A UNIFIED OPERATOR FOR DERIVATIVE AND INTEGRAL IN $\mathbb{R}$ USING THE GAMMA FUNCTION

JIAJIE (JASON) MA

1. Introduction

Higher-order derivatives and integrals of a function $f \in C^k[a,b]$ $(k \geq 0)$ are normally defined iteratively based on the definition of first-order derivative and integral. This inductive nature motivates us to consider the possibility of a generalized equation for the n^{th} derivative or integral of f in terms of itself. Further, given the subtle connection between integral and antiderivative in the Fundamental Theorem of Calculus (FTC), we want to ask:

Question 1.1. *Can we define an operator that unifies differentiation and integration of any order n of a function f with one equation? Like exponentiation, if $n > 0$, it represents the n^{th} derivative of f; if $n \leq 0$, it represents the n^{th} integral of f (i.e. integration is regarded as differentiation in reverse).*

One significance of such operator, clearly, is to enable us to obtain higher-order derivatives and integrals in one computation. Besides, given its similarity to exponentiation, it is of our interest to see if the operator can be extended to take any real order n, e.g. what is a derivative of order $\frac{1}{2}$?

This paper formulates and discusses a unified operator we described above and following [1], we call it *Differintegral.* While [1] gives us a general idea of the derivation with only statements of some intermediate results, our work is to rigorously prove the theory with self-articulated definitions, examples, and proofs. First, we propose an appropriate definition of derivative and integral for this paper based on their *sequential characterization.* From there, we formulate two generalized equations, one for derivative and one for integrals of integer orders, after inspecting two working examples. To synthesize the two equations, inspired by the *combinatorics* intuition, we introduce the *gamma function* $\Gamma : \mathbb{R}\backslash\mathbb{Z}^- \to \mathbb{R}$ as an extension of the integer factorial and we see that the existence of Differintegral follows immediately. The real domain of the Gamma function also yields the possibility of *fractional calculus.*

2. Definitions of First-order Derivative and Integral

Suitable definitions of first-order derivatives and integrals for this paper are to help us define higher-order derivatives and integrals iteratively, and connect the two operations. While [1] adopts the $\delta - \epsilon$ definition of derivative and integral as common knowledge, inspired by the final equation of differintegral in [1], we propose an enhanced definition with complete derivation.

First, notice that for our goal, the typical $\delta - \epsilon$ definition of derivative and that of integral in terms of upper/lower sum may not be preferred. Not only does it seem hard to relate both equations, it may also be difficult to represent the n^{th} integral of f over $[a, b]$ in terms of a series of upper sums with multiple infimums. Another difficulty for connection is that the derivative is a pointwise operation, whereas the integral is with respect to an interval.
To resolve the concerns, notice that both derivative and integral have sequential characterizations. This allows us to represent both in terms of a limit with respect to $N \in \mathbb{N}$. Further, by introducing an endpoint a, the sequential characterization enforces derivative to be an interval-based operation.

In particular, consider a continuous function $g : [a, b] \to \mathbb{R}$. Then, for a point $x_0 \in [a, b]$, since the sequence of even partitions $P_N = \{a, a + \frac{x_0-a}{N}, \dots, x_0\}$ satisfies $\lim_{N\to\infty} U(g, P_N) - L(g, P_N) = 0$, by the Sequential Characterization of Integrability, we define its integral over $[a, x_0]$ to be

$$\int_a^{x_0} g(x)\, dx = \lim_{N\to\infty} \sum_{i=0}^{N-1} g(x_0 - i\frac{x_0 - a}{N})\frac{x_0 - a}{N}. \tag{2.1}$$

Similarly, consider a continuous function $f : [a, b] \to \mathbb{R}$ ($k \in \mathbb{N}$). Given a point $x_0 \in [a, b]$, as we know that the sequence $(\frac{x_0-a}{N}) \to 0$, by the Sequential Characterization of Differentiability, we define the derivative of f at x_0 to be

$$\frac{df}{d(x-a)}(x_0) = \lim_{N\to\infty} (\frac{x_0 - a}{N})^{-1}(f(x) - f(x - \frac{x_0 - a}{N})). \tag{2.2}$$

Notably, we add the lower bound a in the typical notation for derivatives only to recognize it as an essential argument for this way of defining derivative. This idea is based on the final equation of differintegral in [1]. Also, for the sake of simplicity, this paper only considers continuous functions.

