1) at $X_0=[0,0,0,0,0]^T$, $g_0(X_0)=0$,

$U$ is open and its convex hull contains $0$.

$$g_3=[g_0, g_1]=\begin{pmatrix} v\cos\theta\sin\phi \\ v\sin\theta\sin\phi \\ -v\cdot\cos\phi/L \\ 0 \\ 0 \end{pmatrix} \quad g_4=[g_0,[g_0,g_1]]=\begin{pmatrix} -v^2\sin\theta/L \\ v^2\cos\theta/L \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$g_5=[g_0,[g_0,[g_0,g_1]]]=\begin{pmatrix} -v^3\cos\theta\cdot\sin\phi/L^2 \\ -v^3\sin\theta\sin\phi/L^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\det([g_1,g_2,g_3,g_4,g_5])=-\dfrac{v^6\cdot\sin(2\cdot\phi)}{2L^4} \Rightarrow$ so LARC of degree 4

the bad brackets of degree $\leq 4$ are:

$$[g_1,[g_0,g_1]]=\begin{pmatrix} v\cos\theta\cos\phi \\ v\cdot\sin\theta\cos\phi \\ v\cdot\sin\phi/L \\ 0 \\ 0 \end{pmatrix} \triangleq -v\cdot[g_0,g_2] \quad [g_2,[g_0,g_2]]=0$$

Hence, the system is STLC, so LA and controllable

2) $L(w) = \frac{1}{2}(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2) + \frac{1}{2} J_r (w_1+u_1)^2 + \frac{1}{2} J_r (w_2+u_2)^2$

a) $\partial_w L = \begin{pmatrix} J_1 w_1 + J_r(w_1+u_1) \\ J_2 w_2 + J_r(w_2+u_2) \\ J_3 w_3 \end{pmatrix}$

Since $\frac{d}{dt} \partial_w L = \partial_w L \times W = -\hat{W}\, \partial_w L$

so $\partial_w L = C \cdot e^{-\int_0^t \hat{w}(t)\, dt}$. Since $\partial_w L(0) = 0 \Rightarrow C = 0$ so $\partial_w L = 0$

$$\begin{pmatrix} J_1 w_1 + J_r(w_1+u_1) \\ J_2 w_2 + J_r(w_2+u_2) \\ J_3 w_3 \end{pmatrix} = 0 \Rightarrow w_1 = -\frac{J_r}{J_1+J_r} u_1, \quad w_2 = -\frac{J_r}{J_2+J_r} u_2$$

so $$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{J_r}{J_1+J_r} \\ 0 \\ 0 \end{pmatrix}}_{b_1} u_1 + \underbrace{\begin{pmatrix} 0 \\ -\frac{J_r}{J_2+J_r} \\ 0 \end{pmatrix}}_{b_2} u_2$$

b)

Since $R \cdot R^T = I$ and $\dot{R} = R \cdot \hat{w}$, $w = b_1 u_1 + b_2 u_2$, so $R^T \cdot \dot{R} = \hat{w}$

$\dot{R}_{11} = -\sin\beta \cos\gamma \cdot \dot{\beta} - \cos\beta \cdot \sin\gamma \cdot \dot{\gamma}$

$\dot{R}_{12} = \sin\beta \sin\gamma \cdot \dot{\beta} - \cos\beta \cos\gamma \cdot \dot{\gamma}$

$\dot{R}_{13} = \cos\beta \cdot \dot{\beta}$

$\dot{R}_{21} = (\cos\alpha \sin\beta \cos\gamma - \sin\alpha \sin\gamma) \cdot \dot{\alpha} + \sin\alpha \cos\beta \cos\gamma \cdot \dot{\beta} + (\cos\alpha \cdot \cos\gamma - \sin\alpha \sin\beta \sin\gamma) \dot{\gamma}$

$\dot{R}_{22} = (-\sin\alpha \cos\gamma - \cos\alpha \cdot \sin\beta \sin\gamma) \cdot \dot{\alpha} - \sin\alpha \cos\beta \sin\gamma \cdot \dot{\beta} - (\cos\alpha \sin\gamma + \sin\alpha \sin\beta \cos\gamma) \dot{\gamma}$

$\dot{R}_{23} = -\cos\alpha \cos\beta \cdot \dot{\alpha} + \sin\alpha \sin\beta \cdot \dot{\beta}$

