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Homework 1

Due: 2023 Sep. 25

1. Go over Sec. 1.1 to 1.4 of the textbook.

√**2.** Do Exercise 1 on page 7.

3. Consider the function $h(x)$ given by

$$h(x)=\begin{cases}\frac{1}{2}x^2 & 0\le x<0.5\\ \frac{1}{4}-\frac{1}{2}(x-1)^2 & 0.5\le x\le 1\end{cases}$$

(a) Is $h(x)$ in the space C^1?
(b) Is $h(x)$ in the space C^2?
(c) Is $h(x)$ in the space H^2?

4. Consider a boundary-value problem (i.e. strong-form problem) with the Dirichlet boundary conditions imposed on both ends, that is,

$$(S)\begin{cases}u_{,xx}+f=0,\\ u(0)=g_0,\\ u(1)=g_1.\end{cases}$$

(a) State its corresponding weak-form problem.
(b) Prove the equivalence between the strong- and weak-form problems.

2. Solution

Symmetry property

$$\left.\begin{aligned}a(u,v)&=\int_0^1 u_{,x}\cdot v_{,x}\,dx=\int_0^1\frac{du}{dx}\cdot\frac{dv}{dx}\,dx\\ a(v,u)&=\int_0^1 v_{,x}\cdot u_{,x}\,dx=\int_0^1\frac{dv}{dx}\cdot\frac{du}{dx}\,dx\end{aligned}\right\}\Rightarrow a(u,v)=a(v,u)$$

$$\left.\begin{aligned}(u,v)&=\int_0^1 uv\,dx\\ (v,u)&=\int_0^1 vu\,dx\end{aligned}\right\}\Rightarrow(u,v)=(v,u)$$

Bilinearity property

$$\begin{aligned}a(c_1u+c_2v,w)&=\int_0^1(c_1u+c_2v)_{,x}w_{,x}\,dx\\&=\int_0^1\frac{d(c_1u+c_2v)}{dx}\frac{dw}{dx}\,dx\\&=\int_0^1\left(c_1\frac{du}{dx}+c_2\frac{dv}{dx}\right)\frac{dw}{dx}\,dx\\&=\int_0^1\left(c_1\frac{du}{dx}\frac{dw}{dx}+c_2\frac{dv}{dx}\frac{dw}{dx}\right)dx\\&=c_1\int_0^1\frac{du}{dx}\frac{dw}{dx}\,dx+c_2\int_0^1\frac{dv}{dx}\frac{dw}{dx}\,dx\\&=c_1\int_0^1u_{,x}w_{,x}\,dx+c_2\int_0^1v_{,x}w_{,x}\,dx\\&=c_1a(u,w)+c_2a(v,w)\end{aligned}$$

$$\begin{aligned}(c_1u+c_2v,w)&=\int_0^1(c_1u+c_2v)w\,dx\\&=\int_0^1c_1uw\,dx+\int_0^1c_2vw\,dx\\&=c_1\int_0^1uw\,dx+c_2\int_0^1vw\,dx\\&=c_1(u,w)+c_2(v,w)\end{aligned}$$

3. Solution

a) $C^k(\Omega)$: the class of continuous functions possessing k continuous derivative

$$\left.\begin{aligned}\lim_{x\to0.5^-}h(x)&=\lim_{x\to0.5^-}\frac{1}{2}x^2=\frac{1}{8}\\ \lim_{x\to0.5^+}h(x)&=\lim_{x\to0.5^+}\left[\frac{1}{4}-\frac{1}{2}(x-1)^2\right]=\frac{1}{8}\end{aligned}\right\}\Rightarrow\lim_{x\to0.5}h(x)=\frac{1}{8}$$

So $h(x)$ is continuous at $x=0.5$, in $(0,1)$

$$\left.\begin{aligned}\lim_{x\to0.5^-}h'(x)&=\lim_{x\to0.5^-}x=0.5\\ \lim_{x\to0.5^+}h'(x)&=\lim_{x\to0.5^+}(-x+1)=0.5\end{aligned}\right\}\Rightarrow\lim_{x\to0.5}h'(x)=0.5$$

