# LIE GROUPS IN THE CONTEXT OF SMOOTH MANIFOLDS

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## 1. Introduction

**Smooth manifolds** are generalizations of (regular) surfaces in $\mathbb{R}^3$. Intuitively, they are geometric objects that locally behave as $\mathbb{R}^n$ and on which one can perform calculus. The key moral of "smoothness" here is to allow us to obtain (locally) **linear approximations** (models) of manifolds, from which we can study the geometry and topology of the original subjects.

Given a geometric object, one way of studying it is to consider its **symmetries**. That is, the transformations of the object onto itself with some structures invariant. For example, the rotation of an octahedron with angle $\frac{1}{2}\pi$ about the vertical axis $e_1$ is an isometric symmetry with $e_1$ invariant. With an observation that symmetries contain abundant information about their underlying features (structures) of an geometric object, we may encode it by considering its sets of symmetries and studying their interactions. Then, due to the common associativity and invertibility of symmetries, it turns out that many of them form algebraic structures called **groups**,[1] which further encourages these works. In particular, it enables us to gain insights of a geometric object by seeing whether symmetries of another well-studied geometric object operate on it by group actions. For example, the octahedron group (i.e. all isometric symmetries of an octaheron) permutes the faces of a cube in the same way as it permutes the vertices of an octahedron. The intuition is that one can inscribe an octahedron in a cube by connecting the centers of the cube's faces – thus, they admit the same symmetry.

For smooth manifolds, the symmetries of our interests are those that preserve the manifolds' smooth and topological structures. In other words, the

[1]Formally, a group is a set $X$ with an associative binary operation $\circ : X \times X \to X$ under which $X$ has an identity and every element of $X$ has an inverse. The nature of a group is indeed a set of symmetries (of a set, a geometry, etc.) and it is a good intuition to carry forward. For example, every finite group is isomorphic to a subgroup of a permutation group $S_n$, which are symmetries of indices $\{1, 2, \dots, n\}$. However, the converse is not true: not every set of symmetries form a group. The fact that a set of symmetries form a group implies that there is a symmetry of symmetries.

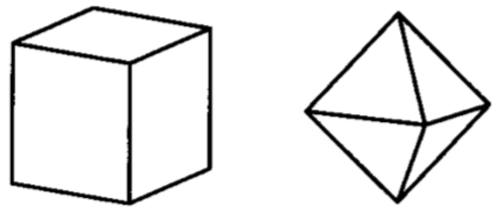

FIGURE 1. A cube (left) and an octahedron (right) [1]

(local) linear approximation (and global arrangement[2]) of a manifold should be equivalent to that of its image under these symmetries. Namely, we may call them **self-diffeomorphisms**. In particular, we consider a group of self-diffeomorphisms whose organization (interaction) also respects the structure of the manifolds so that most information about the manifolds may be attained. Formally, we define a manifold $G$ whose elements form a smooth self-diffeomorphism group to be a **Lie group**. Unlike the octahedron group, a Lie group requires the binary operation $g_1 \circ g_2 \mapsto g_1 g_2$ and the inversion $g \mapsto g^{-1}$ to be smooth so that the elements are naturally self-diffeomorphisms of $G$ and whose interactions respect the structure of $G$. It follows from the definition that Lie groups can be studied as a manifold or a pure algebraic object. This paper focuses on the geometric aspects of Lie groups and discusses its properties as manifolds.

This project is primarily based on [2], with a focus on the understanding and intuitions of ideas more than proofs. First, we provide a historical context of smooth manifolds and Lie groups in Section 2. We start our mathematical discussion by introducing relevant concepts for defining a Lie group with examples in Section 3. As we proceed, an important takeaway is to see some key ideas of smooth manifolds in general. From there, we conclude by investigating some properties of Lie groups as a manifold and how its algebraic nature plays a role in Section 4.

## 2. History

2.1. **Smooth manifold.** The study of (smooth) manifolds is a generalization of curves and surfaces in $\mathbb{R}^3$. A key difference is that manifolds may lie in any **abstract spaces**, motivated by Giovanni Girolamo Saccher's non-Euclidean geometry in 1733. In early Nineteenth Century, Carl Friedrich Gauss is the first to consider abstract spaces as mathematical objects in

[2]Here, what I meant is a topological equivalence (homeomorphism). As we shall see later, since continuity (the essence of a topological equivalence) is implied by a smooth map (the essence of a smooth equivalence), I put it in parentheses.

their own right and his *theorema egregium* gives a method to compute curvature of a surface without considering the ambient space in which the surface lies. Later, mathematicians such as Antoine-Jean Lhuilier generalizes Euler characteristics from a plane to other spaces and shows that it is an **intrinsic property** of a manifold. These led to the study of manifolds, which focuses on the properties (structures) of a geometric object that are irrelevant with the extrinsic properties of the ambient space.

