EN.530.663: Robot Motion Planning
Homework 5 Solution

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problem 1.

$X = \{0, 1, 2, 3, 4\}$.

$\tau_1 = \{X, \phi, \{0\}, \{2,3\}, \{0,2,3\}, \{1,2,3,4\}\}$

① X and ϕ belong to τ_1 ✓.

② Union check:

→ $X \cup$ any other elements in $\tau_1 = X \in \tau_1$ ✓.

→ $\phi \cup$ any other elements in τ_1 = any other elements in $\tau_1 \in \tau_1$ ✓

→ $\{0\} \cup \{2,3\} = \{0\} \cup \{0,2,3\} = \{0,2,3\} \in \tau_1$ ✓

$\{0\} \cup \{1,2,3,4\} = X \in \tau_1$ ✓.

→ $\{0,2,3\} \cup \{1,2,3,4\} = X \in \tau_1$ ✓. Passed ✓

③ Intersection check:

→ $\phi \cap$ any other elements in $\tau_1 = \phi \in \tau_1$ ✓.

→ $X \cap$ any other elements in τ_1 = any other elements in $\tau_1 \in \tau_1$ ✓.

→ $\{0\} \cap \{2,3\} = \phi \in \tau_1$ ✓. $\{0\} \cap \{0,2,3\} = \{0\}$ ✓.

$\{0\} \cap \{1,2,3,4\} = \phi \in \tau_1$ ✓.

→ $\{2,3\} \cap \{0,2,3\} = \{2,3\} \cap \{1,2,3,4\} = \{2,3\} \in \tau_1$ ✓

→ $\{0,2,3\} \cap \{1,2,3,4\} = \{2,3\} \in \tau_1$ ✓.

passed ✓

$\therefore$ τ_1 is a topology.

$\tau_2 = \{X, \phi, \{0\}, \{2,3\}, \{0,2,3\}, \{1,2,3\}\}$.

① X and ϕ belong to τ_2 ✓.

② Union check:

→ X and ϕ follow the same results as in τ_1 ✓.

→ $\{0\} \cup \{1,2,3\} = \{0,1,2,3\} \notin \tau_2$ ⊗ counter-example.

Therefore τ_2 is not a topology

$\tau_3 = \{X, \phi, \{0\}, \{2,3\}, \{0,2,3\}, \{0,1,3,4\}\}$

① X and ϕ belong to τ_3 ✓.

② Union check :

→ X and ϕ follow the same results as in τ_1 ✓.

→ $\{0\} \cup \{2,3\} = \{0\} \cup \{0,2,3\} = \{0,2,3\} \in \tau_3$ ✓.

$\{0\} \cup \{0,1,3,4\} = \{0,1,3,4\}$ ✓.

→ $\{2,3\} \cup \{0,2,3\} = \{0,2,3\} \in \tau_3$ ✓.

$\{2,3\} \cup \{0,1,3,4\} = \{0,1,2,3,4\}$ ✓

→ $\{0,2,3\} \cup \{0,1,3,4\} = \{0,1,2,3,4\}$ ✓.

③ Intersection check :

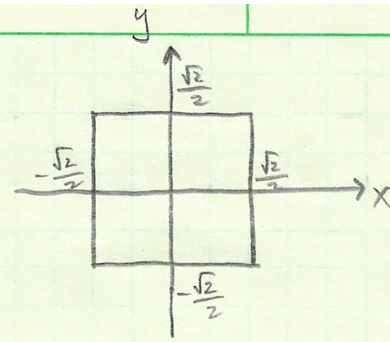
→ ϕ and X follow the same results as in τ_1 ✓.

