

Variance Reduction for Estimating Value at Risk

STAT 428 FA2018 - Group #18

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Abstract

Since risk management is of critical importance in financial system, there is always need to estimate the VAR value with given risk level. It's not hard to estimate the VAR. But the variance of estimating variance is large for general method. Therefore, we indulge into how to reduce the variance of estimating the risk for fixed VAR. In return, it can reduce the variance of estimate the VAR. In this project, we use Control Variate, Importance Sampling and Stratified Sampling Method to reduce the variance. In the result, these three all work well in variance reduction, especially Stratified Sampling. For Statified Sampling, it works better than the other in estimating VAR, especially when you define strata appropriately. Besides, we build a shiny for caculating the VAR and its probability and variance. We hope this can be developed as a assitanship for risk management.

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1 Introduction

Risk management is always an important task in financial field. Financial institutions have to pay attention to the possible risk they will carry on, especially the potential loss and the chances such loss will happen. It is important for banks and insurance companies to estimate and manage the risk to avoid the situation when there is capital deficiency. Therefore, we need to estimate the capital requirement accurately according to the market risk. Value at Risk (VAR) is a measure for the risk of investments loss, which estimates how much value a investment portfolio might lose at given market conditions.

Two types of problems are often encompassed by a company when measuring the risk. The first one is how the value of market risk will change according to risk factors (e.g. interest rate). The other one is what influence the risk will pose on the value of a portfolio.

This project is intended to develop a model to revalue the portfolio with the interest rate by starting from the simple linear assumption to quadratic model and make use of revalued portfolio to estimate Value at Risk (VAR). The technique problem we are facing is how to make our estimate much more precise, which means we should do variance reduction for our estimator of VAR.

1.1 Background

In this project, we applied various methods from the class to improve our estimation accuracy and obtain a more accuracy estimation of the VAR. We can build the model by following steps.

Firstly, we assume the market risk follows multivariate normal distribution with the population mean equals to 0. We denote the vector of market prices and rates as S , $S \sim \mathcal{N}(0, \Sigma_S)$ and Σ_S is the covariance matrix of ΔS .

$$\Delta t = \text{risk-measurement horizon}$$

$$\Delta S = \text{change in } S \text{ over interval } \Delta t$$

Then we will obtain the variance from the history data. To generate random variables following the expected distribution, we use MCMC to sample the objective distribution. After generating random variables with the multivariate normal distribution by MCMC, we can use this statistics to evaluate the change of the portfolio value in certain time intervals.

Since the ΔV is linear in ΔS , we can perform linear transformation to sample the random variable ΔV .

$$\Delta V = \delta^T \Delta S$$

δ^T is the linear transformation matrix.

Then by relevant financial knowledge, we can set up the linear transformation as following, which is also called Delta-Gamma Approximation.

$$\Delta V \approx \frac{\partial V}{\partial t} \Delta t + \delta^T \Delta S + \frac{1}{2} \Delta S^T \Gamma \Delta S$$

where

$$\delta_i = \frac{\partial V}{\partial S_i}, \Gamma_{ij} = \frac{\partial^2 V}{\partial S_i \partial S_j}$$

Without considering the market price changes and other details, the loss over such interval will be the negative of the change of the portfolio value.

Then our task is find the value of L in a certain quantile which represents the loss under certain possibility.

$$L = \text{loss over interval } \Delta t$$

Then our estimator which is the VAR should be obtained by getting the percentile of the portfolio's loss distribution.

$$1 - F_L(x_p) = P(L > x_p)$$

However, the estimation of loss value may have a large variance. Hence we use several methods related to Mento Carlo to reduce the variance which is the main goal of our project. The relevant methods we have learnd form class including importance sampling, stratified sampling, control variable. In this project, we will perform each method and compare their efficiency in reducing the variance and improving accuracy.

1.2 Data

In order to generate the targeted distribution, from the assumption, the covariance os the distribution is important. We use the data of Interest Rate and Bond Rate as the risk factor and assume the paramter of Delta-Gamma function as input to estimate the VAR and its variance.

We process the data by matching their date and generate the difference between each date. The part of the data is shown below:

The Interest Rate data

```
##          DATE INTDSRUSM193N
## 745 2012-01-01          0.75
## 746 2012-02-01          0.75
## 747 2012-03-01          0.75
## 748 2012-04-01          0.75
## 749 2012-05-01          0.75
## 750 2012-06-01          0.75
```

The Bond Rate data

```
##          DATE IRLTLT01USM156N
## 625 2012-01-01          1.97
## 626 2012-02-01          1.97
## 627 2012-03-01          2.17
## 628 2012-04-01          2.05
## 629 2012-05-01          1.80
## 630 2012-06-01          1.62
```

The covariance matrix is shown below:

```
##          delta_Bond delta_Interest
## delta_Bond    0.025250769    0.001486235
## delta_Interest 0.001486235    0.002836682
```

2 Method

In order to estimate the risk, which is $P(L > x)$, we should propose assumptions on the portfolio and the distribution of risk factors.