In mid-19th century, Bernhard Riemann was the first to fomrally define *manifolds*. He described the set of all possible values of a variable with certain constraints as a *Mannigfaltigkeit*, which William Kingdon Clifford translated as *"manifoldness"*. Riemann further distinguised *continuous manifoldness* and *discontinuous manifoldness*, depending on whether the variable change continuously. This idea evolve to Riemann manifolds that we call today.

In 1912, Hermann Weyl formalized Riemann's work in Die Idee der Riemannschen Fläche and gave an intrinsic definition for differentiable manifolds. During the 1930s, mathematicians such as Hassler Whitney further formalized the concept of differentiable manifolds, developing through differential geometry and Lie group theory.

2.2. **Lie group theory.** Lie group is named after the Norwegian mathematician Sophus Lie, who invented and developed the early ideas of the theory in late Nineteenth Century. His work was motivated by that of Carl Gustav Jacobi, on the *geometry of differential equations*. The initiative was to find *symmetries of differential equations* and classify them in terms of group theory just like Évariste Galois had done for algebraic equations. In collaboration with Felix Klein and Henri Poincaré, Lie developed the theory of continuous groups to complement that of discrete groups. Although the Lie Theory holds for some special functions, Lie did not manage to unify the entire field of ordinary differential equations.

At the same time, mathematicians such as Bernhard Riemann started considering *continuous groups on the foundation of geometry*. In 1888, Wilhelm Killing and Élie Cartan classified semisimple Lie algebras, which led to Cartan's theory of symmetric spaces. In 20th century, Hermann Weyl clearly enuciated the distinction between Lie groups and Lie algebras, and started investigations of topology of Lie groups. The Lie theory was systematically reworked in modern mathematical language by Claude Chevalley.

## 3. Definitions and examples

To make sense of Lie groups, the first step is understand the idea of smooth manifold.[3] This not only includes its definition, but also the underlying philosophy to which we adhere in our analyses of smooth manifolds.

### 3.1. Smooth Manifold.

A smooth manifold $M$ is a space with a **topological structure** and a **smooth structure**. Intuitively, a topological structure specifies the *arrangement of a space* (e.g. is it connected? does it have a "hole?" etc.); a smooth structure allows *calculus on a space* and in particular, it specifies *how calculus may be performed.*

To grant a smooth structure on a space $M$, its topological structure has to satisfy the following three constraints:

(1) $M$ is a **Hausdorff space**: for every pair of distinct points $p, q \in G$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$.
(2) $M$ is **second-countable**: there exists a countable basis for the topology of $M$.
(3) $M$ is **locally Euclidean of dimension** $n$: each point of $M$ has a neighborhood that is homeomorphic to an open set (open disk) of $\mathbb{R}^n$

A space that satisfies the three properties are namely **topological manifolds**. For the third property, homeomorphism implies that the local (open) areas of $M$ look like an open set (open disk) of $\mathbb{R}^n$.[4] The motivation is straightforward – since calculus is only defined on $\mathbb{R}^n$ as a local operation, $M$ admits calculus only if we can identify (coordinatize) its local areas as a subspace in $\mathbb{R}^n$. A manifold that locally behaves like $\mathbb{R}^n$ is also called an $n-$**dimensional manifold**. The first two properties are somewhat complicated and they are not central to our discussion. In general, the purpose of their presence is to make $M$ a nicer space by *tearing points of $M$ apart.* For example, a convergent sequence in $M$ converges to a unique limit thanks to these properties. They are also called **Separation Axioms**. Here are some examples of topological manifolds that one can check by the properties above:

**Example 3.1.** *Subspace of $\mathbb{R}^n$ is an $n-$dimensional topological manifold.*

**Example 3.2.** *Real projective space $\mathbb{R}P^n$ is an $n-$dimensional topological manifold.*

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[3] While I attempted to restrict our focus on regular surfaces in $\mathbb{R}^3$, it turned out that most interesting examples of Lie groups are manifolds and would be best understood once concepts such as atlas (smooth structure), etc. are introduced.

[4] Formally, a homeomorphism $\phi : U \to V$ is a bijective continuous map whose inverse is also continuous. A continuous map is one such that for every open set in codomain, its preimage is open in domain. The underlying implication is the common intuition that what's near in codomain should be near in domain. This leads to an equivalence of arrangement for the two spaces.

