Elastodynamics

$$(S)\begin{cases} \text{Find } \vec{u}: \bar{\Omega} \times (0,T) \to R^{nsd} \text{ s.t.} \\ \rho u_{i,tt} = \sigma_{ij,j} + f_i & \text{in } \Omega \times (0,T) \\ u_i = g_i & \text{on } \Gamma_{g_i} \times (0,T) \\ \sigma_{ij} n_j = h_i & \text{on } \Gamma_{h_i} \times (0,T) \\ u_i\big|_{t=0} = u_0 & \text{in } \Omega \\ u_{i,t}\big|_{t=0} = v_0 & \text{in } \Omega \end{cases}$$

In a semi-discrete approach, the spatial derivative is treated by Galerkin finite element method, and the time derivative is left to be treated separately.

$$(W)\begin{cases} \text{Find } u_i(t) \in \mathcal{S}_i \quad t \in (0,T) \quad \text{s.t. for } \forall w \in \mathcal{V} \\ (w, \rho \ddot{u}) + a(w, u) = (w, f) + (w, h)_\Gamma \\ (w, \rho u(0)) = (w, \rho u_0) \\ (w, \rho \dot{u}(0)) = (w, \rho v_0) \end{cases}$$

$$u^h = v^h + g^h \Rightarrow \begin{aligned} (w^h, \rho \ddot{u}^h) &= (w^h, \rho \ddot{v}^h) + (w^h, \rho \ddot{g}^h) \\ a(w^h, u^h) &= a(w^h, v^h) + a(w^h, g^h) \end{aligned}$$

Note, the corresponding (S)-problem is

$$\rho \ddot{u}_i + a\rho \dot{u}_i = \sigma_{ij} + f_i$$

$$\longrightarrow \sigma_{ij} = C_{ijkl}\left(u_{(k,l)} + b\dot{u}_{(k,l)}\right)$$

and the right-hand side of (G) becomes

$$(w^h, f) + (w^h, h)_{\Gamma} - (w^h, \rho \ddot{g}^h) - a(w^h, g^h)$$
$$- a(w^h, b\dot{g}^h) - (w^h, a\dot{g}^h).$$

Consider the semi-discrete problem:

$$\begin{cases} M\ddot{d} + C\dot{d} + Kd = F \\ d(0) = d_0 \\ \dot{d}(0) = v_0 \end{cases}$$

$\Delta t = t_{n+1} - t_n$

$$a_{n+1} \approx \ddot{d}(t_{n+1})$$
$$v_{n+1} \approx \dot{d}(t_{n+1})$$
$$d_{n+1} \approx d(t_{n+1})$$

$$\begin{cases} d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}\left[(1-2\beta)a_n + 2\beta a_{n+1}\right] \\ v_{n+1} = v_n + \Delta t\left[(1-\gamma)a_n + \gamma a_{n+1}\right] \\ M a_{n+1} + C v_{n+1} + K d_{n+1} = F_{n+1}. \end{cases}$$

The matrix problem can be written as :

$$\begin{cases} \text{Find } d(t) : (0,T) \to R^{n_{eq}} \quad \text{s.t.} \\ M\ddot{d} + Kd = F \\ d(0) = d_0 \\ \dot{d}(0) = v_0 \end{cases}$$

$$M = [M_{PQ}] \quad \text{with} \quad M_{PQ} = (N_A \vec{e}_i, \rho N_B \vec{e}_j) = \delta_{ij} \int_\Omega N_A N_B \, d\Omega$$

$$P = ID(i, A) \qquad Q = ID(j, B)$$

Remark: M is the mass matrix. It is symmetric and positive definite.

To introduce dissipative effect, we introduce a <u>viscous damping matrix</u>

$$M\ddot{d}(t) + C\dot{d}(t) + Kd(t) = F(t).$$

A particularily convenient model for C is known as the Rayleigh damping:

$$C = \underbrace{a}_{} M + \underbrace{b}_{} K$$

two parameters.

