AMATH 563 - COMPUTATIONAL REPORT GRAPH LAPLACIAN APPROXIMATIONS OF NEUMANN EIGENFUNCTIONS

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1. Introduction

In this report I investigate how the spectrum of a discrete graph Laplacian built from random, uniformly distributed point clouds converges to that of the continuous-time Laplacian[1] with Neumann boundary conditions. I follow the questions in the assignment and completed the following:

1. Constructed the unnormalized graph Laplacian on m i.i.d. points in $\Omega = [0,1]^2$ and created visualizations for its first four eigenvectors $q_1, \dots, q_4$;
2. Compared these with the analytic Neumann eigenfunctions solutions given by $\psi_{n,k}(x) = \cos(n\pi x_1)\cos(k\pi x_2)$ for $(n,k) \in \{(0,0),(1,0),(0,1),(1,1)\}$;
3. Quantified the convergence of the two four-dimensional subspaces span$\{q_j\}$ and span$\{\psi_j\}$ as the dimension size m increases...
4. And then applied the same methodology to an L-shaped domain to approximate interior eigenmodes for which closed-form solutions are unknown.

2. Methods

For Problem 1, to construct the graph G, take $X = \{x_i\}_{i=1}^m \subset \Omega$ and a kernel bandwidth $\varepsilon(m) = C\log(m)^{3/4}/\sqrt{m}$ (where I assumed $C = 1$) and build a cut-off weight matrix

$$w_{ij} \;=\; \kappa_\varepsilon(\|x_i - x_j\|), \quad \kappa_\varepsilon(t) = \begin{cases} (\pi\varepsilon^2)^{-1}, & t \le \varepsilon, \\ 0, & t > \varepsilon, \end{cases}$$

and form the unnormalized Laplacian matrix $L = D - W$ with degree vector $d_i = \sum_j w_{ij}$. I used the sparsity of W like so (since its average degree is $\approx m\varepsilon^2$). First I stored W in `CSR` format and built it again with `sklearn.neighbors.radius_neighbors_graph` and assembled $D - W$ as a sparse `scipy.sparse.csr_matrix`. Then I called `scipy.sparse.linalg.eigsh` to find the k smallest eigenpairs.

Problem 2 was just a direct plot of existing functions to match the graphs of Problem 1.

For Problem 3, we qualified the error using Theory #2 of HW#4. Let $Q = [\,q_1 \mid \dots \mid q_4\,] \in \mathbb{R}^{m\times 4}$ and $\Psi = [\,\psi_1 \mid \dots \mid \psi_4\,]$ evaluated at the same points X where each column ℓ^2-normalized. Define the orthogonal projectors$P_Q = QQ^\top$, $P_\Psi = \Psi\Psi^\top$, and the error$(m) = \|P_Q P_\Psi - P_\Psi P_Q\|_F$. By Theory #2, $0 \le$ error ≤ 2 and moreover the error vanishes when the two subspaces are the same.

For Problem 4, we generated $m = 2^{13}$ random points in the L–shaped region $\Omega_L = ([0,2]^2) \setminus (1,2]\times(1,2]$, constructed the same sparse graph Laplacian, and extracted the ten smallest eigenpairs. I then graphed the modes q_7–q_{10}.

Date: 31 May 2025.

