力学与航空航天工程系
DEPARTMENT OF MECHANICS AND AEROSPACE ENGINEERING

MECHANICS OF MATERIALS

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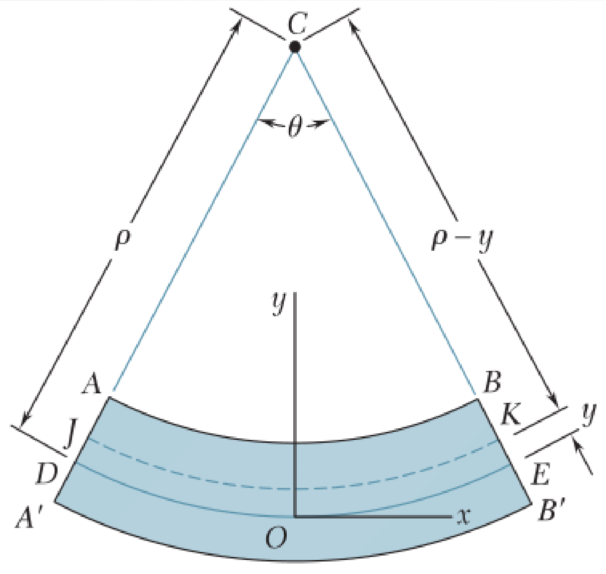
SPRING, 2022

Lesson 10： Deflection of beams

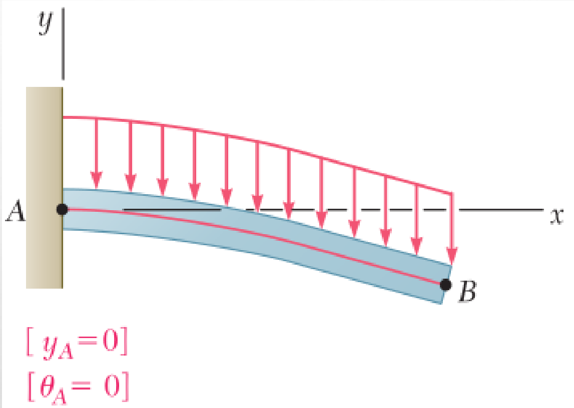
- Determination of the deflection
- Equation of the elastic curve
- Deflections of indeterminate beams
- Method of superposition

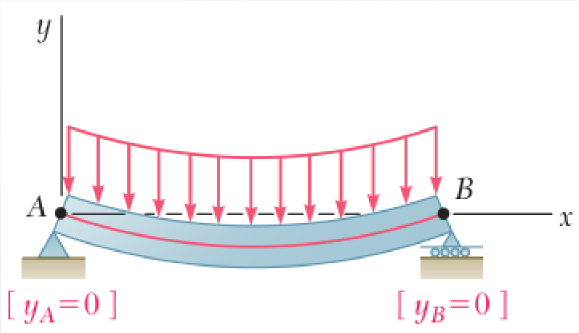
§ 10.1 Introduction

- A prismatic beam subjected to pure bending is bent into an arc of circle.
- When a beam is subjected to a transverse loading, both the bending moment and the curvature of the neutral surface will vary from section to section.



$$\frac{1}{\rho} = \frac{M}{EI}$$



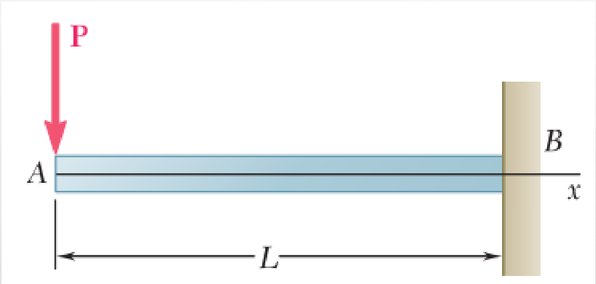


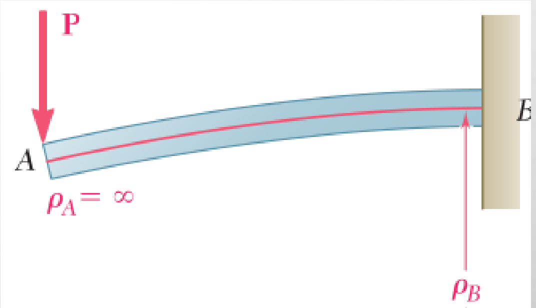
$$\frac{1}{\rho(x)} = \frac{M(x)}{EI}$$

§ 10.2 Deformation of beams under transverse loading

- The bending equation remains valid for any given transverse section of a beam subjected to a transverse loading, provided that Saint-Venant's principle applies.
- Consider a cantilever beam AB of length L subjected to a concentrated load P at its free end A

$$\frac{1}{\rho(x)} = \frac{M(x)}{EI}$$



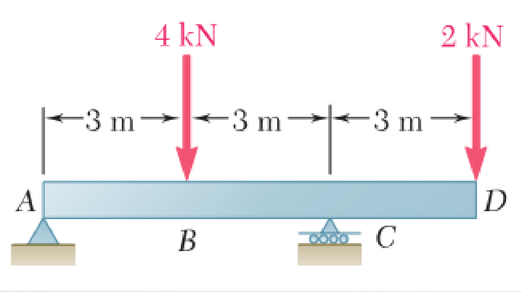


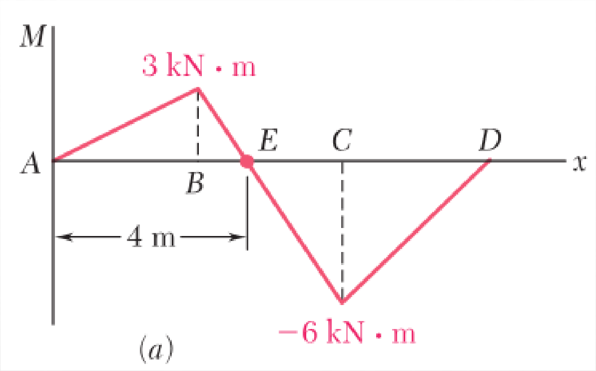
$$M(x) = -Px$$

$$\frac{1}{\rho(x)} = -\frac{Px}{EI}$$

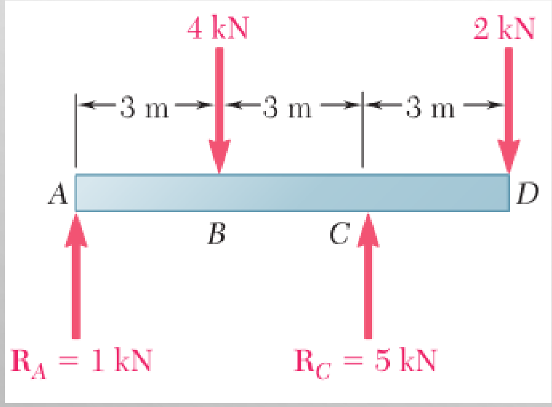
§ 10.2 Deformation of beams under transverse loading

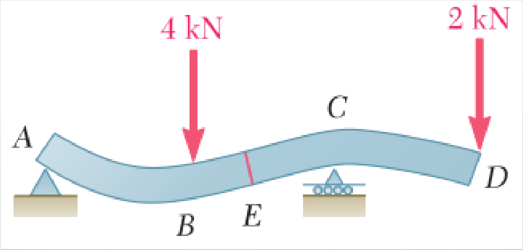
- Consider the overhanging beam that supports two concentrated loads.





$$\frac{1}{\rho(x)} = \frac{M(x)}{EI}$$





$$\left|\frac{1}{\rho}\right|_{\max} = \frac{|M|_{\max}}{EI}$$

§ 10.3 Equation of the elastic curve

- The curvature of a plane curve at a point $Q(x, y)$ of the curve can be expressed as

$$\frac{1}{\rho} = \frac{\dfrac{d^2 y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \xrightarrow{\text{elastic}} \frac{1}{\rho} = \frac{d^2 y}{dx^2}$$

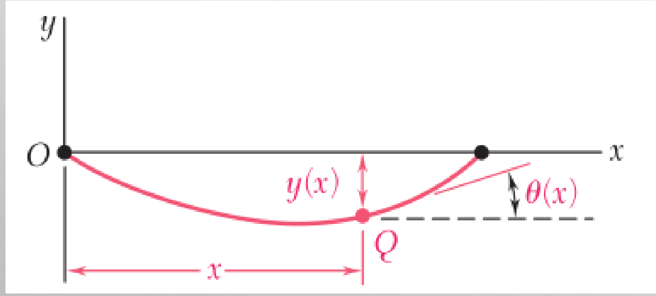
governing differential equation for the elastic curve

$$\boxed{\frac{d^2 y}{dx^2} = \frac{M(x)}{EI}}$$

EI: flexural rigidity

$$EI\frac{dy}{dx} = \int_0^x M(x)dx + C_1 \qquad \frac{dy}{dx} = \tan\theta \cong \theta(x)$$

