常微分方程B HW7

1.解:

1) $y''+2y=0$

$r^2+0r+2=0$

$r=\pm\sqrt{2}i=0\pm\sqrt{2}i$

$y=C_1e^0\cos\sqrt{2}t+C_2e^0\sin\sqrt{2}t$

$=C_1\cos\sqrt{2}t+C_2\sin\sqrt{2}t$

2) $2y''+2y'+y=0$

$2r^2+2r+1=0$

$\Delta=4-4\times2=-4<0$

$r=\lambda+ui \Rightarrow 2(\lambda^2-u+2\lambda ui)+2\lambda+2ui+1=0$

$\lambda=-\frac{1}{2}, u=\frac{1}{4}$

$r=\lambda-ui \Rightarrow \lambda=-\frac{1}{2}, u=-\frac{1}{4}$

$\therefore r=-\frac{1}{2}\pm\frac{1}{4}i$

$y=C_1e^{-\frac{1}{2}t}\cos\frac{1}{4}t+C_2e^{-\frac{1}{2}t}\sin\frac{1}{4}t$

3) $9y''+6y'+y=0$

$9r^2+6r+1=0$

$\Delta=36-4\times9=0 \Rightarrow r=\frac{-1}{3}$

$y=(C_1+C_2t)e^{-\frac{1}{3}t}$

4) $y''+8y'+16y=0$

$r^2+8r+16=0$

$\Delta=64-4\times16=0, \Rightarrow r=-4$

$y=(C_1+C_2t)e^{-4t}$

2.解:

(1) $y''+10y'+25y=0$

$r^2+10r+25=0$

$r=-5$

$y=(C_1+C_2t)e^{-5t}$, $y'=C_2\cdot e^{-5t}+(C_1+C_2t)e^{-5t}(-5)$

$y(0)=2 \Rightarrow 2=C_1$

$y'(0)=-1 \Rightarrow C_2+C_1(-5)=-1$, $C_2=9$

$\therefore y=(2+9t)e^{-5t}$

(2) $y''+2y'+2y=0$

$r^2+2r+2=0$

$r=-1\pm1i$

$y=C_1e^{-t}\cos t+C_2e^{-t}\sin t$

$y'=C_1[e^{-t}\cdot(-1)\cos t+e^{-t}(-\sin t)]+$

$C_2[e^{-t}(-1)\sin t+e^{-t}\cdot\cos t]$

$y(\frac{\pi}{4})=2 \Rightarrow C_1\cdot e^{-\frac{\pi}{4}}\cdot\frac{\sqrt{2}}{2}+C_2e^{-\frac{\pi}{4}}\frac{\sqrt{2}}{2}=0$, $C_1+C_2=0$

$y'(\frac{\pi}{4})=-2 \Rightarrow C_1[e^{-\frac{\pi}{4}}(-\frac{\sqrt{2}}{2})+e^{-\frac{\pi}{4}}(-\frac{\sqrt{2}}{2})]+$

$C_2[e^{-\frac{\pi}{4}}(-\frac{\sqrt{2}}{2})+e^{-\frac{\pi}{4}}\cdot\frac{\sqrt{2}}{2}]=0$, $C_1=C_2=0$

舍去

$y=C_1e^{-t}\cos(-t)+C_2e^{-t}\sin(-t)$

$=C_1e^{-t}\cos t-C_2e^{-t}\sin t=e^{-t}(C_1\cos t-C_2\sin t)$

$y'=C_1[(-1)e^{-t}\cos t+e^{-t}(-\sin t)]-C_2[e^{-t}(-1)\sin t+e^{-t}\cdot\cos t]$

$=e^{-t}\cos t(-C_1-C_2)+e^{-t}\sin t(-C_1+C_2)$

$y(\frac{\pi}{4})=2 \Rightarrow e^{-\frac{\pi}{4}}(\frac{\sqrt{2}}{2}C_1-\frac{\sqrt{2}}{2}C_2)=2$, $C_1-C_2=e^{\frac{\pi}{4}}\cdot2\sqrt{2}$

$y'(\frac{\pi}{4})=-2 \Rightarrow e^{-\frac{\pi}{4}}\cdot\frac{\sqrt{2}}{2}(-2C_1)=-2$, $C_1=e^{\frac{\pi}{4}}\sqrt{2}$

