d) 对于 d 问，请尝试增大单元数目，观察现象，并尝试解释原因。

通过成倍地增加单元个数进行计算，发现：

1）在单元数目较少时，采用不同 tol 计算得到的结果与直接计算得到的结果相同；

2）逐渐增大单元数目，采用 10e-2 tol 计算得到的结果与直接计算结果相比出现明显偏差，此时采用 10e-4 和 10e-6 tol 计算得到的结果与直接计算结果相比误差更小；

3）继续增大单元数目，采用 10e-4 tol 计算得到的结果也出现和直接计算结果相比出现明显偏差，此时直接计算结果是这 4 种结果里面误差最小的。

出现上述情况的原因：

指定 restart = maxit = 10000，总迭代次数不超过其两者之和

指定容差分别为 10e-2,10e-4,10e-6

1）在单元数目较小时，直接计算和不同容差下的 gmres 方法计算结果相同；

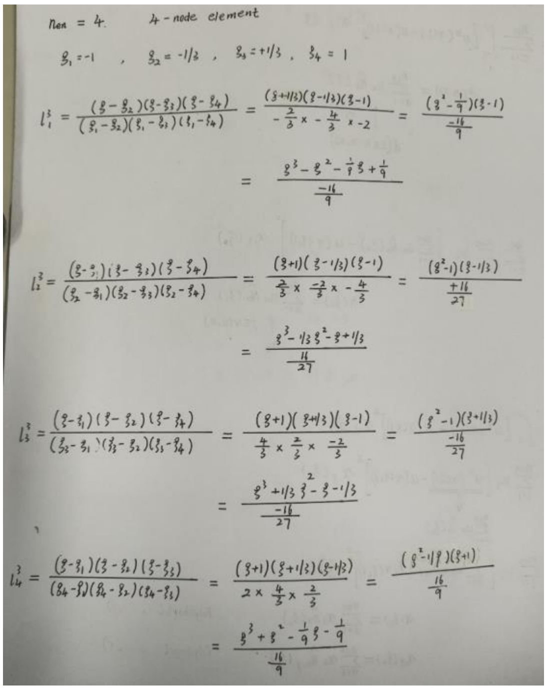
2）逐渐增大单元数目，容差较小的 gmres 方法在不超过总迭代次数下结果最先出现偏差，因为给定的容差为 10e-2；

3）继续增大单元数目，在不超过总迭代次数的情况下，最小容差情况 10e-6 计算所得的结果也会次于直接计算结果，说明在此单元数目下，给定的容差和迭代次数无法满足计算要求，此时可以通过继续缩小容差 tol 的值来实现更高的计算精度。

Bonus 代码已同步更新到 GitHub 上

e) The equivalence between (G) and (M) is contingent upon the accuracy of the quadrature. The effect of numerical quadrature is discussed in Hughes's book p.191. Run the code with the element of degree 3 (i.e. cubic element). Experiment the code with 1,2,3,4,5,6 quadrature points, respectively. Report your observations and make comments.

In the problem definition part, the chosen function f is sin(x).
The element is cubic(element of degree 3), the polyshape function is shown below and attached in the submitted code.


$n_{en} = 4$. 4-node element

$\xi_1 = -1, \quad \xi_2 = -1/3, \quad \xi_3 = +1/3, \quad \xi_4 = 1$

$$l_1^3 = \frac{(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}{(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_1-\xi_4)} = \frac{(\xi+1/3)(\xi-1/3)(\xi-1)}{-\frac{2}{3}\times -\frac{4}{3}\times -2} = \frac{(\xi^2-\frac{1}{9})(\xi-1)}{\frac{-16}{9}} = \frac{\xi^3-\xi^2-\frac{1}{9}\xi+\frac{1}{9}}{\frac{-16}{9}}$$

$$l_2^3 = \frac{(\xi-\xi_1)(\xi-\xi_3)(\xi-\xi_4)}{(\xi_2-\xi_1)(\xi_2-\xi_3)(\xi_2-\xi_4)} = \frac{(\xi+1)(\xi-1/3)(\xi-1)}{\frac{2}{3}\times\frac{-2}{3}\times -\frac{4}{3}} = \frac{(\xi^2-1)(\xi-1/3)}{\frac{+16}{27}} = \frac{\xi^3-1/3\,\xi^2-\xi+1/3}{\frac{16}{27}}$$

$$l_3^3 = \frac{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_4)}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)} = \frac{(\xi+1)(\xi+1/3)(\xi-1)}{\frac{4}{3}\times\frac{2}{3}\times\frac{-2}{3}} = \frac{(\xi^2-1)(\xi+1/3)}{\frac{-16}{27}} = \frac{\xi^3+1/3\,\xi^2-\xi-1/3}{\frac{-16}{27}}$$

