FEM HW6

Exercise 1 on Page 68

解

Solution

(W) : $-\int_\Omega w_{,i}\, q_i \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$

(IBP)

$$\int_\Omega w_{,i}\, q_i \, d\Omega = -\int_\Omega w q_{i,i} \, d\Omega + \int_{\Gamma} w q_i n_i \, d\Gamma$$

(W) ⇒

$$\int_\Omega w q_{i,i} \, d\Omega - \int_{\Gamma} w q_i n_i \, d\Gamma = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$$

$$\int_\Omega w(q_{i,i}-f)\, d\Omega - \int_{\Gamma_h} w(h+q_n)\, d\Gamma = 0$$

$$\sum_{e=1}^{n_{el}} \int_{\Omega^e} w(q_{i,i}-f)\, d\Omega - \sum_{e=1}^{n_{el}} \int_{\Gamma_h^e} w(h+q_n)\, d\Gamma = 0$$

$$\sum_{e=1}^{n_{el}} \int_{\Gamma_h^e} w(h+q_n)\, d\Gamma$$

$$= \int_{\bigcup_{e=1}^{n_{el}} \Gamma_h^e} w(h+q_n)\, d\Gamma \qquad \left(\Gamma_{int} = \bigcup_{e=1}^{n_{el}} \Gamma^e - \Gamma\right)$$

$$= \int_{\Gamma_h} w(h+q_n)\, d\Gamma + \int_{\Gamma_{int}} w(\underbrace{[h]}_{=0=(h-h)n^+}+[q_n])\, d\Gamma$$

$$= \int_{\Gamma_h} w(h+q_n)\, d\Gamma + \int_{\Gamma_{int}} w[q_n]\, d\Gamma$$

⇒

$$\sum_{e=1}^{n_{el}} \int_{\Omega^e} w(q_{i,i}-f)\, d\Omega - \int_{\Gamma_h} w(h+q_n)\, d\Gamma - \int_{\Gamma_{int}} w[q_n]\, d\Gamma = 0$$

Exercise 2 on Page 71

Solution

$$(S)\begin{cases} q_{i,i} = f & \text{in } \Omega \\ u = g & \text{on } \Gamma_g \\ \lambda u - q_i n_i = h & \text{on } \Gamma_h \end{cases}$$

$$\mathcal{S} = \{ u : u \in H^1(\Omega),\ u = g \text{ on } \Gamma_g \}$$

$$\mathcal{V} = \{ w : w \in H^1(\Omega),\ w = 0 \text{ on } \Gamma_g \}$$

$$q_{i,i} = f$$

$$(q_{i,i} - f) = 0$$

$$w(q_{i,i} - f) = 0$$

$$\int_\Omega w(q_{i,i}-f)\, d\Omega = 0$$

(IBP)

$$-\int_\Omega w_{,i} q_i \, d\Omega + \int_{\Gamma_h} w q_i n_i \, d\Gamma - \int_\Omega w f \, d\Omega = 0$$

$$-\int_\Omega w_{,i} q_i \, d\Omega + \int_{\Gamma_h} w \cdot (\lambda u - h)\, d\Gamma - \int_\Omega w f \, d\Omega = 0$$

$a(w,u) = \int_\Omega w_{,i} \underline{k_{ij} u_{,j}} \, d\Omega \qquad -(k_{ij}u_{,j})_{,i} = f$

$(w,f) = \int_\Omega w f \, d\Omega \qquad q_{i,i} = f$

$(w,h)_{\Gamma_h} = \int_{\Gamma_h} w h \, d\Gamma$

$(w,\lambda u)_{\Gamma_h} = \int_{\Gamma_h} \lambda w u \, d\Gamma$

$$\Rightarrow a(w,u) + (w,\lambda u)_{\Gamma_h} = (w,f) + (w,h)_{\Gamma_h}$$

weak-form statement.