$C_2=-\sqrt{2}e^{\frac{\pi}{4}}$

$\therefore y=\sqrt{2}e^{\frac{\pi}{4}-t}\cos t+\sqrt{2}e^{\frac{\pi}{4}-t}\sin t$

3.解:

(1) $t^2y''+2ty'-2y=0$, $t>0$

令 $y(t)=v(t)\cdot t$

$y'(t)=v'(t)\cdot t+v(t)$

$y''(t)=v''(t)\cdot t+2v'(t)$

$\cancel{t^2\cdot[v''(t)\cdot t+2v'(t)]+2t[v'(t)t+v(t)]-2v(t)\cdot t=0}$

$\cancel{v''(t)\cdot t^3+2t\cdot v'(t)+2t^2v'(t)-2v(t)\cdot t=0}$

$\cancel{v''(t)\cdot t^2+v'(t)\cdot(4+2t)-2v(t)=0}$

$t^3\cdot v''(t)+2t^2v'(t)+\cancel{2t^2v'(t)}+2t\cdot v(t)-2tv(t)=0$

$t\,v''(t)+4v'(t)=0$

$\Rightarrow v'(t)=A_1e^{-4\ln t}=A_1\cdot\frac{1}{t^4}$

$v(t)=\frac{-A_1}{3t^3}+C_1$

$\therefore y(t)=\frac{A_2}{3t^2}+C_1t$

(2) $t^2\cdot y''+3ty'+y=0$, $t>0$

令 $y(t)=v(t)\cdot\frac{1}{t}$

$y'(t)=v'(t)\cdot\frac{1}{t}+v(t)\cdot\frac{-1}{t^2}$

$y''(t)=v''(t)\cdot\frac{1}{t}+v'(t)\cdot\frac{-1}{t^2}+v'(t)\cdot\frac{-1}{t^2}+v(t)\cdot\frac{2}{t^3}$

$\underline{t\cdot v''(t)}+v'(t)(-1)+v'(t)(-1)+v(t)\cdot\frac{2}{t}+3v'(t)+\frac{-3}{t}v(t)$
$+v(t)\cdot\frac{1}{t}=0$

$t\cdot v''(t)+v'(t)=0$

$\Rightarrow (v'(t)\cdot t)'=0$, $v'(t)\cdot t=C_1$

$v'(t)=\frac{C_1}{t}$, $v(t)=C_1\ln t+C_2$

$\therefore y(t)=\frac{1}{t}(C_1\ln t+C_2)$

4.解: $y''-y'+\frac{y}{4}=0$

$r^2-r+\frac{1}{4}=0$

$r=\frac{1}{2}$

$y=(C_1+C_2t)e^{\frac{1}{2}t}$

$y'(t)=C_2\cdot e^{\frac{1}{2}t}+(C_1+C_2t)\cdot e^{\frac{1}{2}t}\cdot\frac{1}{2}$

$y(0)=2\Rightarrow C_1=2$

$y'(0)=b\Rightarrow C_2+\frac{1}{2}C_1=b$, $C_2=b-1$

$\therefore y=[2+(b-1)t]e^{\frac{1}{2}t}$

由题: $b-1=0$

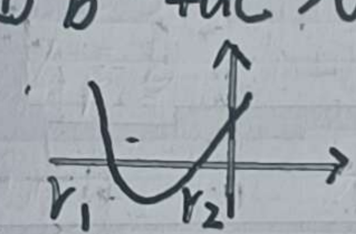
$b=1$, critical value

5.解:

(1) $ay''+by'+cy=0$

$ar^2+br+c=0\Rightarrow r_1, r_2$

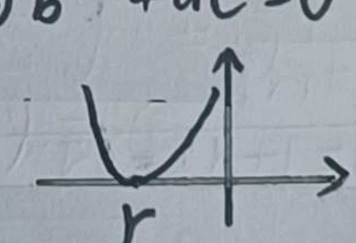
① $b^2-4ac>0$, $r_1<0, r_2<0$



$y=C_1e^{r_1t}+C_2e^{r_2t}$

$\lim_{t\to\infty}y=C_1 0+C_2\cdot 0=0$

② $b^2-4ac=0$



$y=(C_1+C_2t)e^{rt}$

$\lim_{t\to\infty}y=C_1\cdot 0+C_2\cdot 0=0$

③ $b^2-4ac<0$

$r=\lambda\pm ui$

since $r_1,r_2<0$
so $\lambda_1,\lambda_2<0$, $y=C_1e^{\lambda_1t}\cos ut+C_2e^{\lambda_2t}\sin ut$

