力学与航空航天工程系
DEPARTMENT OF MECHANICS AND AEROSPACE ENGINEERING

MECHANICS OF MATERIALS

YAHUI XUE (薛亚辉)

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Lesson 12： Energy methods

- Strain energy and Strain-energy density
- Castigliano’s theorem
- Indeterminate structures
- Virtual work

§ 12.1 Introduction

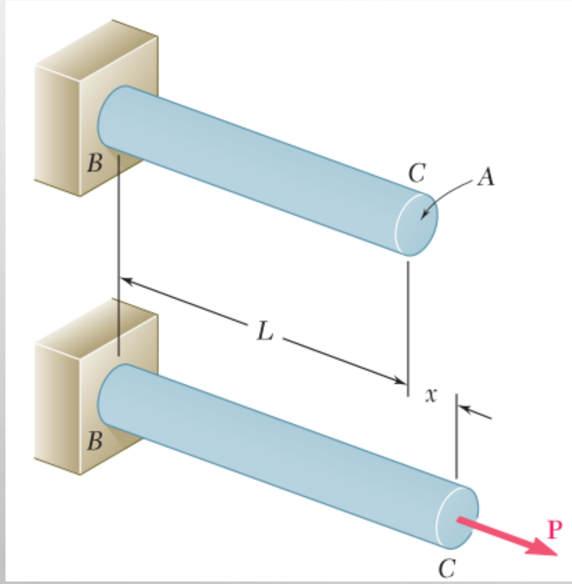
- Our analysis was based on two fundamental concepts, the concept of stress and the concept of strain. A third important concept, the concept of strain energy, will now be introduced.

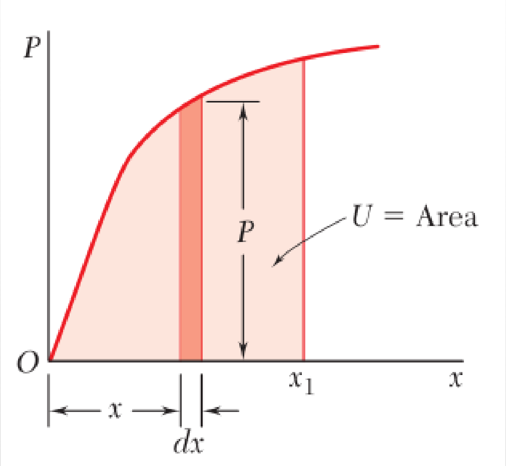
- The strain energy of a member will be defined as the increase in energy associated with the deformation of the member.

As the diver comes down on the diving board the potential energy due to his elevation above the board will be converted into strain energy due to the bending of the board

§ 12.2 Strain energy

- Consider the work dU done by the load P as the rod elongates by a small amount dx.





Elementary work $dU = Pdx$

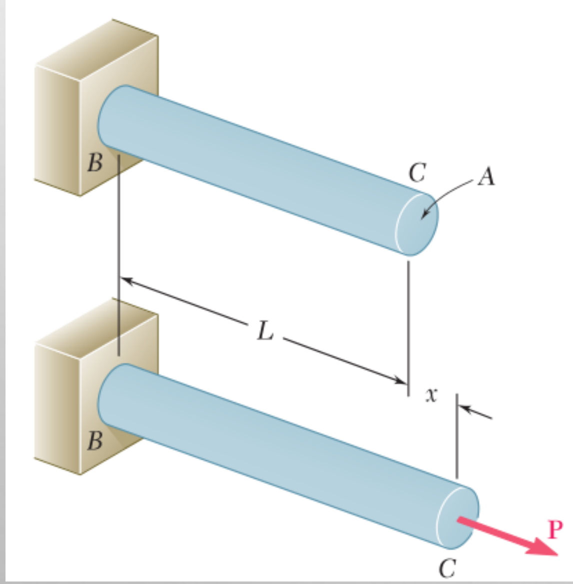
Strain energy $U = \int_0^{x_1} Pdx$

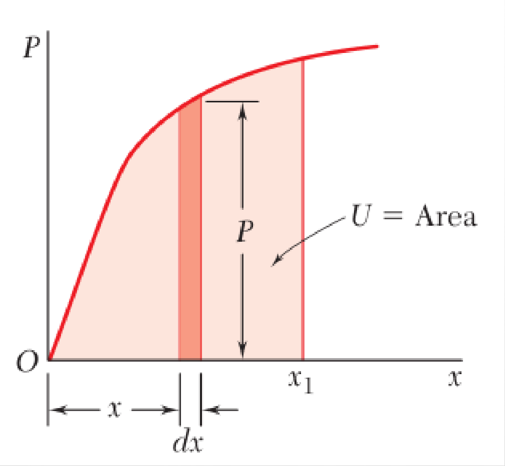
For linear and elastic deformation, $P = kx$

$$U = \frac{1}{2}kx^2$$

§ 12.2 Strain energy

- Consider the work d*U* done by the load *P* as the rod elongates by a small amount d*x*.
- The work done by the load P as it is slowly applied to the rod must result in the increase of some energy associated with the deformation of the rod. This energy is referred to as the **strain energy** of the rod.





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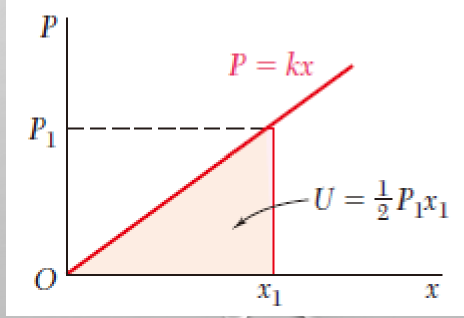


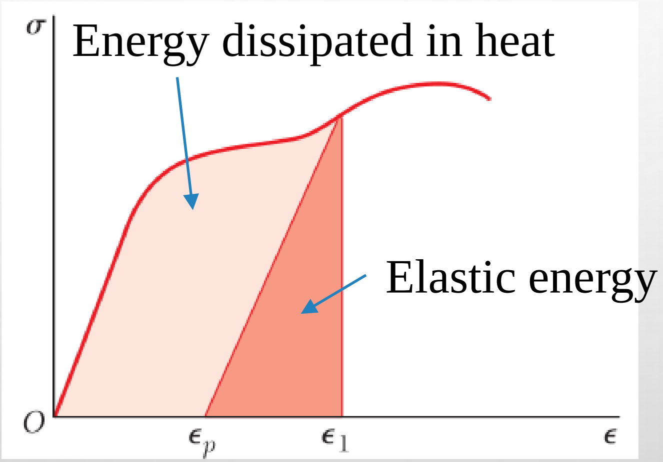
§ 12.3 Strain-energy density

- In order to eliminate the effect of size from our discussion and direct our attention to the properties of the material, the strain energy per unit volume will be considered.

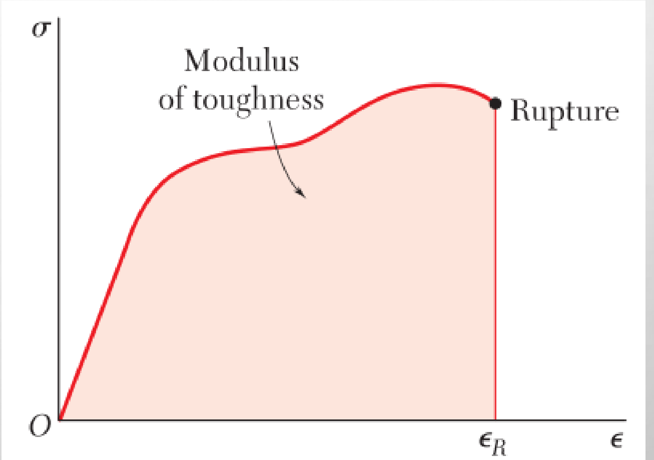
Strain-energy density

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A} \frac{dx}{L}$$

$$u = \frac{U}{V} = \int_0^{\varepsilon_1} \sigma d\varepsilon$$



Strain energy



Modulus of toughness

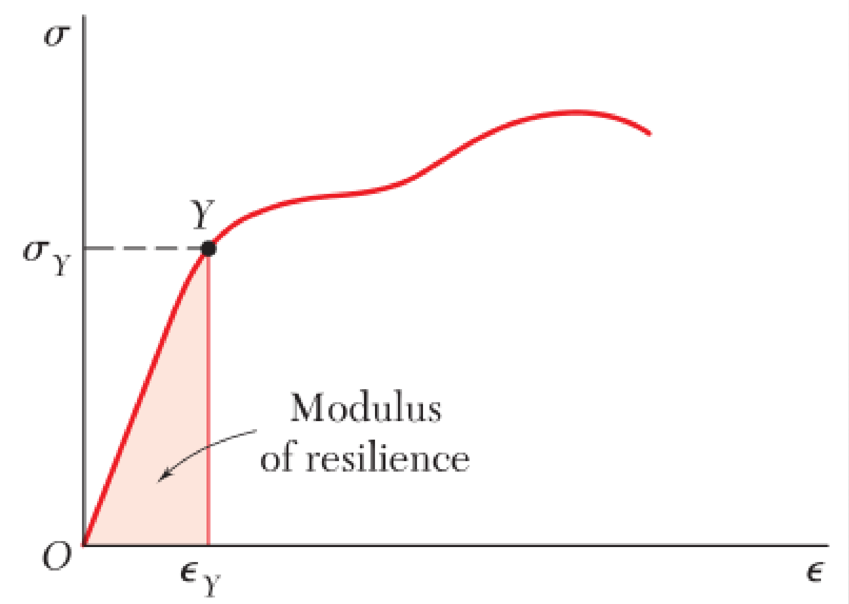
§ 12.3 Strain-energy density

- If the stress σ_x remains within the proportional limit of the material, Hooke's law applies.

