# 5 1D seismic wave propagation

Until now, we dealt with equations that had a first order time derivative (e.g., the diffusion equation) which are called *parabolic equations* or which had no time derivative at all (e.g., steady-state thermal diffusion) called *elliptic equations*. Yet, a third type of partial differential equations exist that have a second order time derivative. These types of equations are called *hyperbolic equations*. In geophysics, the most common equation is the wave equation which governs how seismic waves propagate through an elastic Earth.

In 1-D the governing equation is

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(\sigma) \tag{1}$$

where $u$ is the elastic displacement [m], $\rho$ rock density (between 2500-3300 kg/m$^3$) and $\sigma$ is stress.

In a linear elastic body, stress depends on strain as follows

$$\sigma = E \frac{\partial u}{\partial x} \tag{2}$$

where $E$ is the Young's module (typical values are between $10^{10} - 10^{11}$ Pa.

Equations (1) and (2) can be solved for $u$, with a usual finite difference discretization. One of the nice things of solving elastic wave equations is that it can often be done with explicit methods, which makes the code efficient.

Equation 2 is discretized in a staggered manner as:

$$\sigma_{i+1/2}^n = E \frac{u_{i+1}^n - u_i^n}{\Delta x} \tag{3}$$

And equation 1 in a discretized manner is:

$$\rho \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = \frac{\sigma_{i+1/2}^n - \sigma_{i-1/2}^n}{\Delta x} \tag{4}$$

where $i$ indicates spatial discretization and $n$ the temporal discretization.

In order to solve for $u_i^{n+1}$, we thus need to know both $u_i^n$ and $u_i^{n-1}$ (so the displacement of the current timestep and that of the timestep before). That sounds more difficult than it is in practice, as during the first timestep we simply set $u^{n-1}$ to zero everywhere.

The way wave propagation codes are often written is by (1) defining a displacement field, (2) computing stresses from that displacement field using eq. (3) and (3) computing a new displacement field from equation (4).

## 5.1 Exercises

1. Write a 1D code that solves the wave equation in the upper 100 km of the Earth, using a timestep of 0.1 seconds, a Young's module of 5e10 Pa and a rock density of 3000 kg/m3. To initiate the wave, apply the following function in the middle of the domain for 5 seconds: $u = 10^{-3} sin(t\pi/5)^2$, where $t$ is time. The lower boundary condition is motion-free ($u(-L) = 0$), whereas the upper boundary is stress free ($\sigma(0) = 0$). If you do things correctly, the code should be less than a single page including plotting and you should see a wave that propagates and is deflected.

2. Compute a synthetic seismogram at the surface of your model, which records the vertical displacement there as a function of time.

3. Measure the velocity of the propagating wave. How does this velocity change if we change $E$ and $\rho$?

4. Implement a layer with a different Young's module and density at the bottom of the model. What changes?