$\theta(x)$: the angle

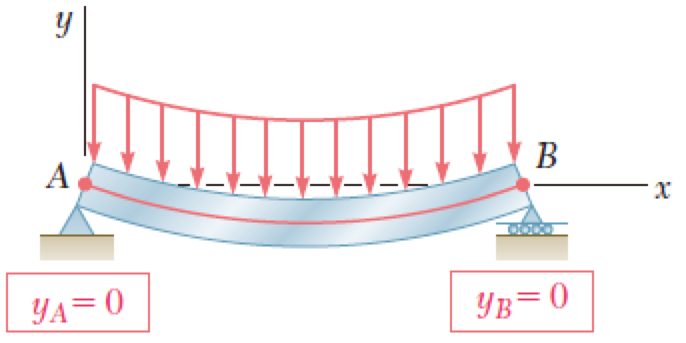


$$EIy = \int_0^x dx \int_0^x M(x)dx + C_1 x + C_2$$

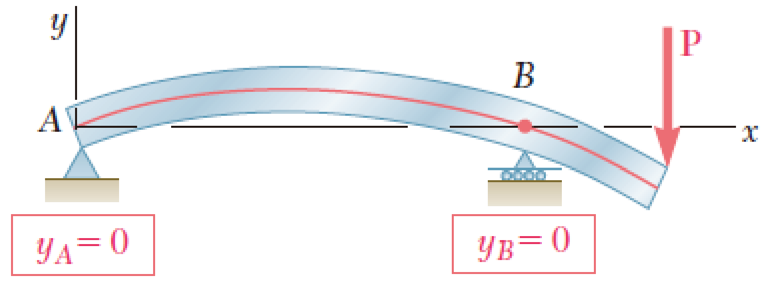
C1, C2, integration constants, determined by the boundary conditions

§ 10.3 Equation of the elastic curve

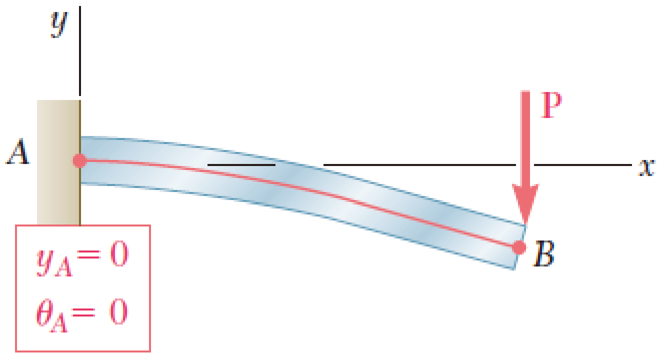
- Three types of statically determinate beams



(a) Simply supported beam



(b) Overhanging beam

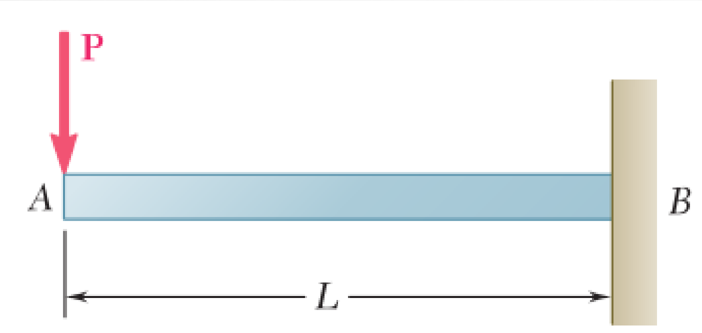


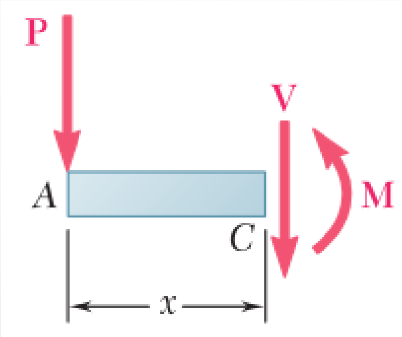
(c) Cantilever beam

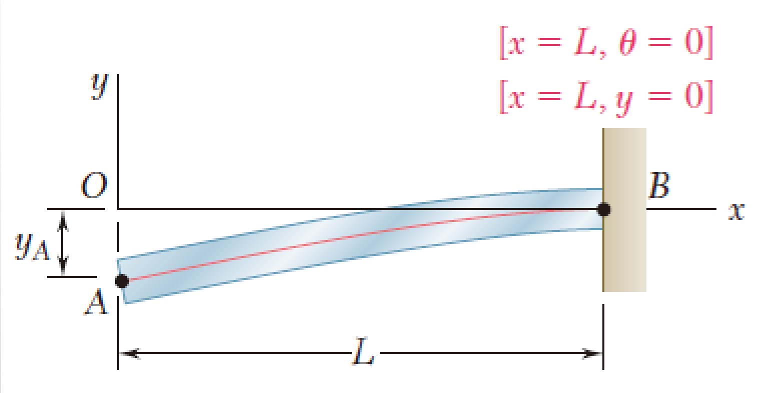
Example 10.1

(Beer, Page 555)

The cantilever beam AB is of uniform cross section and carries a load P at its free end A. Determine the equation of the elastic curve and the deflection and slope at A.



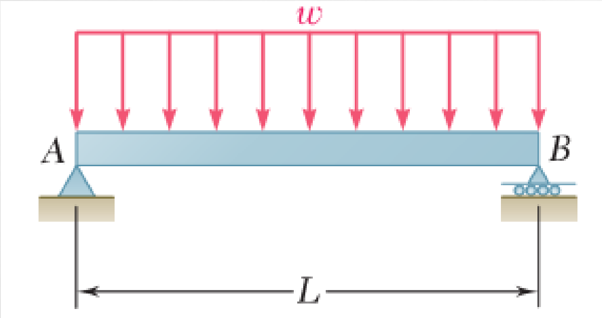


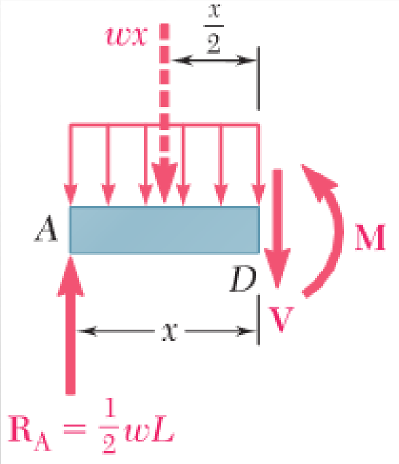


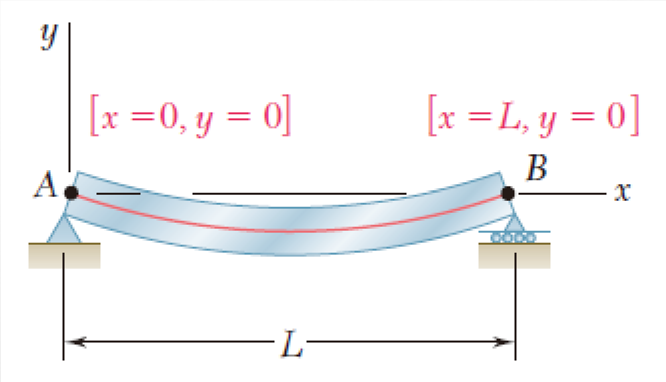
Example 10.2

(Beer, Page 556)

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length. Determine the equation of the elastic curve and the maximum deflection of the beam.



