Error estimates

Context :

① Exact problem $\begin{cases} \text{Find } u \in \mathcal{S}, \text{ for } \forall w \in \mathcal{V} \\ a(w, u) = (w, f) + (w, h)_\Gamma \end{cases}$

② Approximated problem $\begin{cases} \text{find } u^h \in \mathcal{S}^h, \ \forall w^h \in \mathcal{V}^h \\ a(w^h, u^h) = (w^h, f) + (w^h, h)_\Gamma \end{cases}$

Assume : (i) $\mathcal{S}^h \subset \mathcal{S}$ and $\mathcal{V}^h \subset \mathcal{V}$ conforming FE

(ii) $a(\cdot, \cdot)$, $(\cdot, \cdot)$, $(\cdot, \cdot)_\Gamma$ are symmetric bilinear

(iii)

$$C_1 \| w \|_m \leq a(w, w)^{1/2} \leq C_2 \| w \|_m$$

$$\| w \|_m = \left[\int_\Omega w_i w_i + w_{i,j} w_{i,j} + \dots + w_{i,jk\cdots l} w_{i,jk\cdots l} \, d\Omega \right]^{1/2}$$

example : 1D heat eqn.

$a(w, u) = \int_\Omega w_{,x} u_{,x} \, dx$ induces norm w. $m=1$

Euler - Bernoulli beam

$a(w, u) = \int_\Omega w_{,xx} EI \, u_{,xx} \, dx$ induces norm w. $m=2$.

$\| \cdot \|_m$: m-th Sobolev norm (H^m norm)

$a(w, w)^{1/2}$: strain energy norm

$a(\cdot, \cdot)$: strain energy inner product

Theorem : Let $e = u^h - u$ be the error in the finite element approximation,

$$a(w^h, e) = 0 \quad \text{for } \forall\, w^h \in \mathcal{V}^h$$

$$a(e, e) \leq a(U^h - u, U^h - u) \quad \forall\, U^h \in \mathcal{S}^h$$

(best approximation property)

proof : $\mathcal{V}^h \subset \mathcal{V}$ implies $a(w^h, u) = (w^h, f) + (w^h, h)_\Gamma$

then linearity gives $a(w^h, u^h - u) = 0$.

$$a(e + w^h, e + w^h) = a(e, e) + \underbrace{2a(w^h, e)}_{\shortparallel \atop 0} + a(w^h, w^h)$$

$$a(e, e) \leq a(e + w^h, e + w^h) \leq a(U^h - u, U^h - u)$$

$$e + w^h = u^h + w^h - u = U^h - u$$

$\blacksquare$

Corollary : $a(u, u) = a(u^h, u^h) + a(e, e)$ if $\mathcal{S}^h = \mathcal{V}^h$

proof: $a(u, u) = a(u^h - e, u^h - e)$

$$= a(u^h, u^h) + a(e, e)$$

$\blacksquare$

Remark: $a(e,e) = a(u,u) - a(u^h, u^h)$

$$\underset{\text{energy of error}}{\uparrow} \qquad\qquad \underset{\text{error of energy.}}{\uparrow}$$

Remark: $a(u^h, u^h) \leq a(u,u)$

the approximate solution under-estimates the strain energy.

Theorem: Given $u \in H^r$, there exists $U^h \in \mathcal{S}^h$ such that

$$\| u - U^h \|_m \leq c h^\alpha \| u \|_r ,$$

where c is a constant independent of u and h.

$$\alpha = \min(k+1-m, \ r-m)$$

k is the degree of complete polynomial appearing in the element shape functions

h is the mesh parameter, a scalar characterizing the refinement of the mesh.

Theorem: $\| e \|_m \leq \bar{c} h^\alpha \| u \|_r$

where $\bar{c}$ is a constant independent of u and h.

proof: $\| e \|_m \leq \frac{1}{C_1} a(e,e)^{1/2} \leq \frac{1}{C_1} a(u-U^h, u-U^h)^{1/2}$

$\leq \frac{C_2}{C_1} \| u - U^h \|_m \leq \bar{c} h^\alpha \| u \|_r$. $\blacksquare$

Remark: As long as $k+1 > m$ & $r > m$, we have optimal convergence in H_m norm.

Remark: Assume u is sufficiently smooth, $u \in H^{k+1}$, then the error satisfies

$$\| \cancel{x} e \|_m \lesssim c\, h^{k+1-m} \| u \|_{k+1}$$

and this is referred to as the <u>standard error estimate.</u>

Remark: Error estimates in H^s-norm $0 \lesssim s \lesssim m$. can be ~~st~~ established by Aubin-Nitsche method: assume $u \in H^{k+1}$,

$$\| e \|_s \lesssim c\, h^{\beta} \| u \|_{k+1}$$

$$\beta = \min(k+1-s ,\ 2(k+1-m)).$$

Example: 1D heat eqn solved by linear element.

$k = m = 1$

$\Rightarrow \alpha = 1$ $\qquad \beta = \min(2-s,\ 2) = 2.$ for $s=0$

$$\| e \|_1 \lesssim c\, h \ \| u \|_{k+1 = 2}$$

$$\| e \|_0 \lesssim c\, h^2 \| u \|_2 .$$