LOGARITHMIC LAWS

LESSON OBJECTIVES

It's always good practice to check off what you've learnt. By the end of this lesson, you should have completed the following:

- Establish and use logarithmic laws and definitions
- Interpret and use logarithmic scales such as decibels in acoustics, the Richter scale for earthquake magnitude, octaves in music, pH in chemistry
- Solve equations involving indices with and without technology

LOGARITHMIC LAWS

You will remember from earlier in the course that logarithms are defined to be the inverse of indices. The relationship between logarithms and indices is given by the following:

$$a^x = b \Leftrightarrow \log_a(b) = x$$

Given this close relationship, it follows that the laws we apply to indices are similar to those we will develop for logarithms. A useful fact we will use to establish the logarithmic laws is as follows:

$$\log_b(b^c) = c$$

This statement is equivalent, in terms of indices, to

$$b^c = b^c$$

Finally, recall the restrictions on logarithms:

- $a > 0, a \neq 1$ (must be a positive number not equal to one)
- $x > 0$ (must be positive)
- $b \in \mathbb{R}$ (can be any real number, positive or negative)

Logarithm Product Rule

The logarithm product rule is given as follows:

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

We can show this to be true by letting $x = a^i$ and $y = a^j$, meaning also that $\log_a(x) = i$ and $\log_a(y) = j$:

$$\begin{aligned}
\log_a(xy) &= \log_a(a^i a^j) && \text{(substitution)} \\
&= \log_a(a^{i+j}) && \text{(index law)} \\
&= i + j && \text{(previously established fact)} \\
&= \log_a x + \log_a y && \text{(substitution)}
\end{aligned}$$

Logarithm Quotient Rule

The logarithm quotient rule is given as follows:

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

To show this, we use the same setup as beforehand:

$$\begin{aligned}
\log_a\left(\frac{x}{y}\right) &= \log_a\left(\frac{a^i}{a^j}\right) && \text{(substitution)} \\
&= \log_a(a^{i-j}) && \text{(index law)} \\
&= i - j && \text{(previously established fact)} \\
&= \log_a(x) - \log_a(y) && \text{(substitution)}
\end{aligned}$$

Logarithm Power Rule

The logarithm power rule is given as follows:

$$\log_a(x^n) = n\log_a(x)$$

To show this, we again use the same setup, this time only needing $x = a^i$:

$$\begin{aligned}
\log_a(x^n) &= \log_a\left((a^i)^n\right) && \text{(substitution)} \\
&= \log_a(a^{in}) && \text{(index law)} \\
&= in && \text{(previously established fact)} \\
&= n\log_a(x) && \text{(substitution and rearrangement)}
\end{aligned}$$

Logarithm Base Change Rule

The logarithm base change rule is given as follows:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

The proof for this rule is slightly different to the previous ones but follows the same logic of substitution and using preestablished rules. First, let $\log_a(x) = y$, so:

$$\begin{aligned}
a^y &= x && \text{(taking log of both sides)}\\
y\log_b(a) &= \log_b(x) && \text{(using power rule)}\\
\log_a(x)\log_b(a) &= \log_b(x) && \text{(substitution)}\\
\log_a(x) &= \frac{\log_b(x)}{\log_b(a)} && \text{(rearrangement)}
\end{aligned}$$

This rule is particularly important when calculating logarithms since you will often need to use a change of base to input the logarithm on your calculator.

Example:

Simplify $\log_a(x) - \log_b(x) + n\log_a(x)$ in the form $\log_a(x^c)$, where c is some constant.

Solution:

This simplification involves the application of multiple logarithmic laws and some basic algebra:

$$\begin{aligned}
\log_a(x) - \log_b(x) + n\log_a(x) &= \log_a(x) - \frac{\log_a(x)}{\log_a(b)} + n\log_a(x) && \text{(change of base)}\\
&= \left(1 - \frac{1}{\log_a(b)} + n\right)(\log_a(x)) && \text{(collecting terms)}\\
&= \log_a\left(x^{\left(1-\frac{1}{\log_a(b)}+n\right)}\right) && \text{(applying power rule)}
\end{aligned}$$

Questions

Simplify the following:

1. $\log_8(8^{15})$

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2. $8\log_{16}(2)$

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3. $5\log_5(20)$ (Hint: if you are unsure, try letting the equation equal x and solving for x)

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Calculate the following:

4. $\log_{10}(25) + 2\log_{10}(2)$

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5. $\log_3(32) - 5\log_3(2)$

