Problem 1

1, a)

$f(x) = \|x\|^3$: $\nabla f(x) = 3\|x\|^2 \frac{\partial \|x\|}{\partial x} = 3\|x\|^2 \cdot \frac{x}{\|x\|} = 3\|x\| \cdot x$

$\nabla^2 f(x) = \frac{3}{2} \cdot \frac{2x}{(x^T x)^{\frac{1}{2}}} \cdot x^T + 3(x^T x)^{\frac{1}{2}} \cdot I = \frac{3x \cdot x^T}{\|x\|} + 3\|x\| \cdot I$

b) $\dot{V}(x) = \nabla V^T \cdot \dot{x} = \frac{x^T}{\|x\|} \cdot (Ax + b)$

c) let $a = [a_1\ a_2\ a_3]^T$ $b = [b_1\ b_2\ b_3]^T$

$\|a \times b\|^2 = (a_1 b_2 - a_2 b_1)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_2 b_3 - a_3 b_2)^2$

$= a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 + a_1^2 b_3^2 - 2a_1 a_3 b_1 b_3 + a_3^2 \cdot b_1^2 + a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2$

$\|a\|^2 \cdot \|b\|^2 - (a^T b)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$

$= a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + a_3^2 b_3^2$

$- (a_1^2 b_1^2 + 2a_1 a_2 b_1 b_2 + 2a_1 a_3 b_1 b_3 + a_2^2 b_2^2 + 2a_2 a_3 b_2 b_3 + a_3^2 b_3^2)$

$= a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 + a_1^2 b_3^2 - 2a_1 a_3 b_1 b_3 + a_3^2 \cdot b_1^2 + a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2$

$= \|a \times b\|^2$.

Hence, $\|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a^T b)^2$ satisfies for $\forall a, b \in R^3$

d) let $a = [a_1\ a_2\ a_3]^T$ $b = \{b_1\ b_2\ b_3\}^T$ $c = [c_1\ c_2\ c_3]^T$

$$a \times (b \times c) = \begin{bmatrix} a_2 b_1 c_2 - a_2 b_2 c_1 + a_3 b_1 c_3 - a_3 b_3 c_1 \\ a_1 b_2 c_1 - a_1 b_1 c_2 + a_3 b_2 c_3 - a_3 b_3 c_2 \\ a_1 b_3 c_1 - a_1 b_1 c_3 - a_2 b_2 c_3 + a_2 b_3 c_2 \end{bmatrix}$$

$$(a^T c) b = \begin{bmatrix} a_1 b_1 c_1 + a_2 b_1 c_2 + a_3 b_1 c_3 \\ a_1 b_2 c_1 + a_2 b_2 c_2 + a_3 b_2 c_3 \\ a_1 b_3 c_1 + a_2 b_3 c_2 + a_3 b_3 c_3 \end{bmatrix} \quad (a^T b) c = \begin{bmatrix} a_1 b_1 c_1 + a_2 b_2 c_1 + a_3 b_3 c_1 \\ a_1 b_1 c_2 + a_2 b_2 c_2 + a_3 b_3 c_2 \\ a_1 b_1 c_3 + a_2 b_2 c_3 + a_3 b_3 c_3 \end{bmatrix}$$

it is clear that $a \times (b \times c) = (a^T c) b - (a^T b) c$

Hence $a \times (b \times c) = (a^T c) b - (a^T b) c$ satisfies for $\forall a, b, c \in R^3$

Problem 2

Mo Tu We Th Fr Sa Su

Memo No.
Date / /

2) let $V(x) = x^T x$ then $\dot{V}(x) = 2x^T \cdot \dot{x}$

V(x) is a positive definite function $= 2x_1 \cdot \dot{x}_1 + 2x_2 \cdot \dot{x}_2$

a) $\dot{V}(x) = -4x_1^2 + (2x_1 - x_2)2x_2$

$= -4x_1^2 + 4x_1x_2 - 2x_2^2 = -(2x_1 - x_2)^2 - x_2^2$

it is obvious that : $\dot{V}(x) < 0 \quad \forall x \neq 0$. Hence the origin is globally asymptotically stable

b) let $V(x) = x^T x$ then $\dot{V}(x) = 2x^T \cdot \dot{x} = 2x_1 \cdot \dot{x}_1 + 2x_2 \cdot \dot{x}_2$

V(x) is a positive definite function.