By Hooke's law,

$$\sigma_x = E\varepsilon_x$$

$$u = \frac{U}{V} = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$



Modulus of resilience is defined by the yield strength, σ_Y.

$$u_Y = \frac{\sigma_Y^2}{2E}$$

§ 12.4 Elastic strain energy for normal stresses

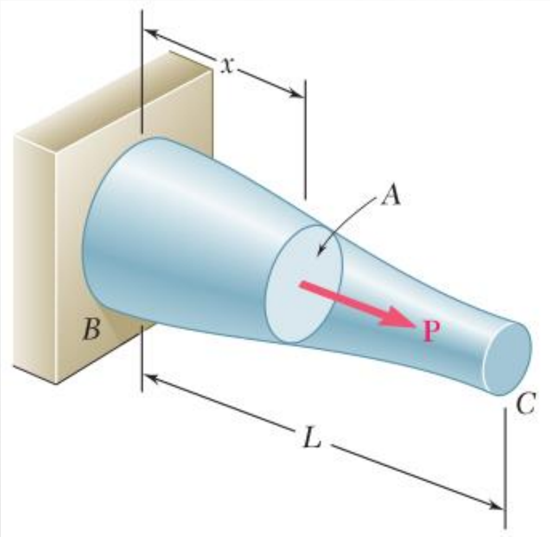
- In a structural element or machine part with a nonuniform stress distribution, the strain-energy density *u* can be defined by considering the strain energy of a small element of material

$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} \rightarrow u = \frac{dU}{dV}$$

$$u = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x$$

$$U = \int \frac{\sigma_x^2}{2E} dV$$

elastic strain energy



Strain Energy under Axial Loading

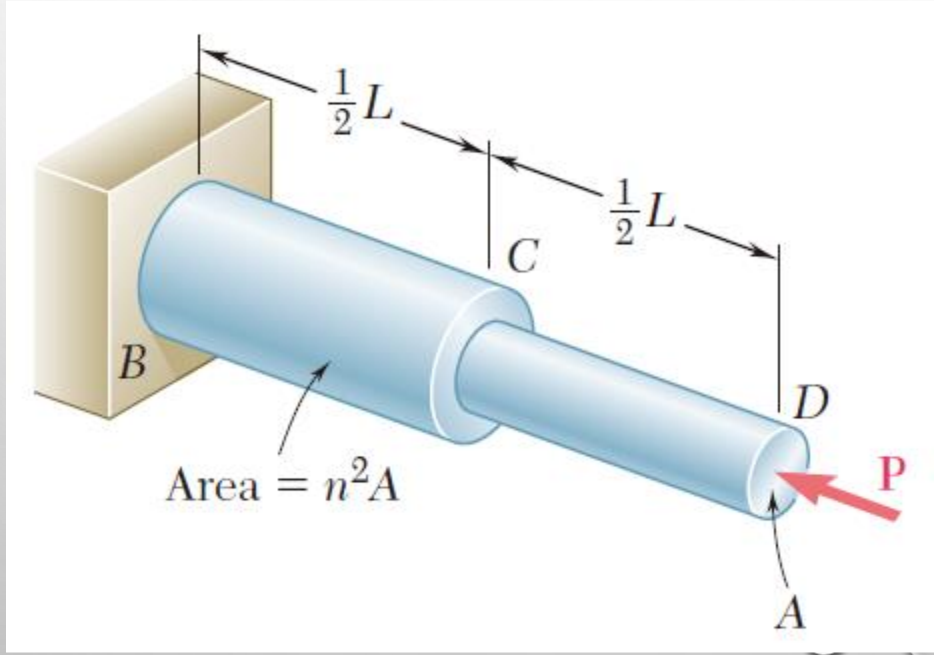
$$\sigma_x = \frac{P}{A}$$

$$U = \int \frac{P^2}{2EA^2} dV$$

Example 12.1

(Beer, Page 699)

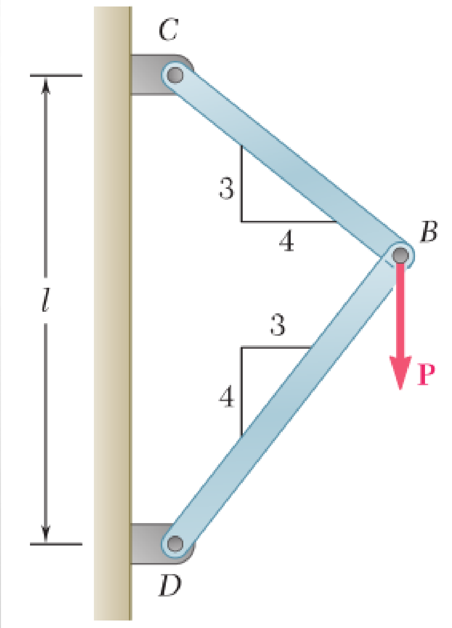
A rod consists of two portions BC and CD of the same material and same length, but of different cross sections. Determine the strain energy of the rod when it is subjected to a centric axial load P, expressing the result in terms of *P*, *L*, *E*, the cross-sectional area *A* of portion CD, and the ratio *n* of the two diameters.



Example 12.2

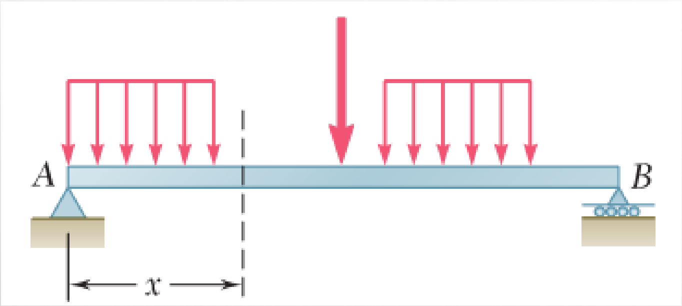
(Beer, Page 700)

A load *P* is supported at B by two rods of the same material and of the same uniform cross section of area A. Determine the strain energy of the system.



§ 12.4 Elastic strain energy for normal stresses

- Strain Energy in Bending.



$$U = \int \frac{M^2 y^2}{2EI^2} dV = \int_0^L \frac{M^2}{2EI^2}\left(\int y^2 dA\right) dx$$

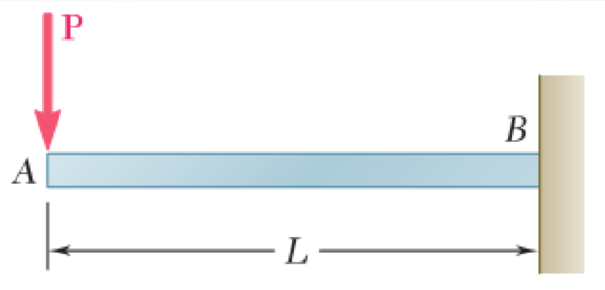
$$\sigma_x = \frac{My}{I}$$

$$U = \int_0^L \frac{M^2}{2EI} dx$$

Example 12.3

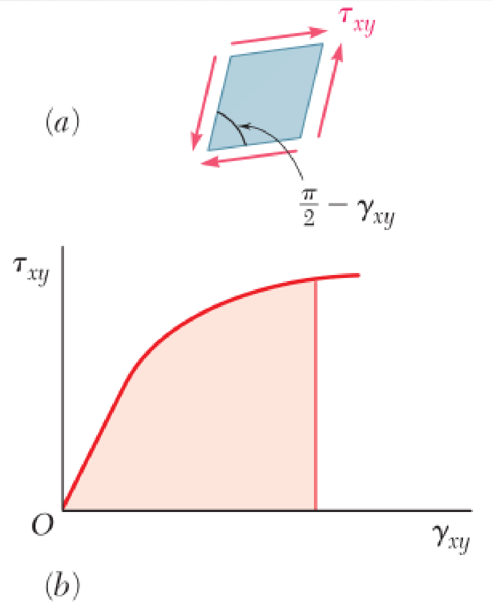
(Beer, Page 701)

Determine the strain energy of the prismatic cantilever beam AB, taking into account only the effect of the normal stresses.



