Galerkin approximation and implementation

Discrete space: $\mathcal{V}_i^h = \Big\{ w_i^h : w_i^h = 0 \text{ on } \Gamma_g ,\ w_i^h = \sum_{A \in \eta - \eta_g} c_{iA} N_A(x) \Big\}$

$$\mathcal{S}_i^h = \Big\{ u_j^h : u_j^h = v_j^h + g_j^h$$

$$v_j^h = \sum_{A \in \eta - \eta_g} d_{jB} N_B(x)$$

$$g_j^h = \sum_{B \in \eta_g} g_{jB} N_B(x) \Big\}$$

$\longrightarrow$ A convenient choice :

$$g_{jB} = g_j(x_B) \quad \text{if } x_B \in \Gamma_g$$

$$(G)\begin{cases} \text{Find } v_i^h \in \mathcal{V}_i^h \text{ such that for } \forall\, w_i^h \in \mathcal{V}_i^h \\ a(\vec{w}^h, \vec{v}^h) = (\vec{w}^h, \vec{f}) + (\vec{w}^h, \vec{h})_\Gamma - a(\vec{w}^h, \vec{g}^h) \end{cases}$$

$$\vec{v}^h = v_i^h \vec{e}_i \qquad \vec{w}^h = w_i^h \vec{e}_i$$

$\longrightarrow$ i-th Euclidean basis vector in $\mathbb{R}^{n_{sd}}$.

$$n_{sd} = 3 \qquad e_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad e_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad e_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

$\Rightarrow$ linearity of the forms gives

$$0 = \sum_{A \in \eta - \eta_g} c_{iA} \left\{ a(N_A \vec{e}_i, \vec{u}^h) - (N_A \vec{e}_i, \vec{f}) - (N_A \vec{e}_i, \vec{h})_\Gamma - a(N_A \vec{e}_i, \vec{g}^h) \right\}$$

for any c_{iA}.

$$\Rightarrow \sum_{B \in \eta - \eta_g} a(N_A \vec{e}_i, N_B \vec{e}_j)\, d_{jB} = (N_A \vec{e}_i, \vec{f}) + (N_A \vec{e}_i, \vec{h})_\Gamma - a(N_A \vec{e}_i, \vec{g}^h)$$

ID Array: $\quad ID(i, A) = \begin{cases} P & \text{if } A \in \eta - \eta_{g_i} \\ 0 & \text{if } A \in \eta_{g_i} \end{cases}$

$1 \leq i \leq n_{dof}$
degrees of freedom per node.

Let $P \leftarrow ID(i, A)$, $Q \leftarrow ID(j, B)$, we have

$$K_{PQ}\, d_Q = F_P$$

or

simply $\underline{\underline{Kd = F.}}$

$$K_{PQ} = a(N_A \vec{e}_i,\ N_B \vec{e}_j)$$

$$F_P = (N_A \vec{e}_i, \vec{f}) + (N_A \vec{e}_i, \vec{h})_\Gamma - \sum_{B \in \eta_{g_j}} a(N_A \vec{e}_i, N_B \vec{e}_j)\, g_{jB}$$

$$d_Q = d_{jB}$$

Moreover,

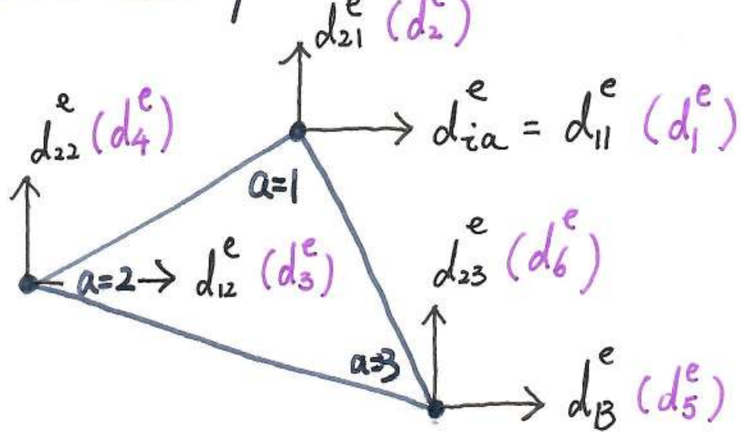
$$(N_A \vec{e}_i, \vec{f}) = \int_\Omega N_A \vec{e}_i \cdot \vec{f}\, d\Omega = \int_\Omega N_A f_i\, d\Omega$$

$$(N_A \vec{e}_i, \vec{h})_\Gamma = \quad \cdots \quad \cdots \quad = \int_{\Gamma_{h_i}} N_A h_i\, d\Gamma$$

$$\varepsilon^{\text{vect}}(N_A \vec{e}_i) = \begin{bmatrix} N_{A,1} & \\ & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix} \begin{Bmatrix} \delta_{1i} \\ \delta_{2i} \end{Bmatrix} = B_A \vec{e}_i \quad \text{for } n_{sd} = 2.$$

$$\Rightarrow \quad K_{PQ} = \vec{e}_i^{\,T} \int_\Omega B_A^T D B_B \, d\Omega\, \vec{e}_j .$$