$\dot{R}_{31} = (\cos\alpha \cdot \sin\gamma + \sin\alpha \sin\beta \cos\gamma) \cdot \dot{\alpha} - \cos\alpha \cos\beta \cos\gamma \cdot \dot{\beta} + (\sin\alpha \cos\gamma + \cos\alpha \sin\beta \sin\gamma) \cdot \dot{\gamma}$

$\dot{R}_{32} = (\cos\alpha \cos\gamma - \sin\alpha \sin\beta \sin\gamma) \cdot \dot{\alpha} + \cos\alpha \cos\beta \sin\gamma \cdot \dot{\beta} + (\cos\alpha \sin\beta \cos\gamma - \sin\alpha \sin\gamma) \cdot \dot{\gamma}$

$\dot{R}_{33} = -\sin\alpha \cos\beta \cdot \dot{\alpha} - \cos\alpha \sin\beta \cdot \dot{\beta}$

$$R^T \cdot \dot{R} = \begin{pmatrix} 0 & -\dot{\gamma} - \dot{\alpha}\cdot\sin\beta & \dot{\beta}\cdot\cos\gamma - \dot{\alpha}\cos\beta\sin\gamma \\ \dot{\gamma} + \dot{\alpha}\cdot\sin\beta & 0 & -\dot{\beta}\sin\gamma - \dot{\alpha}\cos\beta\cos\gamma \\ -\dot{\beta}\cdot\cos\gamma + \dot{\alpha}\cos\beta\sin\gamma & \dot{\beta}\sin\gamma + \dot{\alpha}\cos\beta\cos\gamma & 0 \end{pmatrix}$$

$$\text{so} \begin{pmatrix} \dot{\beta}\sin\gamma + \dot{\alpha}\cos\beta\cos\gamma \\ \dot{\beta}\cdot\cos\gamma - \dot{\alpha}\cos\beta\sin\gamma \\ \dot{\gamma} + \dot{\alpha}\cdot\sin\beta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\beta\cos\gamma & \sin\gamma & 0 \\ -\cos\beta\sin\gamma & \cos\gamma & 0 \\ \sin\beta & 0 & 1 \end{pmatrix}}_{= T} \cdot \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = b_1 u_1 + b_2 u_2$$

$$T^{-1} = \begin{pmatrix} \frac{\cos\gamma}{\cos\beta} & -\frac{\sin\gamma}{\cos\beta} & 0 \\ \sin\gamma & \cos\gamma & 0 \\ -\frac{\sin\beta\cos\gamma}{\cos\beta} & \frac{\sin\beta\sin\gamma}{\cos\beta} & 1 \end{pmatrix}, \text{ then } \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = -\frac{J_r}{J_1+J_r}\underbrace{\begin{pmatrix} \frac{\cos\gamma}{\cos\beta} \\ \sin\gamma \\ -\frac{\sin\beta\cos\gamma}{\cos\beta} \end{pmatrix}}_{g_1} u_1 - \frac{J_r}{J_2+J_r}\underbrace{\begin{pmatrix} -\frac{\sin\gamma}{\cos\beta} \\ \cos\gamma \\ \frac{\sin\beta\sin\gamma}{\cos\beta} \end{pmatrix}}_{g_2} u_2$$

$$[g_1, g_2] = \frac{J_r^2}{(J_1+J_r)(J_2+J_r)} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdots g_3$$

$$\det([g_1, g_2, g_3]) = \frac{J_r^4}{\cos\beta \cdot (J_1+J_r)^2 (J_2+J_r)^2}$$

so $[g_1\ g_2\ g_3]$ is invertible when $\cos\beta \neq 0$,

which means when $\cos\beta \neq 0$ the driftless system is LA $\Leftrightarrow$ controllable $\Rightarrow$ STLC

3) equation of motion:

$$m\ddot{x} = (u_1+u_2)\sin\theta$$

$$m\ddot{y} = (u_1+u_2)\cos\theta - mg$$

$$J\ddot{\theta} = (u_2-u_1)r$$

let $q = \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

then: $\theta = \arcsin\left(\frac{m\ddot{q}_1}{\sqrt{(m\ddot{q}_1)^2+(m\ddot{q}_2+mg)^2}}\right)$ except when $\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$

and we can get $\ddot{\theta}$ except when $\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$

$$u_2 = \frac{1}{2}\left(\frac{J\ddot{\theta}}{r} + \sqrt{(m\ddot{q}_1)^2+(m\ddot{q}_2+mg)^2}\right)$$