§ 12.5 Elastic strain energy for shearing stresses

- When a material is subjected to plane shearing stresses τ_{xy} , the strain-energy density at a given point can be expressed as



$$u = \int_0^{\gamma_{xy}} \tau_{xy} d\gamma_{xy}$$

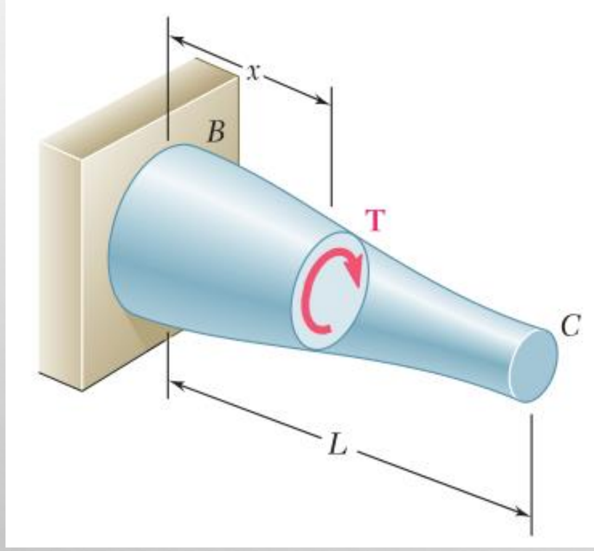
Within the proportional limit，

$$u = \frac{\tau_{xy}^2}{2G}$$

$$U = \int \frac{\tau_{xy}^2}{2G} dV$$

§ 12.5 Elastic strain energy for shearing stresses

- Strain Energy in Torsion



$$\tau_{xy} = \frac{T\rho}{J}$$

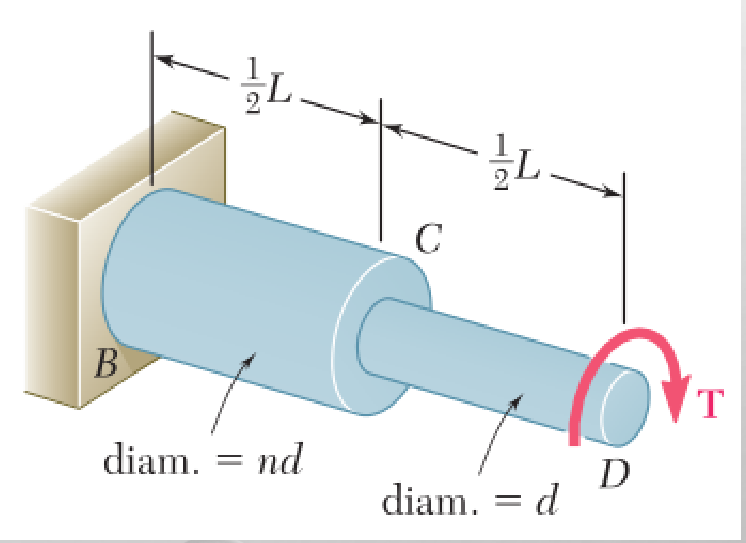
$$U = \int \frac{\tau_{xy}^2}{2G} dV = \int_0^L \frac{T^2}{2GJ^2} \left(\int \rho^2 dA \right) dx$$

$$U = \int_0^L \frac{T^2}{2GJ} dx$$

Example 12.4

(Beer, Page 702)

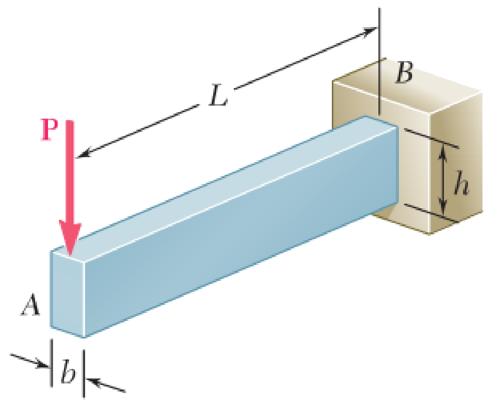
A circular shaft consists of two portions BC and CD of the same material and same length, but of different cross sections. Determine the strain energy of the shaft when it is subjected to a twisting couple T at end D, expressing the result in terms of T, L, G, the polar moment of inertia J of the smaller cross section, and the ratio n of the two diameters.



§ 12.5 Elastic strain energy for shearing stresses

- Strain Energy under Transverse Loading: both normal stresses and shear stresses will be considered.

$$U_\sigma = \frac{P^2L^3}{6EI} \qquad \tau_{xy} = \frac{3}{2}\frac{V}{A}\left(1-\frac{y^2}{c^2}\right)$$



$$U_\tau = \frac{1}{2G}\left(\frac{3}{2}\frac{P}{bh}\right)^2\int\left(1-\frac{y^2}{c^2}\right)^2 dV$$

$$U = U_\sigma + U_\tau = \frac{P^2L^3}{6EI} + \frac{3P^2L}{5GA} = U_\sigma\left(1+\frac{3Eh^2}{10GL^2}\right)$$

Example 12.5

(Beer, Page 703)

§ 12.6 Strain energy for a general state of stress

- In the case of the elastic deformation of an isotropic body, each of the six stress-strain relations involved is linear, and the strain-energy density can be expressed as

$$u=\frac{1}{2}\left(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}\right)$$

$$u=\frac{1}{2E}\left[\sigma_x^2+\sigma_y^2+\sigma_z^2-2\upsilon\left(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x\right)\right]+\frac{1}{2G}\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)$$

If the principal axes at the given point are used as coordinate axes,

$$u=\frac{1}{2E}\left[\sigma_a^2+\sigma_b^2+\sigma_c^2-2\upsilon\left(\sigma_a\sigma_b+\sigma_b\sigma_c+\sigma_c\sigma_a\right)\right]$$

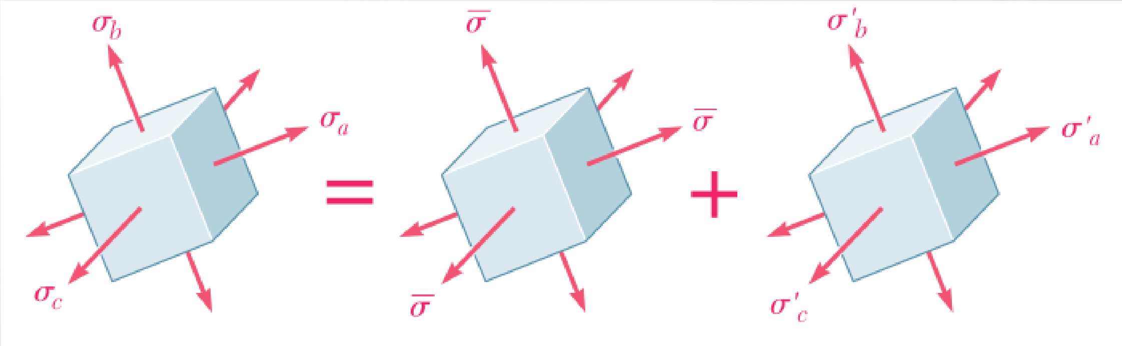
§ 12.6 Strain energy for a general state of stress

- One of the criteria used to predict whether a given state of stress will cause a ductile material to yield, namely, the maximum-distortion-energy criterion, is based on the determination of the energy per unit volume associated with the distortion, or change in shape, of that material.

$$u = u_v + u_d$$ change in volume and distortion, respectively.

$$\bar{\sigma} = \frac{\sigma_a + \sigma_b + \sigma_c}{3}$$ average stress.

$$\sigma'_a + \sigma'_b + \sigma'_c = 0$$



$$e = \frac{1-2\nu}{E}(\sigma'_a + \sigma'_b + \sigma'_c) = 0$$

This state of stress tends to change the shape. Volume change is zero

§ 12.6 Strain energy for a general state of stress

- The portion u_{v} of the strain-energy density corresponding to a change in volume of the element can be obtained by substituting $\bar{\sigma}$ for each of the principal stresses.

$$u_v = \frac{1}{2E}\left[3\bar{\sigma}^2 - 2\nu\left(3\bar{\sigma}^2\right)\right] = \frac{3(1-2\nu)}{2E}\bar{\sigma}^2$$

$$u_v = \frac{1-2\nu}{6E}\left(\sigma_a + \sigma_b + \sigma_c\right)^2 \qquad G = \frac{E}{2(1+v)}$$

$$u_d = u - u_v = \frac{1+\nu}{6E}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right]$$

§ 12.6 Strain energy for a general state of stress

- **The maximum-distortion-energy criterion**: a given structural component is safe as long as the maximum value of the distortion energy per unit volume in that material remains smaller than the distortion energy per unit volume required to cause yield in a tensile-test specimen of the same material.