$\dot{V}(x) = \cancel{-4x_1^2 + (2x_1 - x_2)2x_2} \; 2x_1^4 + 4x_1^2x_2^2 - 2x_1^2 + 2x_2^4 - 2x_2^2$

$= 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)$ when $x_1^2 + x_2^2 \leq 1$, $\dot{V}(x) \leq 0$

Hence the origin is locally asymptotically stable

c) let $V(x) = x^T x$ then $\dot{V}(x) = 2x^T \dot{x} = 2x_1 \cdot \dot{x}_1 + 2x_2 \cdot \dot{x}_2$

V(x) is a positive definite function.

$\dot{V}(x) = 2x_1^2(4x_1^4 - 2x_1^2 - 1)$

$= 2x_1^2((2x_1^2 - \frac{1}{2})^2 - \frac{5}{4})$

Since $x_1, x_2 \in R$ when $x_1^2 = \frac{1+\sqrt{5}}{4}$, $\dot{V}(x) = 0$

When $-\frac{\sqrt{1+\sqrt{5}}}{2} \leq x_1 \leq \frac{\sqrt{1+\sqrt{5}}}{2}$ $\dot{V}(x) \leq 0$

Hence the origin is locally asymptotically stable.

d) let $V(x) = x_1^2 + \frac{5}{2}x_2^2$, then $\dot{V}(x) = 2x_1 \cdot \dot{x}_1 + 5x_2 \cdot \dot{x}_2$ V(x) is a p.d. function.

$\dot{V}(x) = -2x_1^2 - 10x_1^2x_2^2 + 10x_1^2x_2^2 - 5x_2^4$

$= -2x_1^2 - 5x_2^4$

thus, $\dot{V}(x) < 0$; $\forall x \neq 0$

Hence the origin is globally asymptotically stable

Problem 4

Mo | Tu | We | Th | Fr | Sa | Su

Memo No. ______

Date / /

4) proof: $\dot{x} = -a[I_n + S(x) + xx^T]x$, let $x = [x_1\ x_2 \cdots x_n]^T$ $x \in R^n$

let $V(x) = \frac{1}{2}x^Tx$ $V(x)$ is a positive definite function

$$\dot{V}(x) = x^T \cdot \dot{x} = x^T \cdot (-a[I_n + S(x) + x \cdot x^T]x)$$

$$= -a\,x^TAx \qquad A = I_n + S(x) + x \cdot x^T$$

$$x^TAx = \left[\sum_{j=1}^{n} x_j \cdot A_{j1},\ \sum_{j=1}^{n} x_j \cdot A_{j2} \dots\dots,\ \sum_{j=1}^{n} x_j A_{jn}\right] \cdot x$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \cdot A_{ij} \cdot x_j$$

if $i=j$, $(i=j)=1$

if $i \neq j$ $(i=j)=0$

for each element A_{ij} : $A_{ij} = x_i \cdot x_j + S_{ij} + (i=j)$

Since $S(x)$ is a skew-symmetrix matrix $S_{ij} = -S_{ji}$

thus $x^TAx = \sum_{\substack{i=1}}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} x_i^2 x_j^2 + \sum_{i=1}^{n} x_i^2$, which means $x^TAx > 0$ when $x \neq \vec{0}$

and $a > 0$, so that $\dot{V}(x) < 0$, $\forall x \neq 0$

Hence the origin is globally asymptotically stable.