Then, one natural question is that how we can perform calculus on an abstract $n$-dimensional manifold $M$ like $\mathbb{R}P^n$, which lacks a Euclidean coordinate. The idea is **analogous to our definition of (regular) surfaces in** $\mathbb{R}^3$: for each point $p \in M$, we identify its neighborhood $U$ with an (open) subspace $\tilde{U}$ in $\mathbb{R}^n$ via a homeomorphism $\phi : U \to \tilde{U}$; then, a map $f : M \to \mathbb{R}^m$ is smooth at $p$ if $f \circ \phi^{-1}$ is Euclidean **smooth** at $\phi(p)$ [2][3]. The analogy also applies to the smoothness of a map between two abstract manifolds.

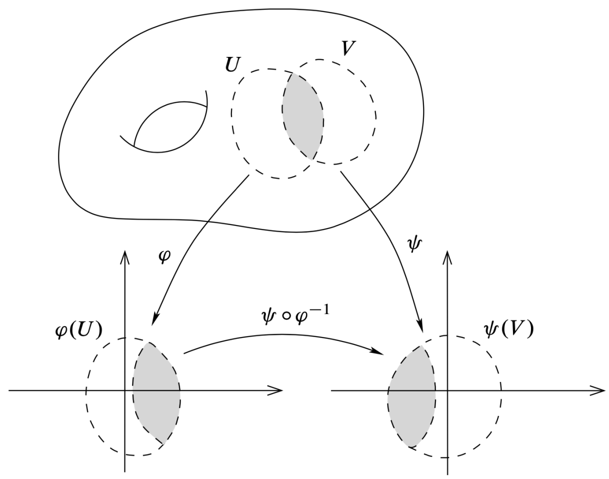


FIGURE 2. Compatibility of charts [2]

However, there is an important **distinction**: note that for a (regular) surface in $\mathbb{R}^3$, we require $\phi^{-1}$ (i.e. patch) to be *differentiable (smooth)*, but this is *not* necessary when defining a smooth structure with $\phi$ merely a homeomorphism for generality.[5] Thus, one problem is that for two neighborhoods $U$ and $V$ of $M$ that intersect, do their identified Euclidean coordinate $\tilde{U}$ and $\tilde{V}$ admit/reject smoothness for the same functions on the part that corresponds $U \cap V$? While we have proved for (regular) surface in $\mathbb{R}^3$ that this is not a problem for patches (for $U \subset M$, every patches admit/reject smoothness for the same functions), it turns out that **for different homeomorphisms, they admit/reject different smooth functions**:

**Example 3.3.** *Consider the topological manifold* $\mathbb{R}$ *and the point* $0$. *First, note that* $\mathbb{R}$ *is a neighborhood of* $0$ *and we want to show that the smoothness of the identity map* $i : \mathbb{R} \to \mathbb{R}$ *depends on how we coordinatize* $\mathbb{R}$.[6]

[5] As we shall see later, (regular) surfaces are indeed a special family of (sub)manifolds.

[6] Here, it would be better to consider the domain of $i$ as a manifold whose smooth structure is to be determined and the codomain as the ordinary Euclidean space.

*First, notice that* $i : \mathbb{R} \to \mathbb{R}$ *itself is a homeomorphism.*[7] *Thus, by identifying* $\mathbb{R}$ *with* $\mathbb{R}$ *via* $i$*, we see that the identity map* $i : \mathbb{R} \to \mathbb{R}$ *is smooth over* $\mathbb{R}$ *with*

$$i \circ i^{-1} = i$$

*Next, observe that* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^3$ *is also a homeomorphism. However, by identifying* $\mathbb{R}$ *with* $\mathbb{R}$ *via* $f$*, we see that*

$$i \circ f^{-1} = x^{\frac{1}{3}},$$

*which is not differentiable at* $0$*. Thus,* $i$ *is not smooth in this case.*

Therefore, the definition of a smooth manifold $M$ requires an explicit specification of a collection of **smooth charts** $\{(U, \phi)\}$, where $U$ is a neighborhood of a point $p \in M$ and $\phi$ is a homeomorphism from $U$ to an open set in $\mathbb{R}^n$. Further, for two charts $(U, \phi)$ and $(V, \psi)$, they should be **smoothly compatible**: $U \cap V = \emptyset$ or $\phi(U \cap V)$ and $\psi(U \cap V)$ admits/rejects smoothness for the same functions.[8] Finally, since we would like to perform calculus on anywhere in $M$, the neighborhoods in $\{(U, \phi)\}$ should *cover* $M$ and it is called a (smooth) **atlas**. An atlas is what we consider the smooth structure of a smooth manifold.

