

1.1

$$\begin{aligned}
\alpha_1(1) &= \pi_1 P(A|Z_1 = 1) = 0.7 * 0.4 = 0.28 \\
\alpha_1(2) &= \pi_2 P(A|Z_1 = 2) = 0.3 * 0.2 = 0.06 \\
\alpha_2(1) &= P(G|Z_2 = 1)(a_{11}\alpha_1(1) + a_{21}\alpha_1(2)) = 0.4 * (0.8 * 0.28 + 0.4 * 0.06) = \\
&0.0992 \\
\alpha_2(2) &= P(G|Z_2 = 2)(a_{12}\alpha_1(1) + a_{22}\alpha_1(2)) = 0.2 * (0.2 * 0.28 + 0.6 * 0.06) = \\
&0.0184 \\
\alpha_3(1) &= P(C|Z_3 = 1)(a_{11}\alpha_2(1) + a_{21}\alpha_2(2)) = 0.4 * (0.8 * 0.0992 + 0.4 * \\
&0.0184) = 8.672 * 10^{-3} \\
\alpha_3(2) &= P(C|Z_3 = 2)(a_{12}\alpha_2(1) + a_{22}\alpha_2(2)) = 0.2 * (0.2 * 0.0992 + 0.6 * \\
&0.0184) = 9.264 * 10^{-3} \\
\alpha_4(1) &= P(G|Z_4 = 1)(a_{11}\alpha_3(1) + a_{21}\alpha_3(2)) = 0.4 * (0.8 * 8.672 * 10^{-3} + 0.4 * \\
&9.264 * 10^{-3}) = 4.25728 * 10^{-3} \\
\alpha_4(2) &= P(G|Z_4 = 2)(a_{12}\alpha_3(1) + a_{22}\alpha_3(2)) = 0.2 * (0.2 * 8.672 * 10^{-3} + 9.264 * \\
&10^{-3}) = 1.45856 * 10^{-3} \\
\alpha_5(1) &= P(T|Z_5 = 1)(a_{11}\alpha_4(1) + a_{21}\alpha_4(2)) = 0.4 * (0.8 * 4.25728 * 10^{-3} + \\
&0.4 * 1.45856 * 10^{-3}) = 3.989248 * 10^{-4} \\
\alpha_5(2) &= P(T|Z_5 = 2)(a_{12}\alpha_4(1) + a_{22}\alpha_4(2)) = 0.2 * (0.2 * 4.25728 * 10^{-3} + \\
&0.61.45856 * 10^{-3}) = 5.179776 * 10^{-4} \\
\alpha_6(1) &= P(A|Z_6 = 1)(a_{11}\alpha_5(1) + a_{21}\alpha_5(2)) = 0.4 * (0.8 * 3.989248 * 10^{-4} + \\
&0.4 * 5.179776 * 10^{-4}) = 2.10532352 * 10^{-4} \\
\alpha_6(2) &= P(A|Z_6 = 2)(a_{12}\alpha_5(1) + a_{22}\alpha_5(2)) = 0.2 * (0.2 * 3.989248 * 10^{-4} + \\
&0.6 * 5.179776 * 10^{-4}) = 7.8114304 * 10^{-5} \\
P(X_{1:6} = O_{1:6}) &= \alpha_6(1) + \alpha_6(2) = 2.10532352 * 10^{-4} + 7.8114304 * 10^{-5} = \\
&2.8865 * 10^{-4}
\end{aligned}$$

1.2

$$\begin{aligned}
\delta_1(1) &= \pi_1 P(A|Z_1 = 1) = 0.7 * 0.4 = 0.28 \\
\delta_1(2) &= \pi_2 P(A|Z_1 = 2) = 0.3 * 0.2 = 0.06 \\
\delta_2(1) &= \max(a_{11}\delta_1(1)P(G|Z_2 = 1), a_{21}\delta_1(2)P(G|Z_2 = 1)) = \max(0.8 * 0.28 * \\
&0.4, 0.4 * 0.06 * 0.4) = 0.0896 \quad \text{Max from s1} \\
\delta_2(2) &= \max(a_{12}\delta_1(1)P(G|Z_2 = 2), a_{22}\delta_1(2)P(G|Z_2 = 2)) = \max(0.2 * 0.28 * \\
&0.2, 0.6 * 0.06 * 0.2) = 0.0112 \quad \text{Max from s1} \\
\delta_3(1) &= \max(a_{11}\delta_2(1)P(C|Z_3 = 1), a_{21}\delta_2(2)P(C|Z_3 = 1)) = \max(0.8 * \\
&0.0896 * 0.1, 0.4 * 0.0112 * 0.1) = 7.168 * 10^{-3} \quad \text{Max from s1}
\end{aligned}$$