3. Derivative and Integral of integer order

We start by formulating the derivative and integral of integer order n individually. To do this, we first work on the case when $n = 2$ to gain the motivating ideas for the general theorems. While [1] states the results of our examples as mere facts, we sketch the proofs and reveal key intuitions.

3.1. **Derivation of the formula for second-order derivative.** Consider an arbitrary function $f \in C^2([a, b])$ with its first and second derivatives, f' and f''. Our goal is to represent $f''(x_0)$ in terms of f.

First, by Equation 2.2, given an $N \in \mathbb{N}$, we may write $f''(x_0)$ as

$$f''(x_0) = \left(\frac{x_0 - a}{N}\right)^{-1} \left(f'(x_0) - f'\left(x_0 - \frac{x_0 - a}{N}\right)\right) + \epsilon_0, \tag{3.1}$$

with some $\epsilon_0 \in \mathbb{R}$. Note that as $N \to \infty$, $\epsilon_0 \to 0$. Similarly, with the same N, $f'(x_0)$ and $f'(x_0 - \frac{x_0-a}{N})$ in can be approximated by $f(x_0)$ and $f(x_0 - \frac{x_0-a}{N})$, $f(x_0 - \frac{x_0-a}{N})$ and $f(x_0 - 2\frac{x_0-a}{N})$, with some $\epsilon_1, \epsilon \in \mathbb{R}$ respectively. Hence, substituting the two approximated equations into Equation 3.1, we see that

$$f''(x_0) = (\frac{x_0-a}{N})^{-2}(f(x_0) - 2f(x_0 - \frac{x_0-a}{N}) + f(x_0 - 2\frac{x_0-a}{N})) + (\frac{x_0-a}{N})^{-1}(\epsilon_1 - \epsilon_2) + \epsilon_0.$$

Now, considering the term $(\frac{x_0-a}{N})^{-1}(\epsilon_1 - \epsilon_2)$, I claim that it converges to 0 when $N \to \infty$. To see why, let $\delta x = \frac{x_0-a}{N}$. Then, we know that $\epsilon_1 = f'(x_0) - (\delta x)^{-1}(f(x_0) - f(x_0 - \delta x))$ and $\epsilon_2 = f'(x_0 - \delta x) - (\delta x)^{-1}(f(x_0 - \delta x) - f(x_0 - 2\delta x))$. To simplify, since f is in $C^2[a,b]$, we may represent f around x_0 and $x_0 - \delta x$ by its Taylor expansion up to the second term:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c)}{2}(x - x_0)^2,$$
$$f(x) = f(x_0 - \delta x) + f'(x_0 - \delta x)(x - (x_0 - \delta x)) + \frac{f''(d)}{2}(x - (x_0 - \delta x))^2,$$

where c is between x and x_0; and d is between x and $x_0 - \delta x$ ($x \in [a,b]$). Hence, substituting $x_0 - \delta x$ into the first expansion and $x_0 - 2\delta x$ into the second, we get $f(x_0 - \delta x) = f(x_0) - f'(x_0)\delta x + \frac{f''(c)}{2}\delta x^2$ and $f(x_0 - 2\delta x) = f(x_0 - \delta x) - f'(x_0 - \delta x)\delta x + \frac{f''(d)}{2}\delta x^2$. Thus, by substitution it follows that