Now, it is possible that for two atlas of a manifold, they admit/reject smoothness for the same functions everywhere. That is, for $\{(U, \phi)\}$ and $\{(U', \phi')\}$ of $M$, **all the charts are compatible and the union** $\{(U, \phi)\} \cup \{(U', \phi')\}$ **is also an atlas of** $M$. In this case, we should consider the two atlas to be *equivalent* as they are essentially the same smooth structures for $M$. To avoid such ambiguity, we may define a smooth structure on $M$ to be the unique **maximal atlas** $\mathcal{A}$ such that it is not properly contained in any larger atlas (i.e. there is no more (smooth) charts that are compatible with $\mathcal{A}$). Hence, the formal definition of a smooth manifold is:

**Definition 3.4.** A **smooth manifold** is a pair $(M, \mathcal{A})$, where $M$ is a topological manifold and $\mathcal{A}$ is a smooth structure on $M$. [2]

An important example of smooth manifold is the general linear group.

**Example 3.5.** *Recall that the* ***general linear group*** $GL(n, \mathbb{R})$ *is an open subset of the* $n^2$*-dimensional vector space* $\mathbb{R}^{n^2}$*. This implies that it is a topological manifold. Further, since* $\mathbb{R}^{n^2}$ *is a smooth manifold (by the global chart* $(\mathbb{R}^{n^2}, i)$*, we see that* $(\mathbb{R}^{n^2} \cap GL(n, \mathbb{R}), i)$ *is an atlas for* $GL(n, \mathbb{R})$ *and it has a smooth structure defined by this atlas.*

[7] Here, it would be better to consider $i$ as a map between two topological spaces.

[8] As we shall see later, this is equivalent to saying that $\psi \circ \phi^{-1}$ is a diffeomorphism.

3.2. **Diffeomorphism.** Although two smooth smooth manifolds may admit/reject smoothness for different functions, it is possible that there is a **bijective correspondence between the smooth functions they admit** – this is considered to be a key equivalence between two smooth manifold. As many other mathematical instances, an identification of such correspondence is to define a morphism.

In particular, suppose $F : M \to N$ is a smooth bijective map whose inverse is also smooth. Then, given a smooth map $f : M \to G$, consider $f \circ F^{-1} : N \to G$. Given a point $p \in N$ and a smooth chart $(U, \phi)$ containing $p$, we can find a smooth chart $(V, \psi)$ of $M$ that contains $F^{-1}(p)$ such that $\psi \circ F^{-1}\phi^{-1}$ is Euclidean smooth at $\psi(p)$. Similarly, since $f$ is smooth, we can find a chart $(W, \chi)$ containing $f(F^{-1}(p))$ such that $\chi \circ f \circ \psi^{-1}$ is Euclidean smooth at $\psi(F^{-1}(p))$. Therefore, we see that

$$\chi \circ (f \circ F^{-1}) \circ \phi^{-1} = (\chi \circ f \circ \psi^{-1}) \circ (\psi \circ F^{-1} \circ \phi^{-1}),$$

is Euclidean smooth as a composition of two Euclidean smooth maps. This implies that $f \circ F^{-1}$ is smooth as a map between manifolds.

In general, we can show that there is a bijective correspondence between the set of smooth functions $f : M \to G$ and that of $g : N \to G$ by the following rule:

$$f \mapsto f \circ F^{-1}, g \mapsto g \circ F.$$

Similarly, there is a bijective correspondence between the set of smooth function $f : G \to M$ and that of $g : G \to N$. Thus, the map $F$ is exactly what we desire as a morphism that identifies correspondent smooth functions on (onto) $M$ and $N$.

**Definition 3.6.** A **diffeomorphism** from $M$ to $N$ is a smooth bijective map $F : M \to N$ that has a smooth inverse. $M$ and $N$ are diffeomorphic if there exists a diffeomorphism between them. [2]

A remaining problem is whether a diffeomorphism identifies **equivalent topological structure between $M$ and $N$** – this turns out to be true by the following proposition.

**Proposition 3.7.** *Every smooth map is continuous. [2]*

Hence, every two diffeomorphic manifolds have equivalent topological and smooth structures.