Assumption:

$$\Delta V \approx \frac{\partial V}{\partial t} \Delta t + \delta^\top \Delta S + \frac{1}{2} \Delta S^\top \Gamma \Delta S$$

where

$$L = -\Delta V$$

$$\delta_i = \frac{\partial V}{\partial S_i}, \Gamma_{ij} = \frac{\partial^2 V}{\partial S_i \partial S_j}$$

$$\Delta S \sim N(\mu, \theta)$$

For the first assumptions, it explains the approximately quadratic relation between risk factor and the loss function, which is called Delta-Gamma Approximation. This is format is deduced from the Taylor Expansion. The another assumption is a basic idea behind the model, which can explain most situation of the risk factor situation.

Therefore, in our model, the random variable is the ΔS , which is multi-normal distribution. We make use of MCMC method to generate it. Then we utilize the Delta_gamma Method to to revalue the portforlio. With the loss function, bootstrap can be used to estimate the VAR at given risk level. Lastly, to reduce the variance of estimating the risk of the estimated VAR, we realize three methods, which are Control Variate, Importance Sampling, Stratified Sampling to estimate the risk and the variance.

2.1 MCMC

Since there could be a huge scope of influencial factors to the market risk, the variable ΔS could be with m dimensional multivariate normal distribution. However, in our project to simplify the question, we just pick 2 dimensions.

$$\Delta S = (X, Y) \sim \mathcal{N}(0, \Sigma_S) = \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\right)$$

Then the conditional distribution is:

$$(X|Y = y) \sim \mathcal{N}\left(\mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y - \mu_2), (1 - \rho^2)\sigma_1^2\right)$$

$$(Y|X = x) \sim \mathcal{N}\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1), (1 - \rho^2)\sigma_2^2\right)$$

Then we are going to generate X and Y by the conditional distribution using gibbs sampling.

2.2 VAR Estimate

In order to estimate the VAR value, we can revalue the Loss function based on the generated ΔS , which is

$$\Delta V \approx \frac{\partial V}{\partial t} \Delta t + \delta^\top \Delta S + \frac{1}{2} \Delta S^\top \Gamma \Delta S$$

After generate the Loss function value, we make use of Bootstrap to estimate the quantile of the Loss function with given risk level.

2.3 Variance Reduction Techniques

There are tree methods we use to estimate the estimated VAR's risk. They are Control Variate, Importance Sampling, Stratified Sampling.

2.3.1 Control Variate

The method of control variates is among the most effective and broadly applicable techniques for improving the efficiency of Monte Carlo simulation. It exploits information about the errors in estimates of known quantities to reduce the error in an estimate of an unknown quantity.

Suppose that the pairs (X_i, Y_i) , $i = 1, 2, 3, \dots, n$ are i.i.d. and that the expectation $E(X)$ of the X_i is known. Then for any fixed b we can calculate

$$Y_i(b) = Y_i - b(X_i - E(X))$$

from the i th replication and then compute the sample mean

$$\bar{Y}(b) = \bar{Y} - b(\bar{X} - E(X)) = \frac{1}{n} \sum_{i=1}^n (Y_i - b(X_i - E(X)))$$

As an estimator of $E(Y)$, the control variate estimator is unbiased. Each $Y_i(b)$ has variance

$$Var[Y_i(b)] = \sigma_Y^2 - 2b\sigma_X\sigma_Y + b^2\sigma_X^2$$

The optimal coefficient b^* minimizes the variance and is given by

$$b^* = Cov(X, Y)Var[X]$$

In our project, we generate CZ with $Z \sim N(0, I)$. Then let (L_i, Q_i) , $i = 1, 2, 3, \dots, n$, be the values recorded on n independent replications. A control variate estimator of $P(L > x)$ is given by

$$1 - \widehat{F}_L^{CV}(x) = \frac{1}{n} \sum_{i=1}^n 1\{L_i > x\} - \hat{\beta} \left(\frac{1}{n} \sum_{i=1}^n 1\{Q_i > x\} - P(Q > x) \right)$$

An estimate $\hat{\beta}$ of the variance-minimizing coefficient can be computed from the (L_i, Q_i) as explained before.