$$\delta_3(2) = \max(a_{12}\delta_2(1)P(C|Z_3 = 2), a_{22}\delta_2(2)P(C|Z_3 = 2)) = \max(0.2 * 0.0896 * 0.3, 0.6 * 0.0112 * 0.3) = 5.376 * 10^{-3} \text{ Max from s1}$$

$$\begin{aligned}\delta_4(1) &= \max(a_{11}\delta_3(1)P(G|Z_4 = 1), a_{21}\delta_3(2)P(G|Z_4 = 1)) = \max(0.8 * 7.168 * 10^{-3} * 0.4, 0.4 * 5.376 * 10^{-3} * 0.4) = 2.29376 * 10^{-3} \text{ Max from s1} \\ \delta_4(2) &= \max(a_{12}\delta_3(1)P(G|Z_4 = 2), a_{22}\delta_3(2)P(G|Z_4 = 2)) = \max(0.2 * 7.168 * 10^{-3} * 0.2, 0.6 * 5.376 * 10^{-3} * 0.2) = 6.4512 * 10^{-4} \text{ Max from s2}\end{aligned}$$

$$\begin{aligned}\delta_5(1) &= \max(a_{11}\delta_4(1)P(T|Z_4 = 1), a_{21}\delta_4(2)P(T|Z_4 = 1)) = \max(0.8 * 2.29376 * 10^{-3} * 0.1, 0.4 * 6.4512 * 10^{-4} * 0.1) = 1.835008 * 10^{-4} \text{ Max from s1} \\ \delta_5(2) &= \max(a_{12}\delta_4(1)P(T|Z_4 = 2), a_{22}\delta_4(2)P(T|Z_4 = 2)) = \max(0.2 * 2.29376 * 10^{-3} * 0.3, 0.6 * 6.4512 * 10^{-4} * 0.3) = 1.376256 * 10^{-4} \text{ Max from s1}\end{aligned}$$

$$\begin{aligned}\delta_6(1) &= \max(a_{11}\delta_5(1)P(A|Z_5 = 1), a_{21}\delta_5(2)P(A|Z_5 = 1)) = \max(0.8 * 1.835008 * 10^{-4} * 0.4, 0.4 * 1.376256 * 10^{-4} * 0.4) = 5.872 * 10^{-5} \text{ Max from s1} \\ \delta_6(2) &= \max(a_{12}\delta_5(1)P(A|Z_5 = 2), a_{22}\delta_5(2)P(A|Z_5 = 2)) = \max(0.2 * 1.835008 * 10^{-4} * 0.2, 0.6 * 1.376256 * 10^{-4} * 0.2) = 1.6515072 * 10^{-5} \text{ Max from s2}\end{aligned}$$

So the end is S1, and trace back we know the most likely path is S1, S1, S1, S1, S1, S1

1.3

$P(X_7 = O_7 | X_{1:6} = O_{1:6}) = \frac{P(X_{1:7}=O_{1:7})}{P(X_{1:6}=O_{1:6})}$ So we just need to maximize the joint probability.

$$a_{11}\alpha_6(1) + a_{21}\alpha_6(2) = 0.8 * 2.10532352 * 10^{-4} + 0.4 * 7.8114304 * 10^{-5} = 1.996 * 10^{-4} \text{ Mark it as } b_1.$$

$$a_{12}\alpha_6(1) + a_{22}\alpha_6(2) = 0.2 * 2.10532352 * 10^{-4} + 0.6 * 7.8114304 * 10^{-5} = 8.897 * 10^{-5} \text{ Mark it as } b_2.$$

$$\text{if } O_7 = A: P(X_{1:7} = O_{1:7}) = \alpha_7(1) + \alpha_7(2) = P(A|Z_7 = 1)b_1 + P(A|Z_7 = 2)b_2 = 9.673 * 10^{-5}$$

$$\text{if } O_7 = C: P(X_{1:7} = O_{1:7}) = \alpha_7(1) + \alpha_7(2) = P(C|Z_7 = 1)b_1 + P(C|Z_7 = 2)b_2 = 4.665 * 10^{-5}$$

$$\text{if } O_7 = G: P(X_{1:7} = O_{1:7}) = \alpha_7(1) + \alpha_7(2) = P(G|Z_7 = 1)b_1 + P(G|Z_7 = 2)b_2 = 9.673 * 10^{-5}$$

$$\text{if } O_7 = T: P(X_{1:7} = O_{1:7}) = \alpha_7(1) + \alpha_7(2) = P(T|Z_7 = 1)b_1 + P(T|Z_7 = 2)b_2 = 4.665 * 10^{-5}$$

So $x^* = A$ or G