**Example 3.8.** *Let $\tilde{\mathbb{R}}$ be the topological manifold $\mathbb{R}$ endowed with the smooth structure defined by the global chart $(\mathbb{R}, \phi)$, where $\phi(x) = x^3$. We have shown in Example 3.3 that $\tilde{\mathbb{R}^3}$ has a different smooth structure from $\mathbb{R}$ with the standard smooth structure (i.e. identify $\mathbb{R}$ as itself). However, consider the map $F : \mathbb{R} \to \tilde{\mathbb{R}}$ by $F(x) = x^{\frac{1}{3}}$. Note that for any $x \in \mathbb{R}$, we have*

$$\phi \circ F \circ i^{-1}(x) = x.$$

*In other words, $\phi \circ F \circ i^{-1}$ is an identity map for $(\mathbb{R}, \phi)$ and $(\mathbb{R}, i)$ so that $F$ is smooth as a map between manifolds. Similarly, since $i \circ F^{-1} \circ \phi^{-1}$ is also an identity map, we see that $F$ has a smooth inverse and $F$ is a diffeomorphism. Hence, although $\tilde{\mathbb{R}}$ and $\mathbb{R}$ have different smooth structures, they are indeed equivalent!*

It follows from work of James Munkres and Edwin Moise that **every topological manifold of dimension less than or equal to** 3 has a smooth structure that is unique up to diffeomorphism. Further, $\mathbb{R}^n$ has a unique smooth structure up to diffeomorphism as long as $n \neq 4$ [2].

3.3. **Differential as the linear approximation.** Recall from Section that the key moral of smoothness is to allow us to study a manifold by its (local) **linear approximations**. Indeed, like (regular) surfaces in $\mathbb{R}^3$, many properties of a smooth manifold $M$ are reflected by the **tangent space** $T_pM$ at each point $p \in M$ (i.e. the linear approximation of the neighborhood of $p$). While the formal definition of $T_pM$ of a smooth manifold $M$ varies from that of a (regular) surface, the key intuition stays the same and there is no harm to simply generalize the surface definition [3] for our purpose.

**Definition 3.9.** Let $p$ be a point in a smooth manifold $M$. A **tangent vector** $v$ to $M$ at a $p$ is tangent to $M$ at $p$ provided $v$ is a velocity of some curve in a chart $(U, \phi)$ of $p$. The set of tangent vectors $v$ to $M$ at $p$ forms the **tangent space** $T_pM$.

Note that $\gamma : [0, 1] \to M$ is a curve in a neighborhood $(U, \phi)$ of $p$ if $\phi \circ \gamma : [0, 1] \to \mathbb{R}^n$ is a curve in $\mathbb{R}^n$.

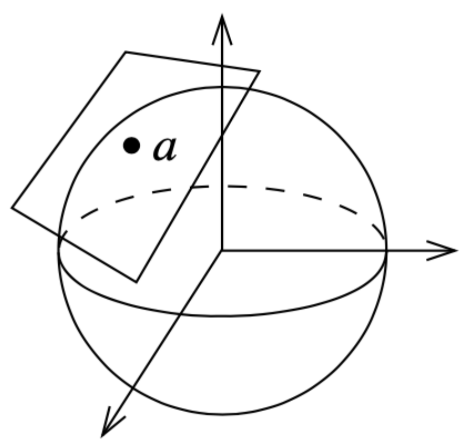


FIGURE 3. Tangent space at a point of $S^2$ [2]

Now, suppose $F : M \to N$ is a smooth map between smooth manifolds. Similarly to the definition above, we may generalize the definition of the differential $dF_p : T_pM \to T_{F(p)}N$ from the surface definition [3]. Intuitively, $dF_p$ is the "best linear approximation" of $F$ near $p$. Thus, it seems plausible that one can learn about $F$ by studying **linear-algebraic properties** of its differential. In fact, by the theorem below, we see that the **rank** (dimension of its image) is the only parameter that distinguishes different linear maps (if we are free to choose bases independently or the domain and codomain):

**Theorem 3.10.** *Suppose $V$ and $W$ are finite-dimensional vector spaces, and $T : V \to W$ is a linear map of rank $r$. Then there are bases for $V$ and $W$ with respect to which $T$ has the following matrix representation (in block form):*

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

We know from linear algebra that the rank of a linear map is fully determined by if it is **injective**, **surjective**, or **bijective**. For a smooth map $F : M \to N$, it turns out that $F$ identifies an equivalence in the (local) smooth structure between $M$ and $N$ if for every point $p \in M$, $dF_p$ is **bijective** (i.e. the differential identifies a bijective correspondence between the tangent spaces at every point $p \in M$ and $F(p) \in N$).