2.3.2 Importance Sampling

In order to realize the Importance Sampling, we should make use of the Delta-Gamma Method and its diagonalized format and the most important thing is to find the importance sampling function. In GHS paper, it gives an optimal importance sampling for estimating the risk. We use its method to realize our algorithm.

The below is the mathematical deduction for the Importance Sampling Method.

$$\begin{aligned} L &\approx a + b^\top Z + Z^\top \Lambda Z \\ &= a + \sum_{j=1}^m (b_j Z_j + \lambda_j Z_j^2 \equiv Q) \end{aligned}$$

In GHS, the importance function shall obey the following rule so that it is proportional to the multi-normal distribution, which can achieve a great effect of variance reduction. $\frac{\partial P_\theta}{\partial P} = e^{\theta Q - \phi(\theta)}$

The solution for the above equation is the distribution $Z \sim N(\mu_\theta, \Sigma_\theta)$, where $\Sigma(\theta)$ is diagnostic matrix. Therefore, we can get the following equation with diagonalization format of Delta_Gamma Method.

$$\mathcal{P}(Q > x) = E[\infty(Q > x)] = E_\theta[(\frac{\partial P}{\partial P_\theta})\infty(Q > x)] = E_\theta[e^{-\theta Q + \phi(\theta)}\infty(Q > x)]$$

so that

$$\mathcal{P}(L > x) = E[e^{-\theta Q + \phi(\theta)}\infty(L > x)]$$

where

$$Z \sim P_\theta$$

The exact distribution of Z is below:

$$Z \sim N(\mu(\theta), \Sigma(\theta))$$

$$\mu_j(\theta) = \frac{\theta b_j}{1 - 2\lambda_j\theta}$$

$$\sigma_j^2(\theta) = \frac{1}{1 - 2\lambda_j\theta}$$

Besides, there are some restrictions for the parameter. $2\lambda\theta < 1$, so that $\phi(\theta) < \infty$

For $e^{-\theta Q + \phi(\theta)}$, we have following expression

$$\phi(\theta) \equiv a\theta + \sum_{j=1}^m \phi_j(\theta) = a\theta + \frac{1}{2} \sum_{j=1}^m \left(\frac{\theta^2 b_j^2}{1 - 2\theta\lambda_j} - \log(1 - 2\theta\lambda_j) \right)$$

In order to decide θ , we can make use of the following equation with uniroot function.

$$\phi'(\theta_x) = x$$

2.3.3 Stratified Sampling

The Stratified Sampling Method is realized based on Importance Sampling with applying it on each strata. The procedure below it's how to decide the best strata that matters in this problem.

Given: \sum, a_0, A (Note: $L \approx a_0 + a' \Delta S + \Delta S' A \Delta S$)

1. Express $Q = b'Z + Z'\Lambda Z$ where $Z = N(0, 1)$, Λ diagonal:
 - Find $\tilde{C}\tilde{C}' = \sum$
 - Solve the eigenvalue problem: $\tilde{C}'A\tilde{C} = U\Lambda U'$, U orthonormal
 - Set $C = \tilde{C}U$, $b' = a'C$
2. Identify the IS distribution
 - Set $\theta = \theta_x$, where $\phi'(\theta_x) = (x - a_0)$
 - For each $i = 1, \dots, m$, define $\sigma_i^2(\theta) = 1/(1 - 2\theta\lambda_i)$, $\mu_i(\theta) = \theta b_i \sigma_i^2(\theta)$
3. Define k strata: given probabilities p_j with $\sum_{j=1}^k p_j = 1$ find s_j such that $P_\theta\{Q \leq s_j\} = \sum_{i=1}^j p_i$, $j = 1, \dots, k-1$ (for equiprobable bins, $p_j \equiv 1/k$)
4. Perform the simulation: For $j = 1, \dots, k, i = 1, \dots, n_j$
 - With parameter θ , generate $Z^{(ij)}, Q^{(ij)}$ in stratum j and set $\Delta S^{(ij)} = CZ^{(ij)}$
 - Using $S^{(ij)}$, evaluate the loss L_{ij} and LR $l_{ij} = \exp\{\phi(\theta) - \theta Q^{(ij)}\}$
 - Estimate $P\{L > x\}$ by $\hat{P} = \sum_{j=1}^k p_j (1/n_j) \sum_{i=1}^{n_j} I(L_{ij} > x) l_{ij}$