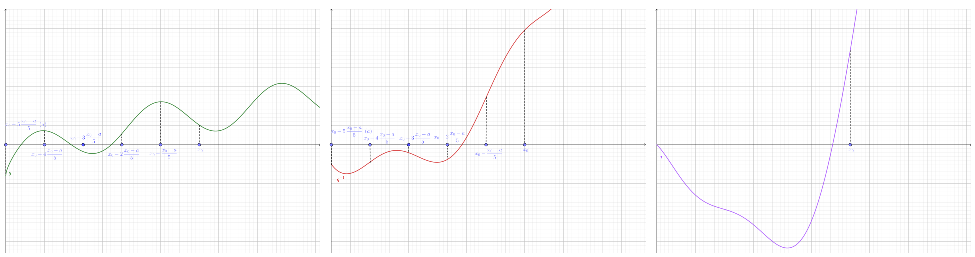
$$\begin{aligned}(\frac{x_0-a}{N})^{-1}(\epsilon_1 - \epsilon_2) &= (\delta x)^{-1}(f'(x_0) - (\delta x)^{-1}(f(x_0) - f(x_0 - \delta x)) \\ &\quad - (f'(x_0 - \delta x) - (\delta x)^{-1}(f(x_0 - \delta x) - f(x_0 - 2\delta x)))) \\ &= \frac{f''(c)}{2} - \frac{f''(d)}{2}.\end{aligned}$$

Now, since c is between x_0 and $x_0 - \delta x$, as $N \to \infty$, $x_0 - \delta x \to x_0$ and by the Squeeze Theorem, $c \to x_0$. Further, as f'' is continuous, it follows that $f''(c) \to f(x_0)$. Similarly, as $N \to \infty$, $f''(d) \to f(x_0)$. Therefore, we have $(\frac{x_0-a}{N})^{-1}(\epsilon_1 - \epsilon_2) \to 0$ as $N \to \infty$ as required. Based on this and the fact that $\epsilon_0 \to 0$ as $N \to 0$, we conclude that

(3.2)
$$f''(x_0) = \lim_{N\to\infty} \left(\frac{x_0-a}{N}\right)^{-2} \left(f(x_0) - 2f(x_0 - \frac{x_0-a}{N}) + f\left(x_0 - 2\frac{x_0-a}{N}\right)\right).$$

Observe that both Equation 2.1 and Equation 3.2 demonstrate a pattern like binomial expansion up to $f(x_0 - n\frac{x_0-a}{N})$, where n is the order of the derivative; and the whole equation keeps dividing by $\frac{x_0-a}{N}$ as n increments. This reflects our key idea of turning every higher-order derivative of f into a calculation in form of f by "snowballing" the $\frac{x_0-a}{N}$-neighborhoods around x_0, $x_0 - \frac{x_0-a}{N}$, etc. As the order n increases, the larger the overall neighborhood around x_0 needs to be in order to complete the n^{th} derivative.

3.2. **Derivation of the formula for second-order integral.** Now, we show that a similar idea holds for integrals of integer order. Again, let's consider an continuous function g (Figure 1). With a lower bound of a, its first and second indefinite integral, $f^{(-1)}$ and $f^{(-2)}$, are plotted as follows:



(A) g (B) g^{-1} (C) g^{-2}

FIGURE 1. Graphs of an arbitrary integrable g with g^{-1} and g^{-2}.

