Final Report

12/18/2020

For $z \in [z_{j-1}, z_j]$ and $j = 1, 2, ..., k$, the upper bound is defined as

$$u_k(x) = h(x_j) + (x - x_j)h'(x_j).$$

The sampling density is

$$s_k(x) = \frac{\exp u_k(x)}{\int_D \exp u_k(x')dx'}.$$

Observations will be sampled as follows. First, the interval to which x belongs will be sampled. The probability p_j that x belongs to interval $[z_{j-1}, z_j]$ is the length of the interval divided by the total size of the domain D, i.e., $p_j = \frac{|z_j - z_{j-1}|}{|D|}$. Then the interval to which x belongs is distributed as Multinomial($p = (p_1, p_2, ..., p_k)$).

The CDF, given that x belongs to a particular interval, is then computed as

$$S_X(x) = P(X \leq x | x \in [z_{j-1}, z_j]) = \frac{\int_{z_{j-1}}^{x} \exp u_k(z)dz}{\int_D \exp u_k(x')dx'}.$$

The numerator can be integrated as follows:

$$\begin{aligned}
\int_{z_{j-1}}^{x} \exp u_k(z)dz &= \int_{z_{j-1}}^{x} \exp(h(x_j) + (z - x_j)h'(x_j))dz \\
&= \exp(h(x_j) - x_j h'(x_j)) \int_{z_{j-1}}^{x} \exp(zh'(x_j))dz \\
&= \exp(h(x_j) - x_j h'(x_j)) \cdot \frac{\exp(zh'(x_j))}{h'(x_j)}\Big|_{z_{j-1}}^{x} \\
&= \frac{\exp(h(x_j) - x_j h'(x_j))}{h'(x_j)} \left(\exp(xh'(x_j)) - \exp(z_{j-1}h'(x_j))\right).
\end{aligned}$$

Meanwhile, the denominator of the CDF can be expressed as

$$\begin{aligned}
\int_D \exp u_k(x')dx' &= \int_{z_0}^{z_1} \exp u_k(x')dx' + \ldots \int_{z_{k-1}}^{z_k} \exp u_k(x')dx' \\
&= \sum_{j=1}^{k} \int_{z_{j-1}}^{z_j} \exp u_k(x')dx' \\
&= \sum_{j=1}^{k} \frac{\exp(h(x_j) - x_j h'(x_j))}{h'(x_j)} \left(\exp(z_j h'(x_j)) - \exp(z_{j-1}h'(x_j))\right).
\end{aligned}$$

Next, the inverse CDF must be computed. Let $Y = P(X \leq x | x \in [z_{j-1}, z_j])$. Then exchanging variables and solving for x, we have

$$\begin{aligned}
x = \frac{\exp(h(x_j) - x_j h'(x_j)) \left(\exp(yh'(x_j)) - \exp(z_{j-1}h'(x_j))\right)}{h'(x_j) \int_D \exp u_k(x') dx'} &\implies \\
\frac{h'(x_j) \int_D \exp u_k(x') dx'}{\exp(h(x_j) - x_j h'(x_j))} x = \exp(yh'(x_j)) - \exp(z_{j-1}h'(x_j)) &\implies \\
\frac{h'(x_j) \int_D \exp u_k(x') dx'}{\exp(h(x_j) - x_j h'(x_j))} x + \exp(z_{j-1}h'(x_j)) = \exp(yh'(x_j)) &\implies \\
\log\left(\frac{h'(x_j) \int_D \exp u_k(x') dx'}{\exp(h(x_j) - x_j h'(x_j))} x + \exp(z_{j-1}h'(x_j))\right) / h'(x_j) = y.&
\end{aligned}$$

Thus, the inverse CDF is

$$S_X^{-1}(x) = \log\left(\frac{h'(x_j) \int_D \exp u_k(x') dx'}{\exp(h(x_j) - x_j h'(x_j))} x + exp(z_{j-1}h'(x_j))\right) / h'(x_j)$$

In summary, we generate from the multinomial distribution to determine the interval $[z_{j-1}, z_j]$ to which x belongs. Following the inverse CDF method of sampling, we then generate x as a Uniform(0, 1) random variable. Finally, we compute $x^* = S_X^{-1}(x)$.