$$\Rightarrow a(w^h, u^h) + (w^h, \lambda u^h)_{\Gamma_h} = (w^h, f) + (w^h, h)_{\Gamma_h}$$

$$u^h = v^h + g^h$$

$$\Rightarrow a(w^h, v^h) + (w^h, \lambda v^h)_{\Gamma_h} = (w^h, f) + (w^h, f)_{\Gamma_h} - a(w^h, g^h) - (w^h, \lambda g^h)_{\Gamma_h}$$

$$w^h(x) = \sum_{A \in \eta - \eta_g} N_A(x)\, c_A$$

$$v^h(x) = \sum_{A \in \eta - \eta_g} N_A(x) d_A$$

$$g^h(x) = \sum_{A \in \eta_g} N_A(x) g_A \quad , \ g_A = g(x_A)$$

$\Rightarrow$

$$\sum_{B \in \eta - \eta_g} \left[a(N_A, N_B) + \lambda (N_A, N_B)_{\Gamma_h} \right] d_B$$

$$= (N_A, f) + (N_A, h)_{\Gamma_h} - \sum_{B \in \eta_g} (N_A, N_B) g_B$$

$$- \lambda \sum_{B \in \eta_g} (N_A, N_B)_{\Gamma_h} g_B \qquad A \in \eta - \eta_g$$

$\Rightarrow$

$$k_{ab}^e = a(N_a, N_b)^e + \lambda (N_a, N_b)_{\Gamma_h}^e$$

$$= \int_{\Omega^e} (\nabla N_a)^T k (\nabla N_b) d\Omega$$

$$+ \lambda \int_{\Gamma_h^e} N_a N_b d\Gamma_h$$

$$K_{AB} = a(N_A, N_B) + \lambda (N_A, N_B)\Gamma_h$$

positive.

$$c^T K c = \sum_{P,Q=1}^{neq} c_P k_{PQ} c_Q$$

$$= \sum_{A,B \in \eta - \eta_g} \bar{c}_A \left[a(N_A, N_B) + \lambda (N_A, N_B)_{\Gamma_h} \right] \bar{c}_B$$

$$= a(w^h, w^h) + \lambda (w^h, w^h)_{\Gamma_h}$$

$$= \int_\Omega w^h_{,i} k_{ij} w^h_{,j} d\Omega$$

$$+ \lambda \int_{\Gamma_h} w^h \cdot w^h d\Gamma_h \quad \geq 0.$$

definite

Assume $c^T K c = 0$

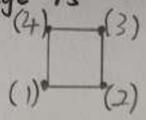
$$\int_\Omega w^h_{,i} k_{ij} w^h_{,j} d\Omega + \lambda \int_{\Gamma_h} (w^h)^2 d\Gamma = 0$$

$$\Rightarrow \int_\Omega w^h_{,i} k_{ij} w^h_{,j} d\Omega = 0 \ , \ \int_{\Gamma_h} (w^h)^2 d\Gamma = 0$$

$$\Rightarrow w^h = 0 \quad , \ c = 0 \ \text{only}.$$

Ex 1 on Page 75

Solution



in one element.

ID:

1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	0	0	0	0	5	6	7	8

$n_{np} = 12$

$n_{eq} = 8$

IEN

	1	2	3	4	5
1	4	6	5	7	9
2	3	5	7	8	10
3	1	3	9	10	12
4	2	4	3	9	11

$n_{el} = 5$

$n_{en} = 4$

LM

	1	2	3	4	5
1	4	0	0	0	5
2	3	0	0	0	6
3	1	3	5	6	8
4	2	4	3	5	7

$n_{el} = 5$

$n_{en} = 4$

$$P = LM(a,e) = ID[IEN(a,e)]$$

Exercise 1 on Page 123

Solution

from the bilinear quadrilateral,

coalescing nodes 3 and 4

namely $x_3^e = x_4^e$

define new shape functions for the triangle N_a'. $a = 1, 2, 3$.

$$x = \sum_{a=1}^{4} N_a x_a^e = N_1 x_1^e + N_2 x_2^e + (N_3 + N_4) x_3^e$$

$$= \sum_{a=1}^{3} N_a' x_a^e$$

$$N_a' = \begin{cases} N_a = \frac{1}{4}\left[1 + (-1)^a \xi\right](1-\eta) & a = 1, 2 \\ N_3 + N_4 = \frac{1}{2}(1+\eta) & a = 3 \end{cases}$$