As in subsection 3.1, our goal is to write out an equation for $g^{(-2)}(x_0)$, the second integral of g over $[a, x_0]$. Thus, given $N \in \mathbb{N}$ and let $\delta x = \frac{x_0 - a}{N}$, by Equation 2.1, we may write $g^{(-2)}(x_0) = \sum_{i=0}^{N-1} \delta x g^{-1}(x_0 - i\delta x) + \epsilon_0$ with some $\epsilon_0 \in \mathbb{R}$. Then, notice that for each i, $g^{-1}(x_0 - i\delta x)$ is the integral of g over $[a, x_0 - i\delta x]$. To approximate, we won't necessarily partition each $[a, x_0 - i\delta x]$ into another N subintervals. Instead, we may partition $[a, x_0 - i\delta x]$ evenly into $(N - i)$ subintervals with every endpoint coincides with those in the even partition of $[a, x_0]$ given by N (Figure 1). The key idea is, again, to snowball the values at those endpoints and thus, we don't lose the alignment even in the iterations for higher-order integral. Formally, for each i, we may write $g^{(-1)}(x_0 - i\delta x) = \sum_{j=0}^{N-i-1} \frac{x_0 - i\delta x - a}{N-i} g(x_0 - i\delta x - j\frac{x_0 - i\delta x - a}{N-i}) + \epsilon_i = \sum_{j=0}^{N-i-1} \delta x g(x_0 - i\delta x - j\delta x) + \epsilon_i$, with some $\epsilon_i \in \mathbb{R}$. To visualize the snowballing process, we may represent it in the modified Pascal's Triangle:

	x_0	$x_0 - \dfrac{x_0 - a}{N}$	$x_0 - 2\dfrac{x_0 - a}{N}$	$\dots$	$x_0 - N\dfrac{x_0 - a}{N}$
g	1	2	3	$\dots$	$N + 1.$
$g^{(-1)}$	1	1	1	$\dots$	1
$g^{(-2)}$	1				

Since $f^{(-n)}(x_0 - i\delta x)$ is approximated by the sum of the values of $f^{(-n+1)}$ between $[a, x_0 - i\delta x]$, the entry $a_{ni} = \sum_{k=0}^{i} a_{(n+1)k}$. Finally, based on the first row of the matrix, we get $g^{(-2)} = (\delta x)^2 \sum_{i=0}^{N-1} ig(x_0 - i\delta x) + \epsilon_i + \epsilon_0$.

Then, as $N \to \infty$, we expect that $\sum_{i=0}^{N} \epsilon_i, \epsilon_0 \to 0$ and it follows that

$$g^{(-2)} = \left(\frac{x_0 - a}{N}\right)^2 \lim_{N\to\infty} \sum_{i=0}^{N-1} ig\left(x_0 - i\frac{x_0 - a}{N}\right).$$

3.3. **General Theorems for the Formulae.** Finally, based on our working examples, we may generalize them to derivatives/integrals of any integer orders. The theorems are phrased originally based on the ideas discussed in [1, pp. 28-30].

Theorem 3.1. *If a function f is in $C^0[a,b]$, then for any point $x_0 \in [a,b]$, provided that the limit exists, the n^{th} integral of f over $[a, x_0]$ is*

$$\int_a^{x_0} \dots \int_a^{x} f(x)\, dx_1 \dots dx_n = \lim_{N\to\infty} \left(\frac{x_0 - a}{N}\right)^n \sum_{i=0}^{N-1} \binom{i+n-1}{i} f\left(x_0 - i\left(\frac{x_0 - a}{N}\right)\right).$$

Theorem 3.2. *If a function f is in $C^k[a,b]$ $(k \geq 0)$, then for any point $x_0 \in [a,b]$, provided that the limit exists, the n^{th} $(n \leq k)$ derivative of f at x_0 is*

$$\frac{d^n f}{d(x_0 - a)^n}(x_0) = \lim_{N\to\infty} \left(\frac{x_0 - a}{N}\right)^{-n} \sum_{i=0}^{N-1} (-1)^i \binom{n}{i} f\left(x_0 - i\left(\frac{x_0 - a}{N}\right)\right).$$

Notably, since $\binom{n}{i} = 0$ for all $i \geq n$, the equation for Theorem 3.2 is equivalent to $\frac{d^n f}{d(x_0-a)^n}(x_0) = \lim_{N\to\infty} \left(\frac{x_0-a}{N}\right)^{-n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\left(x_0 - i\left(\frac{x_0-a}{N}\right)\right)$. The form of the equation in Theorem 3.2 is in accordance with Theorem 3.1, which helps us connect the two equations later. While the proof for Theorem 3.2 follows the proof in subsection 3.1 by induction, that for Theorem 3.1 requires more work. Here's an original proof for Theorem 3.1.