**Theorem 3.11.** *Suppose $M$ and $N$ are smooth manifolds and $F : M \to N$ is a smooth map. Then, for every point $p \in M$, there exists a neighborhood $U$ of $p$ that is diffeomorphic to a neighborhood of $V$ of $F(p)$ if and only if the differential $dF_p$ is bijective. [2]*

Since calculus is a **local operation**, one should be intuitively convinced that these local diffeomorphisms indeed implies that $M$ and $N$ have **equivalent smooth structure**. If they also have *equivalent topological structure*, then they are fully diffeomorphic. To achieve it, it suffices to require $F$ **to be bijective**.

**Proposition 3.12.** *Every bijective local diffeomorphism is a diffeomorphism. [2]*

It is worth noting that if $dF_p$ is only injective at every point $p \in M$, although $M$ may not be locally diffeormophic to $N$, **it is locally diffeomorphic to its image** $F(M)$ (i.e. a subspace of $N$). An important application of this is an **embedded submanifold**. That is, a subspace of $M$ that respects the topological and smooth structures of the original space.

**Definition 3.13.** An **embedded submanifold** of $M$ is a subset $S \subset M$ that is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $F : S \to M$ is a smooth homeomorphism whose differential $dF_p$ is injective at every point $p \in S$.

Similarly, an **immersed submanifold** is an embedded submanifold with a distinctino that **the inclusion map $F$ is not necessarily a homeomorphism** (In other words, it is only required to respect the smooth structure of the original manifold). Now, if we work with the definition of embedded submanifold, we shall see that **embedded submanifolds in $\mathbb{R}^3$ are exactly regular surfaces in $\mathbb{R}^3$**. This gives an alternative explanation of why for a regular surface $S \in \mathbb{R}^3$, it is the same to do calculus on $S$ immediately in $\mathbb{R}^3$ or do so with their patches – **the smooth structure defined by patches respects the standard smooth structure of $\mathbb{R}^3$**.

**Example 3.14.** *Regular surface in $\mathbb{R}^3$ is an equivalent definition as embedded submanifold in $\mathbb{R}^3$.*

We conclude this section by inspecting **smooth maps of constant rank**, which again highlights the importance of differential as linear approximation.

**Definition 3.15.** A smooth map $F : M \to N$ has **constant rank** if its differential $dF_p : T_pM \to T_pN$ has the same rank at each point $p \in M$. [2]

A smooth map of constant rank largely preserves the local Euclidean behavior of $M$ onto its image $F(M)$ with nowhere abruptly collapses. However, there do exist smooth maps without constant ranks.

**Example 3.16.** *Consider the smooth map $\gamma\mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (t^3, 0)$, where $\mathbb{R}$ and $\mathbb{R}^2$ are endowed with their standard smooth and topological structure. Then its differential $d\gamma_p : T_p\mathbb{R} \to T_p\mathbb{R}^2$ is given by $d\gamma_p(v_p) = 3p^2 v_{\gamma(p)}$. This map has rank $1$ (bijective) at every point $p \neq 0$ as a non-zero scalar multiplication. However, it has rank $0$ at $p = 0$ with $d\gamma_p(v_p) = 0$ for all $v_p \in T_p\mathbb{R}$.*

As a corollary of the inverse function theorem, it turns out that smooth maps of constant rank are precisely the ones whose locally behaviors are the same as their differentials, resonating with our intuition above. In particular, from the proof of the following theorem, it can be shown that for an $m-$dimensional manifold $M$ whose image is an $n-$dimensional manifold $N$ under a smooth map $F : M \to N$ of constant rank, **if $dF_p$ is injective, $F$ is locally an inclusion from $\mathbb{R}^m$ to $\mathbb{R}^n$; if $dF_p$ is surjective, $F$ is locally a projection from $\mathbb{R}^m$ to $\mathbb{R}^n$**.

**Theorem 3.17.** *Let $M$ and $N$ be smooth manifolds, let $F : M \to N$ be a smooth map. Then, if $F$ has constant rank, for each $p \in M$ there exist smooth charts containing $p$ and $F(p)$ in which the coordinate representation of $F$ is linear.*

3.4. **Lie group.** Now, we are prepared to define **Lie groups** and explore some of its fundamental examples and properties. With the motivation described in Section 1, we define a Lie group formally as follows.

**Definition 3.18.** A **Lie group** is a smooth manifold $G$ that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G \to G$ and inversion map $i : G \to G$ given by

$$m(g, h) = gh, \ i(g) = g^{-1},$$

are both smooth. [2]

As mentioned in Section 1, elements $g \in G$ (when viewed as a group) can be considered to be **self-diffeomorphisms** $g : G \to G$. This is because the multiplication map $m$, when fixed an element $g$ (either left or right), becomes a smooth map $L_g$ (or $R_g$ depending on whether we fix the left or right element) from $G$ to itself. At the same time, since $g$ has an inverse $g^{-1}$, we see that when $m$ fixes $g^{-1}$ in the same position, it induces a smooth map $L_{g^{-1}}$ (or $R_{g^{-1}}$) that is the inverse of $L_g$ $R_{g^{-1}}$. The induced diffeomorphisms are namely **left translations** or **right translations** depending on which element is fixed in $m$.