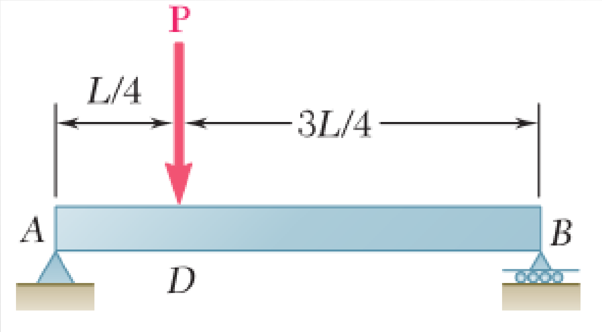


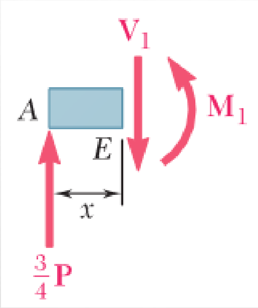


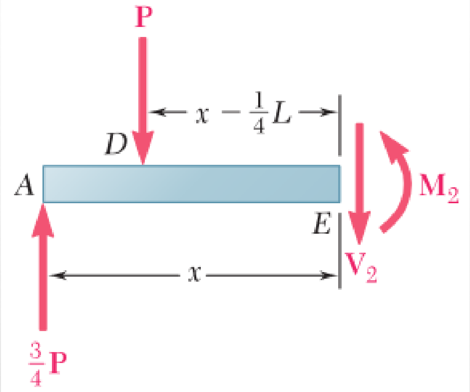
Example 10.3

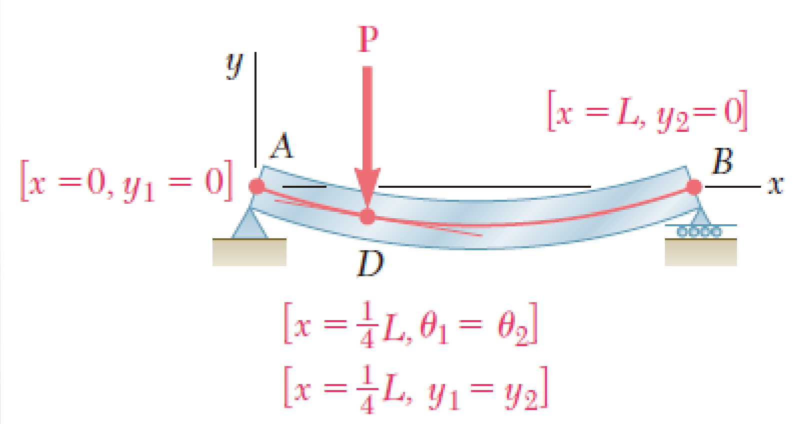
(Beer, Page 557)

For the prismatic beam and the loading, determine the slope and deflection at point D.







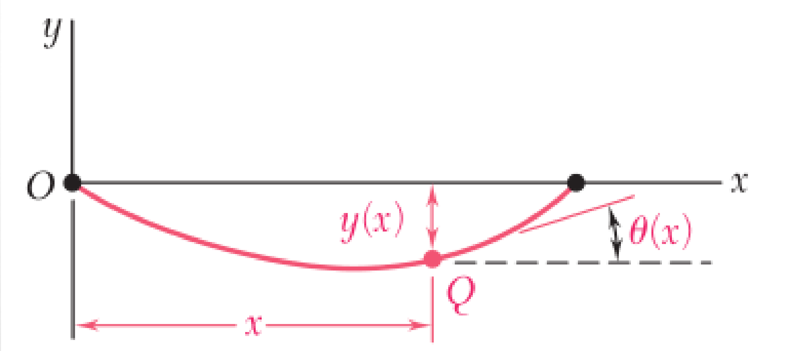


§ 10.4 Direct determination of the elastic curve from the load distribution

- The relation among load, shear, bending moment, curvature, angle and deflection

$$\frac{dV}{dx} = -w(x)$$

$$\frac{dM}{dx} = V(x)$$



$$\frac{1}{\rho} = \frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

$$\longrightarrow$$

$$\boxed{EI\frac{d^4y}{dx^4} = -w(x)}$$

$$\theta(x) = \frac{dy}{dx}$$

§ 10.4 Direct determination of the elastic curve

- The deflection-load relation

$$\boxed{EI\frac{d^4y}{dx^4} = -w(x)}$$

$$EI\frac{d^3y}{dx^3} = V(x) = -\int_0^x w(x)dx + C_1$$

$$EI\frac{d^2y}{dx^2} = M(x) = -\int_0^x dx\int_0^x w(x)dx + C_1x + C_2$$

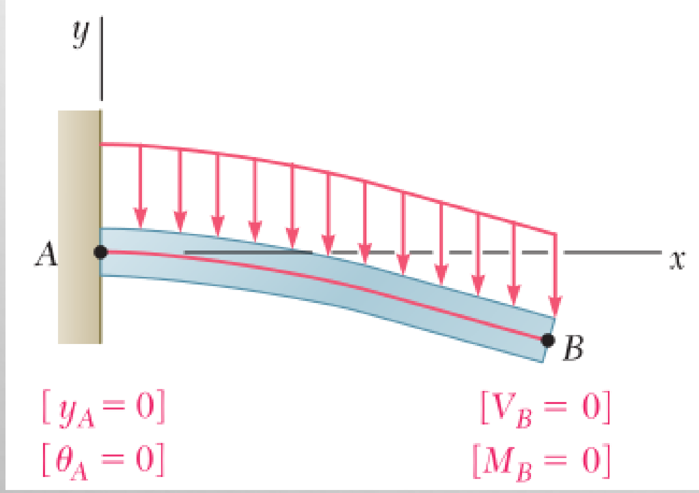
$$EI\frac{dy}{dx} = EI\theta(x) = -\int_0^x dx\int_0^x dx\int_0^x w(x)dx + \frac{1}{2}C_1x^2 + C_2x + C_3$$

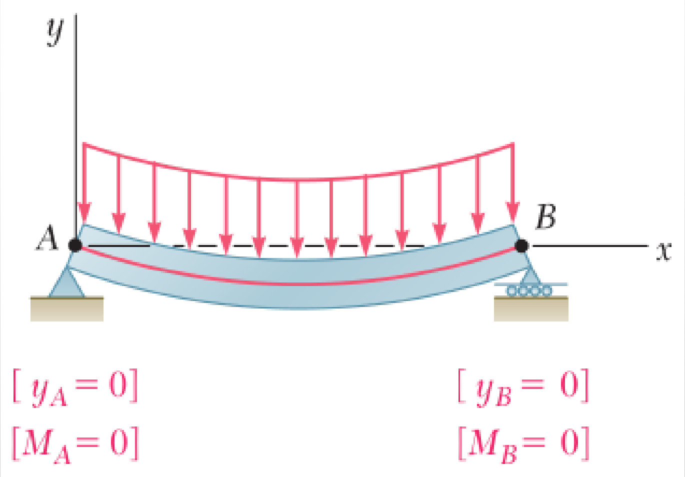
$$EIy = -\int_0^x dx\int_0^x dx\int_0^x dx\int_0^x w(x)dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

§ 10.4 Direct determination of the elastic curve

- The deflection-load relation

$$EIy = -\int_0^x dx \int_0^x dx \int_0^x dx \int_0^x w(x)dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

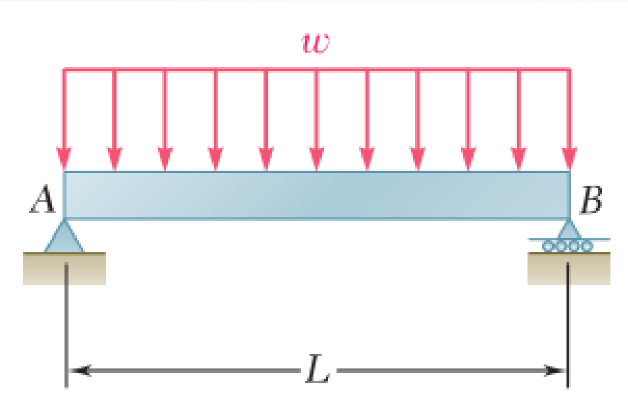


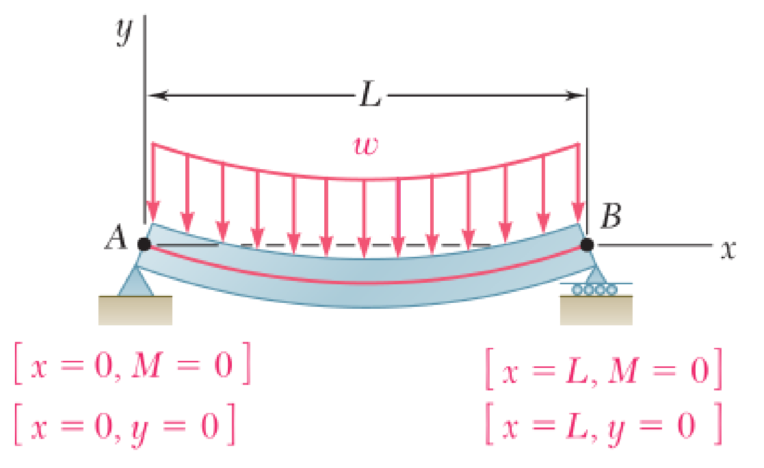


Example 10.4

(Beer, Page 560)

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length. Determine the equation of the elastic curve and the maximum deflection of the beam.

