EN.530.663: Robot Motion Planning
Homework 1 Solution

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Problem 1

$f: [0,1] \to \mathbb{R}$ $\quad \frac{d^2f}{dx^2} + 4\pi^2 n^2 f = 0 \quad f(0) = f(1) = 0$

Solve the equation: → there are several ways.

one way: assume $f(x) = e^{\lambda x}$ to plug in the equation.

$\rightarrow (\lambda^2 + 4\pi^2 n^2) e^{\lambda x} = 0 \quad \rightarrow \lambda = \pm i 2\pi n \quad (i^2 = -1)$

$f(x) = a_1 e^{i 2\pi n x} + a_2 e^{-i 2\pi n x}$

by using the Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$

$\therefore f(x) = c_1 \cos(2\pi n x) + c_2 \sin(2\pi n x)$ //

Now boundary conditions give $\boxed{\therefore f(x) = c_2 \sin(2\pi n x)}$

$\because f(0) = c_1 = f(1) = 0 \quad \therefore c_1 = 0.$

(a) Hence the elements the set $\{f_0, f_1, f_2, \cdots\}$

$\Rightarrow \underline{f_n = \sin(2\pi n x)}$ // $(n = 0, 1, 2, \cdots)$

(b) The set : $\mathcal{X} = \{ y(x) \in \mathbb{R} \mid y(x) = \sum_{n=0}^{\infty} a_n \sin(2\pi n x), a_n \in \mathbb{R} \}$

Consider $y_i(x) = \sum_{n=0}^{\infty} a_n^{(i)} \sin(2\pi n x)$

$y_j(x) = \sum_{n=0}^{\infty} a_n^{(j)} \sin(2\pi n x) \qquad a_n^{(i)}, a_n^{(j)}, a_n^{(k)} \in \mathbb{R}$

$y_k(x) = \sum_{n=0}^{\infty} a_n^{(k)} \sin(2\pi n x)$

'+' ⇒ sum $\qquad$ • scalar (real) multiplication

↳ closed! $\qquad$ ↳ closed!

You can check all 8 properties (skip here)

→ vector space!

(c) since span$\{f_0, f_1, f_2 \cdots\}$ forms any solution to the equation $\{f_0, f_1, f_2 \cdots\}$ is the basis set.

"f_i cannot be expressed with others"

Hence the dimension is <u>infinity</u> //

Problem 2 let $\underset{\sim}{A} = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 2 & 4 \\ 3 & 9 & 4 & 6 \end{pmatrix} = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4]$

(a) $\underset{\sim}{A} \vec{v} = \vec{0}$, $\vec{v} = [v_1, v_2, v_3, v_4]^T$

We can find that

$\vec{a}_2 = 3\vec{a}_1$, $\vec{a}_4 = 2\vec{a}_1$, and $\vec{a}_1, \vec{a}_3$: L.I.

Hence $\underset{\sim}{A}\vec{v} = \vec{0}$

$\Leftrightarrow v_1 \vec{a}_1 + v_2 \vec{a}_2 + v_3 \vec{a}_3 + v_4 \vec{a}_4 = \vec{0}$

$\Leftrightarrow (v_1 + 3v_2 + 2v_4)\vec{a}_1 + v_3 \vec{a}_3 = \vec{0}$

$\Leftrightarrow \begin{cases} v_1 + 3v_2 + 2v_4 = 0 \\ v_3 = 0 \end{cases}$ ✓ $\rightarrow$ 1 eq, 3 var's.

So: let $v_2 = 1$, $v_4 = 0 \rightarrow v_1 = -3$

let $v_2 = 0$, $v_4 = 1 \rightarrow v_1 = -2$

Hence

$$N(A) = \text{span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Note There can be many other forms.
But there should be 2 elements in $N(\underset{\sim}{A})$.

(b) $\text{rank}(\underset{\sim}{A}) = 2$ one way to see is: $\vec{a}_1, \vec{a}_3$ are L.I.

(c) see attached page.

```
>> A = [1 3 0 2; 2 6 2 4; 3 9 4 6]

A =

     1     3     0     2
     2     6     2     4
     3     9     4     6

>> null(A)

ans =

    0.5345   -0.8018
    0.3382    0.4927
    0.0000   -0.0000
   -0.7745   -0.3382

>>
```

Problem 3

$\lambda_i, \vec{v}_i$: eigenpairs of A $\rightarrow$ $A\vec{v}_i = \lambda_i \vec{v}_i$ $(i=1,\dots,n)$

$A = PBP^{-1}$ $\rightarrow$ $PBP^{-1}\vec{v}_i = \lambda_i \vec{v}_i$

since P is invertible, multiply by P^{-1} on both sides.

$\rightarrow$ $BP^{-1}\vec{v}_i = \lambda_i P^{-1}\vec{v}_i$

Let $\vec{u}_i = P^{-1}\vec{v}_i$, then $B\vec{u}_i = \lambda_i \vec{u}_i$.

Hence the eigenvalues of B are λ_i (i.e., the same as A)

and the eigenvectors ~~of~~ B are $P^{-1}\vec{v}_i$ //

Problem 4

(a) $x = f(u,v) \quad y = g(u,v) \qquad (x,y)^T \in \mathbb{R}^2$

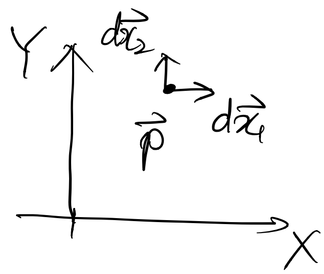
$$J(u,v) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}$$

First, let's consider how we have $dA = dx\,dy$ in Cartesian. tangent vectors that represent the direction of dx and dy direction are calculated as follows.