3 Result

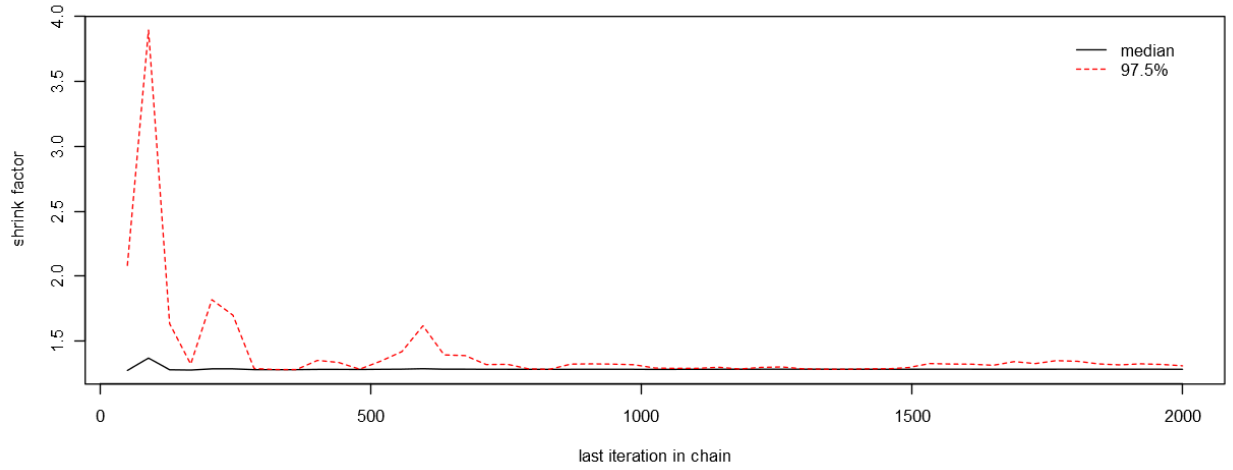


Figure 1: Risk Factor

| Variance Deduction Method | Probability | Variance | Variance Deduction Factor |
|---------------------------|-------------|----------|---------------------------|
| General | 0.5717 | 0.0258 | 1 |
| Control Variates | 0.5803 | 0.0013 | 20.4923 |
| Importance Sampling | 0.6108 | 1e-04 | 303.8453 |
| Stratified Sampling | 0.5713 | 1e-04 | 306.9635 |

Figure 2: Probability and Variance Factor Table

The Figure 1 shows whether the MCMC we generated converges. After generating the MCMC, we try to choose the 1000 elements in the tail and make use of it to analyze the loss function.

The Figure 2 shows the analysis result. In the output, the general method is to generate the VAR by Bootstrap and also estimate the variance with Bootstrap. Then in each method, we all output the probability each method estimates on the VAR point. Variance is also computed for each method.

The Variance Deduction Factor is computed as $(\text{Variance of General Method}) / (\text{Variance of Specific Method})$. In this way, we can judge the reduction effect from the factor. When the factor is higher, the result is better.

4 Discussion

Firstly, the Figure 1 shows that the MCMC of 2-dimensional Multi-Normal converges.

From the Figure 2, we can discover that for all the methods, the estimated probability is almost the same. Therefore there is no bias in estimating the risk. But from the risk factor column, we can find that the variance of Importance Sampling and Stratified Sampling is much larger than that of other methods, which shows that we have realized our target to reduce the variance of estimating risk of VAR and it's significant.

To compare the effect, we can find the Importance Sampling and Stratified Sampling are the best. Actually, Stratified Sampling shall be much better than Importance Sampling. Since we just choose the few points to cut the region so that the effect is not so significant.

To analyze our model assumption, although it's a raw model which is based on quadratic assumption. But according to GHS paper, the Delta-Gamma Method can approximate most of portfolio combination. As for the normal assumption of ΔS , in the future research, we can realize the model based on the heavy tail distribution, like student distribution, which fits the truth much better.

In order to realize our model and method, we decide the parameter of Delta-Gamma Approximation by ourselves. In fact, these are decided by other methods, like B-S equation or the knowledge of the financial market. But in order to compare the variance reduction effect. There is no difference in how accurate these parameters are. Besides, in the process of realizing the variance reduction process, we didn't stick strictly to the optimal strata selection and θ . Therefore, there can be some improvement for our model, which means these three methods can be better.

5 Appendix

5.1 Data Source

1. Interest Rate, Discount Rate for the U.S

This dataset is reported by the International Monetary Fund, comprising of 808 observations from 1/1/1950 to 4/1/2017. There are two variables: Date with 1-month intervals and INT-DS Rate.

2. Long-Term U.S. Government Bond Yields

This dataset is reported by the Organization for Economic Co-operation and Development, comprising of 705 observations from 1/1/1960 to 9/1/2018. There are two variables: Date with 1-month intervals and IRLTLT01USM156N (Long-term Bond Yields).

