EN530.678 Nonlinear Control and Planning in Robotics
Midterm #2

due May 12, 2021, 8:30am

Prof: Marin Kobilarov

1. Consider the system

$$\dot{x}_1 = -x_1 + (1 + 3x_2^2)u,$$
$$\dot{x}_2 = x_1 x_2 + u,$$
$$\dot{x}_3 = x_2,$$

with output $y = h(x) = x_3$. What is the system relative degree? Transform the system into a normal form, i.e. by finding the transformation $(z, \eta) = \Phi(x)$ and determine whether it is minimum phase.

2. Consider an underactuated spacecraft with two control torques, with dynamics given by:

$$\dot{\omega}_1 = a_1 \omega_2 \omega_3 + u_1,$$
$$\dot{\omega}_2 = a_2 \omega_1 \omega_3 + u_2,$$
$$\dot{\omega}_3 = a_3 \omega_1 \omega_2,$$

for some known constants a_1, a_2, a_3 where $a_3 \neq 0$.

(a) Using feedback linearization, design a control law that asymptotically stabilizes the system to given desired outputs $y_1 = \omega_1$ and $y_2 = \omega_2$ (the actual desired values are constant but can be non-zero). Put the system in normal form, define the zero dynamics, determine its stability properties.

(b) Using dynamic feedback linearization or backstepping, design a control law that asymptotically stabilizes the system to desired outputs $y_1 = \omega_1$ and $y_2 = \omega_3$ (the actual desired values are constant but can be non-zero). Under what conditions is such controller valid?

(c) While it is not possible to drive all degrees of freedom to desired values simultaneously, it is possible to design a sequence of *maneuvers*, i.e. different control law executions for finite time durations, that in fact could bring the full state $\omega = (\omega_1, \omega_2, \omega_3)$ close to a desired value. Could you suggest such a sequence of maneuvers relying on the control laws derived in a) and b).

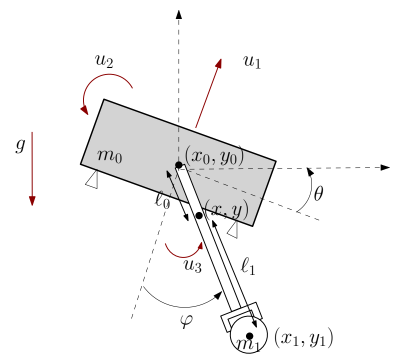

Figure 1: Vehicle with manipulator.

3. Vehicle with manipulator. Consider an aerial vehicle consisting of a main "base" body and a manipulator attached at its center of mass (analogous problem can be defined for an underwater vehicle). The base has mass m_0 and moment of inertia J_0 while the arm is ℓ meters long, has negligible mass, but is carrying a load with mass m_1. The vehicle has configuration $(x_0, y_0, \theta, \varphi)$ where (x_0, y_0) is the position of the base body, θ is its orientation, and φ is the manipulator joint angle. The vehicle is controlled by a lift force u_1 aligned with the body-fixed vertical axis, torque u_2 around the main body, and torque u_3 controlling the manipulator. The position of the tip is denoted by (x_1, y_1) and given by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \ell \begin{bmatrix} \sin(\theta + \varphi) \\ -\cos(\theta + \varphi) \end{bmatrix}. \tag{1}$$

It turns out the dynamics can be written in a simple form, using the coordinates of the instantaneous center of mass given by

$$x = \frac{m_0}{m} x_0 + \frac{m_1}{m} x_1, \qquad y = \frac{m_0}{m} y_0 + \frac{m_1}{m} y_1,$$

where $m = m_0 + m_1$, as follows:

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ mg \end{bmatrix} + \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} u_1, \tag{2}$$

$$J_0 \ddot{\theta} = u_2 - u_3, \tag{3}$$

$$a \ddot{\varphi} = u_3 - \ell_0 u_1 \sin\varphi - \frac{a}{J_0}(u_2 - u_3), \tag{4}$$

where $a = \frac{m_0 m_1}{m} \ell^2$ and $\ell_0 = \frac{m_1}{m} \ell$ are constants.

(a) Design feedback linearizing controller to track desired outputs $(x(t), y(t), \varphi(t))$.

(b) (Optional: for extra credit) How would you design a similar controller to track a given desired manipulator tip spatial configuration, i.e. when the output is $(x_1(t), y_1(t), \theta(t) + \varphi(t))$?

4. A typical model for crab-steered ground vehicles (i.e. with non-zero lateral motion) is given by the dynamics:

$$
\begin{aligned}
\dot{x}_1 &= (\cos x_3 + \beta \sin x_3)x_4 \\
\dot{x}_2 &= (\sin x_3 - \beta \cos x_3)x_4 \\
\dot{x}_3 &= u_2 x_4 \\
\dot{x}_4 &= u_1
\end{aligned}
$$

where (x_1, x_2) is the position, x_3 is the orientation, x_4 is the velocity, u_1 is the acceleration control, and u_2 is the curvature control. The constant β defines the lateral motion bias.

(a) Derive a control law $u = \psi(t, x)$ to track a desired reference trajectory $y_d(t) = (x_{1d}(t), x_{2d}(t))$

(b) Assume that β is uncertain, i.e. it can vary around a known nominal value $\hat{\beta}$. Such variation generates an error in the position dynamics:

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (\cos x_3 + \hat{\beta} \sin x_3)x_4 \\ (\sin x_3 - \hat{\beta} \cos x_3)x_4 \end{pmatrix} + \delta(x_4),
$$

where the error can be bounded as $\|\delta(x_4)\| \leq a_1 |x_4|$. What is a good estimate of the error constant a_1 in terms of β and $\hat{\beta}$?

(c) Design a robust controller that can handle the uncertainty defined in b). For simplicity it is fine to assume $y_d(t) = 0$.

Hint: note that it is possible to express the system dynamics as:

$$
\begin{aligned}
\dot{\eta} &= \xi + \delta_\eta(\xi), \\
\dot{\xi} &= G_a(\eta, \xi)u,
\end{aligned}
$$

where $\delta_\eta(\xi) = \delta(x_4)$ by employing the change of variables:

$$
\eta = (x_1, x_2), \qquad \xi \triangleq \begin{pmatrix} (\cos x_3 + \hat{\beta} \sin x_3)x_4 \\ (\sin x_3 - \hat{\beta} \cos x_3)x_4 \end{pmatrix}.
$$

It would be useful to design a *desired* controller in the form $\xi = \phi(\eta)$ that stabilizes robustly the η-dynamics first.

Last problem below is optional, for extra credit:

5. Is the unicycle with drift (e.g. corresponding to an airplane subject to wind) with dynamics given by

$$
\begin{aligned}
\dot{x}_1 &= \cos x_3 u_1 \\
\dot{x}_2 &= x_2 + \sin x_3 u_1 \\
\dot{x}_3 &= u_2
\end{aligned}
$$

small-time locally controllable (STLC)? Use the definition of good and bad brackets to support your claim.