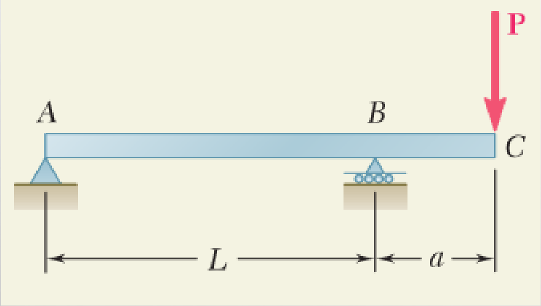


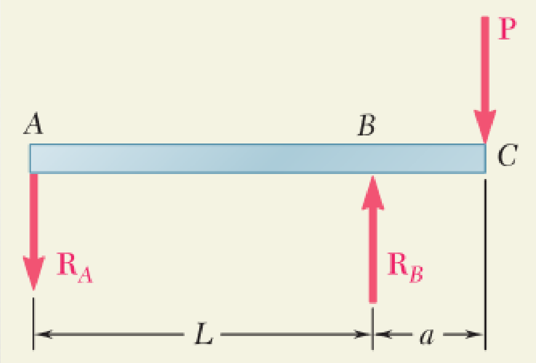


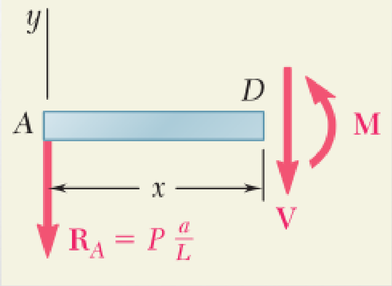
Example 10.5

(Beer, Page 563)

The overhanging steel beam ABC carries a concentrated load P at end C. For portion AB of the beam, (a) derive the equation of the elastic curve, (b) determine the maximum deflection.



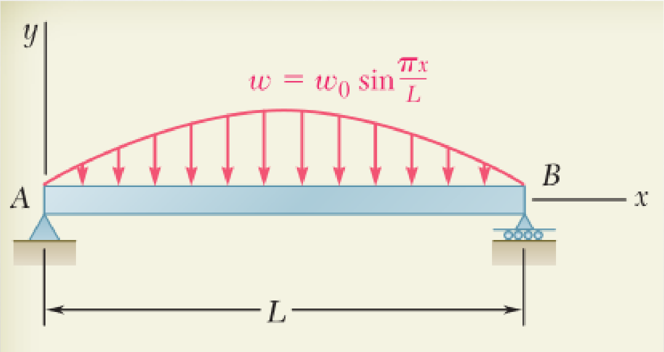


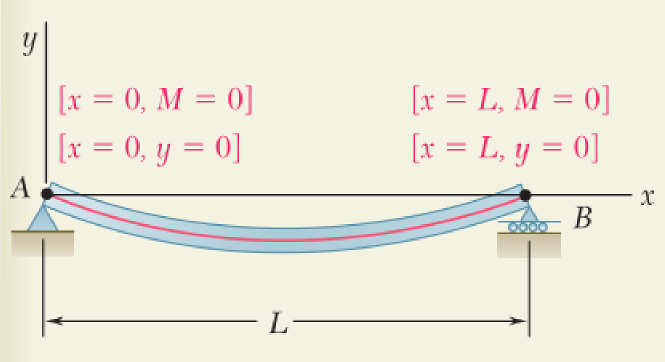


Example 10.6

(Beer, Page 564)

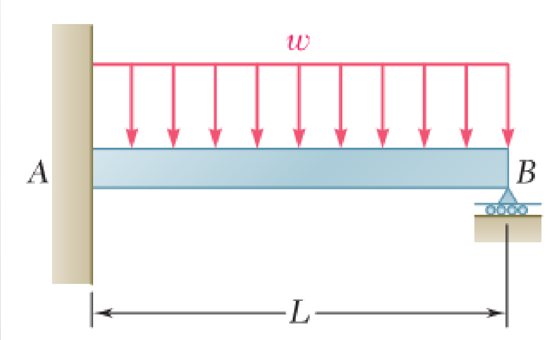
For the beam and loading shown, determine (a) the equation of the elastic curve, (b) the slope at end A, (c) the maximum deflection.

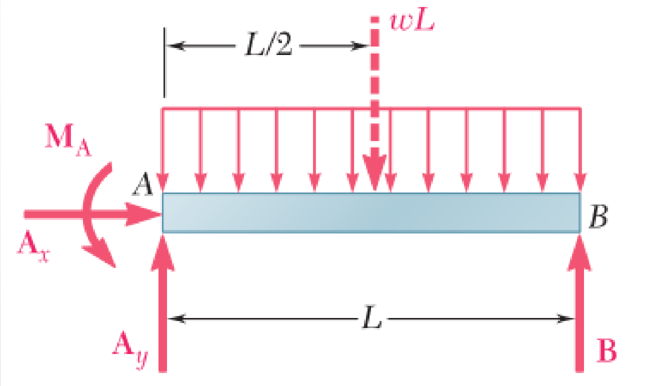




§ 10.5 Statically indeterminate beams

- Consider now the prismatic beam AB, which has a fixed end at A and is supported by a roller at B.
- There was one redundant reaction, corresponding to statically indeterminate to the first degree.





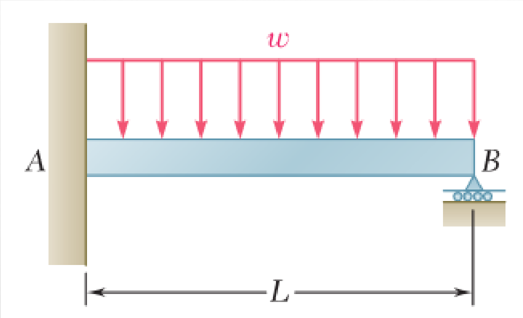
Four unknown reactions but only three equilibrium equations

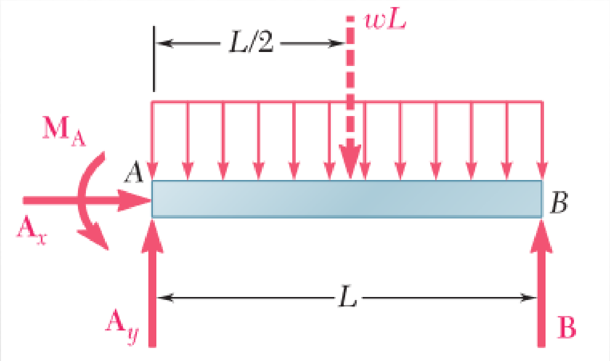
$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0$$

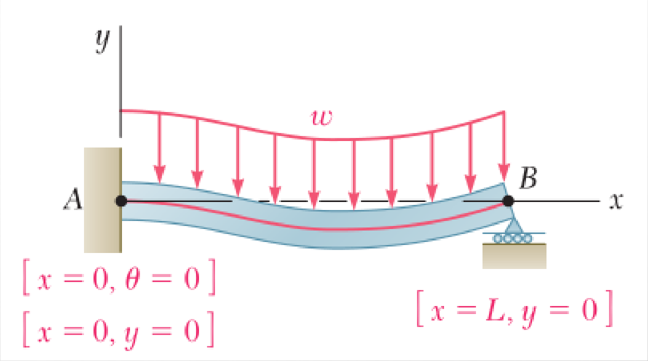
Example 10.7

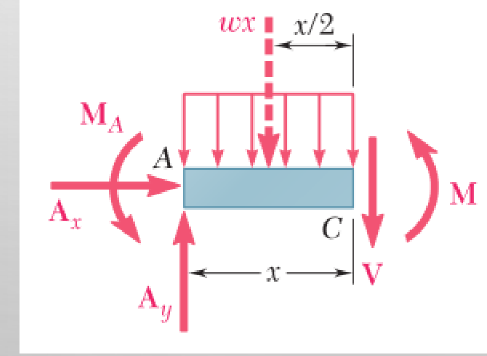
(Beer, Page 561)

Determine the reactions at the supports for the prismatic beam.