For tensile test,

$$u_d = \frac{\sigma_a^{\ 2}}{6G} < \left(u_d\right)_Y = \frac{\sigma_Y^{\ 2}}{6G}$$

For plane stress,

$$u_d = \frac{1}{6\mathrm{G}}\left[\sigma_a^{\ 2} - \sigma_a\sigma_b + \sigma_b^{\ 2}\right] < \frac{\sigma_Y^{\ 2}}{6G}$$

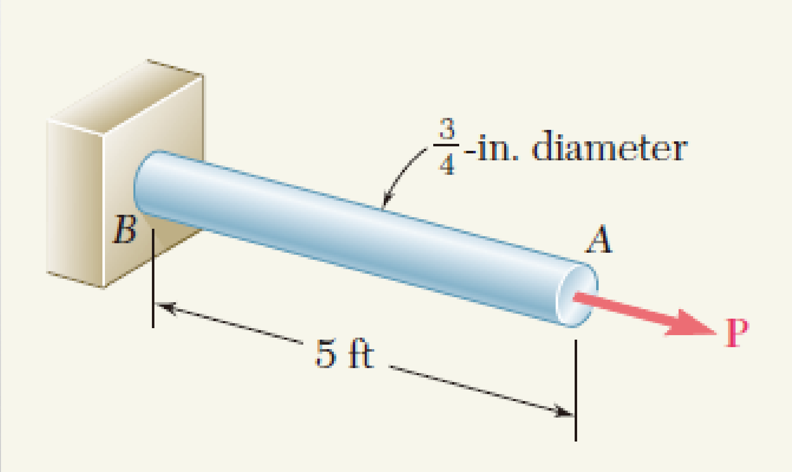
For general state,

$$u_d = u - u_v = \frac{1}{12G}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right] < \frac{\sigma_Y^{\ 2}}{6G}$$

Example 12.6

(Beer, Page 707)

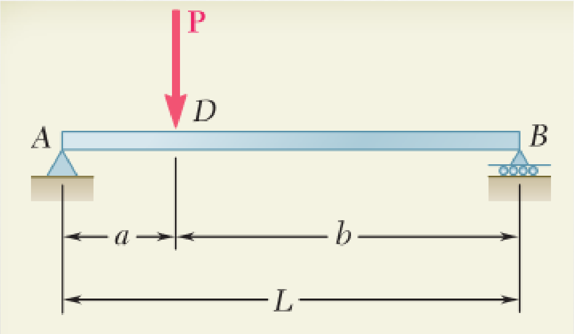
During a routine manufacturing operation, rod AB must acquire an elastic strain energy of 120 in ·lb. Using $E = 29 \times 10^6$ psi, determine the required yield strength of the steel if the factor of safety with respect to permanent deformation is to be five.

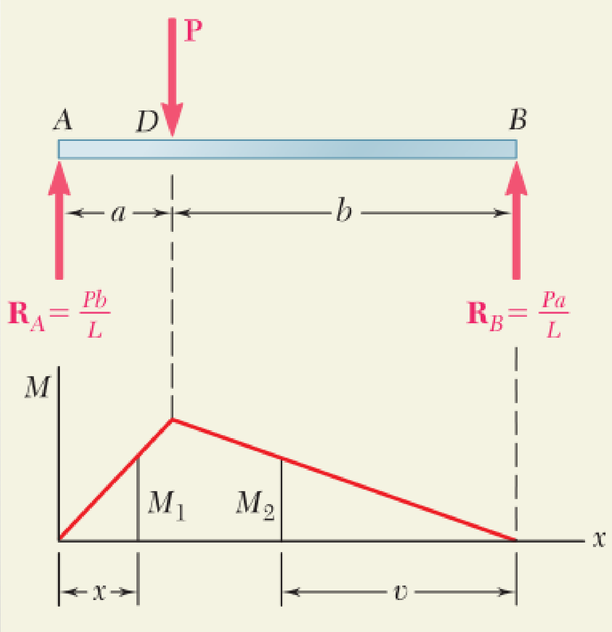


Example 12.7

(Beer, Page 708)

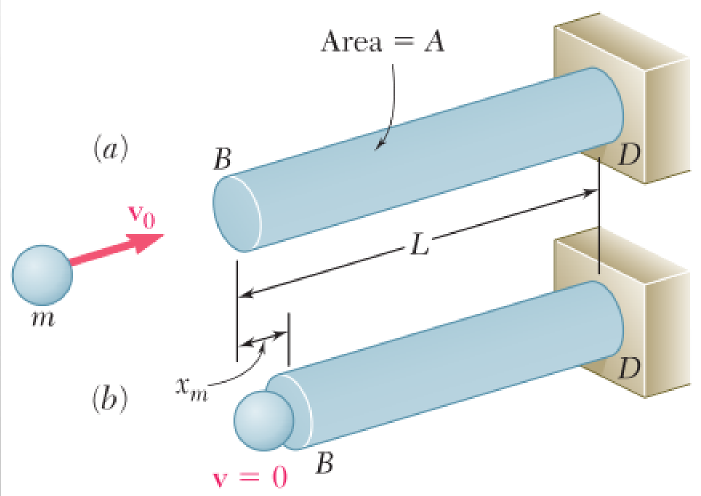
Taking into account only the effect of normal stresses due to bending, determine the strain energy of the prismatic beam AB for the loading shown.





§ 12.7 Impact loading

- Consider a rod BD of uniform cross section which is hit at its end B by a body of mass *m* moving with a velocity v_0



Kinetic energy, $$T = \frac{1}{2} m v_0^2$$

Strain energy, $$U_m = \frac{1}{2} m v_0^2$$

No dissipation and no bounce off

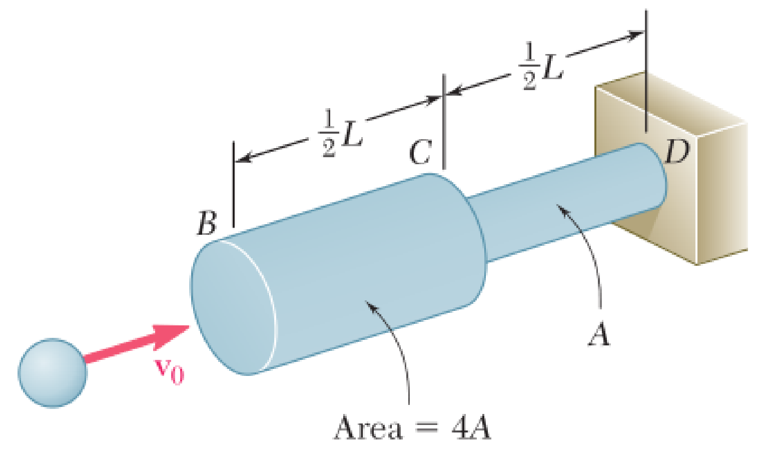
$$U_m = \int \frac{\sigma_m^2}{2E} dV = \frac{V\sigma_m^2}{2E}$$

$$\sigma_m = \sqrt{\frac{m v_0^2 E}{V}}$$

Example 12.8

(Beer, Page 717)

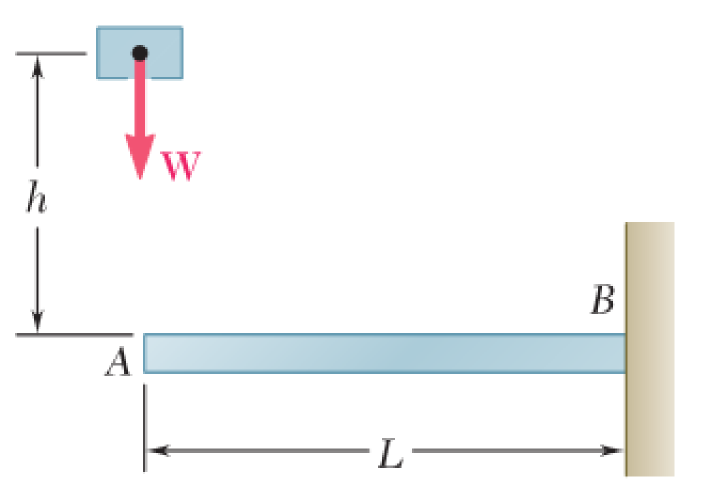
A body of mass m moving with a velocity v_0 hits the end B of the non-uniform rod BCD. Knowing that the diameter of portion BC is twice the diameter of portion CD, determine the maximum value σ_m of the stress in the rod.



Example 12.9

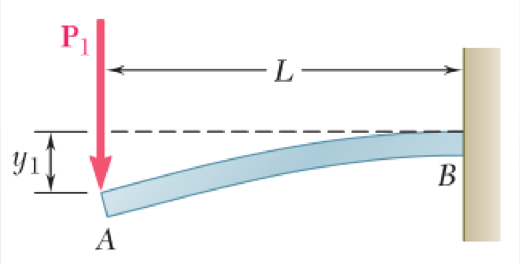
(Beer, Page 717)

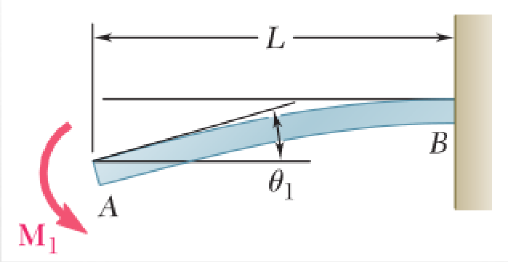
A block of weight W is dropped from a height h onto the free end of the cantilever beam AB. Determine the maximum value of the stress in the beam.