→ $\{2,3\} \cap \{0,1,3,4\} = \{3\} \notin \tau_3$ ⊗ counter-example

$\boxed{\therefore\ \tau_3 \text{ is not a topology.}}$

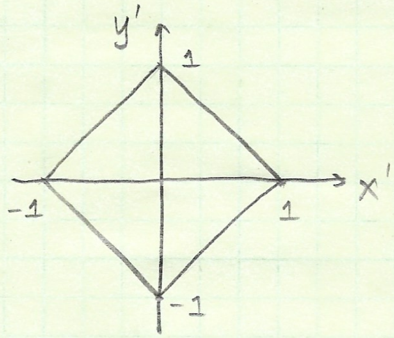
problem 2:

a) τ :



Homeomorphic transform τ by rotating 45° ccw.

τ' :



$$f: \begin{bmatrix} y' \\ x' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\theta = 45^\circ$.

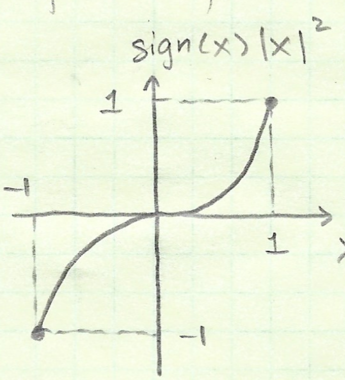
it's bijective and continuous.

τ' can be represented by line functions also:

$$\tau' = \{ (x,y) \in \mathbb{R}^2 \mid |x| + |y| = 1 \}.$$

also, $$S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$$

Define a function: $f: [-1, 1] \mapsto [-1, 1]$: $\begin{cases} x = \text{sign}(x)|x|^2 \\ y = \text{sign}(y)|y|^2 \end{cases}$



where $\text{sign}(a) = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -1, & a < 0 \end{cases}$.

it's continuous and bijective.

$\therefore$ f^{-1} exists.

Therefore, S^1 and τ' are homeomorphic to each other, or equivalently S^1 and τ are also homeomorphic to each other.

b) $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \le 4x^2+y^2 \le 4\}$.

$B = \{(x,y) \in \mathbb{R}^3 \mid x^2+y^2 = 1,\ 0 \le z \le 1\}$.

Homeomorphic transform of A first:

$$A' = \{(\theta, \rho) \in \mathbb{R}^2 \mid 0 < \theta \le 2\pi,\ 1 \le \rho \le 2\} \quad ①$$

$$f_1 : \begin{cases} x = (\rho \cdot \cos\theta)/2 \\ y = \rho \cdot \sin\theta \end{cases}$$ it's bijective and continuous.

$\therefore$ A' and A are equivalent.

Homeomorphic transform of B:

$$B' = \{(\theta, z) \in \mathbb{R}^2 \mid 0 \le \theta \le 2\pi,\ 0 \le z \le 1\} \quad ②$$

$$f_2 = \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$$ it's bijective and continous.

$\therefore$ B' and B are equivalent.

Show ① & ② are homeomorphic:

$$f_3 : \begin{array}{lll} \theta \mapsto \theta & : & \theta = \theta \\ \rho \mapsto z & : & \rho = z+1 \end{array}$$

it's bijective and continuous.

Therefore, A and B are homeomorphic to each other.

problem 3.

a) Check if f is bijective.

Define $\begin{cases} x_1' = e^{x_1} & \text{where } x_1 \in \mathbb{R},\ x_1' \in (0, \infty) \\ x_2' = 3x_2 + 1 & \text{where } x_2 \in \mathbb{R},\ x_2' \in \mathbb{R} \end{cases}$

$\hookrightarrow \begin{cases} x_1 = \ln(x_1') \\ x_2 = (x_2' - 1)/3. \end{cases}$ is f^{-1} with $\begin{cases} \text{domain} & \begin{cases} x_1' \in (0, \infty) \\ x_2' \in \mathbb{R} \end{cases} \\ \text{codomain} & \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R}. \end{cases} \end{cases}$

so f is surjective.

Check injective ?

If $\exists (a, b) \neq (a', b')$ s.t. $f(a', b') = f(a, b)$, then it's not injective

$\hookrightarrow$ Let $\begin{cases} e^a = e^{a'} \\ 3b + 1 = 3b' + 1 \end{cases} \Leftrightarrow \begin{cases} a = a' \\ b = b' \end{cases}$ this does not hold

note that in the current definition f: R –> R, f is not bijective.
But once you change the domain from R to R>=0, then f is bijective.