At the same time, when inputs of $m$ are both considered self-diffeomorphisms, the smoothness of $m$ implies that the self-diffeomorphisms in $G$ interact (arrange) in a way that **respects the structure of** $G$. Thus, $G$ as a symmetry group contains information about itself as a manifold to the greatest extent.

**Example 3.19.** *We have seen in Example that the **general linear group** $GL(n, \mathbb{R})$ is a smooth manifold with standard smooth structure inherited from $\mathbb{R}^{n^2}$. Simultaneously, one can show that under matrix multiplication and inversion, it is indeed a **group**. Further, matrix multiplication is smooth because the matrix entries of a product matrix $AB$ are polynomials in the entries of $A$ and $B$. The matrix inversion is also smooth by Cramer's rule from linear algebra. Similarly, $GL(n, \mathbb{C})$ is a Lie group.*

**Example 3.20.** *A unit circle $S^1$ can be represented as a subset $\{(r, \phi)\}$ in $\mathbb{C} = GL(1, \mathbb{C})$ with the same radial of $1$. It is a $1$-dimensinoal smooth manifold because it can be identified as a circle curve in $\mathbb{R}^2$. Simultaneously, notice that it is also a group under complex multiplication and inversion, both of which are smooth.*

It is worth noting that for a smooth manifold, it may admit **different Lie group structures**, each specifying a **(potentially) different symmetry of the manifold**.

**Example 3.21.** *Since $GL(n, \mathbb{C})$ is a Lie group under matrix multiplication, $\mathbb{C}$ is a Lie group under **complex multiplication**. At the same time, $\mathbb{C}$ is also a Lie group under **addition**.*

However, we should point that **not all smooth manifolds admit a Lie group structure**. A well-known example is that of a unit sphere $S^2$.

**Example 3.22.** *$S^2$ does not admit a Lie group structure. [4]*

While the argument is beyond our scope, it is interesting to introduce the underlying mechanism we provide a sketch of the proof.

First, similarly to tangent space as a linear model for smooth manifolds, there is an analogy for Lie group called **Lie algebra**. Informally, consider a **vector field** (same as that for surface in $\mathbb{R}^3$ [3]) $X : M \to TM$ such that when it applies to the self-diffeomorphism induced by $g \in G$, it is **invariant**. That is, once $X(p)$ is determined for any $p \in M$, all other $X(p')$ is determined via actions of $g$. Now, the Lie algebra of a Lie group $G$ is precisely the set of all such vector fields for $G$ called **left-invariant vector fields** on $G$. Overall, the Lie algebra is a **linear model of the Lie group** so that we can learn about the Lie group by studying its Lie algebra. In fact, one can show that the Lie algebra of $G$ is a vector space and that it indeed **spans the tangent spaces of** $G$. This implies that for every Lie group, it has a **global frame field** (same as that for surface in $\mathbb{R}^3$ [3]) which can be any **basis of the Lie algebra**. However, by the **Hairy Ball Theorem**, it follows that $S^2$ cannot have a global frame field. Therefore, $S^2$ cannot be a Lie group! In fact, one can show that the **only spheres that admit a Lie group structure are** $S^0$**,** $S^1$**, and** $S^3$.

With a group structure, we are able to discuss how (symmetries of) a manifold acts on another in the same way as octahedron group acting on a cube. The **orbits** and **stabilizers** of the actions reflect some shared structural properties of the two manifolds.

## 4. Properties of Lie groups

We conclude by introducing some properties of Lie groups and in particular, how its **algebraic nature plays a role in geometry**.

### 4.1. Lie homomorphism.

**Group homomorphism** is the equivalence between groups. Since Lie groups are also groups, we are able to connect them via homomorphisms. Further, to preserve some smooth structure, we require such homomorphisms to be **smooth**.

**Definition 4.1.** If $G$ and $H$ are Lie groups, a **Lie group homomorphism** from $G$ to $H$ is a smooth map $F : G \to H$ that is also a group homomorphism. It is called a **Lie group isomorphism** if it is also a diffeomorphism. [2]

Here are some examples of Lie group homomorphisms.