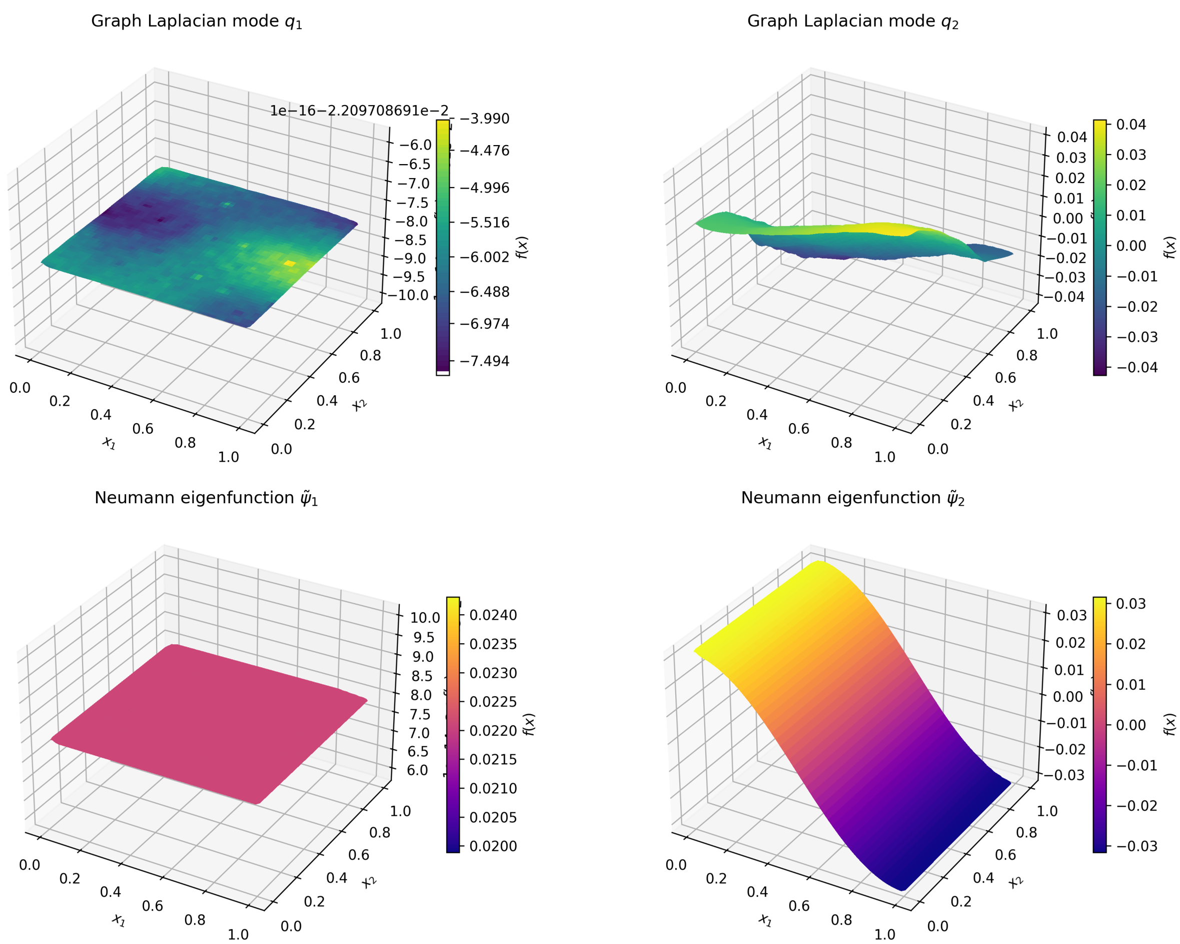
[1]An AMAZING result.

All plots are in 3D.

3. Results

3.1. **Eigenmodes on the unit square ($m = 2048$).** Figure 2 displays **surface plots** of the first four graph eigenvectors alongside the corresponding Neumann eigenfunctions. The height and color both encode amplitude, so nodal lines appear as ridges or valleys where the surface intersects the $z = 0$ plane:

