ME335A Finite Element Analysis
Final Examination
Saturday, June 7, 2019
3 hours
Open notes and open textbook only; No internet access except to access Canvas

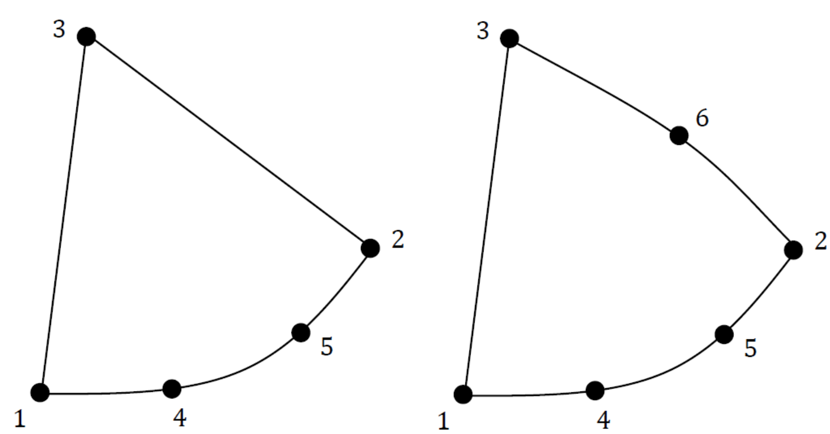
Problem 1 (20 points)

(a) Consider the transition (non-Lagrangian) element shown. Using triangular coordinates[1] (ξ, η, ζ), derive shape functions $N_4(\xi,\eta,\zeta)$ and $N_5(\xi,\eta,\zeta)$ and then express shape functions N_1, N_2 and N_3 in terms of triangular coordinates and N_4 and N_5. To save time, you should not simplify the expressions obtained.

(b) Now consider an additional node 6 as shown. Derive shape function $N_6(\xi,\eta,\zeta)$ and then express shape functions N_1 through N_5 in terms of triangular coordinates and $\widehat{N}_4$, $\widehat{N}_5$ and N_6, where $\widehat{N}_4$ and $\widehat{N}_5$ are the shape function found in Part (b). To save time, you should not simplify the expressions obtained.

(c) What is the degree of completeness of the final transition element? (Hint: no calculations are needed).

(d) A Lagrangian approximation of polynomial order p on a triangle has three nodes at the vertices and $p-1$ nodes along each edge that are not at the vertices (thus $p+1$ nodes in total along each edge). How many additional interior nodes are needed if the element is to be complete to order p? Note that the Pascal triangle for completeness to order k contains $(k+1)(k+2)/2$ monomials.



Problem (a) *Problem (b)*

[1] The task is more complicated if the canonical coordinates (ξ, η) are used.

Problem 2 (15 points)

(a) Consider the problem $u_{,xxxx} = f$. The standard 2-node element for approximating this equation is based on Hermite interpolation and gives $\left\|u - u^h\right\|_{H^m} = \mathcal{O}\left(h^2\right)$. Carefully describe how you would create an element that solves this problem with $\left\|u - u^h\right\|_{H^m} = \mathcal{O}\left(h^4\right)$. You will need to consider an element of length h and describe the number of nodes, the number of degrees of freedom and describe how you would obtain the element shape functions (points will be awarded for clarity of discussion).

(b) What will be the rate of convergence of the new element measured in the L_2 norm?

(c) Set up, but do not solve, the problem of finding the Barlow points (points will be awarded for clarity of discussion).

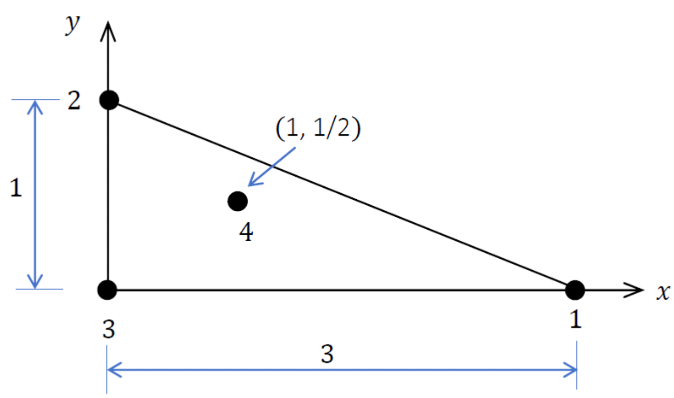
Problem 3 (20 points)
Consider the 4-node (non-Lagrangian) triangular element shown. The element is to be used in a problem of plane strain.

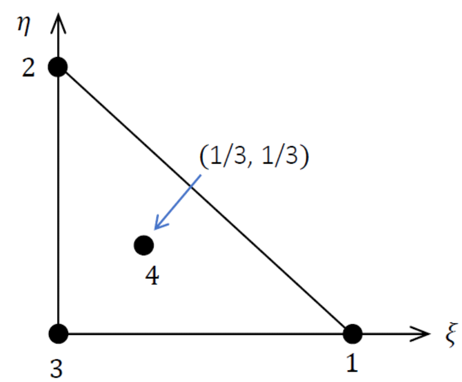
(a) It is easy to show, using triangular coordinates[2], that $N_4^e = 27\xi\eta\zeta$. What are the shape functions N_1^e, N_2^e and N_3^e expressed in triangular coordinates?

(b) Determine $\mathbf{J}^e$ and $j^e = \det \mathbf{J}^e$.