**Example 4.2.** *The* ***inclusion map*** $S^1 \to \mathbb{C}$ *is a Lie group homomorphism.*

**Example 4.3.** *The* ***determinant function*** $\Delta : GL(n, \mathbb{R}) \to \mathbb{R}$ *is a group homomorphism with* $\Delta(AB) = (\Delta A)(\Delta B)$*. Further, since* $\Delta A$ *is a polynomial in the matrix entries of* $A$*, it is smooth. Thus,* $\Delta : GL(n, \mathbb{R}) \to \mathbb{R}$ *is a Lie group homomorphism. Similarly, the determinant of* $GL(n, \mathbb{C})$ *is a Lie group homomorphism.*

The speciality of Lie group homomorphism is that it is a **locally linear map**. This shouldn't be too surprising because for two Euclidean spaces, a group homomorphism is (similar to) a linear map. Since two Lie groups are first of all manifolds that locally behave as Euclidean spaces, the result follows. Formally, we have:

**Theorem 4.4.** *Every Lie group homomorphism has constant rank. [2]*

It follows from our discussion of smooth manifolds that smooth maps of constant rank are precisely those that locally appear as a projection (inclusion) from $\mathbb{R}^m$ to $\mathbb{R}^n$.

4.2. **Lie subgroup.** A number of new examples of Lie groups with interesting properties arise from **Lie subgroups**.

**Definition 4.5.** A **Lie subgroup** of a Lie group $G$ is an algebraic subgroup endowed with a topology and smooth structure making it into a Lie group and an immersed submanifold of $G$. [2]

In other words, a Lie subgroup is a submanifold of $G$ that **respects its smooth structure and Lie group structure** (not necessarily the topology). Besides that it is a group, since a Lie group structure depends only on the smooth and topological structures of a manifold, an immediate consequence from the intuition is that any subgroup of a Lie group that is also an **embedded submanifold** should be a Lie subgroup.

**Proposition 4.6.** *Let $G$ be a Lie group, and suppose $H \subset G$ is a subgroup that is also an embedded submanifold. Then $H$ is a Lie subgroup. [2]*

Conversely, it is also possible to obtain an embedded Lie subgroup by imposing some constraints on a subgroup. The simplest examples of these are the **open subgroups**.

**Lemma 4.7.** *Suppose $G$ is a Lie group and $H \subset G$ is an open subgroup. Then $H$ is an embedded Lie subgroup. In addition, $H$ is (topologically) closed, so it is a union of connected components of $G$. [2]*

In other words, every open subgroup of a Lie group $G$ is an embedded subgroup that is the union of **maximally connected manifolds** in $G$. Here, a manifold is maximally connected if it is not connected to any other points in $G$ (i.e. it itself stands as "an island!"). This is an exciting result as it turns out that open subgroups of a Lie group have very nice topological properties.

**Example 4.8.** *The subgroup $GL^+(n, \mathbb{R}) \subset GL(n, \mathbb{R})$ (**invertible matrices with positive determinants**) is an open subgroup and thus an embedded Lie subgroup.*

Now, since every subgroup of a group contains the identity of $G$. One natural question is to ask **which neighborhoods of the identity are open subgroups of** $G$. It turns out that every **subgroup generated by a neighborhood of the identity** (i.e. the smallest subgroup containing a neighborhood of the identity) **is automatically open**!

**Proposition 4.9.** *Suppose $G$ is a Lie group, and $W \subset G$ is any neighborhood of the identity. [2]*

(1) *$W$ generates an open subgroup of $G$.*
(2) *If $W$ is connected, it generates a connected open subgroup of $G$.*
(3) *If $G$ is connected, then $W$ generates $G$.*

The latter two properties follow immediately from the fact that an open subgroup is a union of connected components of $G$.

Finally, in the same way as group homomorphisms identify normal subgroups though its **kernel**, the kernel of a Lie group homomorphism is indeed a properly embedded Lie subgroup of $G$. This enables us to produce many more examples of embedded Lie subgroups:

**Proposition 4.10.** *Let $F : G \to H$ be a Lie group homomorphism. The kernel of $F$ is a properly embedded Lie subgroup of $G$. [2]*

In general, we see that the smooth group structure on the manifold helps us study its geometry in a rather algebraic way! We end by providing one application of the above proposition.

**Example 4.11.** *The **special linear group** $SL(n, \mathbb{R})$ of degree $n$ is the group of $n \times n$ real matrices with determinant equal to $1$. Since $SL(n, \mathbb{R})$ is the kernel of the Lie group homomorphism $\Delta : GL(n, \mathbb{R}) \to \mathbb{R}$ (as mentioned above), it is a properly embedded Lie subgroup.*

## References

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