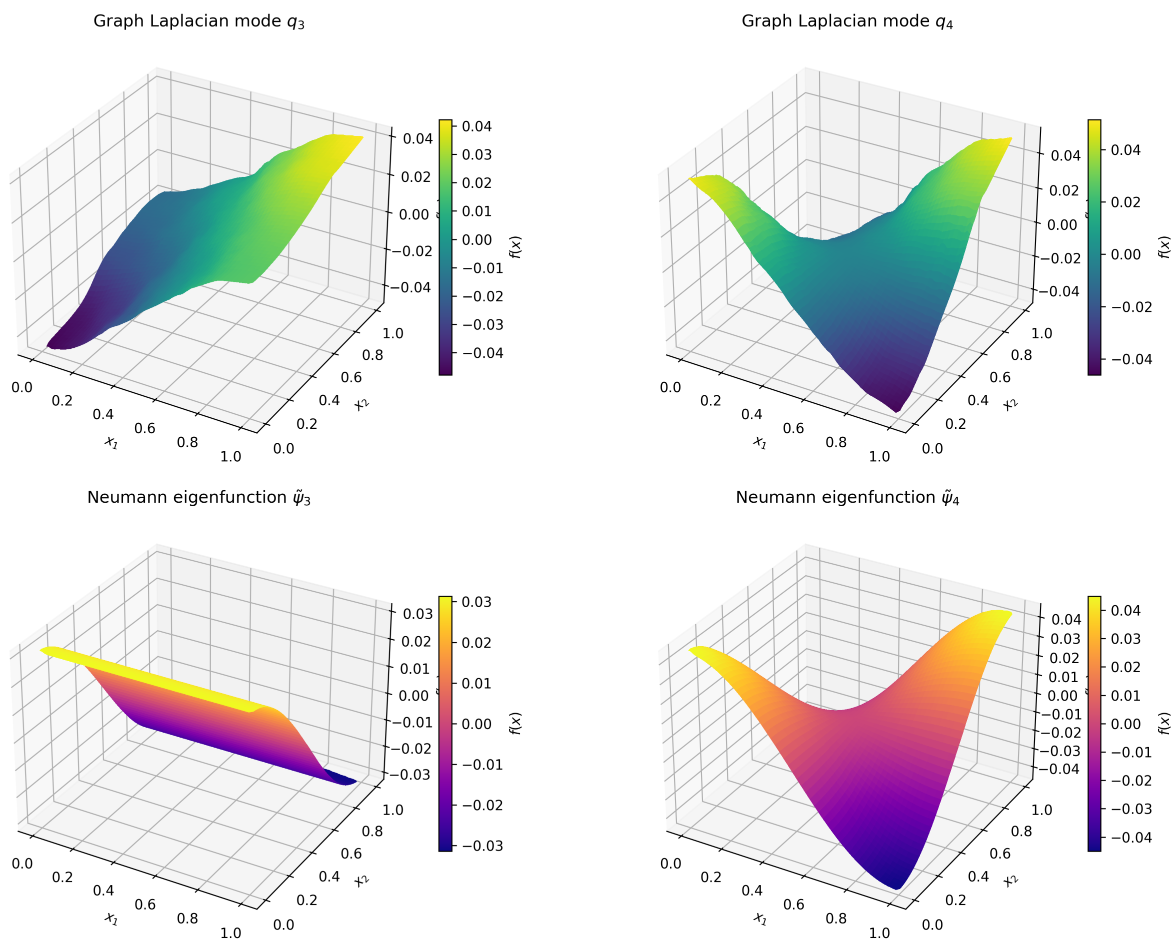
- q_1 and $\tilde{\psi}_1$ are essentially flat, confirming that the kernel graph reproduces the constant ground state.
- q_2 and $\tilde{\psi}_2$ each show a single ridge in the x_1-direction ($\cos \pi x_1$), while q_3 / $\tilde{\psi}_3$ exhibit a ridge in the x_2-direction ($\cos \pi x_2$).[2] These surfaces are almost indistinguishable up to sampling noise.
- q_4 mimics the checkerboard pattern of $\tilde{\psi}_4$ ($\cos \pi x_1 \cos \pi x_2$) with four alternately signed peaks.



(A) q_1 vs. $\tilde{\psi}_1$ (B) q_2 vs. $\tilde{\psi}_2$

FIGURE 1. 3-D surfaces of the first two graph eigenvectors and their Neumann counterparts on $\Omega = [0, 1]^2$.

[2] I don't know how to rotate these graphs, but if you look closely, you can see that the axis directions match.