(c) Suppose the element is subject to a body force (corresponding to a centrifugal force) $\mathbf{b} = \rho\omega^2 x\mathbf{e}_1 + 0\mathbf{e}_2$ where ρ and ω are given constants. Obtain the 2×1 element force vector component for node 4, that is $[\mathbf{f}_b^e]_4$. Note that since this element has straight sides, integrals can be exactly integrated using formula (7.86) in Ch. 7.

(d) Express the integral in Part (c) as an integral over the parent domain $\triangle$ and state what order of Gaussian quadrature is needed to exactly evaluate $[\mathbf{f}_b^e]_4$?





[2] Triangular coordinates are denoted here as (ξ, η, ζ) but are sometimes denoted (ξ_1, ξ_2, ξ_3) (for example in Ch. 7).

Problem 4 (25 points)
Consider a tapered beam in plane stress and modeled with 4 linear quadrilateral elements as shown. The EBCs (fixed on left edge) and NBCs are homogeneous as shown.

(a) The beam is subject to time-harmonic motion with angular frequency ω. In this case, the equation of equilibrium is,

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{f} = -\rho\omega^2\mathbf{u}$$

where $-\rho\omega^2\mathbf{u}$ describes the inertial force. Let us take $\mathbf{f} = \mathbf{0}$. The finite element problem is then $\mathbf{Zd} = \mathbf{0}$, where the element impedance matrix $\mathbf{z}^e$ takes the form,

$$\mathbf{z}^e = \mathbf{k}^e - \rho\omega^2\mathbf{q}^e$$

and where $\mathbf{k}^e$ is the standard stiffness matrix and $-\rho\omega^2\mathbf{q}^e$ is the additional term arising from the inertial force. For the mesh shown, find an explicit expression for q^4_{13} of the form $\int_\square$ [*function of* ξ *and* η *- fill this in*] $d\xi d\eta$ and state what order of Gaussian quadrature would exactly evaluate the integral. Take,

$$\mathbf{IEN} = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 4 & 6 & 8 & 10 \end{bmatrix}$$

(b) The element stiffness matrix $\mathbf{k}^e$ is full but $\mathbf{q}^e$ is sparse. Describe the pattern of fill in $\mathbf{q}^e$; can you explain why this pattern occurs?

(c) Write the column ℓ^4 of the location array $\mathbf{L}$ for element 4 (assuming degrees of freedom are assigned sequentially)

(d) Now consider the plane elastic problem for the static case and with a concentrated load applied at any point $\overline{\mathbf{x}} \in \boldsymbol{\Omega}$, so that $\mathbf{f}(\mathbf{x}) = \delta(\mathbf{x} - \overline{\mathbf{x}})$ and

$$\operatorname{div}\boldsymbol{\sigma} + \delta(\mathbf{x} - \overline{\mathbf{x}}) = \mathbf{0}$$

Note for this problem that $\mathcal{T}_i^h = \mathcal{V}_i^h$ for $i = 1, 2$ (see Figure). We wish to prove that $\mathbf{u}^h(\overline{\mathbf{x}}) \leq \mathbf{u}(\overline{\mathbf{x}})$, that is, the finite element solution is "stiffer" than the exact solution.

(i) If the variational equation is $a(\mathbf{w}, \mathbf{u}) = \ell(\mathbf{w})$, define $\ell(\mathbf{w})$.

(ii) Setting $\mathbf{e} = \mathbf{u} - \mathbf{u}^h \Rightarrow a(\mathbf{u}, \mathbf{u}) = a\left(\mathbf{e} + \mathbf{u}^h, \mathbf{e} + \mathbf{u}^h\right)$, prove that $a(\mathbf{u}, \mathbf{u}) \geq a\left(\mathbf{u}^h, \mathbf{u}^h\right)$.

(iii) Use (i) and (ii) to prove that $\mathbf{u}(\overline{\mathbf{x}}) \geq \mathbf{u}^h(\overline{\mathbf{x}})$ for any $\overline{\mathbf{x}} \in \boldsymbol{\Omega}$.

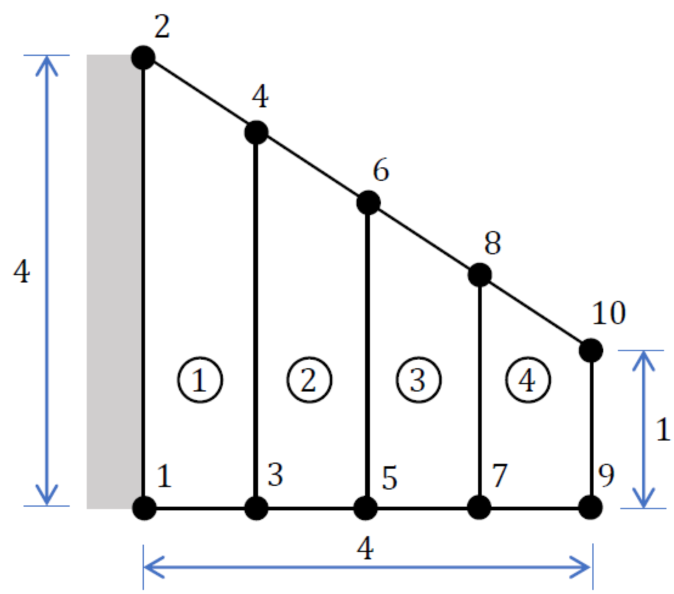


Figure 1: Problem 4.