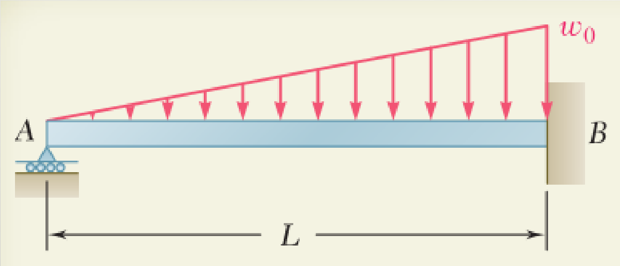


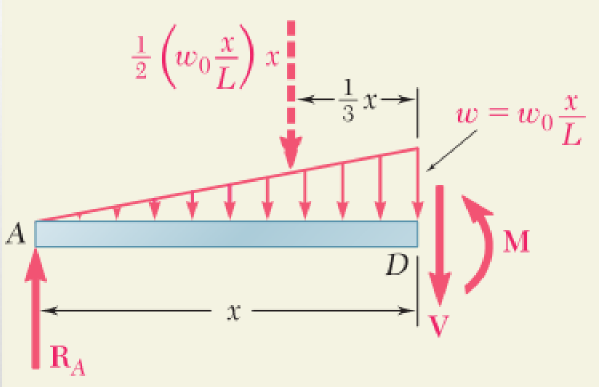


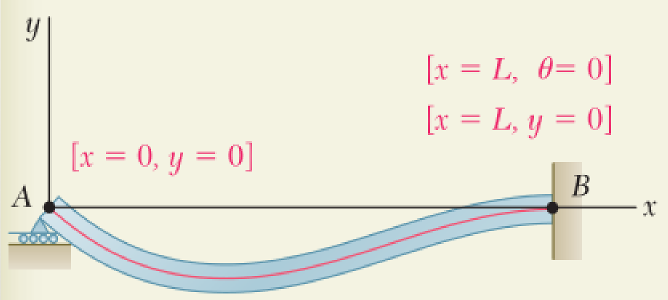
Example 10.8

(Beer, Page 565)

For the uniform beam AB, (a) determine the reaction at A, (b) derive the equation of the elastic curve, (c) determine the slope at A. (Note that the beam is statically indeterminate to the first degree.)

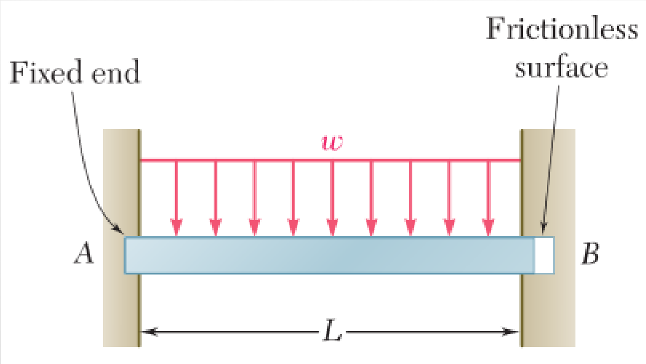


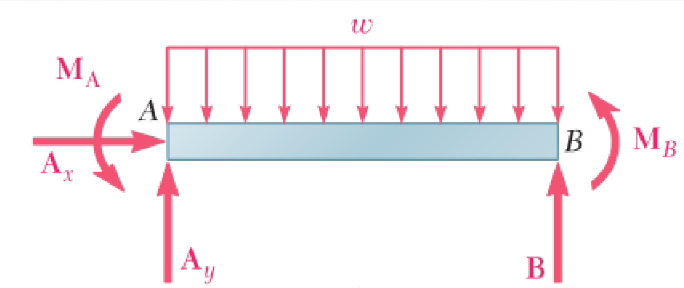


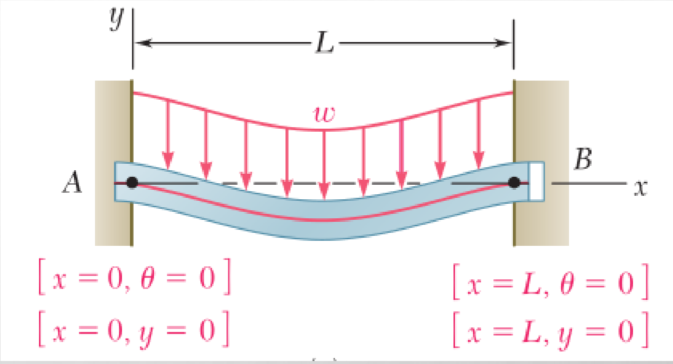


§ 10.5 Statically indeterminate beams

- In a statically indeterminate problem (to the second degree), the reactions can be obtained by considering the deformations of the structure involved.
- Altogether seven equations are available to determine the five reactions and the two constants of integration.



