EN530.678 Nonlinear Control and Planning in Robotics
Homework #1

February 3, 2021

Due: February 10, 2021 (before class)

Prof: Marin Kobilarov

1. Perform the following calculations (2 pts each):
   (a) Let $f(x) = \|x\|^3$ where $x \in \mathbb{R}^n$. Compute the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$, ideally using vector calculus as opposed to individual coordinates x_i.
   (b) Let $\dot{x} = Ax + b$ where $x \in \mathbb{R}^n$ and $V(x) = \|x\|$. What is $\dot{V}(x)$?
   (c) Show that $\|a \times b\|^2 = \|a\|^2\|b\|^2 - (a^T b)^2$ for some $a, b \in \mathbb{R}^3$
   (d) Show that $a \times (b \times c) = (a^T c)b - (a^T b)c$ for some $a, b, c \in \mathbb{R}^3$

2. (Khalil) For each of the following systems, use a quadratic Lyapunov function candidate (e.g. $V(x) = \frac{1}{2}x^T x$ or more generally $V(x) = x^T Ax$ for some matrix $A > 0$) to show that the origin is asymptotically stable. (3 pts each)
   (a) $\dot{x}_1 = -2x_1, \qquad \dot{x}_2 = 2x_1 - x_2$
   (b) $\dot{x}_1 = -x_1(1 + x_2^2)(1 - x_1^2 - x_2^2), \qquad \dot{x}_2 = -x_2(1 - x_1^2)(1 - x_1^2 - x_2^2)$
   (c) $\dot{x}_1 = -(x_1 + x_2)(1 + 2x_1^2 - 4x_1^4), \qquad \dot{x}_2 = x_1(1 + 2x_1^2 - 4x_1^4)$
   (d) $\dot{x}_1 = -x_1 - 5x_1x_2^2, \qquad \dot{x}_2 = 2x_1^2x_2 - x_2^3$

3. **Implementation:** For 1(a)-1(d), write a Matlab or Python script which plots $x(t)$ and your Lyapunov candidate $V(t)$ for $t \in [0, 5]$ starting from $x(0) = (0.5, 0.5)$. You can use the `ode45` function for integrating the dynamics. *An example script is provided on the course web page for reference. See file* `hw1_lyapunov_example.m`. (2 pts each)

   Note: Upload your code and plots as a .zip file using `https://forms.gle/Z2AYx3FRNJHtXTR47`; in addition attach a printout of the code and all plots to your homework solutions.

4. (Khalil) Consider the system

$$\dot{x} = -a[I_n + S(x) + xx^T]x,$$

   where a is a positive constant, I_n is the nxn identity matrix, and $S(x)$ is an x-dependent skew symmetric matrix. Show that the origin is *globally* asymptotically stable. (10 pts)