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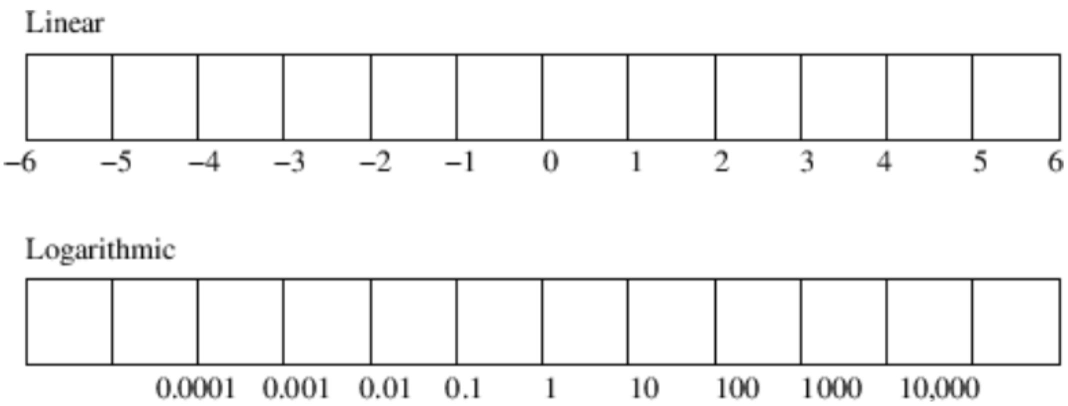
LOGARITHMIC SCALES

Logarithms are important in a range of different areas, particularly where we want to compare things that vary by multiple orders of magnitude. For example, whilst a linear scale might change in increments of 1, a logarithmic scale might change in factors of 10:



Decibel Scale

The decibel scale is a type of logarithmic scale that is used to describe sound intensity. Because of the way that the ear works, the difference in intensity (measured in watts per metre square) between two sounds might be orders of magnitude apart whilst sounding quite similar. Hence, the decibel scale compares sound intensity in a way that is more relatable for practical use.

The scale is given below, where L represents the level in decibels (dB) that corresponds to a given sound wave with intensity I in W/m^2. I_0 is the reference intensity and corresponds to 0 dB, which is the intensity of a 1000 Hz wave at the threshold of hearing ($\sim 10^{-12} W/m^2$).

$$L = 10 \log_{10} \left(\frac{I}{I_0} \right)$$

The table below shows some examples of different sounds, their associated level decibels, and their linear intensity.

Decibels (dB)	**Intensity** (W/m^2)	**Example of sound**
130	10	artillery fire at close proximity (threshold of pain)
120	1	amplified rock music; near jet engine
110	10^{-1}	loud orchestral music, in audience
100	10^{-2}	electric saw
90	10^{-3}	bus or truck interior
80	10^{-4}	automobile interior
70	10^{-5}	average street noise; loud telephone bell
60	10^{-6}	normal conversation; business office

Richter Scale

The Richter scale is a type of logarithmic scale that is used to measure an earthquake's magnitude. The scale is based off of the amplitude of the earthquake's largest seismic wave (the vibration caused by the earthquake).
The scale is given below, where M is the magnitude of the earthquake, I is the amplitude, or intensity of the seismic wave, and I_N is the comparison intensity. The comparison intensity is calculated depending on a range of factors.

$$M = \log_{10} \left(\frac{I}{I_N} \right)$$

SOLVING EQUATIONS INVOLVING INDICES

Here is a quick recap of the index laws:

- $a^m \cdot a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^m = a^m \cdot b^m$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Some other important properties are as follows:

- $a^0 = 1$
- $a^{-x} = \frac{1}{a^x}$
- $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

With these laws and properties, you should have everything you need to start solving equations involving indices.

Example:

Solve for x in the equation $10^x = 50$
Convert to logarithm form and calculate:

$$x = \log_{10}(50) = 1.70$$

Question:

11. $2^x = 32$

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12. $3^x = 81$

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13. $343x^3 = 7$

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Convert the following to logarithmic form with x as the subject:

14. $(ab)^{\frac{x}{z}} = z$

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15. $n\left(\frac{a}{b}\right)^{\sqrt{x}} = m$

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16. Simplify $\log(2) - (\log(10) + 2\log(4))$

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17. Simplify $\log(15) + \log(10)$

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18. Write $\log_a\left(\frac{1}{a^{\frac{1}{2}}}\right) + 3\log_a\sqrt{x}$ in terms of $\log_a(x)$

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19. Write $\log_a\left(\frac{x^2}{\sqrt{y}}\right)$ as a summation of $\log_a(x)$, $\log_a(y)$, and $\log_a(z)$

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20. Given that $a^3 = 15$, find a^6.

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21. Given that $a^5 = 34$, find a^3.

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23. Given that $a^6 = 50$, find $(6a)^3$.

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24. Solve $2^{x+6} = 16$

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25. Solve $\left(\frac{1}{10}\right)^{5x} = 10$

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26. Solve $8^{1/3} \log_8(64x^3) = 8$

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HOMEWORK QUESTIONS

Solve the following:

1. $2^x > 12$

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2. $3^{x-2} > 27$

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3. $2 \leq 2^x \leq 128$

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4. $7 \cdot 3^{-x} \leq 60$

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HOMEWORK QUESTIONS

5. $\log_x(81) = \frac{2x}{2^3}$

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6. By solving $2^x < 10^{20}$, find how many positive integer powers of 2 are less than 10^{20}.

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7. By solving $10^5 < 5^x < 10^{30}$, find how many integer powers of 5 are between 10^5 and 10^{30}.

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8. Explain why the majority of the logarithmic scales we looked at used a base of 10.

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