Let $\vec{p} = \begin{pmatrix} x \\ y \end{pmatrix}$, (position of a point in $\mathbb{R}^2$, Cartesian coordinates)

$$d\vec{x}_1 = \frac{\partial \vec{p}}{\partial x} dx = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dx = \vec{e}_1\, dx$$

$$d\vec{x}_2 = \frac{\partial \vec{p}}{\partial y} dy = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dy = \vec{e}_2\, dy$$

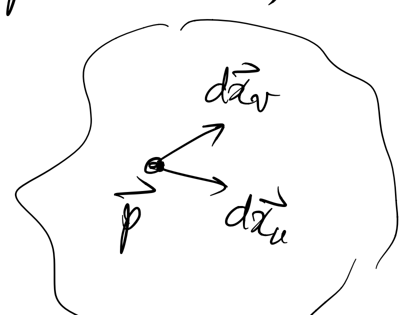


Then

$$dA = \| d\vec{x}_1 \times d\vec{x}_2 \| = dx\,dy\, \| \vec{e}_1 \times \vec{e}_2 \| = dx\,dy.$$

likewise for the same point $\vec{p}$ in (u,v) coordinates,

$$d\vec{x}_u = \frac{\partial \vec{p}}{\partial u} du = \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial g}{\partial u} \end{pmatrix} du$$



$$d\vec{x}_v = \frac{\partial \vec{p}}{\partial v} dv = \begin{pmatrix} \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial v} \end{pmatrix} dv$$

Then

$$dA = \| d\vec{x}_u \times d\vec{x}_v \| = du\,dv \left\| \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial g}{\partial u} \\ 0 \end{pmatrix} , \begin{pmatrix} \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial v} \\ 0 \end{pmatrix} \right\|$$

$$= \Big| \underbrace{\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u}}_{=: \det(\underset{\sim}{J})} \Big| \, du\,dv.$$

$$\underline{= \left| \det \underset{\sim}{J}(u,v) \right| du\,dv} \;//$$

(b) $u = r,\ v = \theta$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}} \Leftarrow \text{mapping from } (r,\theta) \text{ to } (x,y)$$

//

$$\underset{\sim}{J}(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$|\det \underset{\sim}{J}| = | r\cos^2\theta + r\sin^2\theta | = |r| = r$$

$(\because\ r \geq 0)$

$$\therefore\ \underline{dA = r\,dr\,d\theta} \;//$$

(c) Let $\vec{f} : (r, \theta) \to (x, y)$

s.t. $\vec{f}(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

$\therefore \left.\begin{array}{l} r \in [0, \infty) \\ \theta \in [0, 2\pi) \end{array}\right]$ makes $\vec{f}$ surjective.

<u>Note</u>: $\begin{array}{l} r \in [0, \infty) \\ \theta \in [0, 2\pi] \end{array}$ also possible

Problem 5

(a) $G=(V, E)$: simple graph.

→ only one edge per pair of nodes, no loops.

The maximum case is the one wher all pairs of nodes are adjacent, in which case the number of edges $|E|$ should be

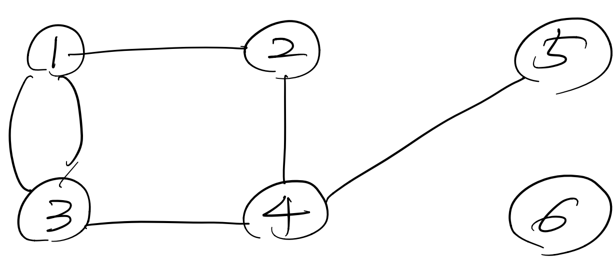
$$\binom{|V|}{2} = C_2^{|V|}$$: binomial coefficient.

↳ # possible ways to select 2 out of $|V|$ without considering orders.

Hence $|E| \le \binom{|V|}{2}$ //

(b) $G=(V, E)$ $V=\{1, 2, 3, 4, 5\}$

$E=\{(1,2), (1,3), (2,4), (3,4), (4,5), (3,1)\}$

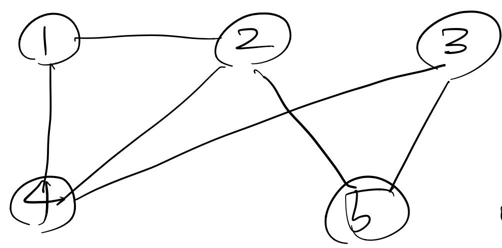


← diagram

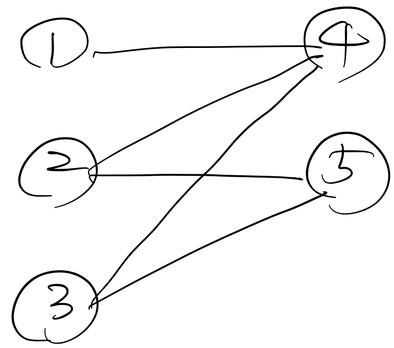
$G' = (V', E')$ $\quad V' = \{1, 2, 3, 4, 5\}$

$E' = \{(1,4), (2,4) (2,5) (3,4), (3,5)\}$

diagram



or.



(c) G is <u>not</u> a simple graph ($\because$ (1,3) : parallel edges)

G' is a simple graph

o G and G' are <u>not</u> isomorphic $|V| \neq |V'|$

(G is NOT a simple graph but G' is simple)

(d) $$A_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A_{G'} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$