$\lim_{t\to\infty}y=C_1\cdot 0+C_2\cdot 0=0$

(2) $ay''+cy=0$

$ar^2+c=0$

$r^2=\frac{-c}{a}$, $r=\pm\sqrt{\frac{c}{a}}\,i$

$y=C_1\cos\sqrt{\frac{c}{a}}\,t+C_2\sin\sqrt{\frac{c}{a}}\,t$

As $t\to\infty$, y is bounded.

(3) $ay''+by'=0$

$ar^2+br=0$

$r_1=0$

$r_2=\frac{-b}{a}$

$y=C_1+C_2e^{-\frac{b}{a}t}$, $y'=C_2\cdot\frac{b}{-a}\cdot e^{-\frac{b}{a}t}$

As $t\to\infty$, $y\to C_1$

$y(0)=y_0 \Rightarrow C_1+C_2=y_0$

$y'(0)=y_0' \Rightarrow -\frac{b}{a}C_2=y_0'$

$\Rightarrow C_2=\frac{-a}{b}\cdot y_0'$

$C_1=y_0+\frac{a}{b}y_0'$

6. 解:

(1) $y''-2y'-3y=3e^{2t}$

let $y=Ae^{2t}$

$y'=2Ae^{2t}$

$y''=4A\cdot e^{2t}$

$e^{2t}(4A-4A-3A)=3e^{2t}$

$A=-1$

$\therefore y=-e^{2t}+C_1e^{-t}+3e^{3t}$ $\boxed{\text{Ans}}$

for $y''-2y'-3y=0$

$r^2-2r-3=0$

$r_1=-1,\ r_2=3$

$y=C_1e^{-t}+3e^{3t}$

(2) $y''-2y'+y=te^t+4$

let $y=Ate^t+C$

$y'=A\cdot e^t+At\cdot e^t$

$y''=2Ae^t+At\cdot e^t$

$2A\cdot e^t+At\cdot e^t-2Ae^t-2At\cdot e^t \Rightarrow$

~~let $y=Ate^{rt}+C$~~

~~$y'=A\cdot e^{rt}+At\cdot r\cdot e^{rt}$~~

~~$y''=A\cdot r e^{rt}+Ar\cdot e^{rt}+Ar^2t\cdot e^{rt}$~~

~~$e^{rt}(2Ar+Ar^2t-2A-2Art+At)+C=te^t+4$~~

~~$\Rightarrow C=4,\ r=1,\ A=?$~~

$\therefore y=4+t\cdot e^t$

$y'=e^t+t\cdot e^t$

$y''=e^t+e^t+t\cdot e^t=2e^t+t\cdot e^t$

$\cancel{2e^t}+\cancel{t\cdot e^t}-\cancel{2e^t}-\cancel{2t\cdot e^t}+4+\cancel{t\cdot e^t}=4$

for $y''-2y'+y=0$

$r^2-2r+1=0$

$r=1$, $y_c=(C_1+C_2t)e^t$

令 $y=At^ne^t+4$

$y'=An\cdot t^{n-1}\cdot e^t+A\cdot t^n\cdot e^t$

$y''=An\cdot(n-1)\cdot t^{n-2}\cdot e^t+An\cdot t^{n-1}\cdot e^t+A\cdot n\cdot t^{n-1}\cdot e^t+A\cdot t^n\cdot e^t$

$An(n-1)\cdot t^{n-2}+\cancel{2An\,t^{n-1}}+\cancel{A\,t^n}-\cancel{A\,t^n}+\cancel{1}=t$