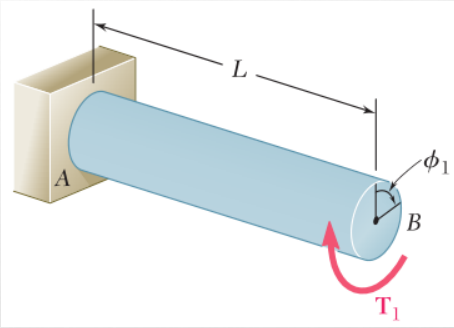


§ 12.8 Work and energy under a single load

- When a structure or member is subjected to a single concentrated load, it is possible to evaluate its elastic strain energy, provided that the relation between the load and the resulting deformation is known.







$$U_m = \frac{1}{2} P_1 y_1$$

$$y_1 = \frac{P_1 L^3}{3EI}$$

$$U_m = \frac{1}{2} M_1 \theta_1$$

$$\theta_1 = \frac{M_1 L}{EI}$$

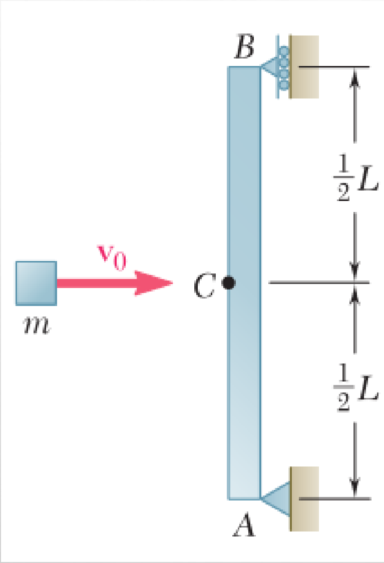
$$U_m = \frac{1}{2} T_1 \phi_1$$

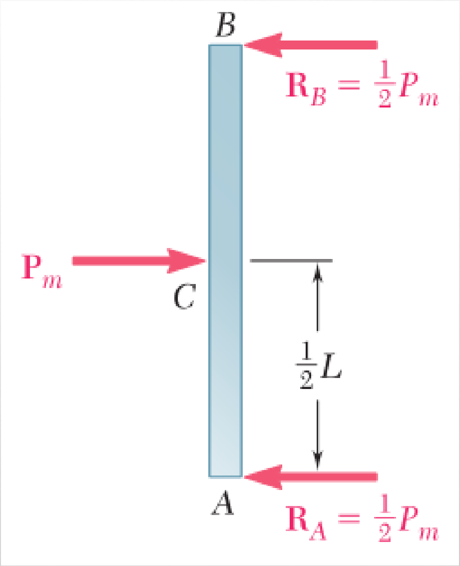
$$\phi_1 = \frac{T_1 L}{JG}$$

Example 12.10

(Beer, Page 717)

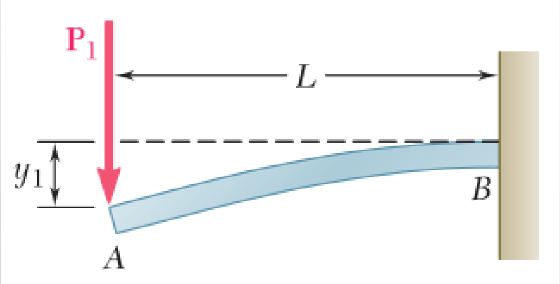
A block of mass *m* moving with a velocity v_0 hits squarely the prismatic member AB at its midpoint C. Determine (a) the equivalent static load P_m , (b) the maximum stress σ_m in the member, and (c) the maximum deflection x_m at point C.

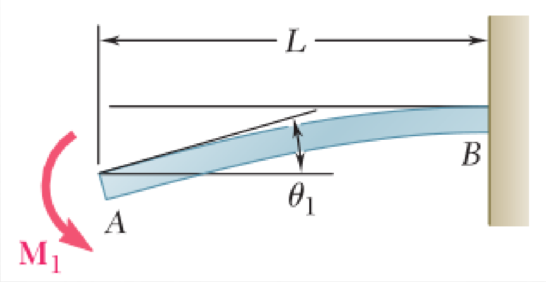


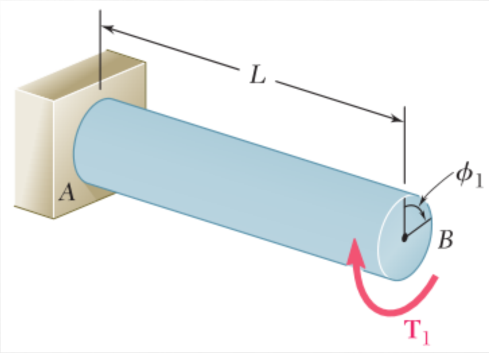


§ 12.9 Deflection under a single load

- If the strain energy U of a structure or member subjected to a single concentrated load P_1 or couple M_1 is known, the corresponding deflection x_1 or angle u_1 can be used to determine the corresponding deflection or angle.







$$U_m = \frac{1}{2} P_1 y_1$$

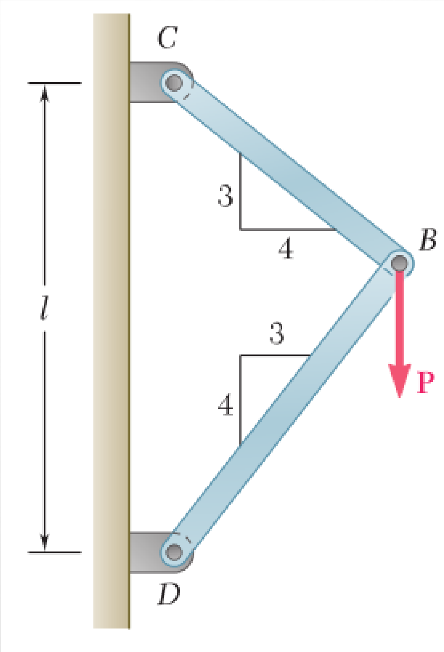
$$U_m = \frac{1}{2} M_1 \theta_1$$

$$U_m = \frac{1}{2} T_1 \phi_1$$

Example 12.11

(Beer, Page 723)

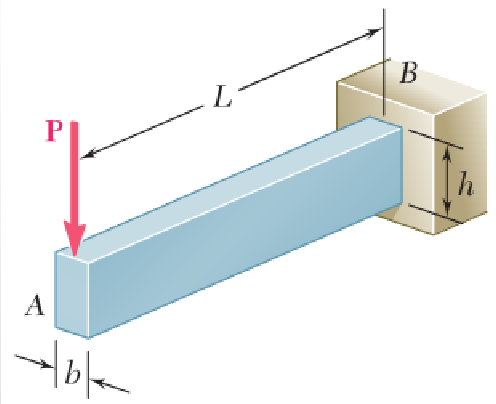
A load P is supported at B by two uniform rods of the same cross-sectional area A Determine the vertical deflection of point B.



Example 12.12

(Beer, Page 723)

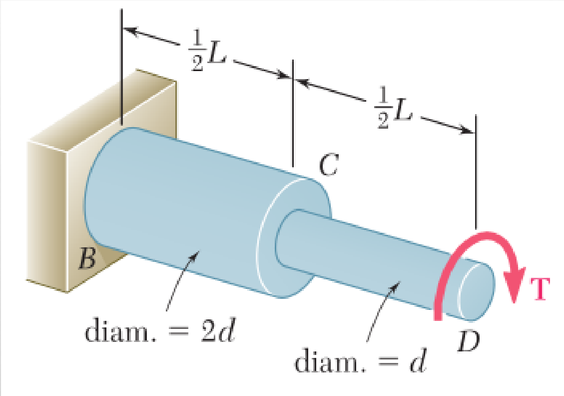
Determine the deflection of end A of the cantilever beam AB,taking into account the effect of (a) the normal stresses only, (b) both the normal and shearing stresses.



Example 12.13

(Beer, Page 724)

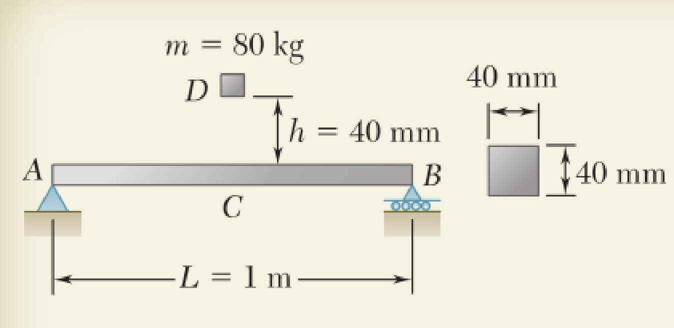
A torque T is applied at the end D of shaft BCD. Knowing that both portions of the shaft are of the same material and same length, but that the diameter of BC is twice the diameter of CD, determine the angle of twist for the entire shaft.