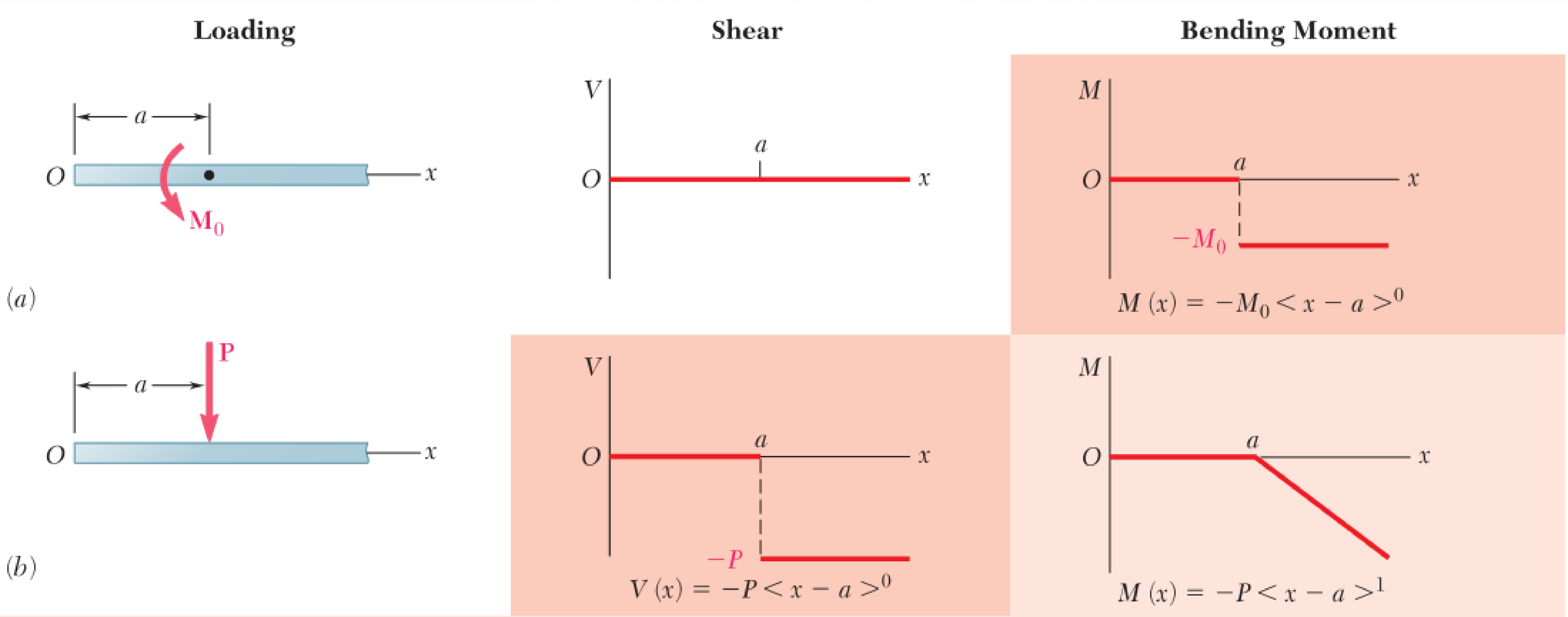




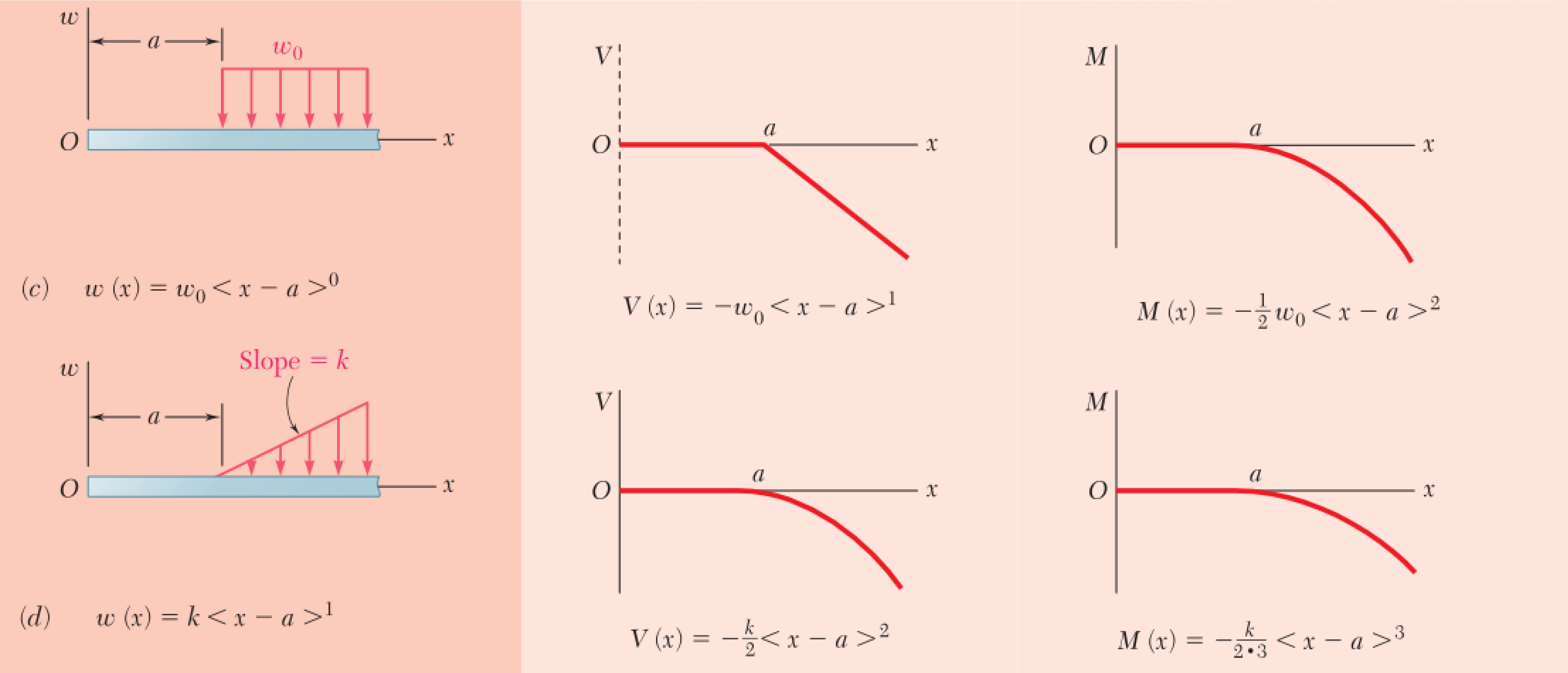
Four unknown reactions
but only three equilibrium equations

four equations may be obtained
from the boundary conditions

§ 10.6 Using singularity equation

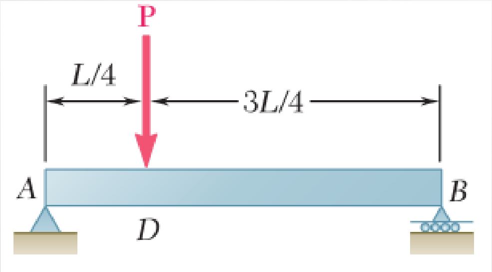


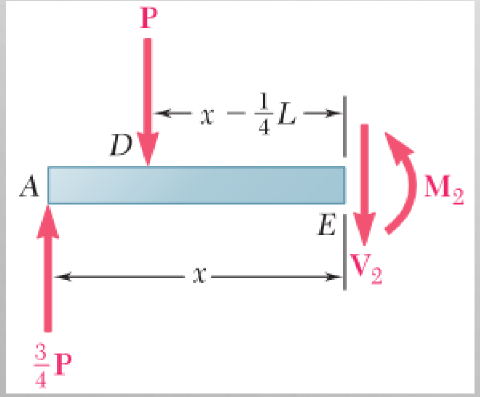
§ 10.6 Using singularity equation



§ 10.6 Using singularity equation

- The integration method provides a convenient and effective way of determining the slope and deflection at any point of a prismatic beam, as long as the bending moment can be represented by a single analytical function M(x).





$$V(x)=\frac{3P}{4}-P\left\langle x-\tfrac{1}{4}L\right\rangle^0 \qquad M(x)=\frac{3P}{4}x-P\left\langle x-\tfrac{1}{4}L\right\rangle$$

$$EI\frac{d^2y}{dx^2}=M(x)=\frac{3P}{4}x-P\left\langle x-\tfrac{1}{4}L\right\rangle$$

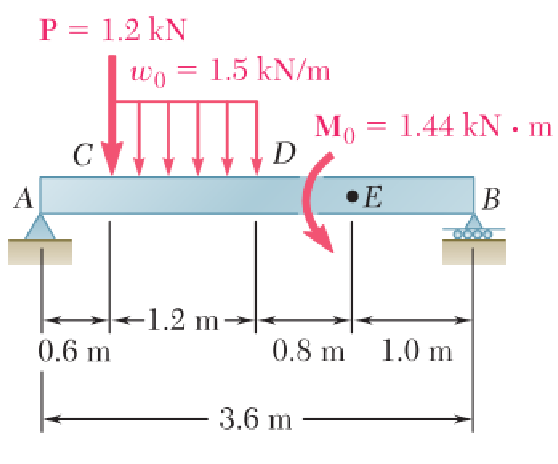
$$EI\theta=\frac{3P}{8}x^2-\frac{1}{2}P\left\langle x-\tfrac{1}{4}L\right\rangle^2+C_1$$

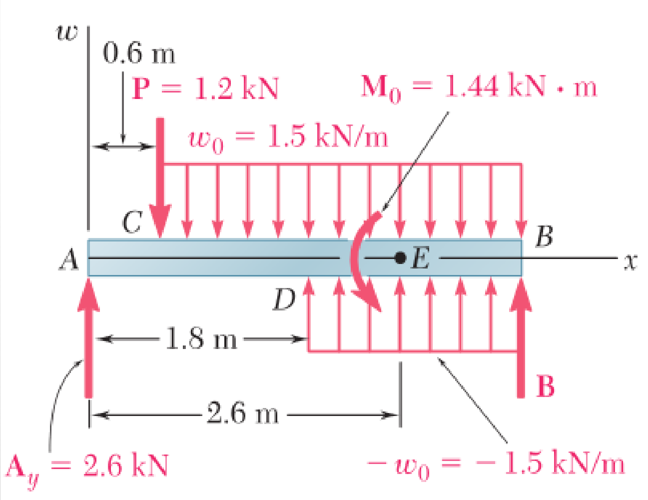
$$EIy=\frac{P}{8}x^3-\frac{1}{6}P\left\langle x-\tfrac{1}{4}L\right\rangle^3+C_1x+C_2$$

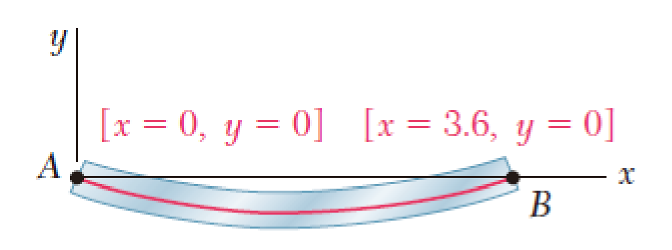
Example 10.9

(Beer, Page 573)

For the beam and loading and using singularity functions, (a) express the slope and deflection as functions of the distance x from the support at A, (b) determine the deflection at the midpoint D.