Proof. To show that the equation holds for all $n \in \mathbb{N}$, let's prove by induction.
Base Case: When $n = 1$,
Proof for Base Case. Given a function $f \in C^0[a,b]$ and a point $x_0 \in [a,b]$, by definition, given , we see that

$$\int_a^{x_0} f(x), dx = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_0 - i\frac{x_0 - a}{N})\frac{x_0 - a}{N} = \lim_{N\to\infty} \frac{x_0 - a}{N} \sum_{i=0}^{N-1} \binom{i+1-1}{i} f(x_0 - i\frac{x_0 - a}{N}).$$

Inductive Step: Suppose the Theorem 3.1 holds for $n = k$, we want to show that it holds for $n = k + 1$.
Proof for Inductive Step. Given a function $f \in C^0[a,b]$ and a point $x_0 \in [a,b]$, consider $f^{(-k-1)}(x_0)$, the $(k+1)^{th}$ integral of f over $[a, x_0]$ and $f^{(-1)} = \int_a^x f(x), dx$, the first indefinite integral of f with the lower bound a. Then, by inductive hypothesis, since $f^{(-k-1)}(x_0)$ is the k^{th} integral of f^{-1} over

$[a, x_0]$, it follows that $f^{(-k-1)}(x_0) = \lim_{N\to\infty}(\frac{x_0-a}{N})^k \sum_{i=0}^{N-1} \binom{i+k-1}{i} f^{(-1)}(x_0 - i\frac{x_0-a}{N})$. Hence, to prove this limit equals the limit in Theorem 3.1, it suffices to show that the sequence $(\frac{x_0-a}{N})^k \sum_{i=0}^{N-1} \binom{i+k-1}{i} f^{(-1)}(x_0 - i\frac{x_0-a}{N}) - (\frac{x_0-a}{N})^{k+1} \sum_{i=0}^{N-1} \binom{i+k}{i} f(x_0 - i\frac{x_0-a}{N})$ converges to 0.

Hence, given an $\epsilon > 0$, as f is uniformly continuous on $[a, b]$, there exists a $\delta > 0$ such that if $h < \delta$, for all $x \in [a, b]$, we have $|f(x)-f(x-h)| < \frac{\epsilon}{(x_0-a)^{k+1}}$. By Archimedean Property, there exists an $N_1 \in \mathbb{N}$ such that $N_1 > \frac{x_0-a}{\delta}$. Then, since the sequence $\frac{n+c}{n} \to 1$ for all $c \in \mathbb{R}$, by Algebraic Limit Theorem, the sequence $\frac{(n+k-1)...(n-1)}{n^{k+1}} \to 1$. Thus, we may find an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, it satisfies that $\frac{(n+k-1)...(n-1)}{n^{k+1}} \leq 1 + k$. Choose $N_0 = \max(N_1, N_2)$.

To verify, suppose $N \geq N_0$, for each $0 \leq i \leq N - 1$, consider the partition P_{N_i} that divides $[a, x_0 - i\delta x]$ evenly into N_i subintervals. Then, for each $N_i \in \mathbb{N}$, let $f(x_{N_{i,j}})$ be the supremum over the subinterval $[x_0 - i\delta x - (j + 1)(\frac{x_0-i\delta x-a}{N_i}), x_0 - i\delta x - j(\frac{x_0-i\delta x-a}{N_i})]$ $(0 \leq j \leq N_i - 1)$ and let $U(f, P_{N_i}) = (\frac{x_0-i\delta x-a}{N_i}) \sum_{j=0}^{N_i-1} f(x_{N_i,j})$. We see that $U(f, P_{N_i})$ is a sequence of upper sums of f over $[a, x_0 - i\delta x]$ for even partitions (with respect to N_i). Since $f^{(-1)}(x_0 - i\delta x)$ is the integral of f over $[a, x_0 - i\delta x]$, for all $N_i \in \mathbb{N}$, we know that $f^{(-1)}(x_0 - i\delta x) \leq U(f, P_{N_i})$. Hence, let $N_i = N - i$, we have $f^{(-1)}(x_0 - i\delta x) \leq \delta x \sum_{j=0}^{N-i-1} f(x_{N_i,j})$.

