• Small-strain elastostatics

The strong-form BVP is given as :

$$\begin{cases} \text{Given } f_i : \Omega \to \mathbb{R},\ g_i : \Gamma_{g_i} \to \mathbb{R},\ h_i : \Gamma_{h_i} \to \mathbb{R}, \text{ find} \\ u_i : \bar{\Omega} \to \mathbb{R} \text{ such that} \\ \qquad \sigma_{ij,j} + f_i = 0 \qquad \text{in } \Omega \\ \qquad u_i = g_i \qquad \text{on } \Gamma_{g_i} \\ \qquad \sigma_{ij} n_j = h_i \qquad \text{on } \Gamma_{h_i} \end{cases}$$

- u_i : displacement
- ε_{ij} : strain $:= u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i})$
- σ_{ij} : Cauchy stress ($\sigma_{ij} = \sigma_{ji}$)
- f_i : prescribed body force per volume
- g_i & h_i : prescribed boundary displacements & tractions.

Remark:
The above are Cartesian components of the vectors & tensors.
Here the indices i, j, k, l run from 1 to n_{sd}.

Remark: $\Gamma = \overline{\Gamma_{g_i} \cup \Gamma_{h_i}}$ $\quad \emptyset = \Gamma_{g_i} \cap \Gamma_{h_i}$.

In the following, we assume $\underline{\Gamma_{g_i} = \Gamma_g}$ and $\underline{\Gamma_{h_i} = \Gamma_h}$.
for $i = 1, \cdots, n_{sd}$.

Constitutive relation: $\sigma_{ij} = c_{ijkl}\, \varepsilon_{kl}$
is known as the generalized Hooke's law, where $c_{ijkl}(x)$ are the elastic coefficients.

properties: (Symmetry)

$$c_{ijkl} = c_{klij} \qquad \text{major symmetry}$$

$$\left.\begin{aligned} c_{ijkl} &= c_{jikl} \\ c_{ijkl} &= c_{ijlk} \end{aligned}\right\} \text{minor symmetry}$$

(positive definiteness)

$$c_{ijkl}\, \psi_{ij}\, \psi_{kl} \geqslant 0$$

$$c_{ijkl}\, \psi_{ij}\, \psi_{kl} = 0 \quad \text{implies} \quad \psi_{ij} = 0$$

for all $x \in \Omega$ and all ψ_{ij} with $\psi_{ij} = \psi_{ji}$.

- If c_{ijkl} does not depend on x, we say the material is <u>homogeneous.</u>
- If $c_{ijkl}(x) = \lambda(x)\, \delta_{ij}\delta_{kl} + \mu(x)\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
the body is <u>isotropic.</u>

Weak-form problem:

Trial solution space: $\mathcal{S}_i := \{ u_i : u_i = g_i \text{ on } \Gamma_{g_i}, \cdots \}$

Test function space: $\mathcal{V}_i := \{ w_i : w_i = 0 \text{ on } \Gamma_{g_i}, \cdots \}$

$$\begin{cases} \text{Given } f_i,\ g_i,\ h_i, \text{ find } u_i \in \mathcal{S}_i \text{ such that for } \forall\, w_i \in \mathcal{V}_i \\ \displaystyle\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega = \int_\Omega w_i f_i \, d\Omega + \int_{\Gamma_h} w_i h_i \, d\Gamma \end{cases}$$

Lemma 1 (Euclidean decomposition of a rank-two tensor)

Let S_{ij} denote a rank-two tensor, then

$$S_{ij} = S_{(ij)} + S_{[ij]},$$

where $S_{(ij)}$ is symmetric (i.e., $S_{(ij)} = S_{(ji)}$)

$S_{[ij]}$ is skew-symmetric (i.e., $S_{[ij]} = - S_{[ji]}$).