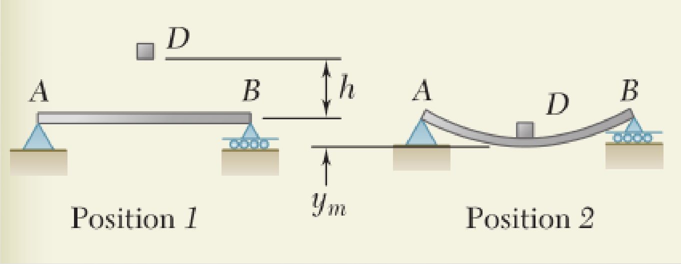


Example 12.14

(Beer, Page 725)

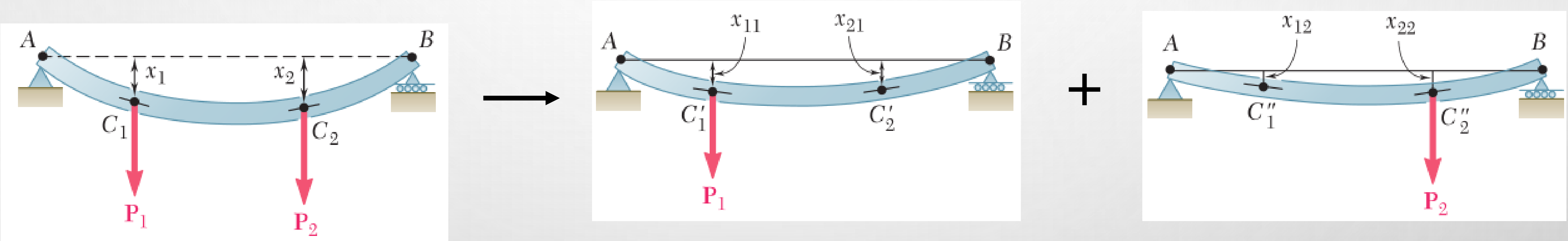
The block D of mass m is released from rest and falls a distance h before it strikes the midpoint C of the aluminum beam AB. Using E = 73 GPa, determine (a) the maximum deflection of point C, (b) the maximum stress that occurs in the beam.





§ 12.10 Work and energy under several loads

- Consider an elastic beam AB subjected to two concentrated loads P_1 and P_2. The strain energy of the beam is equal to the work of P_1 and P_2 as they are slowly applied to the beam at C_1 and C_2 , respectively.



$$x_{11} = \alpha_{11} P_1$$

$$x_{21} = \alpha_{21} P_1$$

influence coefficients

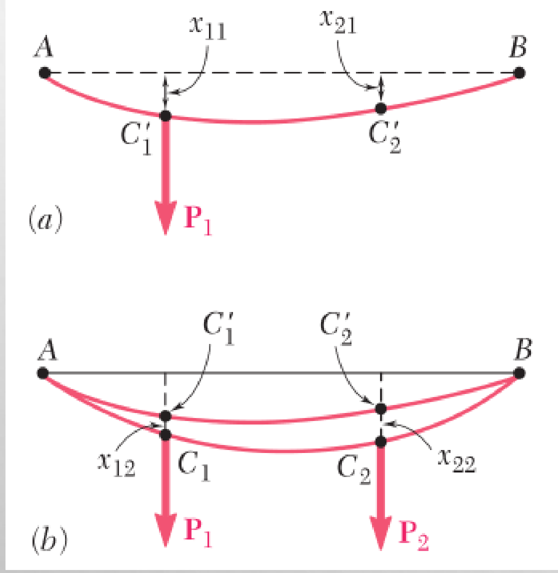
$$x_{12} = \alpha_{12} P_2$$

$$x_{22} = \alpha_{22} P_2$$

influence coefficients

§ 12.10 Work and energy under several loads

- To compute the work done by P_1 and P_2 , and thus the strain energy of the beam,



$$x_1 = x_{11} + x_{12} = \alpha_{11} P_1 + \alpha_{12} P_2$$

$$x_2 = x_{21} + x_{22} = \alpha_{21} P_1 + \alpha_{22} P_2$$

Assume P_1 is first applied, Then P_2 is applied.

$$\frac{1}{2} P_1 x_{11} = \frac{1}{2} \alpha_{11} P_1^2 \qquad \frac{1}{2} P_2 x_{22} = \frac{1}{2} \alpha_{22} P_2^2$$

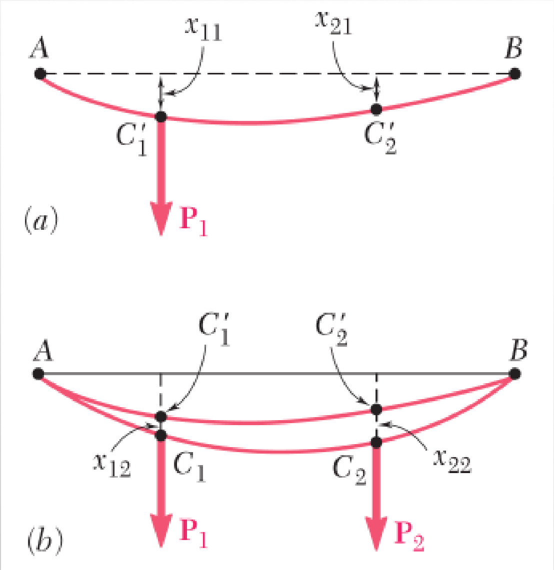
$$U = \frac{1}{2} \alpha_{11} P_1^2 + \alpha_{12} P_1 P_2 + \frac{1}{2} \alpha_{22} P_2^2 \qquad P_1 x_{12} = \alpha_{12} P_1 P_2$$

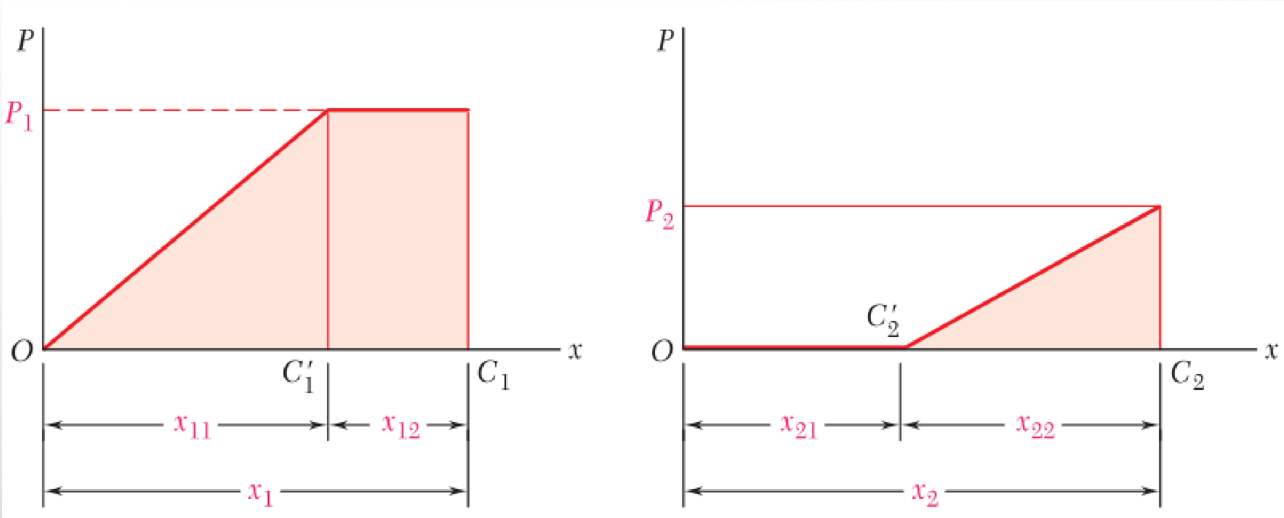
Maxwell’s reciprocal theorem $\longrightarrow$ $\alpha_{12} = \alpha_{21}$

P_1 is fully applied.

