ME335A Finite Element Analysis

Final Examination

Open notes and textbook (and approved computers to read digital notes)

Three hours

March 15, 2016

Problem 1 (30 points)

Consider a heat transfer problem over $\Omega \subset \mathbb{R}^2$ with Robin radiation condition expressed by the following variational principle: Find $u \in \mathcal{S} = H^1(\Omega)$ such that

$$\Pi(u) \leq \Pi(v)$$

for all $v \in S$ with

$$\Pi(v) = \frac{1}{2}\int_\Omega \kappa v_{,i} v_{,i} d\Omega + \frac{1}{2}\int_\Gamma \lambda (v - u_0)^2 d\Gamma - \int_\Omega v f d\Omega$$

and $\kappa > 0$ and $\lambda > 0$ and $u_0 \in \mathbb{R}$ a given (ambient) temperature.

1. Obtain the condition of stationarity, define the bilinear operator $a(\delta u, u)$, and state the weak form of the problem.
2. Obtain the Euler-Lagrange equation(s).
3. Will the finite element system stiffness matrix $\mathbf{K}$ be positive-definite? Prove your answer.

Problem 2 (30 points)

Consider the isotropic heat transfer problem given by

$$\begin{aligned} -\kappa u_{,ii} &= f \quad \text{in } \Omega & (1)\\ u &= g \quad \text{on } \Gamma_u & (2)\\ \kappa u_{,i} n_i &= h \quad \text{on } \Gamma_h & (3) \end{aligned}$$

Consider the problem domain, boundary conditions and mesh shown in Fig. 1.

1. For element $e = 4$, obtain an integral expression (in a natural coordinate over $[-1, 1]$) for the element load vector due to h.
2. Evaluate the integral found in Part 1 by exact Gaussian quadrature.
3. For the given mesh, define the **IEN**, **ID** and **L** data arrays

Problem 3 (30 points)

It is required to determine the shape functions for a complex transition element shown in Fig. 2b.

1. Consider a 4−node parent domain in $1-D$ as shown in Fig. 2a. Determine the Lagrange interpolation functions $\ell_2^3(\xi)$ and $\ell_3^3(\xi)$ for points $\xi = -\frac{1}{3}$ and $\frac{1}{3}$, respectively.

2. The problem (over $\square$) can be set up in tabular form. In the Table below the shape functions for nodes 1, 2, 3, 4, 9, and 5 have been defined and suitable adjustments have been introduced. Extend this table systematically for nodes 6, 7 and 8. First introduce shape function N_6 and indicate the necessary adjustments that are needed. Then introduce N_7 and N_8 together (using your ℓ_2^3 and ℓ_3^3 functions from Part 1) and make further adjustments. In the Table, X indicates a possible entry which needs to be defined.

N_1	$=$	$\frac{1}{4}(1-\xi)(1-\eta)$	$-\frac{1}{4}N_9$	$-\frac{1}{2}N_5$	X	X	X
N_2	$=$	$\frac{1}{4}(1+\xi)(1-\eta)$	$-\frac{1}{4}N_9$	$-\frac{1}{2}N_5$	X	X	X
N_3	$=$	$\frac{1}{4}(1+\xi)(1+\eta)$	$-\frac{1}{4}N_9$		X	X	X
N_4	$=$	$\frac{1}{4}(1-\xi)(1+\eta)$	$-\frac{1}{4}N_9$		X	X	X
N_9	$=$		$\left(1-\xi^2\right)\left(1-\eta^2\right)$				
N_5	$=$			$\frac{1}{2}\left(1-\xi^2\right)(1-\eta)-\frac{1}{2}N_9$			
N_6	$=$				X		
N_7	$=$					X	
N_8	$=$						X

3. What is the degree of completeness of the element in the (ξ, η) coordinates? *Hint: For this element, it may be useful to simply observe the finite element approximation around the boundary of the parent domain – do not attempt to construct the Pascal triangle.*

4. If this element is used to solve a plane strain problem, what will be the expected rate of convergence for $\left\|u-u^h\right\|_{H^1}$?

Problem 4 (30 points)

In this problem, the Bernoulli-Euler beam meets plane stress. Consider a transition element that allows a beam element to be connected to a plane stress mesh. Such a transition element is shown in Fig. 3. It is assumed that the beam rotation d_{31} causes the left side of the transition element to rotate by the same angle. In this case, the kinematics of the transition element is given by

$$\left\{ \begin{array}{c} u_1^h(\xi,\eta) \\ u_2^h(\xi,\eta) \end{array} \right\} = \left\{ \begin{array}{c} u_1^{beam}(\xi,\eta) \\ u_2^{beam}(\xi,\eta) \end{array} \right\} + \sum_{a=2}^{3} N_a^e(\xi,\eta) \left\{ \begin{array}{c} d_{1a} \\ d_{2a} \end{array} \right\}$$

where $N_2^e(\xi,\eta)$ and $N_3^e(\xi,\eta)$ are the standard bilinear shape functions with $(\xi,\eta) \in \square$, and where the displacements imposed by the beam are modeled by

$$\left\{ \begin{array}{c} u_1^{beam}(\xi,\eta) \\ u_2^{beam}(\xi,\eta) \end{array} \right\} = \left[\begin{array}{ccc} N_1^e(\xi) & 0 & -\frac{b}{2} N_1^e(\xi)\,\eta \\ 0 & N_1^e(\xi) & 0 \end{array} \right] \left\{ \begin{array}{c} d_{11} \\ d_{21} \\ d_{31} \end{array} \right\}$$

with $N_1^e(\xi) = \frac{1}{2}(1-\xi)$.