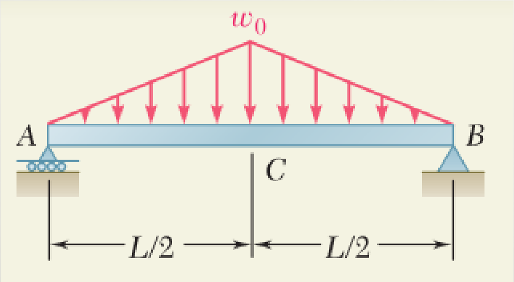


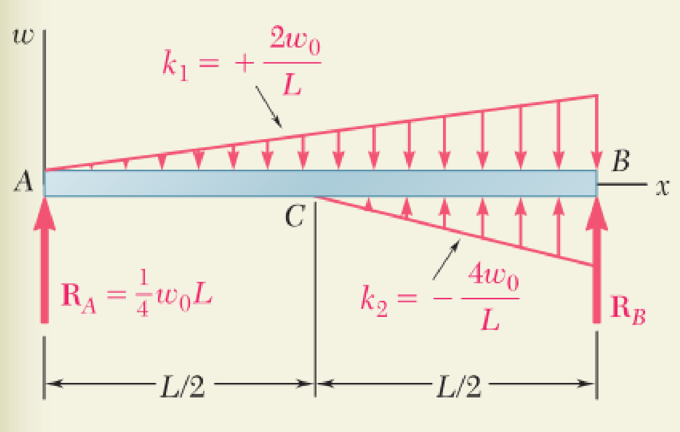


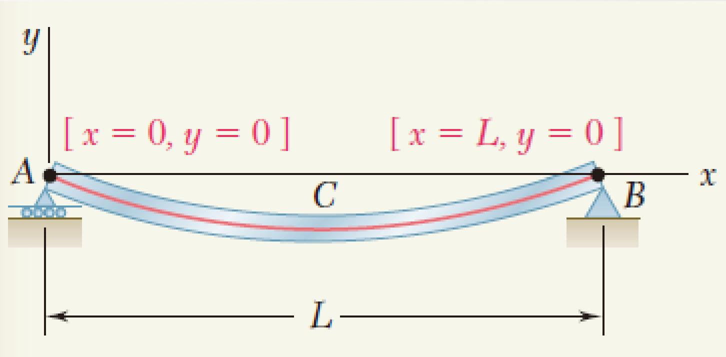
Example 10.10

(Beer, Page 574)

For the prismatic beam and loading shown, determine (a) the equation of the elastic curve, (b) the slope at A, (c) the maximum deflection.



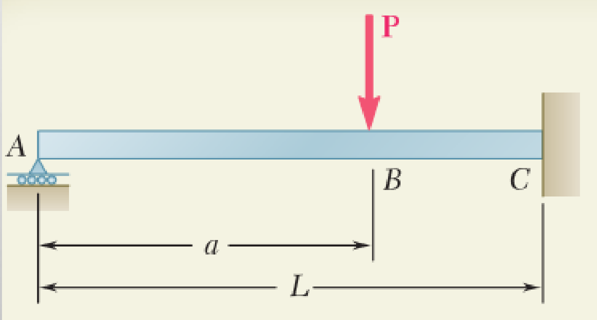


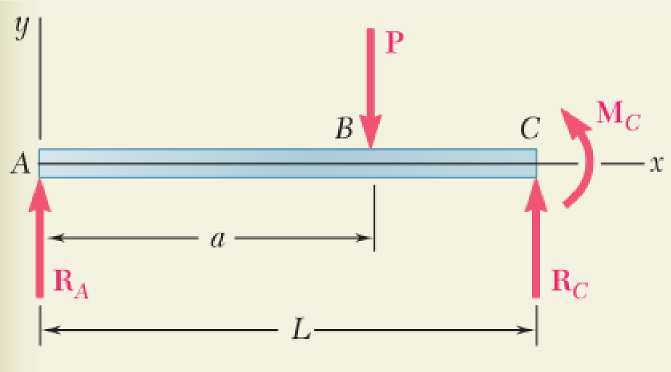


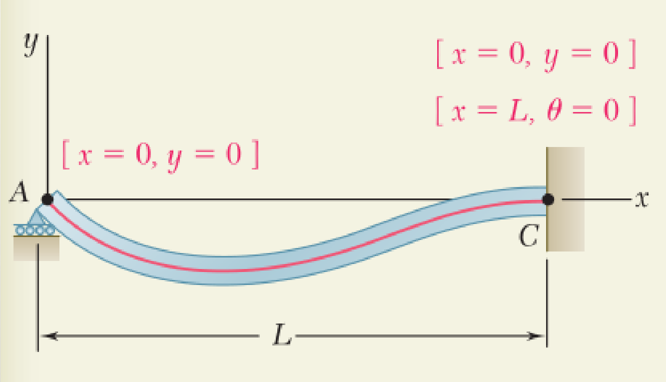
Example 10.11

(Beer, Page 576)

For the uniform beam ABC, (a) express the reaction at A in terms of P, L, a, E, and I, (b) determine the reaction at A and the deflection under the load when a = L/2.

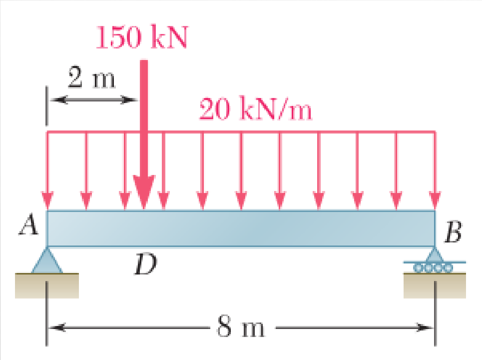


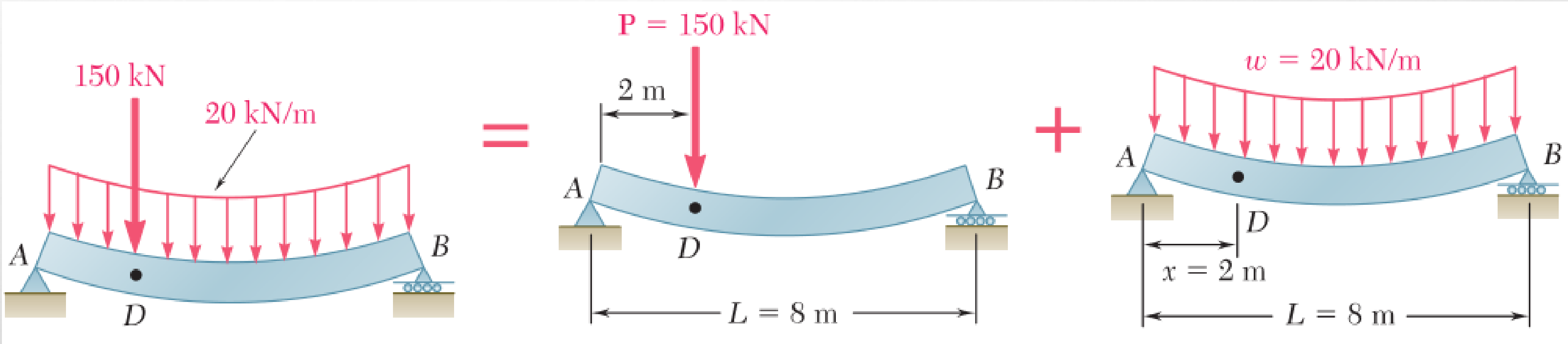




§ 10.7 Method of superposition

- When a beam is subjected to several concentrated or distributed loads, it is often found convenient to compute separately the slope and deflection caused by each of the given loads.



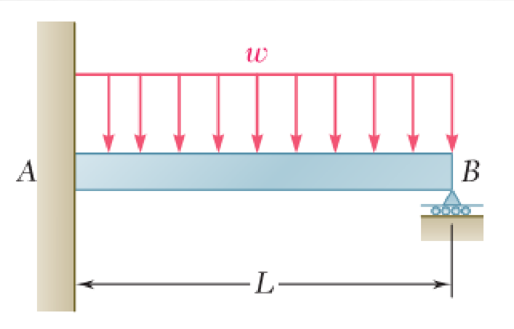


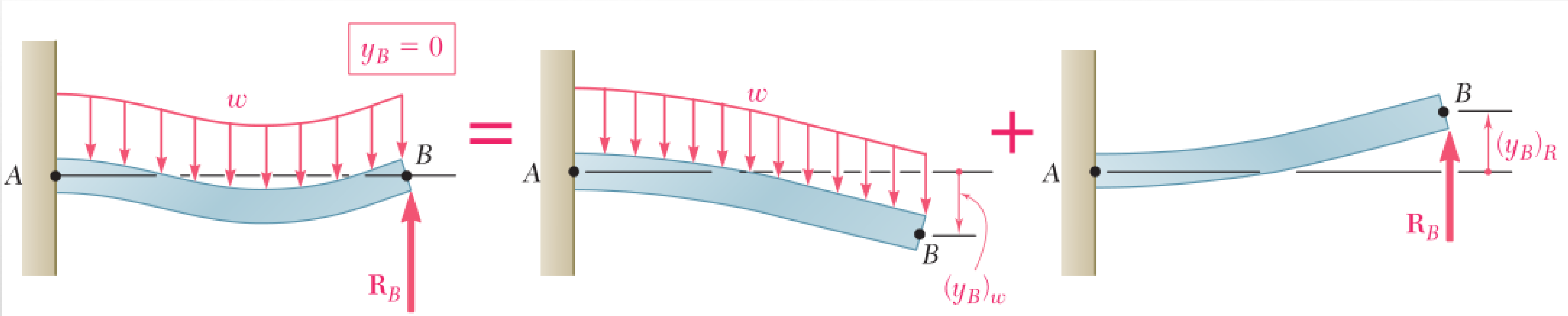
(Beer, Page 576)

Example 10.12

§ 10.8 Application of superposition

- We often find it convenient to use the method of superposition to determine the reactions at the supports of a statically indeterminate beam.