§ 12.10 Work and energy under several loads

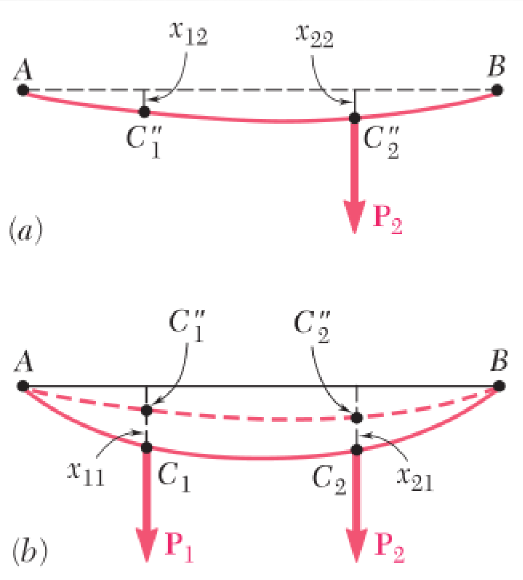
- Load-displacement diagrams: P_1 is first applied

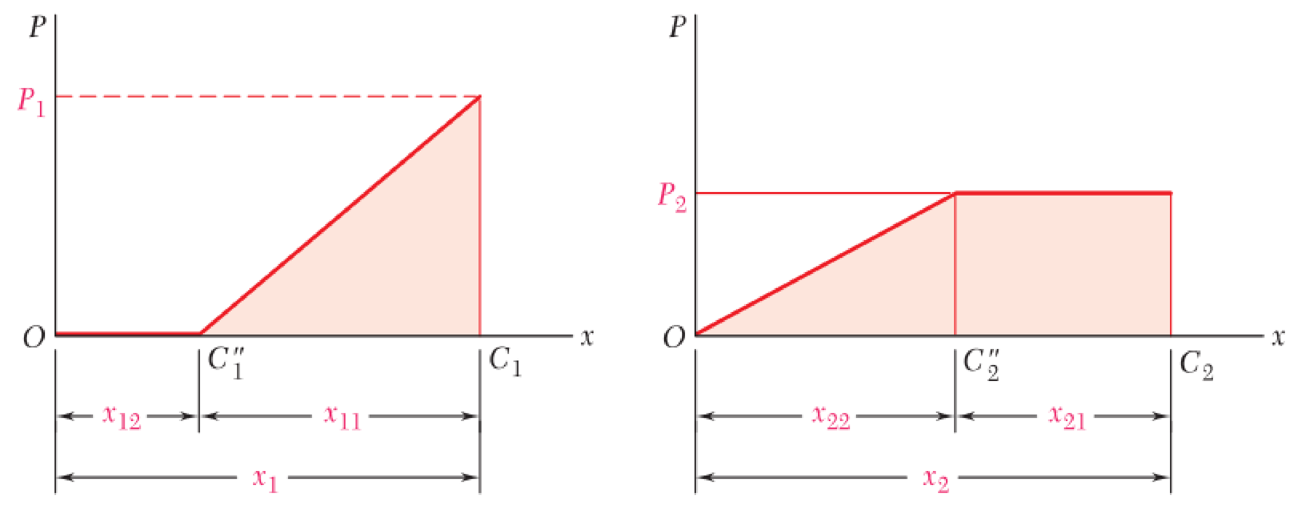




§ 12.10 Work and energy under several loads

- Load-displacement diagrams: P_2 is first applied





§ 12.11 Castigliano’s Theorem

- If an elastic structure is subjected to *n* loads P_1 , P_2 , . . ., P_n , the deflection x_j of the point of application of P_j , measured along the line of action of P_j , can be expressed as the partial derivative of the strain energy of the structure with respect to the load P_j . This is Castigliano’s theorem.

$$x_j = \frac{\partial U}{\partial P_j} \qquad \theta_j = \frac{\partial U}{\partial M_j} \qquad \phi_j = \frac{\partial U}{\partial T_j}$$

§ 12.11 Castigliano's Theorem

- Prove

$$x_j = \sum_j \alpha_{jk} P_k$$

$$U = \frac{1}{2}\sum_i \sum_k \alpha_{ik} P_i P_k \ (\alpha_{ik} = \alpha_{ki})$$

$$\frac{\partial U}{\partial P_j} = \frac{1}{2}\sum_k \alpha_{jk} P_k + \frac{1}{2}\sum_i \alpha_{ij} P_i = \sum_j \alpha_{jk} P_k$$

$$\frac{\partial U}{\partial P_j} = x_j$$

$$x_1 = \alpha_{11} P_1 + \alpha_{12} P_2$$

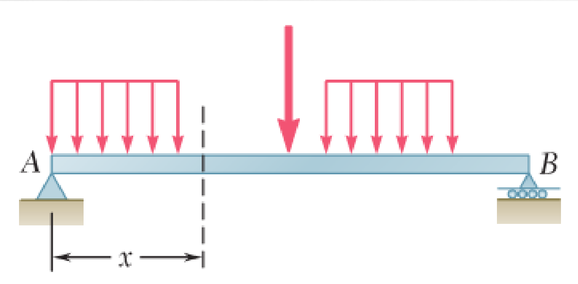
$$U = \frac{1}{2}\alpha_{11} P_1^2 + \alpha_{12} P_1 P_2 + \frac{1}{2}\alpha_{22} P_2^2$$

$$\frac{\partial U}{\partial P_1} = \alpha_{11} P_1 + \alpha_{12} P_2$$

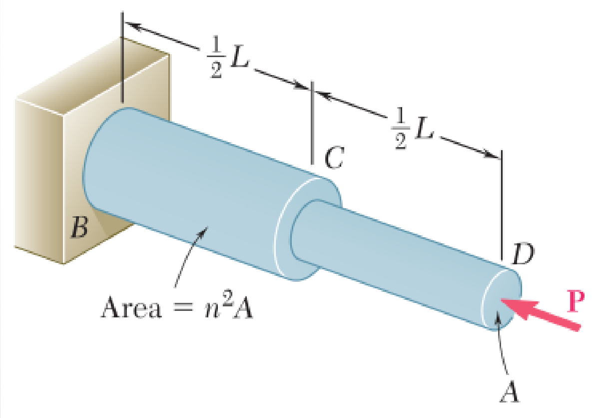
$$\frac{\partial U}{\partial P_1} = x_1$$

§ 12.12 Deflections by Castigliano's Theorem

- The calculation by Castigliano's theorem of the deflection x_j is simplified if the differentiation with respect to the load P_j is carried out before the integration or summation.



$$U = \int_0^L \frac{M^2}{2EI}\,dx$$



$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_i E}$$

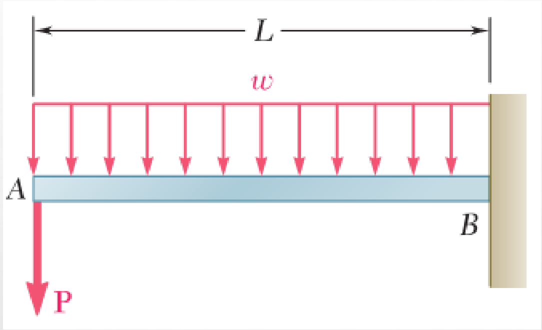
$$x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{2EI}\frac{\partial M}{\partial P_j}\,dx$$

$$x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E}\frac{\partial F_i}{\partial P_j}$$

Example 12.15

(Beer, Page 736)

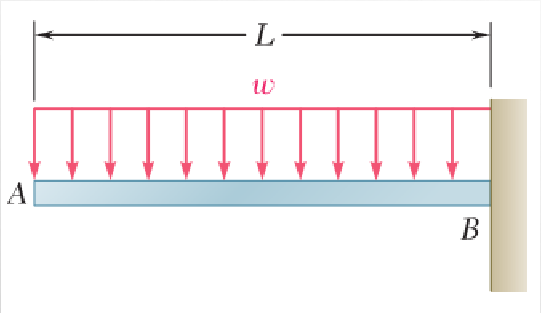
The cantilever beam AB supports a uniformly distributed load *w* and a concentrated load *P*, determine the deflection at A.

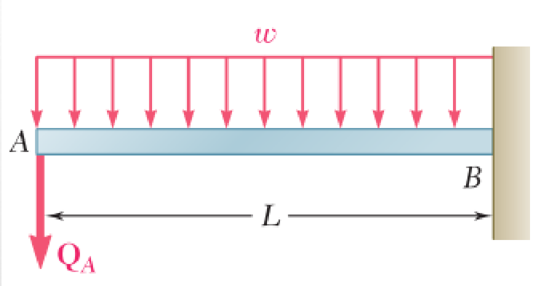


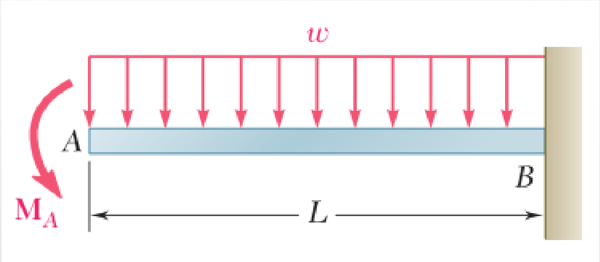
Example 12.16

(Beer, Page 738)

The cantilever beam AB supports a uniformly distributed load *w*. Determine the deflection and slope at A.