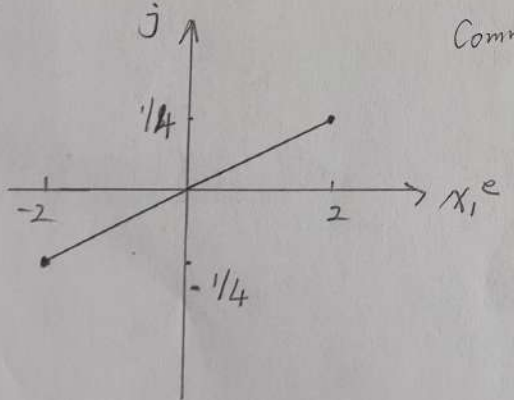
$$x(\xi, \eta) = \sum_{a=1}^{3} N_a'(\xi, \eta) x_a^e = N_1' x_1^e + 0 + 0 = \frac{1}{4}(1-\xi)(1-\eta) x_1^e$$

$$y(\xi, \eta) = \sum_{a=1}^{3} N_a'(\xi, \eta) y_a^e = 0 + N_2' y_2^e + 0 = \frac{1}{4}(1+\xi)(1-\eta) \cdot 1$$

$$x_{,\xi} = \frac{1}{4}(1-\eta) x_1^e (-1) \quad , \quad x_{,\eta} = \frac{1}{4}(1-\xi) x_1^e (-1)$$

$$y_{,\xi} = \frac{1}{4}(1-\eta) \quad , \quad y_{,\eta} = \frac{1}{4}(1+\xi)(-1)$$

$$j\big|_{\xi=\eta=0} = \det \begin{bmatrix} -\frac{1}{4} x_1^e & -\frac{1}{4} x_1^e \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} = \frac{1}{16} x_1^e + \frac{1}{16} x_1^e = \frac{1}{8} x_1^e$$



Commet from Sec 3.3:

① $j(\xi) > 0$ for all $\xi \in \square$ is one of the conditions ensuring the existing inverse mapping $\xi = x^{-1} : \Omega^e \to \square$

② the degeneration is performed, so the Jacobian det vanishes at certain nodal points within the element. Away from these points it is positive, mapping remains smooth.

③ if zero or negative j is encountered. computations will be terminated.

Exercise 3 on Page 145

Solution

three-point Gauss rule in each direction

Similar to Table 3.8.1 :

$n_{int} = n_{int}^{(1)} n_{int}^{(2)} = 9$, $\tilde{\xi}_L = \tilde{\xi}^{(1)}_{L^{(1)}}$, $\tilde{\eta}_L = \tilde{\eta}^{(2)}_{L^{(2)}}$

$W_L = W^{(1)}_{L^{(1)}} W^{(2)}_{L^{(2)}}$

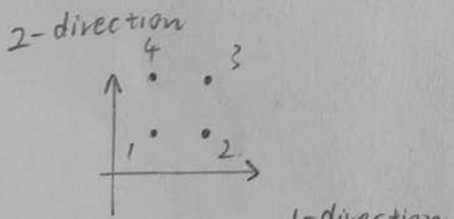
L	$L^{(1)}$	$L^{(2)}$	$\tilde{\xi}_L$	$\tilde{\eta}_L$	W_L
1	1	1	$\tilde{\xi}_L = -\sqrt{\frac{3}{5}}$	$\tilde{\eta}_L = -\sqrt{\frac{3}{5}}$	$W_L = \frac{25}{81}$
2	2	1	0	$-\sqrt{\frac{3}{5}}$	$\frac{40}{81}$
3	3	1	$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$\frac{25}{81}$
4	3	2	$\sqrt{\frac{3}{5}}$	0	$\frac{40}{81}$
5	2	2	0	0	$\frac{64}{81}$
6	1	2	$-\sqrt{\frac{3}{5}}$	0	$\frac{40}{81}$
7	1	3	$-\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\frac{25}{81}$
8	2	3	0	$\sqrt{\frac{3}{5}}$	$\frac{40}{81}$
9	3	3	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\frac{25}{81}$

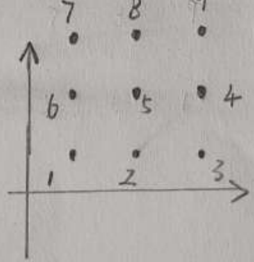
one-dimension

$n_{int} = 3$, $\tilde{\xi}_1 = -\sqrt{\frac{3}{5}}$ $\quad W_1 = W_3 = \frac{5}{9}$

$\tilde{\xi}_2 = 0$ $\quad W_2 = \frac{8}{9}$

$\tilde{\xi}_3 = \sqrt{\frac{3}{5}}$





$\Downarrow$

L	$L^{(1)}$	$L^{(2)}$
1	1	1
2	2	1
3	2	2
4	1	2