Isotropic elasticity, plane strain, and plane stress

Consider isotropic elasticity:

$$C_{ijkl}(x) = \mu(x)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \lambda(x)\,\delta_{ij}\delta_{kl}$$

Lamé's 1st parameter $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$ $\qquad$ ν: Poisson's ratio

E: Young's modulus

Shear modulus $\mu = \dfrac{E}{2(1+\nu)}$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (*)$$

Note: $$\begin{bmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{bmatrix}^{-1} = \frac{1}{b^2+3ab}\begin{bmatrix} 2a+b & -a & -a \\ -a & 2a+b & -a \\ -a & -a & 2a+b \end{bmatrix}$$

$$b^2+3ab\Big|_{a=\lambda \quad b=2\mu} = 4\mu^2+3\mu\lambda = \frac{E^2}{(1-2\nu)(1+\nu)} = \frac{E}{\nu}\lambda .$$

$$2a+b = 2\lambda+2\mu = \frac{E}{(1+\nu)(1-2\nu)} = \frac{\lambda}{\nu} .$$

$$\begin{bmatrix} 2\mu+\lambda & \lambda & \lambda \\ \lambda & 2\mu+\lambda & \lambda \\ \lambda & \lambda & 2\mu+\lambda \end{bmatrix}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} .$$

$$\Downarrow$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \qquad (\Delta)$$

In the setting of plane strain (with plane being the $e_1 - e_2$ plane),
$\varepsilon_{33} = \varepsilon_{23} = \varepsilon_{13} = 0 \Rightarrow$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \\ \\ & \text{From } (*) & \\ \\ \\ \end{bmatrix}\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 0 \\ 0 \\ 0 \\ \varepsilon_{12} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} \qquad \underline{\underline{\varepsilon^{vect} = D\, \varepsilon^{vect}}}$$

$$\sigma_{33} = \lambda\,(\varepsilon_{11} + \varepsilon_{22}).$$ out-of-plane normal stress

In the setting of <u>plane stress</u> : $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$. From (Δ)

We may deduce that

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu}{E}\,(\sigma_{11} + \sigma_{22}).$$ out-of-plane normal strain

$$\underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}}_{\sigma^{vect}} = \underbrace{\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}}_{\varepsilon^{vect}.}$$