Further, since each $x_{N_i,j}$ is within a subinterval $[x_0 - (i + j + 1)\delta x, x_0 - (i + j)\delta x]$ of length $\delta x = \frac{x_0-a}{N} < \frac{x_0-a}{N_0} < \delta$, it follows that

$$f^{-1}(x_0 - i\delta x) \leq \delta x \sum_{j=0}^{N-i-1} (f(x_0 - (i + j)\delta x) + \frac{\epsilon}{(x_0 - a)^{k+1}}).$$

Hence, by substitution, this result above implies that

$$\delta x^k \sum_{i=0}^{N-1} \binom{i+k-1}{i} f^{(-1)}(x_0 - i\delta x) \leq \delta x^{k+1} \sum_{i=0}^{N-1} \binom{i+k-1}{i} \sum_{j=0}^{N-i-1} (f(x_0-(i+j)\delta x) + \frac{\epsilon}{(x_0 - a)^{k+1}}).$$

To simplify, observe that for each $0 \leq i \leq N-1$, we sum $\binom{i+k-1}{i}(f(x_0-i\delta x) + \frac{\epsilon}{(x_0-a)^{k+1}})$ up to $\binom{i+k-1}{i}(f(a) + \frac{\epsilon}{(x_0-a)^{k+1}})$. Thus, for each $0 \leq i \leq N - 1$, we may put all the terms containing $(f(x_0 - i\delta x) + \frac{\epsilon}{(x_0-a)^{k+1}})$ together: $\binom{k-1}{0}(f(x_0 - (0 + i)\delta x) + \frac{\epsilon}{(x_0-a)^{k+1}}) + \ldots \binom{i+k-1}{i}(f(x_0 - i\delta x) + \frac{\epsilon}{(x_0-a)^{k+1}})$. Since $\binom{k-1}{0}+\ldots\binom{i+k-1}{i} = \binom{i+k}{i}$, this may be further simplified as $\binom{i+k}{i}(f(x_0 - i\delta x) + \frac{\epsilon}{(x_0-a)^{k+1}})$. Finally, we compute that

$$\delta x^k \sum_{i=0}^{N-1} \binom{i+k-1}{i} f^{(-1)}(x_0 - i\delta x) - \delta x^{k+1} \sum_{i=0}^{N-1} \binom{i+k}{i} f(x_0 - i\delta x)$$

$$\leq \delta x^{k+1}\sum_{i=0}^{N-1}\binom{i+k}{i}(f(x_0-i\delta x)+\frac{\epsilon}{(x_0-a)^{k+1}})-\delta x^{k+1}\sum_{i=0}^{N-1}\binom{i+k}{i}f(x_0-i\delta x)$$
$$\leq \delta x^{k+1}\sum_{i=0}^{N-1}\binom{i+k}{i}\frac{\epsilon}{(x_0-a)^{k+1}}$$
$$=(\frac{x_0-a}{N})^{k+1}\binom{N+k}{N-1}\frac{\epsilon}{(x_0-a)^{k+1}}$$
$$=\frac{(x_0-a)^{k+1}}{(x_0-a)^{k+1}}\frac{1}{N^k}\frac{(N-k+1)\dots(N-1)}{(k+1)!}\epsilon$$
$$<\epsilon.$$

