Quiz 14

Date: 2022-05-20 Name: SID:

Find the solution of the given initial value problems.

$$(1)\ x' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} x,\ x(0) = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix};$$

$$(2)\ x' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} x,\ x(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}.$$

Chapter 7.

1.

The characteristic equation of the coefficient matrix is $-r^3+3r^2-4=0$, with roots $r_1=-1$ and $r_{2,3}=2$. Setting $r=-1$, we have

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations $2\xi_1+\xi_2+\xi_3=0$ and $\xi_2-2\xi_3=0$. A corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)}=(-3,4,2)^T$. Setting $r=2$, the system of equations is reduced to the equations $-\xi_1+\xi_2+\xi_3=0$ and $\xi_2+\xi_3=0$. An eigenvector vector is given by $\boldsymbol{\xi}^{(2)}=(0,1,-1)^T$. The second solution corresponding to the double eigenvalue will have the form specified by Eq.(13), which yields

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \boldsymbol{\eta} e^{2t}.$$

Substituting this into the given system, or using Eq.(24), we find that

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Using row reduction we find that $\eta_1=1$ and $\eta_2+\eta_3=1$. If we choose $\eta_2=0$, then $\boldsymbol{\eta}=(1,0,1)^T$ and thus

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

$c_1=c_2=c_3=1$

Therefore the general solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \right].$$

2.

9.(a) The characteristic equation of the system is $(r-1)^2(r-2)=0$. The eigenvalues are $r_1=2$ and $r_{2,3}=1$. The eigenvector associated with r_1 is $\boldsymbol{\xi}^{(1)}=(0,0,1)^T$. Setting $r=1$, the system of equations is reduced to the equations $\xi_1=0$ and $6\xi_2+\xi_3=0$. An eigenvector vector is given by $\boldsymbol{\xi}^{(2)}=(0,1,-6)^T$. The second solution corresponding to the double eigenvalue will have the form specified by Eq.(13), which yields

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^{t} + \boldsymbol{\eta} e^{t}.$$

Substituting this into the given system, or using Eq.(24), we find that

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}.$$

Using row reduction we find that $\eta_1=-1/4$ and $6\eta_2+\eta_3=-21/4$. If we choo $\eta_2=0$, then $\boldsymbol{\eta}=(-1/4,0,-21/4)^T$ and thus

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^{t} + \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} e^{t}.$$

Therefore the general solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} e^{t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^{t} + \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} e^{t} \right].$$

The initial conditions then yield $c_1=3$, $c_2=2$ and $c_3=4$ and hence

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} e^{2t} + 4 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^{t} + \begin{pmatrix} -1 \\ 2 \\ -33 \end{pmatrix} e^{t}.$$