Implementation : (a- form)

predictor :

$$\tilde{d}_{n+1} := d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}\left[(1-2\beta)\, a_n + 2\beta\, \tilde{a}_{n+1} \right]$$

$$\tilde{v}_{n+1} := v_n + \Delta t \left[(1-\gamma)\, a_n + \gamma\, \tilde{a}_{n+1} \right].$$

Corrector :

$$(M + \gamma \Delta t\, C + \beta \Delta t^2 K)\, \Delta a_{n+1} = F_{n+1} - M \tilde{a}_{n+1} - C \tilde{v}_{n+1} - K \tilde{d}_{n+1}.$$

$$a_{n+1} = \tilde{a}_{n+1} + \Delta a_{n+1}$$

$$v_{n+1} = \tilde{v}_{n+1} + \gamma \Delta t\, \Delta a_{n+1}$$

$$d_{n+1} = \tilde{d}_{n+1} + \beta \Delta t^2 \Delta a_{n+1}.$$

- Constant displacement predictor :

$$\tilde{d}_{n+1} = d_n$$

$$\tilde{v}_{n+1} = v_n + \Delta t \left[(1-\gamma)\, a_n - \frac{\gamma}{\beta \Delta t} v_n - \frac{1-2\beta}{2\beta} \gamma\, a_n \right]$$

$$\tilde{a}_{n+1} = -\frac{1}{\beta \Delta t} v_n - \frac{1-2\beta}{2\beta} a_n$$

- Constant velocity predictor :

$$\tilde{v}_{n+1} = v_n$$

$$\tilde{a}_{n+1} = \frac{\gamma - 1}{\gamma} a_n$$

$$\tilde{d}_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}\left[(1-2\beta)\, a_n + 2\beta \frac{\gamma-1}{\gamma} a_n \right].$$

- Zero acceleration predictor

$$\hat{a}_{n+1} = 0$$

$$\hat{d}_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}(1-2\beta)\, a_n$$

$$\hat{v}_{n+1} = v_n + \Delta t\,(1-\gamma)\, a_n \,.$$

Remark: $M^* := M + \gamma \Delta t\, C + \beta \Delta t^2 K$ is known as the effective mass matrix.

$M^* = M^{\ell} :=$ row-sum lumped mass matrix.

$$M^{\ell}_{PQ} = \begin{cases} 0 & P \neq Q \\ \sum_{Q=1}^{n_{eq}} M_{PQ} & P = Q \end{cases}$$

can be used, but we need multi-ple correctors.

start.

↓

predictor $i=0$

$$d^i_{n+1} = \hat{d}$$

$$v^i_{n+1} = \hat{v}$$

$$a^i_{n+1} = \hat{a}$$

↓

$$M^* \Delta a^i_{n+1} = \underbrace{F_{n+1} - M a^i_{n+1} - C v^i_{n+1} - K d^i_{n+1}}_{\Delta F^i_{n+1}}$$

Corrector:

$$a_{n+1}^{i+1} = a_{n+1}^{i} + \Delta a_{n+1}^{i}$$

$$v_{n+1}^{i+1} = v_{n+1}^{i} + \gamma \Delta t \, \Delta a_{n+1}^{i}$$

$$d_{n+1}^{i+1} = d_{n+1}^{i} + \beta \Delta t^2 \Delta a_{n+1}^{i} .$$

Test if $\| \Delta F_{n+1}^{i} \| \leq \varepsilon \| \Delta F_{n+1}^{0} \|$ — No → $i = i+1$

Yes ↓

$n \leftarrow n+1$

Accuracy: The newmark family achieves 2nd-order accuracy if and only if $\gamma = 1/2$.

Stability: Unconditional: $2\beta \geq \gamma \geq 1/2$

Conditional: $\beta < \gamma/2$

$$\omega^h \Delta t \leq \Omega_{crit} = \frac{\xi(\gamma - 1/2) + \left[\gamma/2 - \beta + \xi^2 (\gamma - 1/2)^2\right]^{1/2}}{\gamma/2 - \beta}$$

Reduction to SDOF problem:

$$(K - \lambda M)\psi = 0 \quad \text{'undamped eigenproblem'}$$

$$\psi_\ell^T M \psi_m = \delta_{\ell m} \qquad \psi_\ell^T K \psi_m = \lambda_\ell \, \delta_{\ell m} \quad \text{(no sum)}$$

$$\Rightarrow \quad \psi_\ell^T C \psi_m = (a + b\,\lambda_\ell)\, \delta_{\ell m} \quad \text{(no sum)}$$

$\omega_\ell := \lambda_\ell^{1/2}$ the ℓ-th undamped frequency of vibration

$\xi_\ell := (\frac{a}{\omega_\ell} + b\,\omega_\ell)/2$ the ℓ-th modal damping ratio

$$d = \sum_{m=1}^{n_{eq}} \underbrace{d_{(m)}}_{\text{Fourier coefficients}} \psi_m$$

$$\psi_\ell^T \left\{ M\ddot{d} + C\dot{d} + Kd = F \right\}$$

$$\Rightarrow \quad \ddot{d}_{(\ell)} + 2\,\xi_\ell\,\omega_\ell\,\dot{d}_{(\ell)} + \omega_\ell^2\, d_{(\ell)} = F_{(\ell)}. \quad \text{(no sum)}$$

