Problem 1

a) when $\dot{x} = \vec{0}$ $\dot{X}_3 = 0$, then $X_3 = 1$ or 0

if $X_3 = 0$ then $\dot{x}_1 = 1$. so X_3 can only be 1.

$X_3 = 1$, then $\dot{X}_1 = -X_2 + 1$, so $X_2 = 1 \Rightarrow \dot{X}_2 = X_1 - 1 \Rightarrow X_1 = 1$

so the system only has one equilibrium point $(1, 1, 1)$

b) when $x = (1, 1, 1)$

$$A = Df\Big|_{x=(1,1,1)} = \begin{bmatrix} 0 & -X_2 & -X_2 \\ X_3 & -1 & X_1 \\ 0 & 0 & 2X_3 - 3X_3^2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

the eigenvalues are: $-1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Since $Re\ \lambda_i(A) < 0$, then the equilibrium point is asymptotically stable

when $X_3 = 0$, $\dot{X}_1 = 1$ $\dot{X}_2 = -X_2$. Since $\dot{x}_1 = 1$, which is a positive constant so X_1 will keep increasing. Therefore, the equilibrium is not globally asymptotically stable

2) With $u_1 = u_2 = u_3 = 0$ $w = 0$

a) let $V(w) = \frac{1}{2}(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2)$ $V(w)$ is p.d.

$$\dot{V} = \left[(J_2 - J_3) + (J_3 - J_1) + (J_1 - J_2)\right] w_1 w_2 w_3 = 0$$

So, the origin is stable

b) Since $\dot{V}$ is always zero, it is not asymptotically stable

c) let $V(w) = \frac{1}{2}(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2)$ $V(w)$ is p.d

with $u_i = -k_i w_i$

$\dot{V}(w) = -k_1 w_1^2 - k_2 w_2^2 - k_3 w_3^2$, so $\dot{V}$ is negative definite

Hence, the origin of the system is globally asymptotically stable.

3) with $u=0$

a) $V=\frac{1}{2}\dot{q}^T M(q)\dot{q}+P(q)$, since $M(q)$ is p.d. and $P(q)$ is p.d. then V is p.d.

$\dot{V}=\dot{q}^T M(q)\ddot{q}+\frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q}+g(q)^T\cdot\dot{q}$

$=\frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q}-\dot{q}^T\left(C(q,\dot{q})\dot{q}+D\dot{q}+g(q)\right)+g(q)^T\cdot\dot{q}$

$=\frac{1}{2}\dot{q}^T(\dot{M}(q)-2C)\dot{q}-\dot{q}^T D\dot{q}-\dot{q}^T g(q)+\dot{q}^T g(q)$

Since $\dot{M}-2C$ is skew-symmetric, then $\dot{q}^T(\dot{M}(q)-2C)\dot{q}=0$

so $\dot{V}=-\dot{q}^T D\dot{q}$. and D is p.s.d.

Hence, the origin is stable

b)

with $u=-K_d\dot{q}$, use V in (a) V is p.d.

then, we have $\dot{V}=-\dot{q}^T(K_d+D)\dot{q}$

$=-\dot{q}^T K_d\dot{q}-\dot{q}^T D\dot{q}$, so $\dot{V}\leq 0$

when $\dot{V}=0$, then $\dot{q}=0$, so $\ddot{q}=0$

according to dynamics $g(q)=0$, so $q=0$

Since the $\dot{V}$ is 0 only when $q=0$

by Lasalle's principle, the origin is asymptotically stable

3 c) With $u = g(q) - k_p(q - q^*) - k_d\dot{q}$

We have: $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + k_p(q-q^*) - k_d\dot{q} = 0$

When $q = q^*$, $\dot{q} = 0$, the dynamic system is 0

so $(q^*, 0)$ is an equilibrium point.

let $e = q - q^*$, then $\dot{e} = \dot{q}$ (q^* is constant)

let $V = \frac{1}{2}\dot{e}^T M(q)\dot{e} + \frac{1}{2}e^T k_p e$, which is p.d

$$\dot{V} = \dot{q}^T M(q)\cdot\ddot{q} + \frac{1}{2}\dot{q}^T\dot{M}(q)\cdot\dot{q} + e^T k_p\dot{q}$$

$$= \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} - \dot{q}^T C\dot{q} - \dot{q}^T k_p e - \dot{q}^T k_d e - \dot{q}^T D\dot{q} + e^T k_p\dot{q}$$

$$= \frac{1}{2}\dot{q}^T(\dot{M} - 2C)\dot{q} - \dot{q}^T k_p e - \dot{q}^T(k_d + D)\dot{q} + \dot{q}^T k_p e$$

$$= -\dot{q}^T(k_d + D)\dot{q} = -\dot{e}^T(k_d + D)\dot{e} \quad \text{so } \dot{V} \leq 0$$

When $\dot{V} = 0$, $\dot{e} = 0 = \dot{q}$ so $\ddot{q} = 0$

from the dynamics: $k_p(q - q^*) = 0$ so $q = q^*$

Since $\dot{V} = 0$ only when $\dot{q} = 0$ and $q = q^*$

according to Lasalle's principle, the point $(q^*, 0)$ is an asymptotically stable equilibrium point.