5.2 Code

You can refer to the code.R in the zip file or turn to our Github

5.3 Shiny

These three figures are illustration of our interactive application developed with Shiny. This application is used for risk management assistantship. There are three main functions. The first is to detect the convergence of the MCMC generated. The second is to visualize the scatterplot of the risk factor. The most important part is the result of probability estimated and its variance, which is the target of our project.

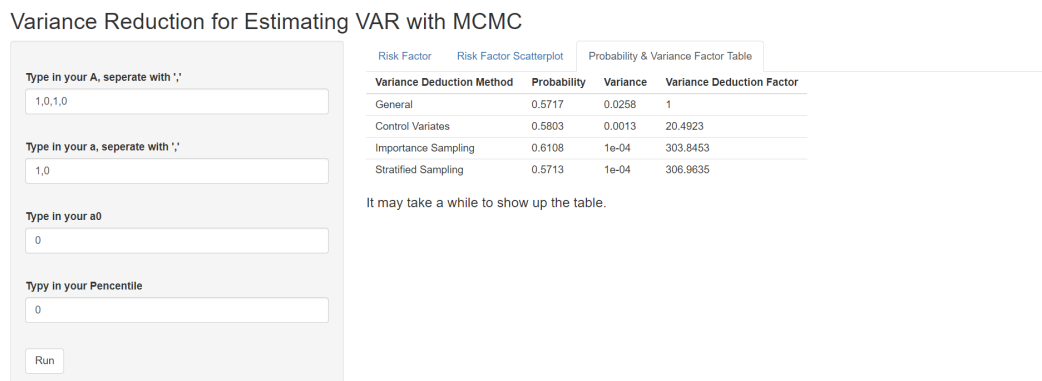


Figure 3: Shiny: Probability and Variance Factor Table

Variance Reduction for Estimating VAR with MCMC

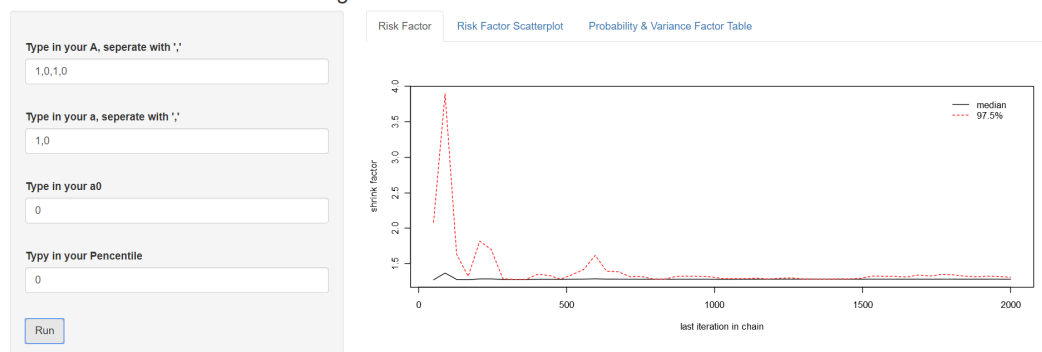


Figure 4: Shiny: Plot of the Risk Factors

Variance Reduction for Estimating VAR with MCMC

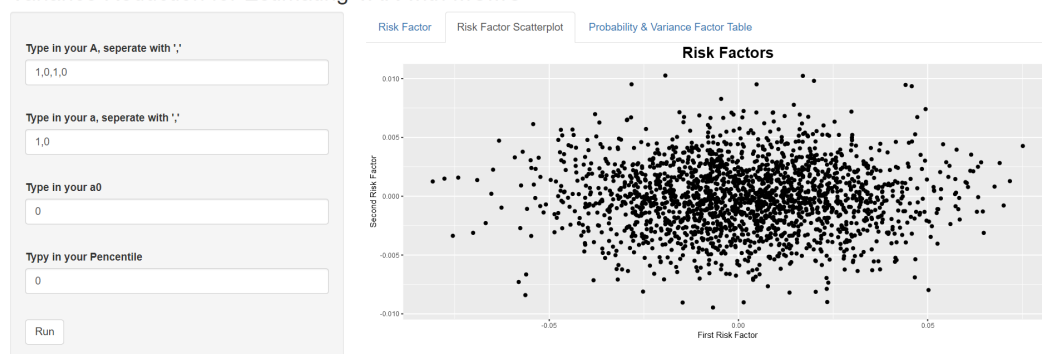


Figure 5: Shiny: Scatterplot of the Risk Factors

6 References

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