Check if f is C^∞ differentiable:

1st: $\dfrac{\partial f_1}{\partial x_1} = e^{x_1}$; $\dfrac{\partial f_1}{\partial x_2} = 0$. | $\dfrac{\partial f_2}{\partial x_1} = 0$ $\quad \dfrac{\partial f_2}{\partial x_2} = 3$.

2nd: $\dfrac{\partial^2 f_1}{\partial x_1^2} = e^{x_1}$; $\dfrac{\partial^2 f_1}{\partial x_2^2} = 0$; | $\dfrac{\partial^2 f_2}{\partial x_1^2} = 0$ $\quad \dfrac{\partial^2 f_2}{\partial x_2^2} = 0$.

$\dfrac{\partial^2 f_1}{\partial x_1 \partial x_2} = 0$ $\quad\quad \dfrac{\partial^2 f_2}{\partial x_1 \partial x_2} = 0$

$\vdots \quad\quad\quad \vdots$

$\therefore \dfrac{\partial^{n+m} f_1}{\partial x_1^n \partial x_2^m} = \begin{cases} e^{x_1}, & m = 0. \\ 0, & m \neq 0, \end{cases}$ $\quad \dfrac{\partial^{n+m} f_2}{\partial x_1^n \partial x_2^m} = \begin{cases} 3, & m = 1. \\ 0, & m \neq 1 \end{cases}$

where m, n are non-negative integers.

$\boxed{\text{From above, we see } f \text{ is } C^\infty \text{ differentiable.}}$

Check if f is diffeomorphism with Inverse Function Theorem.

$\forall (x_1, x_2) \in \mathbb{R}^2, \quad J = \begin{bmatrix} e^{x_1} & 0 \\ 0 & 3 \end{bmatrix}$

$\det(J) = 3e^{x_1} > 0 \quad \forall x_1 \in \mathbb{R}$

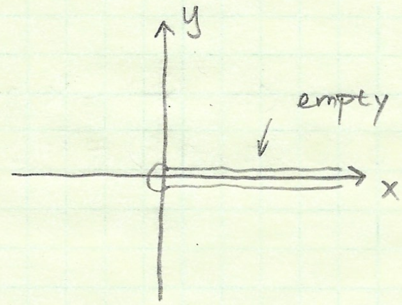
According to inverse function theorem, $\boxed{f \text{ is a diffeomorphism.}}$

again, if you use the different codomain of f.

b). $f: (0,\infty)\times(0,2\pi) \to \mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\} : (r,\theta) \to (x,y)$

Find the mapping first:

$\mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\}$ is visualized as:



re-represent as:

$\{(r,\theta) \mid r>0,\ 0<\theta<2\pi\}$

$$f: \begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases} \qquad f^{-1}: \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \operatorname{atan2}(y,x) \end{cases}$$

↳ it's surjective

Check injective?

Let $(r',\theta') \in (0,\infty),(0,2\pi)$

↳ set $\begin{cases} r\cdot\cos\theta = r'\cos\theta' \\ r\cdot\sin\theta = r'\cos\theta' \end{cases} \iff \begin{cases} \theta = \theta' \\ r = r' \end{cases}$ due to atan2 property.

↳ it's injective.

$\therefore$ $\boxed{f \text{ is bijective}}$

Check f is C^∞ differentiable

1st: $\frac{\partial f_1}{\partial r} = \cos\theta \qquad \frac{\partial f_1}{\partial\theta} = -r\sin\theta. \qquad \frac{\partial f_2}{\partial r} = \sin\theta \qquad \frac{\partial f_2}{\partial\theta} = r\cos\theta.$

2nd: $\frac{\partial^2 f_1}{\partial r^2} = 0 \qquad \frac{\partial^2 f_1}{\partial^2\theta} = -r\cos\theta. \qquad \frac{\partial^2 f_2}{\partial r^2} = 0 \qquad \frac{\partial^2 f_2}{\partial\theta^2} = -r\sin\theta.$

