EN.530.663: Robot Motion Planning
Homework 3 Solution

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Problem 1 Suppose 2 convex set S_1 and S_2.

$\because S_1, S_2$ convex $\Rightarrow$ for any $x_1, x_2 \in S_1$ and $x_1', x_2' \in S_2$

$$x = \varepsilon x_1 + (1-\varepsilon)x_2 \in S_1, \text{ for } \forall \varepsilon \in [0,1] \quad ①$$
$$x' = \varepsilon x_1' + (1-\varepsilon)x_2' \in S_2, \text{ for } \forall \varepsilon \in [0,1] \quad ②$$

Then for any $x_{k_1}, x_{k_2} \in S_1 \cap S_2 \subseteq S_1 \Rightarrow x_{k_1}, x_{k_2} \in S_1$

$① \Rightarrow$ for $\forall \varepsilon \in [0,1]$, $x = \varepsilon x_{k_1} + (1-\varepsilon)x_{k_2} \in S_1$

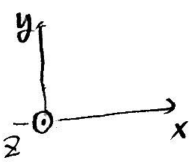
Similarly $x_{k_1}, x_{k_2} \in S_1 \cap S_2 \subseteq S_2 \Rightarrow x_{k_1}, x_{k_2} \in S_2$.

$② \Rightarrow$ for $\forall \varepsilon \in [0,1]$, $x = \varepsilon x_{k_1} + (1-\varepsilon)x_{k_2} \in S_2$

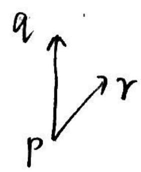
$\Rightarrow$ for $\forall \varepsilon \in [0,1]$, $x = \varepsilon x_{k_1} + (1-\varepsilon)x_{k_2} \in S_1 \cap S_2$

$\Rightarrow S_1 \cap S_2$ is convex.

Problem 2 Establish an coordinate system O-x,y,z as follow (right-handed) (so that the cross-product is well defined.), and suppose that all the points p, q, r lies in the x, y-plane



CASE I. $(q_y > p_y)$



$$\vec{pq} \times \vec{pr} = \begin{bmatrix} q_x - p_x \\ q_y - p_y \\ 0 \end{bmatrix} \times \begin{bmatrix} r_x - p_x \\ r_y - p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \begin{pmatrix} q_x r_y - q_y p_x - p_x r_y - r_x q_y \\ + r_x p_y + p_x q_y \end{pmatrix} \end{bmatrix}$$

$$(\vec{pq} \times \vec{qr})_z = \begin{bmatrix} q_x & r_x \\ q_y & r_y \end{bmatrix} - \begin{bmatrix} p_x & r_x \\ p_y & r_y \end{bmatrix} + \begin{bmatrix} p_x & q_x \\ p_y & q_y \end{bmatrix} = D$$

$\therefore$ when $D > 0$, r lies to the left of line (p, q)

$D < 0$, r lies to the right of line (p, q)

CASE II $(q_y < p_y)$

Similarly, when $D > 0$, r lines to the right

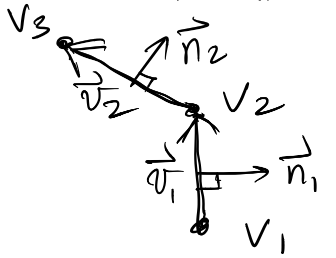
$D < 0$, r lines to the left.

CASE III $(q_y = p_y)$ and CASE IV extend to determine whether r is above or below the horizontal line.

You can verify by yoreself.

