Transient analysis of the heat equation

Recall that the (S)- problem is :

$$
(S)\left\{
\begin{array}{l}
\text{Given } f,\ g,\ h, \text{ and } u_0, \text{ find } u : \bar{\Omega}\times[0,T]\to\mathbb{R} \\
\text{such that} \\
\qquad \rho c\, u_{,t} + q_{i,i} = f \quad \text{on} \quad \Omega\times[0,T] \\
\qquad\qquad u = g \quad \text{on} \quad \Gamma_g\times[0,T] \\
\qquad\quad -q_i n_i = h \quad \text{on} \quad \Gamma_h\times[0,T] \\
\qquad\quad u(x,0) = u_0(x) \quad x\in\Omega
\end{array}
\right.
$$

$$\Updownarrow$$

$$
(W)\left\{
\begin{array}{l}
\text{Given the data, find } u(t)\in \mathcal{S}_t := \{u(\cdot,t) : u(x,t) = g(x,t) \text{ on } \Gamma_g,\ u(\cdot,t)\in H^1(\Omega)\} \\
\text{s.t. for } \forall\, w\in\mathcal{V}, \\
(w, \rho c\, \dot{u}) + a(w,u) = (w,f) + (w,h)_{\Gamma_h} \\
\qquad (w, \rho c\, u(0)) = (w, \rho c\, u_0)
\end{array}
\right.
$$

Let $u^h = v^h + g^h$

$$v^h(t) = \sum_{A\in\eta-\eta_g} N_A(x)\, d_A(t)$$

$$g^h(x,t) = \sum_{A\in\eta_g} N_A(x)\, g_A(t)$$

then the galerkin formulation is.

$$\begin{cases} (w^h, \rho c \dot{v}^h) + a(w^h, v^h) = (w^h, f) + (w^h, h)_{\Gamma_h} \\ \qquad\qquad - (w^h, \rho c \dot{g}^h) - a(w^h, g^h). \\ (w^h, \rho c v^h(0)) = (w^h, \rho c u_0) - (w^h, \rho c g^h(0)) \end{cases}$$

$\Rightarrow$ Matrix problem is

$$(*) \begin{cases} M \dot{d} + K d = F & \text{for } t \in (0, T) \\ d(0) = d_0 \end{cases}$$

$$M_{PQ} = (N_A, \rho c N_B) \qquad K_{PQ} = a(N_A, N_B)$$

$\longrightarrow$ Mass matrix or capacity matrix

$$F = \mathop{A}_{e=1}^{n_{el}} f^e \qquad f_a^e = \int_{\Omega^e} N_a f \, d\Omega + \int_{\Gamma_h^e} N_a h \, d\Gamma - \sum_{b=1}^{n_{en}} (k_{ab}^e g_b^e + m_{ab}^e \dot{g}_b^e)$$

$$M d_0 = \mathop{A}_{e=1}^{n_{el}} \hat{d}^e$$

$$\rightarrow \hat{d}_a^e = \int_{\Omega^e} N_a \rho c u_0 \, d\Omega - \sum_{b=1}^{n_{en}} m_{ab}^e g_b^e(0)$$

We refer to the (*) problem the "semi-discrete" problem.

It is computable after we discretize the time derivative.

$\Rightarrow$ generalized trapezoidal family of methods.

$$(\bigstar)\begin{cases} M v_{n+1} + K d_{n+1} = F_{n+1} \\ d_{n+1} = d_n + \Delta t\, v_{n+\alpha} \\ v_{n+\alpha} = (1-\alpha) v_n + \alpha v_{n+1} \end{cases}$$

We use d_n & v_n to represent the approximations to $d(t_n)$ and $\dot{d}(t_n)$; $F_{n+1} = F(t_{n+1})$; $\alpha \in [0, 1]$.

Remark: We call ($\bigstar$) the fully discrete problem.

$\alpha = 0$, we get the forward Euler

$\alpha = 1/2$, " " " mid-point rule

$\alpha = 1$, " " " backward Euler.

Implementation:

At time $t=0$, d_0 is known. v_0 is determined from

$$M v_0 = F_0 - K d_0.$$

at time t_{n+1}, d_n & v_n are given,

we make a prediction as

$$\tilde{d}_{n+1} = d_n + (1-\alpha)\Delta t \, v_n .$$

then
$$d_{n+1} = \tilde{d}_{n+1} + \alpha \Delta t \, v_{n+1} .$$

and v_{n+1} is solved by

$$(M + \alpha \Delta t K) v_{n+1} = F_{n+1} - K \tilde{d}_{n+1} .$$

Remark: If $\alpha = 0$, the method is <u>explicit</u>, otherwise it is <u>implicit</u>.

Remark: The method is unconditionally stable if $\alpha \geq 1/2$.

It is conditionally stable if $\alpha < 1/2$, and the condition is posed as a restriction on the time step size:

$$\Delta t < \frac{2}{(1-2\alpha)\lambda^h_{eq}}$$

→ largest mode of the heat discrete problem. (see 8.2.1).