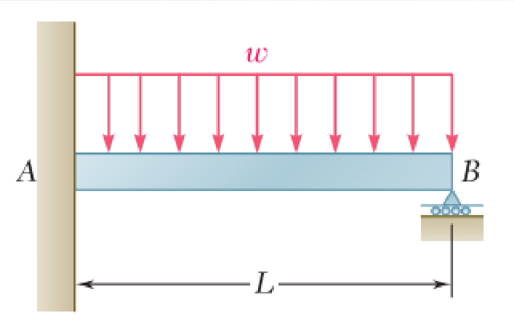


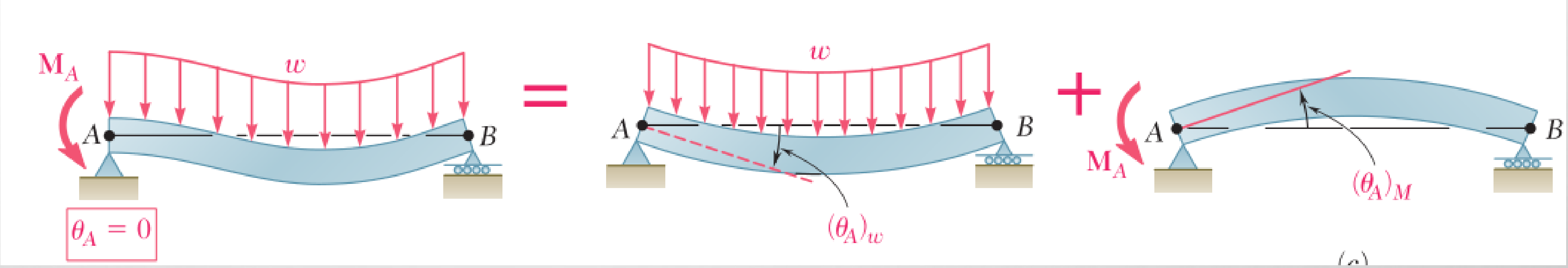
(Beer, Page 583)

Example 10.13

§ 10.8 Application of superposition

- Alternative Solution: We may consider the couple exerted at the fixed end A as redundant and replace the fixed end by a pin-and-bracket support.

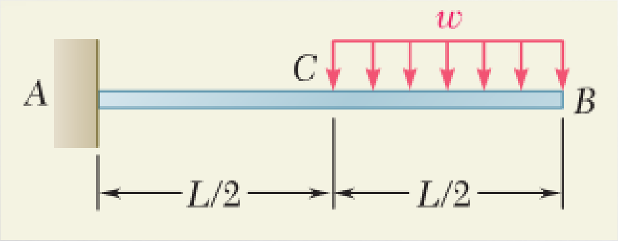


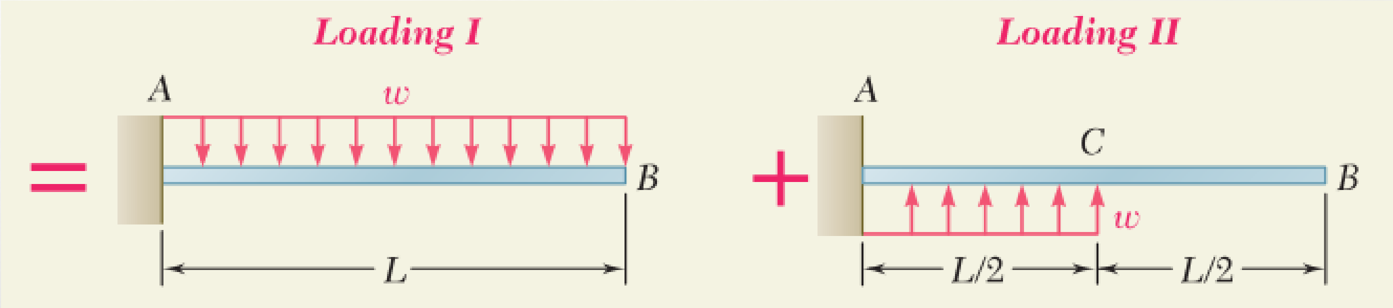


Example 10.14

(Beer, Page 585)

For the beam and loading shown, determine the slope and deflection at point B.

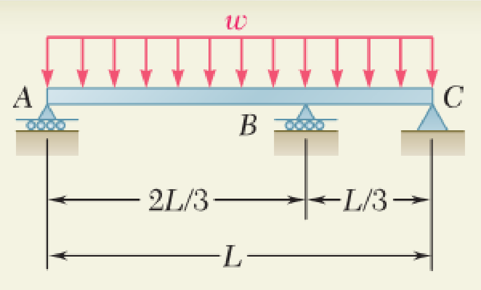


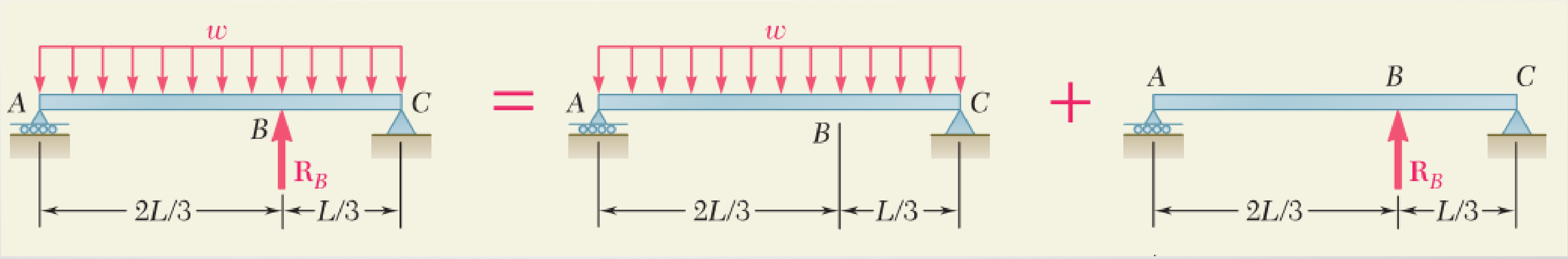


Example 10.15

(Beer, Page 586)

For the uniform beam and loading shown, determine (a) the reaction at each support, (b) the slope at end A.

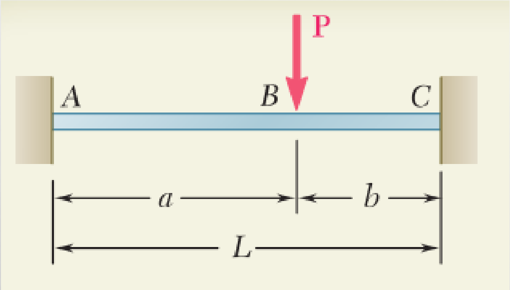


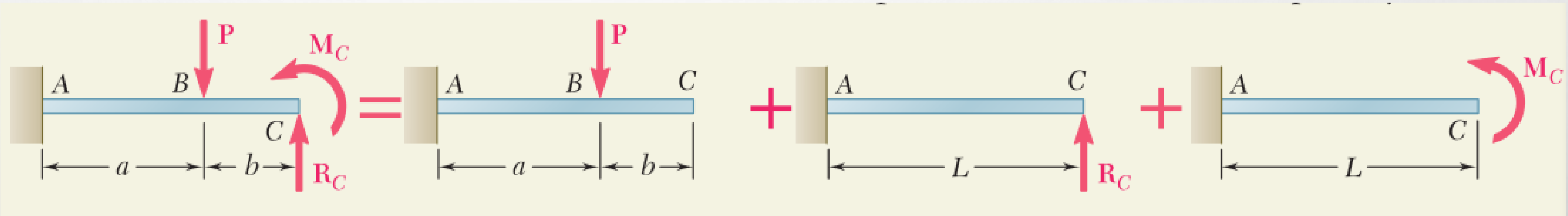


Example 10.16

(Beer, Page 587)

For the beam and loading shown, determine the reaction at the fixed support C.





§ 10.9 Summary

- **Deformation of a beam under transverse loading**
- **Elastic curve defined by different functions**
- **Statically indeterminate beams**
- **Use of singularity functions**
- **Method of superposition**