Problem 3

Consider three consecutive vertices



$V_1 : (x_1, y_1) = \vec{P}_1^T$

$V_2 : (x_2, y_2) = \vec{P}_2^T$

$V_3 : (x_3, y_3) = \vec{P}_3^T$

$$\vec{v}_1 = \vec{P}_2 - \vec{P}_1 = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

$$\vec{v}_2 = \vec{P}_3 - \vec{P}_2 = \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix}$$

by rotating $\vec{v}_i$ CW, we have $\vec{n}_i$

$$\vec{n}_1 = \begin{pmatrix} \cos(-90^\circ) & -\sin(-90^\circ) \\ \sin(-90^\circ) & \cos(90^\circ) \end{pmatrix} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

$$= \begin{pmatrix} y_2 - y_1 \\ -x_2 + x_1 \end{pmatrix}$$

$$\vec{n}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix} = \begin{pmatrix} y_3 - y_2 \\ -x_3 + x_2 \end{pmatrix}$$

Compute $\begin{pmatrix} \vec{n}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} \vec{n}_2 \\ 0 \end{pmatrix} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ y_2 - y_1 & x_1 - x_2 & 0 \\ y_3 - y_2 & x_2 - x_3 & 0 \end{vmatrix}$

z-coordinate $= (y_2 - y_1)(x_2 - x_3) - (x_1 - x_2)(y_3 - y_2)$ — ①

We want to show that ① > 0.

Use the result from Problem 2.

If V_3 is on the left of the line from V_1 to V_2

Then $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} > 0$

$$\Rightarrow x_2y_3 - x_3y_2 - (x_1y_3 - x_3y_1) + x_1y_2 - x_2y_1 > 0$$

$$x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1 > 0 \quad -②$$

Going back to ①, expand:

$$\cancel{x_2y_2} - x_2y_1 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 + x_2y_3 - \cancel{x_2y_2}$$

$$= x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1 \underline{> 0}$$

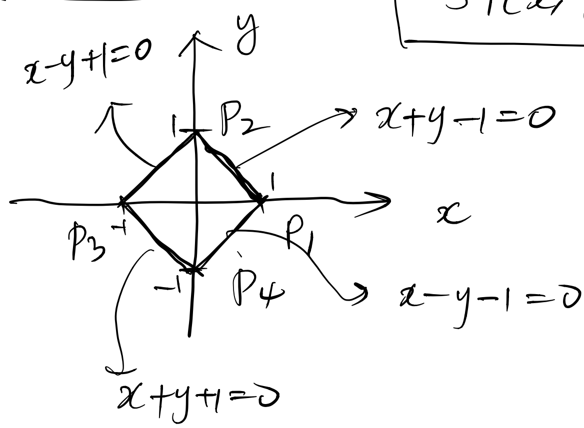
from ②

$\Rightarrow \vec{n}_1 \times \vec{n}_2$ always positive $\rightarrow$ convex!

$\therefore \vec{P}_{i+2}$ is always located on the left side of the line from $\vec{P}_i$ to $\vec{P}_{i+1}$

(WLOG $\vec{P}_i \leftrightarrow \vec{P}_1$, $\vec{P}_{i+1} \leftrightarrow \vec{P}_2$, $\vec{P}_{i+2} \leftrightarrow \vec{P}_3$)

Problem 4



$\boxed{f_1(x, y) = x+y-1}$

$\boxed{f_2(x, y) = -x+y-1}$

$\boxed{f_3(x, y) = -x-y-1}$

$\boxed{f_4(x, y) = x-y-1}$

$\therefore \; H_i = \{(x, y) \in \mathbb{R}^2 \mid f_i(x, y) \leq 0\} \quad (i = 1, 2, 3, 4)$

and the solid representation

$= H_1 \cap H_2 \cap H_3 \cap H_4$ //

The above is just the answer.
You can use any method from the class.
Just show your work.

Problem 5

$$f_1 = x^2 + y^2 - r_1^2 \le 0$$

$$f_2 = -\left[(x - x_2)^2 + (y - y_2)^2 - r_2^2 \right] \le 0$$

$$f_3 = -\left[(x - x_3)^2 + (y - y_3)^2 - r_3^2 \right] \le 0$$

$$f_4 = -\left[\frac{x^2}{a^2} + \frac{(y - y_4)^2}{b^2} - 1 \right] \le 0$$

$$f_5 = \frac{3\pi}{2} - y + 2 \le 0$$

$$f_6 = -\frac{3}{2}x - y + 2 \le 0$$

$$f_7 = 1 + y \le 0$$