So $h'(x)$ is continuous at $x=0.5$, in $(0,1)$

$\Rightarrow h(x)$ in space C^1

b)
$$\left.\begin{aligned}\lim_{x\to0.5^-}h''(x)&=\lim_{x\to0.5^-}1=1\\ \lim_{x\to0.5^+}h''(x)&=\lim_{x\to0.5^+}(-1)=-1\end{aligned}\right\}\Rightarrow h''(x)\text{ is not continuous at }x=0.5$$

$\Rightarrow h(x)$ is not in space C^2

c) $H^2=H^2(\Omega)=\{w: w\in L_2,\ w_{,x}\in L_2,\ w_{,xx}\in L_2\}$

$L_2=L_2(\Omega)=\{w: \int_\Omega w^2dx=\int_0^1w^2dx<\infty\}$

$$\int_0^1h^2(x)\,dx=\int_0^{0.5}\frac{1}{4}x^4dx+\int_{0.5}^1\left[\frac{1}{4}-\frac{1}{2}(x-1)^2\right]^2dx=\frac{23}{960}<+\infty$$

$$\int_0^1[h'(x)]^2dx=\int_0^{0.5}x^2dx+\int_{0.5}^1(1-x)^2dx=\frac{1}{12}<+\infty$$

$$\int_0^1[h''(x)]^2dx=\int_0^{0.5}1\,dx+\int_{0.5}^1 1\,dx=1<+\infty$$

$\therefore h(x)$ is in space H^2

4. Solution

$\mathcal{S}=\{u: u\in H^1,\ u(0)=g_0,\ u(1)=g_1\}$

$\mathcal{V}=\{w: w\in H^1,\ w(0)=0,\ w(1)=0\}$

a) Given f, g_0, g_1 ($f\in L_2$, $g_0\in\mathbb{R}$, $g_1\in\mathbb{R}$), find $u\in\mathcal{S}$ such that

$$(W)\begin{cases}\int_0^1 w_{,x}u_{,x}\,dx=\int_0^1 wf\,dx\\ \text{for all } w\in\mathcal{V}\end{cases}$$

b) ① $(S)\Rightarrow(W)$: strong solution solves (W)

$u_{,xx}+f=0$

$w(u_{,xx}+f)=0$ $\quad w\in\mathcal{V}=\{w: w\in H^1,\ w(0)=0,\ w(1)=0\}$

$\int_0^1wu_{,xx}\,dx+\int_0^1wf\,dx=0$

$\int_0^1wu_{,xx}\,dx=w(1)u_{,x}(1)-w(0)u_{,x}(0)-\int_0^1w_{,x}u_{,x}\,dx$ (IBP)

$\Rightarrow\int_0^1w_{,x}u_{,x}\,dx=\int_0^1wf\,dx$ ∎

$u\in\mathcal{S}=\{u: u\in H^1,\ u(0)=g_0,\ u(1)=g_1\}$ ∎

turn over →

② $(W) \Rightarrow (S)$: weak solution solves (S)

$$\int_0^1 w_{,x} u_{,x} \, dx = \int_0^1 w f \, dx \qquad w \in \mathcal{V}$$

IBP: $\int_0^1 w_{,x} u_{,x} \, dx = \underbrace{w(1)}_{0} u(1) - \underbrace{w(0)}_{0} u(0) - \int_0^1 w u_{,xx} \, dx = \int_0^1 w f \, dx$

$\Rightarrow \int_0^1 w (u_{,xx} + f) \, dx = 0 \qquad (1)$

$w(0) = w(1) = 0$

set $w = \phi (u_{,xx} + f)$ with $\phi > 0$, $\phi(0) = \phi(1) = 0$

$(1) \Rightarrow \int_0^1 \underset{0}{\underset{\vee}{\phi}} \underbrace{(u_{,xx} + f)^2}_{\substack{\vee\!\shortparallel \\ 0}} \, dx = 0$

$\therefore \; u_{,xx} + f = 0$, with $u(0) = g_0$, $u(1) = g_1$

$(u \in \mathcal{S})$