Y = [y_m, \dot{y}_m, y_f, \dot{y}_f] = \begin{bmatrix} 5 & -1 & 0 & -1 \\ 2.5 & 0 & 0 & 0 \end{bmatrix}$

so $A = \begin{bmatrix} -0.01 & 0.15 & -1 & 5 \\ 0.005 & -0.075 & 0 & 2.5 \end{bmatrix}$

$y(t) = \begin{pmatrix} -0.01t^3 + 0.15t^2 - t + 5 \\ 0.005t^3 - 0.075t^2 + 2.5 \end{pmatrix}$ for $X_m \to X_f$

2.
a) $A = \begin{pmatrix} 0 & 0 & -u_1 \cdot \sin(x_3) \\ 0 & 0 & u_1 \cdot \cos(x_3) \\ 0 & 0 & 0 \end{pmatrix}$ $B_0 = \begin{pmatrix} \cos(x_3) & 0 \\ \sin(x_3) & 0 \\ \tan(u_2)/L & u_1 \cdot (\tan^2 u_2 + 1)/L \end{pmatrix}$

$$B_1 = \dot{B}_0(t) - A(t)B(t) = \begin{pmatrix} 0 & u_1^2 \sin x_3 \cdot (\tan^2 u_2 + 1)/L \\ 0 & -u_1^2 \cos x_3 \cdot (\tan^2 u_2 + 1)/L \\ \dot{u}_2 (\tan^2 u_2 + 1)/L & \dot{u}_1 (\tan^2 u_2 + 1)/L + 2\dot{u}_2 u_1 \tan u_2 (\tan^2 u_2 + 1)/L \end{pmatrix}$$

obviously $\text{rank}(\{B_0\ B_1\}) = 3$ when $u_1 \neq 0$

So $\text{rank}(\{B_0\ B_1\ B_2\}) = 3$ when $u_1 \neq 0$

which means the error dynamics is controllable, when $u_1(t) \neq 0$