$n=3$, $A=\frac{1}{6}$

$\therefore y=\frac{1}{6}t^3e^t+4$

$\therefore y=y_c+y=(C_1+C_2t)e^t+\frac{t^3}{6}\cdot e^t+4$ $\boxed{\text{Ans}}$

3) $y''+4y=2\sin2t$

let $y=A\sin2t+B\cos2t$

$y'=A\cdot2\cdot\cos2t+B\cdot2(-1)\cdot\sin2t$

$y''=-4A\cdot\sin2t+(-4B)\cdot\cos2t$

$-4A\cdot\sin2t-4B\cos2t+4A\sin2t+4B\cos2t=0$

let $y=Ct\cdot(A\sin2t+B\cos2t)$

$y'=C\cdot(A\sin2t+B\cos2t)+Ct[2A\cdot\cos2t+(-2B\sin2t)]$

$y''=C[A\cdot2\cos2t+B\cdot2\cdot(-\sin2t)]+C[2A\cos2t-2B\sin2t)+Ct[-4A\sin2t-4B\cos2t]$

$=4C(A\cos2t-B\sin2t)+4Ct(-A\sin2t-B\cos2t)$

$4C\cdot(A\cos2t-B\sin2t)+4Ct(-A\sin2t-B\cos2t)+4Ct(A\sin2t+B\cos2t)=2\sin2t$

$\Rightarrow A=0$

$BC=-\frac{1}{2}$

$\therefore y=Ct\cdot B\cos2t=-\frac{1}{2}t\cdot\cos2t$

for $y''+4y=0$

$r^2+4=0$

$r=\pm2i$

$y_c=C_1\cos2t+C_2\sin2t$

$\therefore y=C_1\cos2t+C_2\sin2t-\frac{1}{2}t\cos2t$ $\boxed{\text{Ans}}$

4) $y'' + 4y' + 4y = e^{-2t} + \sin 2t$

let $y = A \cdot e^{-2t} + B \sin 2t + C \cos 2t$

$y' = A \cdot (-2) \cdot e^{-2t} + B \cdot 2 \cdot \cos 2t + C \cdot 2 (-\sin 2t)$

$y'' = +4Ae^{-2t} + 4B \cdot (-\sin 2t) + 4C \cdot (-\cos 2t)$

$e^{-2t}(4A - 8A + 4A) = 0 \quad \times$

let $y = A \cdot t \cdot e^{-2t} + B \sin 2t + C \cdot \cos 2t$

$y' = A \cdot e^{-2t} + A \cdot t \cdot (-2) \cdot e^{-2t} + 2B \cos 2t + (-2C) \cdot \sin 2t$

$y'' = A(-2) \cdot e^{-2t} + (-2A) \cdot e^{-2t} + (+4A) \cdot t \cdot e^{-2t} + (-4B) \sin 2t + (-4C) \cdot \cos 2t$

$e^{-2t}(\underline{-4A + 4At - 4A + 4At}) = e^{-2t}$

$8A(t-1) = 1 \quad \times.$

let $y = A \cdot t^n \cdot e^{-2t} + B \sin 2t + C \cdot \cos 2t$

$y' = A \cdot n \cdot t^{n-1} \cdot e^{-2t} + A \cdot t^n (-2) \cdot e^{-2t} + 2B \cos 2t - 2C \sin 2t$

$y'' = An(n-1) t^{n-2} \cdot e^{-2t} + An \cdot t^{n-1} \cdot (-2) \cdot e^{-2t} + A(-2) \cdot n t^{n-1} \cdot e^{-2t} + A(+4) \cdot t^n \cdot e^{-2t}$
$- 4B \sin 2t - 4C \cos 2t$

$An(n-1) \cdot t^{n-2} \underline{+ Ant^{n-1}(-2)} \underline{-2A \cdot n \cdot t^{n-1}} \underline{+4At^n} \underline{+4Ant^{n-1}} \underline{-8At^n} \underline{+4At^n} = 1$

$n = 2, \quad A = \frac{1}{2}$

$\underline{-4B \sin 2t} - \underline{4C \cos 2t} + 8B \cos 2t - 8C \sin 2t \underline{+4B \sin 2t} + \underline{4C \cos 2t} = \sin 2t$

$B = 0, \quad C = -\frac{1}{8}$

$\therefore y = \frac{1}{2} t^2 \cdot e^{-2t} - \frac{1}{8} \cos 2t$

for $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$

$r = -2$

$y_c = (C_1 + C_2 t) e^{-2t}$

$\therefore y = (C_1 + C_2 t) e^{-2t} + \frac{1}{2} t^2 \cdot e^{-2t} - \frac{1}{8} \cos 2t$