proof: $S_{(ij)} = \frac{1}{2} (S_{ij} + S_{ji})$

$$S_{[ij]} = \frac{1}{2} (S_{ij} - S_{ji}).$$

$\blacksquare$

Lemma 2: Let S_{ij} be a general rank-two tensor and t_{ij} be a symmetric rank-two tensor,

$$S_{ij} t_{ij} = S_{(ij)} t_{ij}.$$

proof: $S_{ij} t_{ij} = S_{(ij)} t_{ij} + S_{[ij]} t_{ij}$

$$S_{[ij]} t_{ij} = S_{[ji]} t_{ji} = - S_{[ij]} t_{ij}$$

$$\Rightarrow \quad S_{[ij]} t_{ij} = 0.$$

Now we may show the equivalence between (S) and (W)

(S) $\Rightarrow$ (W)

$$0 = \int_\Omega w_i (\sigma_{ij,j} + f_i) d\Omega = - \int_\Omega w_{i,j} \sigma_{ij} d\Omega + \int_{\partial\Omega} \sigma_{ij} w_i n_j \, d\Gamma + \int_\Omega w_i f_i d\Omega$$

$$= - \int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega + \int_\Omega w_i f_i \, d\Omega + \int_{\Gamma_h} w_i h_i \, d\Gamma.$$

(W) $\Rightarrow$ (S)

Euler-Lagrange condition:

$$0 = \int_\Omega w_i (\sigma_{ij,j} - f_i) d\Omega - \int_{\Gamma_h} w_i (\sigma_{ij} n_j - h_i) d\Gamma$$

from which we may recover the equilibrium in interior & on surface Γ_h using the fundamental lemma of the calculus of variations. See p. 80.

Abstract notation

$$a(w, u) := \int_\Omega w_{(i,j)} \, C_{ijkl} \, u_{(k,l)} \, d\Omega$$

$$(w, f) := \int_\Omega w_i f_i \, d\Omega$$

$$(w, h)_\Gamma := \int_{\Gamma_h} w_i h_i \, d\Gamma$$

They are symmetric, bilinear forms.

Voigt notation

$$\left.\begin{array}{l} \text{rank-two tensor} \rightarrow \text{"vector" or array} \\ \text{rank-four tensor} \rightarrow \text{"matrix"} \end{array}\right\} \text{collapse a pair of indices into a single index.}$$

Example in 2D:

strain vector $$\epsilon^{vect}(w) = \begin{Bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \end{Bmatrix} = \{ \epsilon_I^{vect} \}$$

stress vector $$\sigma^{vect} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \{ \sigma_I^{vect} \}$$

$$D = [D_{IJ}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{symm} & & D_{33} \end{bmatrix},$$

$$D_{IJ} = C_{ijkl}$$

with

$I \,/\, J$	$i \,/\, k$	$j \,/\, l$
1	1	1
2	2	2
3	1	2
3	2	1 .

e.g. $D_{11} = C_{1111}$

$$D_{22} = C_{2222}$$

$$D_{33} = C_{1212} = C_{2121} = C_{1221} = C_{2112}$$

$$D_{13} = C_{1112} = C_{1121} .$$

$$D_{23} = C_{2212} = C_{2221}$$

$$D_{12} = C_{1122} .$$

$$D_{IJ}\, \varepsilon_J^{vect}(u) \overset{I=1}{=} D_{11}\, \varepsilon_1^{vect} + D_{12}\, \varepsilon_2^{vect} + D_{13}\, \varepsilon_3^{vect}$$

$$= C_{1111}\, u_{1,1} + C_{1122}\, u_{2,2} + C_{1112}\, (u_{1,2} + u_{2,1})$$

$$= \sigma_{11}$$

$$= \sigma_1^{vect}$$

We may verify that $\sigma^{vect} = D\, \varepsilon^{vect}$

matrix-vector multiplication

Moreover, we may verify that

$$w_{(i,j)} C_{ijkl} u_{(k,l)} = \varepsilon^{vect}(w)^T D \, \varepsilon^{vect}(u).$$

Then we have

$$a(w, u) = \int_{\Omega} \varepsilon^{vect}(w)^T D \, \varepsilon^{vect}(u) \, d\Omega.$$