$$l_4^3 = \frac{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)}{(\xi_4-\xi_1)(\xi_4-\xi_2)(\xi_4-\xi_3)} = \frac{(\xi+1)(\xi+1/3)(\xi-1/3)}{2\times\frac{4}{3}\times\frac{2}{3}} = \frac{(\xi^2-1/9)(\xi+1)}{\frac{16}{9}} = \frac{\xi^3+\xi^2-\frac{1}{9}\xi-\frac{1}{9}}{\frac{16}{9}}$$

PolyShape Function in this Case

Solving the problem of Kd=F with different numbers of quadrature points,
the solutions (d = [uh; g])are listed as follows:

	1
1	[-1.1618e+18;-2.1460e+16;2.1456e+16;-3.1027e+15;1.7097e+14;2.8588e+14;0.8415]
2	[-0.1663;-1.2560e+15;1.2560e+15;0.4187;-4.9029e+14;4.9029e+14;0.8415]
3	[4.0818e-08;0.1659;0.3272;0.4794;0.6184;0.7402;0.8415]
4	[-1.5304e-11;0.1659;0.3272;0.4794;0.6184;0.7402;0.8415]
5	[5.9893e-15;0.1659;0.3272;0.4794;0.6184;0.7402;0.8415]
6	[3.1071e-17;0.1659;0.3272;0.4794;0.6184;0.7402;0.8415]

The six rows correspond to the cases with 1 to 6 quadrature points.
The first two rows correspond to the 1 and 2 quadrature points cases, which result in one warning that “警告: 矩阵接近奇异值，或者缩放错误。结果可能不准确”.

The following is the reason for this phenomenon.
We solve the matrix problem using the Matlab code in the element view.
Gaussian quadrature is needed to calculate the integral.
For the element stiffness:

$$k_{ab}^e = \int_{-1}^{1} \widehat{N_{a,\xi}}\,\widehat{N_{b,\xi}}\,\xi_{,x} d\xi \approx \sum_{l=1}^{n_{int}} w_l \widehat{N_{a,\xi}}(\xi_l)\widehat{N_{b,\xi}}(\xi_l)\left[x_{,\xi}(\xi_l)\right]^{-1} \tag{1}$$

Similarly:

$$\int_{-1}^{1} \widehat{N_a} f\big(x(\xi)\big) x_{,\xi} d\xi \approx \sum_{l=1}^{n_{int}} w_l \widehat{N_a}(\xi_l) f\big(x(\xi_l)\big) x_{,\xi}(\xi_l) \tag{2}$$

The n_{int}-point Gaussian quadrature can integrate polynominals with degrees up to $2n_{int} - 1$

The degrees of $\widehat{N_a}$ and $\widehat{N_{a,\xi}}$ are 3 and 2.

The degree of $x_{,\xi}$ is same to $\widehat{N_{a,\xi}}$, which is 2. $(x(\xi)$ is a mapping from ξ to x)

Note that:

$$\xi_{,x} = x_{,\xi}^{-1} \tag{3}$$

So the degree of $\widehat{N_{a,\xi}}\widehat{N_{b,\xi}}\xi_{,x}$ is 2, which can be obtained through **2 quadrature points.**

We choose f to be sin(x), and the degree of $\widehat{N_a} f\big(x(\xi)\big) x_{,\xi}$ is 5, which can be obtained through **3 quadrature points.**

In conclusion, we need at least 3 quadrature points to integrate the above two polynominals, which can explain the warning that occurs in the case with only 1 and 2 quadrature points.

	1
1	[-1.1618e+18;-2.1460e+16;2.1456e+16;-3.1027e+15;1.7097e+14;2.8588e+14;0.8415]
2	[-0.1663;-1.2560e+15;1.2560e+15;0.4187;-4.9029e+14;4.9029e+14;0.8415]
3	[4.0818e-08;0.1659;0.3272;0.4794;0.6184;0.7402;0.8415]
4	[-1.5304e-11;0.1659;0.3272;0.4794;0.6184;0.7402;0.8415]
5	[5.9893e-15;0.1659;0.3272;0.4794;0.6184;0.7402;0.8415]
6	[3.1071e-17;0.1659;0.3272;0.4794;0.6184;0.7402;0.8415]

n_int is not enouth

Solutions with different number of quadrature points

With increasing the number of quadrature points, the values of the results are nearly the same with very small tolerances, increasing accuracy.