EN.553.761: Nonlinear Optimization I

Homework Assignment #1

Exercise 1.1: Compute $\nabla f(x)$ and $\nabla^2 f(x)$ for the following functions $f : \mathbb{R}^n \to \mathbb{R}$.

(a) $f(x) = \frac{1}{2}x^T H x$, where $H \in \mathbb{R}^{n\times n}$ is a fixed matrix. What if H is symmetric?

(b) $f(x) = b^T A x - \frac{1}{2}x^T A^T A x$, where $A \in \mathbb{R}^{m\times n}$ is a fixed matrix and $b \in \mathbb{R}^m$ is a fixed vector.

(c) $f(x) = \|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$

(d) $f(x) = \|Ax - b\|_2$, where $A \in \mathbb{R}^{m\times n}$ is a fixed matrix and $b \in \mathbb{R}^m$ is a fixed vector.

Exercise 1.2: Let $\mathcal{S} \subseteq \mathbb{R}^n$, $f : \mathcal{S} \to \mathbb{R}$. Let $x \in \mathcal{S}$ and $s \in \mathbb{R}^n$ be such that $[x, x+s] \in \mathcal{S}$.

(a) By defining $\phi(\alpha) = f(x + \alpha s)$ and using the Fundamental Theorem of Calculus:

$$\phi(1) = \phi(0) + \int_0^1 \phi'(\alpha)d\alpha,$$

show that

$$|f(x+s) - f(x) - g(x)^T s| \le \tfrac{1}{2}\gamma^L \|s\|_2^2$$

whenever f has a Lipschitz continuous gradient with Lipschitz constant γ^L on $\mathcal{S}$.

(b) Justify the formula

$$\phi(1) = \phi(0) + \phi'(0) + \int_0^1 \int_0^\alpha \phi''(t)\; dt\; d\alpha.$$

Hence, show that

$$|f(x+s) - f(x) - g(x)^T s - \tfrac{1}{2}s^T H(x) s| \le \frac{1}{6}\gamma^Q \|s\|_2^3,$$

whenever f has a Lipschitz continuous Hessian with Lipschitz constant γ^Q on $\mathcal{S}$.

Exercise 1.3* Write a Matlab m-function that performs Newton's Method for finding a zero of a function $F : \mathbb{R}^n \to \mathbb{R}^n$. The function call should have the form

```
[x,F,J,iter,status] = newton( Fun,x0,maxit,printlevel,tol )
```

where `Fun` is of type *string* that holds the name of a Matlab m-function, `x0` is an initial guess at a zero, `maxit` is the maximum number of iterations allowed, `printlevel` determines the amount of printout required, and `tol` is the final stopping tolerance. The Matlab m-function `Fun` should have the form

```
[F,J] = Fun( x )
```

where `F` and `J` should contain the value and Jacobian of a desired function at the point `x`. In the code, if the parameter `printlevel` has the value zero, then no printing should occur; otherwise, a single line of output is printed (in column format) per iteration. On output, the parameters `x`, `F`, and `J` should contain the final iterate, function value, and Jacobian matrix computed by the algorithm. The paramter `iter` should contain the total number of iterations performed. Finally, `status` should have the value 0 if the final stopping tolerance was obtained and the value 1 otherwise.

Exercise 1.4* Let A be a given real symmetric matrix.

(a) Define an iteration of Newton's Method for solving the $n+1$ nonlinear equations

$$(A - \lambda I)x = 0 \quad \text{and} \quad x^T x = 1$$

in the $n+1$ unknowns (x, λ). Note that a zero (x, λ) is an eigenpair of the matrix A.

(b) Use the code you wrote for Exercise 2.1 to find an eigenpair (x, λ) for the matrix

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{with starting point} \quad x_0 = \begin{pmatrix} 1/5 \\ -1/5 \\ 4/5 \end{pmatrix} \quad \text{and} \quad \lambda_0 = 1.$$