EN.530.663: Robot Motion Planning
Homework 4 Solution

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Problem 2

(a) $R_{euler}(\alpha, \beta, \gamma) = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)$

$$= \begin{bmatrix} C\alpha C\beta C\gamma - S\alpha S\gamma & -S\alpha C\gamma - C\alpha C\beta S\gamma & C\alpha S\beta \\ C\alpha S\gamma + S\alpha C\beta C\gamma & C\alpha C\gamma - S\alpha C\beta S\gamma & S\alpha S\beta \\ -S\beta C\gamma & S\beta S\gamma & C\beta \end{bmatrix}$$

(b) $\sin(\alpha - \pi) = -\sin\alpha, \quad \cos(\alpha - \pi) = -\cos\alpha$

$\sin(-\beta) = -\sin\beta, \quad \cos(-\beta) = \cos\beta$

$\sin(\gamma - \pi) = -\sin\gamma, \quad \cos(\gamma - \pi) = -\cos(\gamma)$

$R_{euler}(\alpha - \pi, -\beta, \gamma - \pi)$

$$= \begin{bmatrix} -C\gamma C\beta(-C\gamma) - (-S\alpha) - S\gamma & C\alpha C\beta(-S\gamma) + S\alpha(-C\gamma) & (-C\alpha)(-S\beta) \\ (-S\alpha)C\beta(-C\gamma) + (-S\gamma)(-C\alpha) & (-C\alpha)(-C\gamma) - (-S\alpha)C\beta(-S\gamma) & (-S\alpha)(-S\beta) \\ -(-S\beta)(-C\gamma) & (-S\beta)(-S\gamma) & C\beta \end{bmatrix}$$

$= R_{euler}(\alpha, \beta, \gamma)$

(c) $\dfrac{r_{23}}{r_{13}} = \dfrac{S\alpha S\beta}{C\alpha S\beta} = \tan(\alpha) \Rightarrow \alpha = \operatorname{atan2}(r_{23}, r_{13})$

$r_{33} = \cos\beta, \quad \sqrt{1 - r_{33}^2} = |\sin\beta|, \quad \beta = \operatorname{atan2}(\sqrt{1 - r_{33}^2}, r_{33})$

$r_{32} = \sin\beta \sin\gamma, \quad r_{31} = -\sin\beta \cos\gamma, \quad \gamma = \operatorname{atan2}(r_{32}, -r_{31})$

Problem 3

$H = \{(x,y) \mid x^2+y^2 \le 1\}$

Transformed coordinates is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C\theta & -S\theta \\ S\theta & C\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xC\theta - yS\theta \\ xS\theta + yC\theta \end{bmatrix}$$

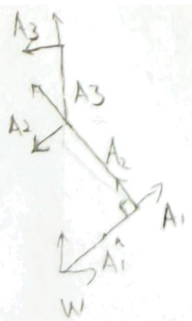
$$x'^2 + y'^2 = (xC\theta - yS\theta)^2 + (xS\theta + yC\theta)^2 = x^2 + y^2 \le 1$$

Therefore $x^2+y^2 \le 1 \iff x'^2 + y'^2 \le 1$

$H' = \{(x',y') \mid x'^2+y'^2 \le 1\}$ is equivalent to H

So the primitive is unchanged when rotation is applied.

Problem 4



(a) Consider homogeneous transformation to A_1, A_2, A_3 frame

$$R_{A_1W} = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) & 5\sqrt{2} \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) & 5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$a = R_{A_1W}\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5\sqrt{2} \\ 2.5\sqrt{2} \\ 1 \end{bmatrix} \qquad (X_a, Y_a) = (2.5\sqrt{2},\ 2.5\sqrt{2})$$

$$R_{A_2A_1} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & 0 \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 10 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b = R_{A_1W} R_{A_2A_1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5\sqrt{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 10\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5\sqrt{2} \\ 10.5\sqrt{2} \\ 1 \end{bmatrix}$$

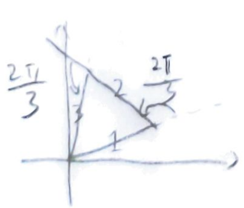
$$(x_b, y_b) = (-0.5\sqrt{2},\ 10.5\sqrt{2})$$

$$R_{A_3A_2} = \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) & 5\sqrt{2} \\ \sin(-\frac{\pi}{4}) & \cos(\frac{\pi}{4}) & -5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 5\sqrt{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$c = R_{A_1W} R_{A_2A_1} R_{A_3A_2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 10+10\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 11+10\sqrt{2} \\ 1 \end{bmatrix}$$

$$(x_c, y_c) = (-1,\ 11+10\sqrt{2})$$

(b)

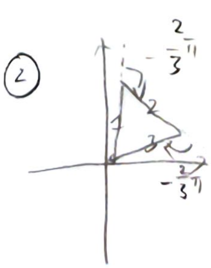


① $\theta_1 \in \mathbb{R}$

$\theta_2 = \frac{2}{3}\pi + 2k\pi,\ k = 0, \pm 1, \ldots$

$\theta_3 = \frac{2}{3}\pi + 2k\pi,\ k = 0, \pm 1, \ldots$

②



$\theta_1 \in \mathbb{R}$

$\theta_2 = -\frac{2}{3}\pi + 2k\pi,\ k = 0, \pm 1, \ldots$

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