For each mode, we have the Newmark scheme:

$$\begin{cases} a_{n+1} + 2\,\xi\,\omega\, v_{n+1} + \omega^2 d_{n+1} = F_{n+1} \\ d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}\left[(1-2\beta)\,a_n + 2\beta\, a_{n+1}\right] \\ v_{n+1} = v_n + \Delta t\left[(1-\gamma)\,a_n + \gamma\, a_{n+1}\right] \end{cases}$$

Then the discrete SDOF problem can be written as

$$y_{n+1} = A y_n + L_n \qquad y_n = \begin{Bmatrix} d_n \\ v_n \end{Bmatrix}.$$

refer to p. 496 for the formula of A.

Stability comes by

(i) $\rho(A) \leq 1$, (ii) eigenvalues of multiplicity greater than 1 are strictly less than 1,

where $\rho(A) = \max_i |\lambda_i(A)|$.

A list of well-known members of Newmark family.

Name	Type	β	γ	stability	accuracy
Trapezoidal rule (Ave. acceleration)	Implicit	1/4	1/2	unconditional	2
Linear acceleration	Implicit	1/6	1/2	$\Omega_{crit} = 2\sqrt{3}$	2
Fox-Goodwin (royal road)	Implicit	1/12	1/2	$\Omega_{crit} = \sqrt{6}$	2
Central diff.	Explicit	0	1/2	$\Omega_{crit} = 2$	2

An unstable algorithm.

$$y_n = \int_0^1 \frac{x^n}{x+5}\,dx$$

$$\to y_n = \frac{1}{n} - 5y_{n-1}, \quad \text{with } y_0 = \ln(x+5)\Big|_0^1 = \ln 6 - \ln 5.$$

Correct value:

$y_0 = 1.82\times10^{-1}$ $\quad y_1 = 8.84\times10^{-2}$ $\quad y_2 = 5.80\times10^{-2}$ $\quad y_3 = 4.31\times10^{-2}$

$y_4 = 3.43\times10^{-2}$

If we use the recurrence formula with a decimal numerical system with 3-digits:

$\hat{y}_0 = 1.82\times10^{-1}$ $\quad \hat{y}_1 = 9.00\times10^{-2}$ $\quad \hat{y}_2 = 5.00\times10^{-2}$ $\quad \hat{y}_3 = 8.30\times10^{-2}$

$\hat{y}_4 = -1.65\times10^{-1}$.

Analysis: $\hat{y}_n = y_n + e_n$

$$\hat{y}_n = \frac{1}{n} - 5\hat{y}_{n-1} = \frac{1}{n} - 5y_{n-1} - 5\varepsilon_n = y_n + 5\varepsilon_{n-1}$$

$$\Rightarrow \varepsilon_n = 5\,\varepsilon_{n-1}$$

error grows exponentially!

The modern integration scheme for structural dynamics: generalized-α scheme

$$M a_{n+\alpha_m} + C v_{n+\alpha_f} + K d_{n+\alpha_f} = F_{n+\alpha_f}$$

$$d_{n+\alpha_f} = (1-\alpha_f)\, d_n + \alpha_f\, d_{n+1}$$

$$v_{n+\alpha_f} = (1-\alpha_f)\, v_n + \alpha_f\, v_{n+1}$$

$$a_{n+\alpha_m} = (1-\alpha_m)\, a_n + \alpha_m\, a_{n+1}$$

$$v_{n+1} = v_n + \Delta t\left((1-\gamma) a_n + \gamma a_{n+1} \right)$$

$$d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}\left[(1-2\beta) a_n + 2\beta a_{n+1} \right].$$

Accuracy: 2-nd order accurate if

$$\gamma = \frac{1}{2} - \alpha_f + \alpha_m$$

$$\beta = \frac{1}{4}(1-\alpha_f+\alpha_m)^2$$

Unconditional stability: $\alpha_m \geq \alpha_f \geq 1/2$.

Chung & Hulbert 1993 gives

$$\alpha_m = \frac{1-\rho_\infty}{1+\rho_\infty} \qquad \alpha_f = \frac{1}{1+\rho_\infty}.$$

ρ_∞: spectral radius of A at highest fregency.