1. Take the Jacobian matrix to be constant

$$\mathbf{J}^e = \left[\begin{array}{cc} \frac{h}{2} & 0 \\ 0 & \frac{b}{2} \end{array} \right]$$

   and determine $\left\{ \begin{array}{c} N_{2,x}^e \\ N_{2,y}^e \end{array} \right\}$ and $\left\{ \begin{array}{c} N_{3,x}^e \\ N_{3,y}^e \end{array} \right\}$ (for use below):

2. Let

$$\left[\begin{array}{c} \varepsilon_{11}^{beam} \\ \varepsilon_{22}^{beam} \\ 2\varepsilon_{12}^{beam} \end{array} \right] = \left[\begin{array}{c} u_{1,1}^{beam} \\ u_{2,2}^{beam} \\ u_{1,2}^{beam} + u_{2,1}^{beam} \end{array} \right] = \left[\mathbf{B}^{beam}(\xi,\eta) \right] \left\{ \begin{array}{c} d_{11} \\ d_{21} \\ d_{31} \end{array} \right\}$$

   and find $\left[\mathbf{B}^{beam}(\xi,\eta) \right]$.

3. The plane stress transition element stiffness matrix may be expressed in standard form by $\mathbf{k}_{\text{transition}} = \int_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega$, where $\mathbf{D}$ is the (3×3) plane stress elasticity matrix. Obtain the (3×7) $\mathbf{B}$ matrix (using the results found above).

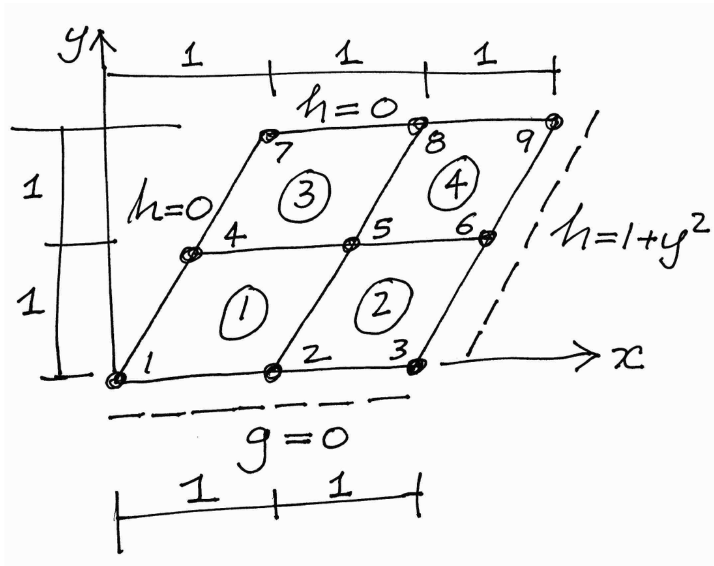

Figure 1: Problem 2

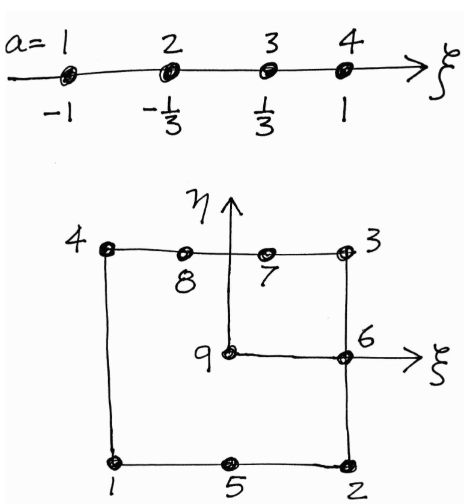

Figure 2: Problem 3

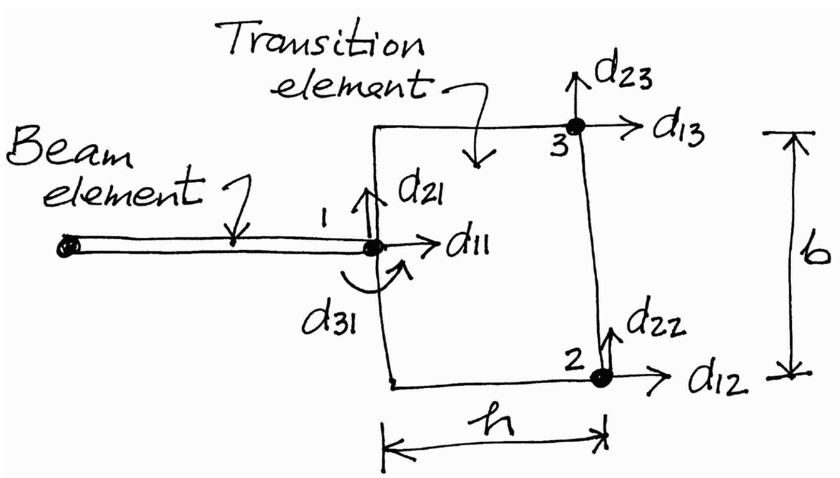

Figure 3: Problem 4