$$u_1 = \frac{1}{2}\left(\sqrt{(m\ddot{q}_1)^2+(m\ddot{q}_2+mg)^2} - \frac{J\ddot{\theta}}{r}\right)$$

thus we have $q = \begin{bmatrix} x \\ y \end{bmatrix}$, $\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \varphi = (q, \ddot{q})$, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha(\ddot{q}, q^{(4)})$

so the system is differentially flat

4) we denote the point between the rear wheels as $\underline{q}$

$$\underline{q}=\begin{pmatrix} q_1 = x-d\cdot\cos\theta_1 \\ q_2 \quad y-d\cdot\sin\theta_1 \end{pmatrix} \text{ then } \dot{\underline{q}}=\begin{pmatrix} \dot{x}+d\cdot\sin\theta_1\cdot\dot{\theta}_1 \\ \dot{y}-d\cdot\cos\theta_1\cdot\dot{\theta}_1 \end{pmatrix}=\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$\ddot{\underline{q}}=\begin{pmatrix} \ddot{x}+d(\sin\theta_1\cdot\ddot{\theta}_1+\cos\theta_1\cdot\dot{\theta}_1^2) \\ \ddot{y}-d(\cos\theta_1\cdot\ddot{\theta}_1-\sin\theta_1\cdot\dot{\theta}_1^2) \end{pmatrix}=\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix}$$

$$\dot{q}_1=\dot{x}+d\cdot\sin\theta_1\cdot\dot{\theta}_1=\cos\theta u_1+\sin\theta_1\cdot\sin(\theta-\theta_1)\cdot u_1=\cos(\theta-\theta_1)\cos\theta_1 u_1$$

$$\dot{q}_2=\dot{y}-d\cdot\cos\theta_1\cdot\dot{\theta}_1=\sin\theta u_1-\cos\theta_1\cdot\sin(\theta-\theta_1)\cdot u_1=\sin\theta_1\cos(\theta-\theta_1)u_1$$

so $\theta_1=\operatorname{atan}(\dot{q}_1/\dot{q}_2)$, and we can get $\dot{\theta}_1$ and $\ddot{\theta}_1$ when $\dot{q}_2\neq 0$

$$x=q_1+d\cdot\cos\theta_1 \qquad y=q_2+d\cdot\sin\theta_1$$

$$\dot{x}=\dot{q}_1-d\cdot\sin\theta_1\cdot\dot{\theta}_1 \qquad \dot{y}=\dot{q}_2+d\cdot\sin\theta_1\dot{\theta}_1$$

$$\ddot{x}=\ddot{q}_1-d(\sin\theta_1\cdot\ddot{\theta}_1+\cos\theta_1\cdot\dot{\theta}_1^2) \qquad \ddot{y}=\ddot{q}_2+d\cdot(\cos\theta_1\cdot\ddot{\theta}_1-\sin\theta_1\cdot\dot{\theta}_1^2)$$

$$u_1=\sqrt{\dot{x}^2+\dot{y}^2} \quad \dot{u}_1=\frac{1}{2}\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\cdot(2\dot{x}\ddot{x}+2\dot{y}\ddot{y}) \text{ when } \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}\neq\begin{pmatrix}0\\0\end{pmatrix}$$

$$\dot{x}=\cos\theta\cdot u_1 \Rightarrow \theta=\arccos\left(\frac{\dot{x}}{u_1}\right) \Rightarrow \dot{\theta}=-\frac{1}{\sqrt{1-(\frac{\dot{x}}{u_1})^2}}\cdot\frac{\ddot{x}u_1-\dot{x}\dot{u}_1}{u_1^2}$$

when $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}\neq\begin{pmatrix}0\\0\end{pmatrix}$

$$\dot{\theta}=\frac{\tan\phi}{L}\cdot u_1 \Rightarrow \tan\phi=\frac{\dot{\theta}\cdot L}{u_1}$$

$$\text{so } \phi=\operatorname{atan}\left(\frac{\dot{\theta}\cdot L}{u_1}\right) \Rightarrow u_2=\dot{\phi}=\frac{1}{1+\left(\frac{\dot{\theta}\cdot L}{u_1}\right)^2}\cdot\frac{L(\ddot{\theta}u_1-\dot{\theta}\dot{u}_1)}{u_1^2}$$

so $(x,y,\theta)$ could be represented by $\varphi(q,\dot{q})$

$(u_1,u_2)$ could be represent by $\alpha(q,\dot{q},\ddot{q},q^{(3)})$

Hence the system is differentially flat