Local description :



$$d_p^e = d_{ia}^e$$

for $p = n_{sd}(a-1) + i$

$$K_{pq}^e = a(N_a \vec{e}_i, N_b \vec{e}_j)^e = \vec{e}_i^{\,T} \int_{\Omega^e} B_a^T D B_b \, d\Omega \, \vec{e}_j$$

$$= \vec{e}_i^{\,T} K_{ab}^e \vec{e}_j$$

$$f_p^e = \int_{\Omega^e} N_a f_i \, d\Omega + \int_{\Gamma_{h_i}^e} N_a h_i \, d\Gamma - \sum_{q=1}^{n_{ee}} K_{pq}^e g_q^e$$

$\Gamma_{h_i}^e \rightarrow \Gamma_{h_i} \cap \Gamma^e$

$n_{ee} \rightarrow$ # of element equations $= n_{en} \cdot n_{ed} = n_{en} \cdot n_{dof}$.

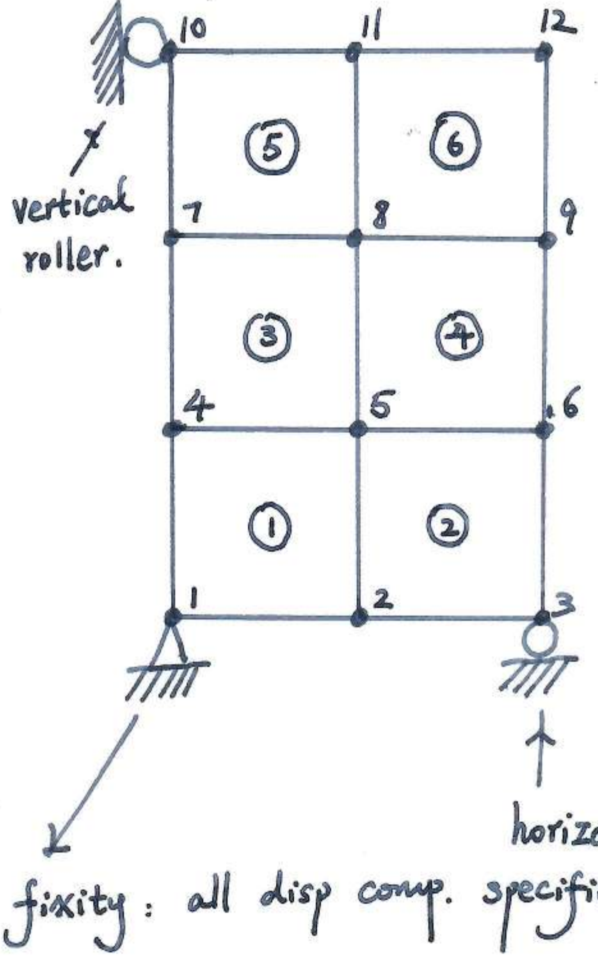
Data structure :

$$LM(\underbrace{i, a}_{p}, e) = ID(i, IEN(a, e))$$

$$\| $$

$$LM(p, e)$$

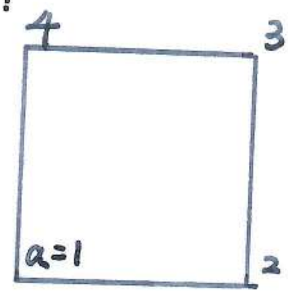
Example:



ID:

$A =$	1,	2,	....	,	12
$i = 1$	0	1	3	....	
2	0	2	0	....	

IEN:



Implementation 1 (p. 151)

$$k^e = \int_{\Omega^e} B^T D B \, d\Omega = \int_{\square} B^T D B \, j \, d\square \approx \sum_{l=1}^{n_{int}} (B^T D B)\Big|_{\tilde{\xi}_l} W_l$$

$$= \sum_{l=1}^{n_{int}} (B^T \tilde{D} B)_l$$

Implementation 3. (p. 154)

$$k^e_{pq} = k^e_{iajb} = \int_{\Omega^e} (N_a \vec{e}_i)_{,k} \, C_{i\cancel{jk}l} \, (N_b \vec{e}_j)_{,l} \, d\Omega$$

$$\qquad\qquad\qquad\qquad kj$$

$$= \int_{\Omega^e} N_{a,k} \, C_{ikjl} \, N_{b,l} \, d\Omega$$

isotropic & homogeneous:

$$C_{ikjl} = \mu (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj}) + \lambda \delta_{ik} \delta_{jl}$$

$$\Rightarrow \quad K^e_{iajb} = \mu \left(\delta_{ij} \int_{\Omega^e} N_{a,k} N_{b,k} \, d\Omega + \int_{\Omega^e} N_{a,j} N_{b,i} \, d\Omega \right)$$

$$+ \lambda \underbrace{\int_{\Omega^e} N_{a,i} N_{b,j} \, d\Omega}_{\approx \sum_{l=1}^{n_{int}} (N_{a,i} N_{b,j}) \big|_{\tilde{\xi}_l} W_l .}$$