$\frac{\partial^2 f_1}{\partial r\partial\theta} = -\sin\theta. \qquad \frac{\partial^2 f_1}{\partial r\partial\theta} = \cos\theta.$

$\vdots \qquad \vdots$

$\boxed{\text{From above, we can see } f \text{ is } C^\infty \text{ differentiable.}}$

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \quad \forall \begin{matrix} r\in(0,\infty) \\ \theta\in(0,2\pi) \end{matrix} \qquad \det(J) = r\in(0,\infty) \neq 0$$

$\boxed{\text{Therefore, } f \text{ is a diffeomorphism according to Inverse Function Thm}}$

Problem 4

(a) $D = \{(r,\theta) \in \underbrace{\mathbb{R}_{\geq 0} \times [0,2\pi]/\sim}_{\text{It can be } \mathbb{R}_{\geq 0} \times S^1} \mid r \in (1,2),\ \theta \in [-\pi,\pi)\}$

It can be $[0,2\pi)$, ...

(b) $$f(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}, \quad r \in (1,2),\ \theta \in [-\pi,\pi)$$

(c) Let

$$\gamma(t) = \begin{cases} (r_1 \cos\theta_1(t),\ r_1 \sin\theta(t))^T, & 0 \leq t \leq 1/2 \\ (r(t)\cos\theta_2,\ r(t)\sin\theta_2)^T, & 1/2 < t \leq 1 \end{cases}$$

with $\theta(t) = \theta_1 + 2(\theta_2 - \theta_1)t$

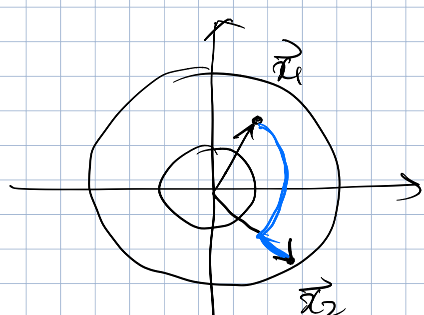
$r(t) = r_1 + 2(r_2 - r_1)(t - \frac{1}{2})$

and $\vec{x}_1 = \begin{pmatrix} r_1\cos\theta_1 \\ r_1 \sin\theta_1 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} r_2\cos\theta_2 \\ r_2\sin\theta_2 \end{pmatrix}$

$\forall\, \vec{x}_1, \vec{x}_2 \in S$

$\gamma(t)$ exists such that $\gamma(0) = \vec{x}_1$ and $\gamma(1) = \vec{x}_2$

$\Rightarrow$ path-connected!



Problem 5

(a) $\phi_1(x,y) = (x,y)$

$\phi_2(x,y) = (\sqrt{x^2+y^2},\ \mathrm{atan2}(y,x))$

(b) $\phi_1(U_1 \cap U_2) = \mathbb{R}^2 \setminus \{(x,0) \mid x \le 0\}$

$\phi_2(U_1 \cap U_2) = \{(r,\theta) \mid r>0,\ \theta \in (-\pi,\pi)\}$

$\phi_2(r,\theta) = (r\cos\theta,\ r\sin\theta)$

(c) $\phi_2(x,y) = (\phi_2 \circ \phi_1^{-1})(x,y)$ in $U_1 \cap U_2$

$= (\sqrt{x^2+y^2},\ \mathrm{atan2}(y,x))$

$$\underset{\sim}{J} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

$$\det(\underset{\sim}{J}) = \frac{1}{\sqrt{x^2+y^2}}$$

since $(x,y) \in U_1 \cap U_2 \rightarrow (0,0) \notin U_1 \cap U_2$

$\therefore \underline{\det(\underset{\sim}{J}) \neq 0}$ //

$\therefore$ locally diffeomorphism, since $\phi_{12}(x,y)$ is bijective, it is globally diffeomorphism