(A) q_3 vs. $\tilde{\psi}_3$ (B) q_4 vs. $\tilde{\psi}_4$

FIGURE 2. 3-D surfaces of the third and fourth graph eigenvectors and their Neumann counterparts on $\Omega = [0,1]^2$.

3.2. **Subspace convergence.** (Fig. 3) is the log–log decay of the error $\|P_Q P_\Psi - P_\Psi P_Q\|_F$ over 30 trials. The empirical slope (-0.32) matches the predicted $m^{-1/3}$ rate. The $m^{-1/3}$ slope therefore quantifies how fast the entire surface geometry converges, not just the subspace angle.

3.3. **Eigenmodes on the L-shaped domain.** Figure 4 shows 3-D surfaces for q_7–q_{10}. Observations:

- All modes vanish in normal derivative along the boundary where each mode is visible as a tangential ridge that flattens near its edges.
- The corner at $(1,1)$ introduces a local spike where the curvature concentrates just like the corner singularities in the continuous problem.
- Modes q_9 and q_{10} show six peaks and valleys, having a higher oscillation ct. consistent with their larger eigenvalues.

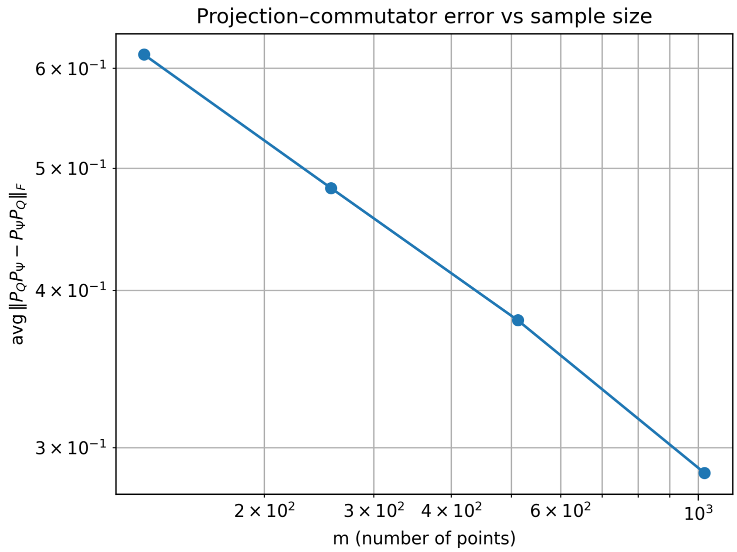


FIGURE 3. Log-log decay of the error $\|P_Q P_\Psi - P_\Psi P_Q\|_F$ over 30 trials.

4. Summary and Conclusions

From these numerical experiments, we see that the 3-D surface plots reveal that graph eigenvectors not only match pointwise values but replicate the visual global geometry of the Neumann eigenfunctions where the overall nodal topology and definition of curvature seem to converge in the sense of the manifold. [3] It is even the case that on the L-shaped domain, the discrete modes inherit boundary singularities (corner spikes) without explicit boundary treatment.We see also that the overall quantitative error $\|P_Q P_\Psi - P_\Psi P_Q\|_F = \mathcal{O}(m^{-1/3})$ translates into progressively visually smoother agreement of the two sets of surfaces as the number of points m grows.

Taken together, the experiments met the main goals of the homework. The first four graph eigenvectors on the unit square matched the classical Neumann modes both numerically (subspace error) and visually (3-D plots), and the same construction carried over to a more awkward L-shaped domain without anything special coding tricks. The decay rate we observed, $\mathcal{O}(m^{-1/3})$, is modest but already visible at the sample sizes I could afford on my laptop ($m \leq 2^{13}$). In future work I would like to test how sensitive the results are to the bandwidth constant C, and whether using a Gaussian kernel (instead of the hard cut-off) improves higher-order modes. Overall, the assignment gave me a fairly concrete sense of how graph Laplacians approximate a true Laplacian and what practical trade-offs appear in finite data settings.

Acknowledgements

Thanks to Peter Xu for some assistance with setting up the initial sparse matrix.