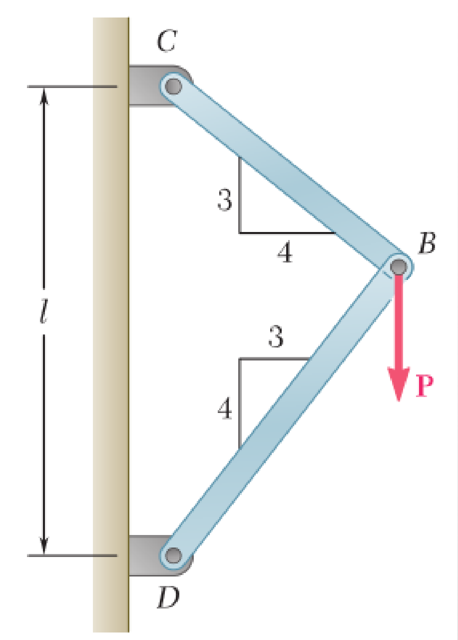


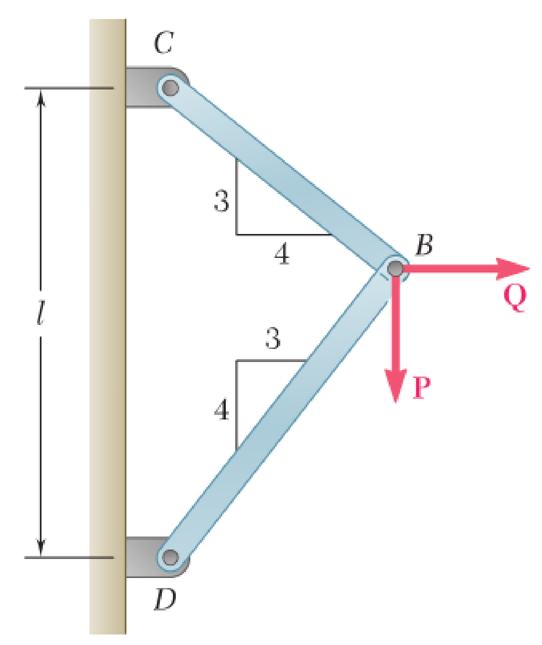
Dummy Q_A and M_A

Example 12.17

(Beer, Page 739)

A load *P* is supported at B by two rods of the same material and of the same cross-sectional area A. Determine the horizontal and vertical deflection of point B.





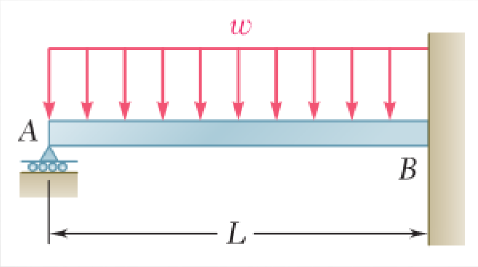
§ 12.13 Statically indeterminate structures

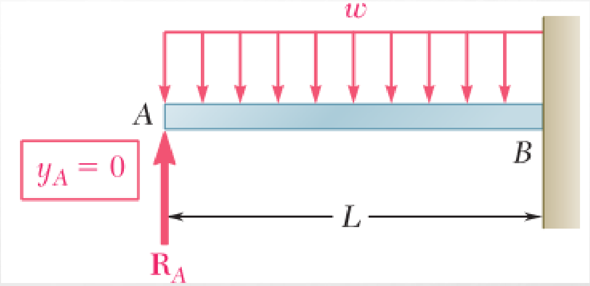
- In the case of a structure indeterminate to the first degree, for example, we designate one of the reactions as redundant and eliminate or modify accordingly the corresponding support.
- The redundant reaction is then treated as an unknown load that, together with the other loads, must produce deformations that are compatible with the original supports.
- We first calculate the strain energy U of the structure due to the combined action of the given loads and the redundant reaction. Observing that the partial derivative of U with respect to the redundant reaction represents the deflection (or slope) at the support that has been eliminated or modified, we then set this derivative equal to zero and solve the equation obtained for the redundant reaction. The remaining reactions can be obtained from the equations of statics.

Example 12.18

(Beer, Page 740)

Determine the reactions at the supports for the prismatic beam and loading.

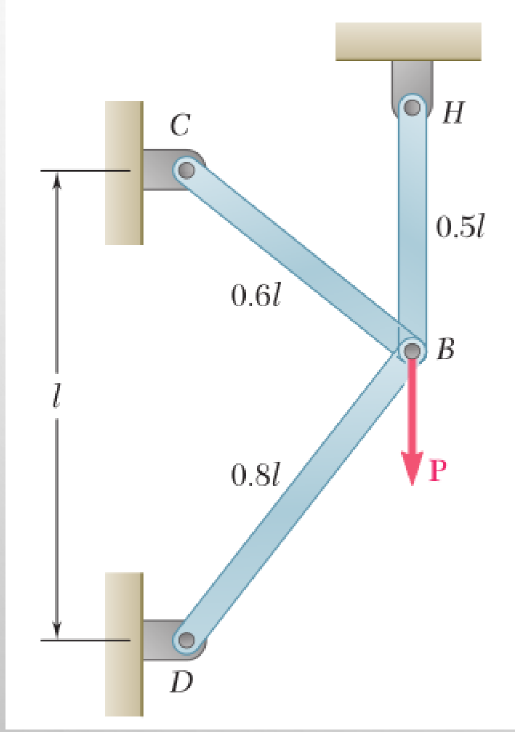


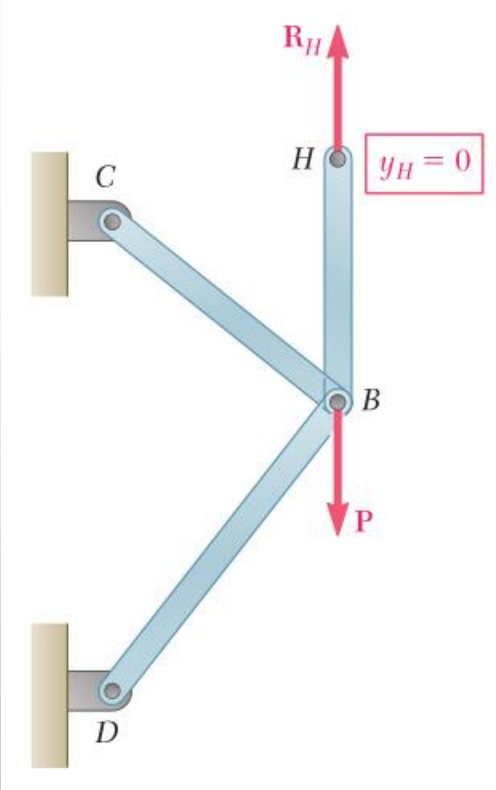


Example 12.19

(Beer, Page 741)

A load *P* is supported at B by three rods of the same material and the same cross-sectional area A. Determine the force in each rod.

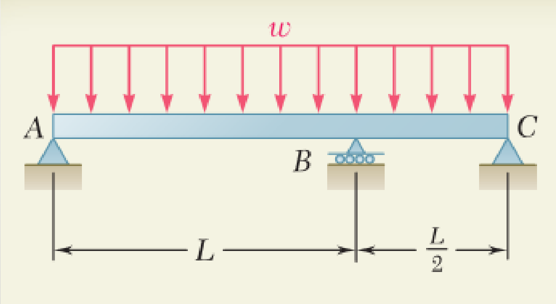


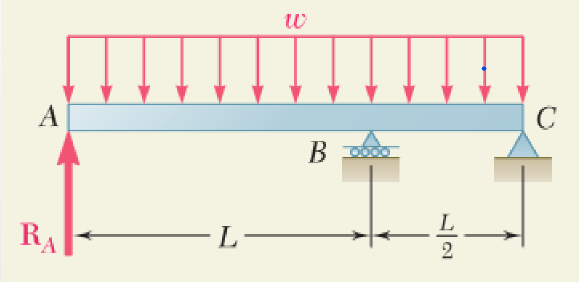


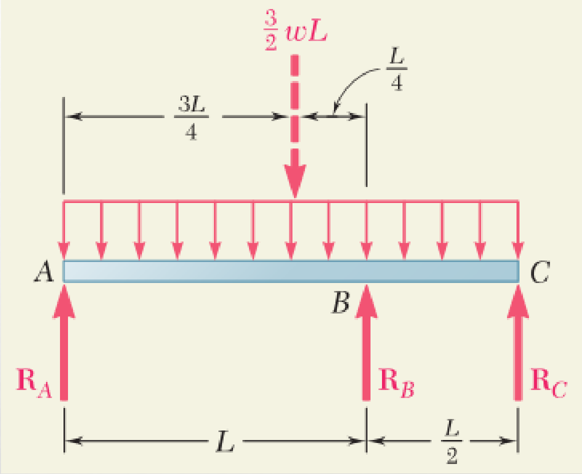
Example 12.20

(Beer, Page 744)

For the uniform beam and loading shown, determine the reactions at the supports.







§ 12.14 Summary

- **Strain energy and Strain-energy density**
- **Modulus of resilience**
- **Strain energy under axial load, due to bending, shearing stresses, and torsion.**
- **General state of stress**
- **Castigliano’s theorem**
- **Indeterminate structures**