4) a) from equation of motion: $\ddot{q} = M^{-1} \cdot B(q) \cdot u - M^{-1} \cdot D \cdot \dot{q}$

with $q^* = (0,0)$, then, $e = q - q^* = q$

the error dynamic: $\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -kp & -kd \end{bmatrix}}_{\text{denote as } A} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$

let $kd = M^{-1}D$, since $M^{-1}D$ is p.d and symmetric.

then we need let $-kp \cdot q = M^{-1}B(q)u$.

$$M^{-1}B(q) = \begin{pmatrix} \frac{\cos\theta}{m} & \frac{\cos\theta}{m} & \frac{-\sin\theta}{m} \\ \frac{\sin\theta}{m} & \frac{\sin\theta}{m} & \frac{\cos\theta}{m} \\ \frac{-r}{J} & \frac{r}{J} & 0 \end{pmatrix} \quad B = \begin{pmatrix} \cos\theta & \cos\theta & -\sin\theta \\ \sin\theta & \sin\theta & \cos\theta \\ -r & r & 0 \end{pmatrix}$$

$$\text{let } u = -\begin{pmatrix} \frac{\cos\theta}{2} & \frac{\sin\theta}{2} & -\frac{1}{2r} \\ \frac{\cos\theta}{2} & \frac{\sin\theta}{2} & \frac{1}{2r} \\ -\sin\theta & \cos\theta & 0 \end{pmatrix} \cdot q$$

$$\text{then } M^{-1}B(q) \cdot u = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{J} \end{bmatrix} \cdot (-q)$$

this matrix is p.d and symmetric

Since $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$, $\dot{x} = Ax$

Hence the controller is exponentially stable (theorem 7)

obviously. $q^*(0,0)$ is an equilibrium point

and A is Hurwitz (theorem 7), so the system can stabilize at the origin $q^*(0,0)$

2) b

denote $B(q)u$ as $k \cdot q$

denote the combination of D and steering force as matrix k_d (they are related to $\dot{q}$)

let $V = \frac{1}{2}(q^T k q + \dot{q}^T M \dot{q})$ V is p.d. $X = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$

$$\dot{V} = \dot{q}^T k q + \dot{q}^T M \ddot{q}$$
$$= -\dot{q}^T k q + \dot{q}^T(-kq - k_d \cdot \dot{q}) = -\dot{q}^T (k_d) \cdot \dot{q}$$

when the robot doesn't apply the 'steering' force

$k_d = D$ which is p.d and symmetric so $\dot{V} \leq 0$

when steering force applies: k_d becomes: $\begin{pmatrix} d_x & \frac{-k_0}{d(q)} & 0 \\ \frac{k_0}{d(q)} & d_y & 0 \\ 0 & 0 & d_\theta \end{pmatrix}$

since k_d is on diagonal

and $\frac{k_0}{d(q)} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is skew-symmetric

$X^T S X = 0$ for $\forall X$ if S is skew-symmetric

so for $\forall x' \in \mathbb{R}^3$ $x'^T k_d x'$ will remain the same

so $\dot{V} \leq 0$

from a) we know that $B(q) \cdot u = -q$

then the motion function is: $M \cdot \ddot{q} + D \cdot \dot{q} + q = 0$

when $\dot{q} = 0$, $\ddot{q} = 0 \Rightarrow q = 0$

according to Lasalle's principle, the origin is asymptotically stable

Since V is radically unbounded, the system is globally AS stable

5 let $V(x) = \frac{1}{2} x^T x$, then $\frac{1}{2}\|x\|^2 \leq V(x) \leq \frac{1}{2}\|x\|^2 \cdots\cdots$ ①

$\dot{V} = x^T \dot{x} = -a\left[x^T x + x^T S(x) x + (x^T x)^2\right]$

Since $S(x)$ is skew-symmetric, then $x^T S(x) \cdot x = 0$

so $\dot{V} = -a\left[I + x^T x\right] \cdot x^T x$

$= -a x^T x - a x^T x \cdot x^T x$

Since $x^T x \geq 0$, then $\dot{V} \leq -a x^T x \cdots\cdots$ ②

$\left\|\frac{\partial V}{\partial x}\right\| = \|x\| \leq \|x\| \cdots\cdots$ ③

from ① : let $\alpha_1 = \alpha_2 = \frac{1}{2}$

② : let $\alpha_3 = a$

③ : let $\alpha_4 = 1$

Using theorem 6: exponential stability theorem

the origin is globally exponentially stable.