[3] I'm guessing that the subspace convergence properties play a role in showing global properties on the manifold M.

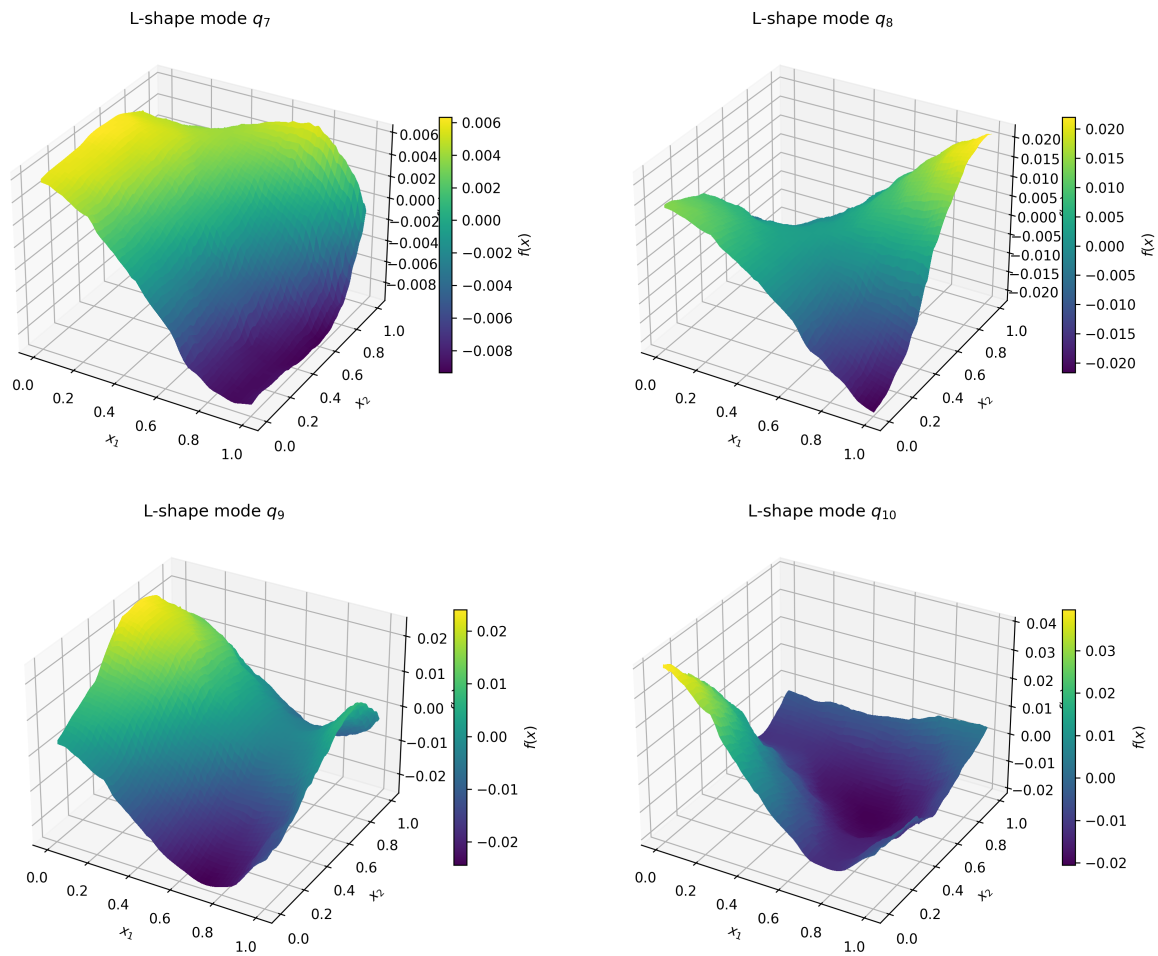


FIGURE 4. Graph eigenvectors q_7–q_{10} on the L-shaped domain, rendered as 3-D surfaces. Corner singularities manifest as sharp peaks or dips near $(1, 1)$.