At the same time, using the sequence of *lower sums* of $f^{-1}(x_0-i\delta)$, it follows that the equation above is greater than $-\epsilon$ and the proof is complete. □

4. Gamma Function and Differintegral

Notice that the two equations in Theorem 3.1 and Theorem 3.2 are mostly identical except for the coefficients $(-1)^i\binom{n}{i}$ and $\binom{i+n-1}{i}$, where $i, n \in \mathbb{Z}$ and $i \geq 0$. To synthesize the two equations into one so that if $n > 0$, Theorem 3.2 follows or otherwise, Theorem 3.1 follows; we want to write $(-1)^i\binom{n}{i}$ and $\binom{i-n-1}{i}$ into one expression. Since both coefficients are in essence combinations, this motivates us to consider representing them with factorials. However, we want the expression to be alternating if and only if n is non-negative and this seems not natural in our narrow definition of factorial operation. To do this, we introduce the *gamma function*. The material is based on [2, pp. 270-281] and [3], though the follow-up discussions and proofs are original to suit this paper. First, based on [2, pp. 270-281] and [3], I phrase the definition of the *gamma function* below.

Definition 4.1. $\Gamma : \mathbb{R} \to \mathbb{R}$ is a function such that $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(1) = 1$, and for two intervals $[a, b]$,$[a', b']$ in $\mathbb{R}$, if $a \leq a'$ and $b \leq b'$, the Γ satisfies the inequality $\frac{\Gamma(b)-\Gamma(a)}{b-a} \leq \frac{\Gamma(b')-\Gamma(a')}{b'-a'}$.

For the purpose of this paper, we only use the first two properties of the gamma function. However, it is also important to add the third property such that the function is unique by Bohr-Mollerup Theorem (see [2, p. 29]).

Now, to understand the definition of the gamma function, given an $n \in \mathbb{N}$, by first property above, we have $\Gamma(n) = (n-1)\Gamma(n-1) = \dots = (n-1)\dots 2\Gamma(1) = (n-1)!$. Therefore, we see that the gamma function is like factorial in the sense that for any positive integer n, $n! = \Gamma(n-1)$.

At the same time, let us consider the gamma function at a non-positive integer. Suppose $n \in \mathbb{Z}$ and $n \leq 0$. As above, we have $\Gamma(n) = (n-1)\Gamma(n-1) = \dots = (x-1)\dots(x-n)\Gamma(x-n) = \dots$. Observe that x will never be reduced to 1, the gamma function becomes an infinite product of negative integers and thus can not be defined. On the other hand, given two non-positive integer n, N and without loss of generality, suppose $n \geq N$, we see that $\frac{\Gamma(n)}{\Gamma(N)} = \frac{(n-1)(n-2)...N(N-1)(N-2)...}{(N-1)(N-2)...} = (-1)^{n-N}\frac{(-N)!}{(-n)!}$. Thus, the ratios of the gamma functions of negative integers can be properly defined and the value is alternating with respect to $(n-N)$. Also, note that when $n, N > 0$, $\frac{\Gamma(n)}{\Gamma(N)}$ is constantly positive, this corresponds the pattern demonstrated on our coefficients. Thus, following this intuition, we want to prove the following modified proposition that is based on [1, p. 20]:

Proposition 4.2. *For any $i, n \in \mathbb{Z}$ and $i \geq 0$,*

$$\frac{\Gamma(i-n)}{\Gamma(-n)\Gamma(i+1)} = \begin{cases} (-1)^i\binom{n}{i} & n > 0, \\ \binom{i-n-1}{i} & n \leq 0. \end{cases}$$

Proof. We start by proving the first case. Let $i, n \in \mathbb{Z}$ with $i \geq 0$ and $n > 0$ be given. Then, if $i > n$, we have $\frac{\Gamma(i-n)}{\Gamma(-n)\Gamma(i+1)} = \frac{(i-n-1)!}{(\prod_{i=n+1}^{\infty}(-i))i!}$. Since $\frac{1}{\prod_{i=n+1}^{\infty}} = 0$ and $\frac{(i-n-1)!}{-i}$ is defined, $\frac{\Gamma(i-n)}{\Gamma(-n)\Gamma(i+1)} = 0 = (-1)^i\binom{n}{i}$. Otherwise, if $i \leq n$, we compute that $\frac{\Gamma(i-n)}{\Gamma(-n)\Gamma(i+1)} = \frac{\Gamma(i-n)}{\Gamma(-n)}\frac{1}{\Gamma(i+1)} = (-1)^{i-n+n}\frac{n!}{(i-n)!i!} = (-1)^i\binom{n}{i}$ as required. Now, if $n \leq 0$, we see that $\frac{\Gamma(i-n)}{\Gamma(-n)\Gamma(i+1)} = \frac{(i-n-1)!}{(-n-1)!i!} = \binom{i-n-1}{i}$. □

Finally, based on Theorem 3.1, Theorem 3.2 and Proposition 4.2, we have verified the following theorem that is phrased based on [1, pp. 28-30]:

Theorem 4.3. *For a function $f \in C^k[a, b]$ $(k \geq 0)$, define the* Differintegral $\frac{d^n f}{d(x-a)^n}$ *over $[a, b]$ to be*

$$\frac{d^n f}{d(x-a)^n} = \lim_{N\to\infty} \frac{\left(\frac{x-a}{N}\right)^{-n}}{\Gamma(-n)} \sum_{i=0}^{N-1} \frac{\Gamma(i-n)}{\Gamma(i+1)} f\left(x - i\left(\frac{x-a}{N}\right)\right).$$

Then, given a point $x_0 \in \mathbb{R}$, if $n > 0$, $\frac{d^n f}{d(x-a)^n}(x_0)$ is the n^{th} derivative of f at x_0. If $n \leq 0$ $\frac{d^n f}{d(x-a)^n}(c)$ is the n^{th} integral of f over $[a, x_0]$.

Notably, since the Gamma function is defined on $\mathbb{R}\backslash\mathbb{Z}^-$, it is valid to have the differintegral of f with any fractional order such as order $\frac{1}{2}$ derivative of f. In those cases, notice that the Gamma function at a non-integer value is an infinite product and thus $\frac{\Gamma(i-n)\Gamma(-n)}{\Gamma(i+1)}$ never equals 0 as i increments; therefore, derivatives behave like integrals in the sense that they are computed with respect to the whole interval $[a, x_0]$ instead of a local neighborhood around x_0. For some applications, [4] shows that a nowhere-differentiable function may have fractional derivative. Using the idea of [5], we see that more functions may be categorized as analytic in this way.

References

[1] Keith B. Oldham and Jerome Spanier. *The fractional calculus*. Mathematics in Science and Engineering. Academic Press, New York-London, 1974, pp. xiii+234. ISBN: 0-12-525550-0.

[2] Stephen Abott. *Understanding analysis*. Undergraduate Texts in Mathematics. Springer, New York, 2015, pp. xii+312. ISBN: 978-1-4939-2711-1.

[3] Philip J. Davis. "Leonhard Euler's integral: A historical profile of the gamma function". In: *Amer. Math. Monthly* 66.10 (1959), 849869. ISSN: 0002-9890. DOI: 0.2307/2309786. URL: https://www.jstor.org/stable/2309786.

[4] Stefan G. Ross Bertram; Samko and E. Russel Love. "Functions that have no first order derivative might have fractional derivatives of all orders less than one". In: *Real Anal. Exchange* 20.2 (1994/95), 140157. ISSN: 0147-1937.

[5] G. H Hardy. "Riemann's form of Taylor's series". In: *J. London Math. Soc.* 20.2 (1945), 4857. URL: https://doi.org/10.1112/jlms/s1-20.1.48.