Exercise 1 on page 46

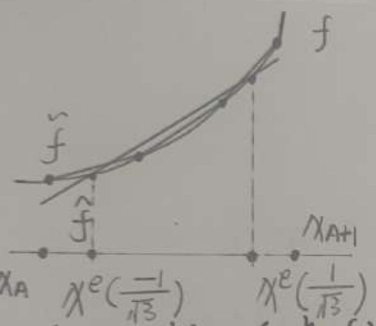
Solution

$f(x) = a_2x^2 + a_1x + a_0$ quadratic

$a(w,u) = (w,f) + w(0)h$ for $\forall w \in \mathcal{V}$

$a(w^h,u) = (w^h,f) + w^h(0)h$ for $w^h \in \mathcal{V}^h \subset \mathcal{V}$

$a(w^h,u^h) = (w^h,f) + w^h(0)h$ for (G) problem



$a(w^h, u-u^h) = (w^h, f) - (w^h, \hat{f})$

if nodally exact : $(w^h, f) = (w^h, \hat{f})$

$$\int_{-1}^{1} w^h f\, dx = \int_{-1}^{1} w^h \hat{f}\, dx \quad (*).$$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1}(a_2x^2 + a_1x + a_0)dx$$

$$= \frac{2}{3}a_2 + 2a_0$$

Choose two Gaussian points

$$w_1(a_2x_1^2 + a_1x_1 + a_0) + w_2(a_2x_2^2 + a_1x_2 + a_0) = \frac{2}{3}a_2 + 2a_0$$

$\Rightarrow w_1 = w_2 = 1$

$x_1 = \frac{1}{\sqrt{3}}, \; x_2 = -\frac{1}{\sqrt{3}}$

the two-point Gaussian rule can exactly integrate quadratic function.

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

$$\int_{-1}^{1} g(\xi)\,d\xi \simeq \sum_{l=1}^{n_{int}} w_l\, g(\xi_l)$$

for $n_{int} = 2$.

$$\int_{-1}^{1} g(\xi)\,d\xi = w_1 \cdot g(\xi_1) + w_2\, g(\xi_2)$$

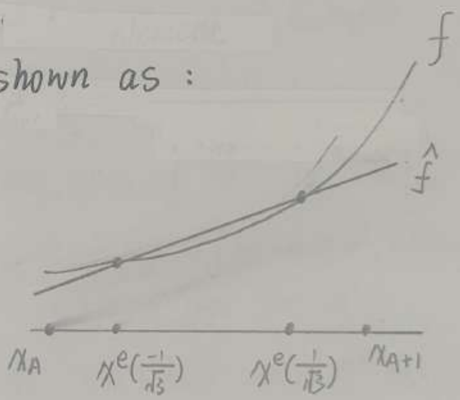
$$= 1 \cdot g(\tfrac{-1}{\sqrt{3}}) + 1 \cdot g(\tfrac{1}{\sqrt{3}})$$

$$x^e(\xi) = \sum_{a=1}^{2} x_a^e \hat{N}_a(\xi) = \frac{1}{2}(x_A + x_{A+1}) + \frac{1}{2}h_A\,\xi$$

$$x^e(\tfrac{-1}{\sqrt{3}}) = \frac{1}{2}(x_A + x_{A+1}) - \frac{1}{2\sqrt{3}}h_A$$

$$x^e(\tfrac{1}{\sqrt{3}}) = \frac{1}{2}(x_A + x_{A+1}) + \frac{1}{2\sqrt{3}}h_A$$

shown as :



Exercise 2 on Page 46

$$(S)\begin{cases} u_{,xx} - \lambda u + f = 0 \\ u(1) = g \\ -u_{,x}(0) = h \end{cases}$$

$$(W): \int_\Omega (w_{,x}u_{,x} + w\lambda u)\,dx = \int_\Omega wf\,dx + w(0)h$$

where $u \in \mathcal{S}$, $w \in \mathcal{V}$

$(S) \Rightarrow (W)$:

$$u_{,xx} - \lambda u + f = 0$$

$$w\cdot(u_{,xx} - \lambda u + f) = 0$$

$$wu_{,xx} - w\lambda u + wf = 0$$

$$\int_\Omega wu_{,xx}\,dx - \int_\Omega w\lambda u\,dx + \int_\Omega wf\,dx = 0$$

$$\left(-\int_\Omega w_{,x}u_{,x}\,dx + wu_{,x}\Big|_0^1\right) - \int_\Omega w\lambda u\,dx + \int_\Omega wf\,dx = 0$$

Since $w \in \mathcal{V}$, $w(1) = 0$

$$\int_\Omega (w_{,x}u_{,x} + w\lambda u)\,dx = \int_\Omega wf\,dx + w(1)\underline{u_{,x}(1)} - w(0)\underbrace{u_{,x}(0)}_{-h}$$

$$\int_\Omega (w_{,x}u_{,x} + w\lambda u)\,dx = \int_\Omega wf\,dx + w(0)h$$

where $w \in \mathcal{V}$, $u \in \mathcal{S}$ $(u(1) = g)$

$$a(w,u) + (w,\lambda u) = (w,f) + w(0)h$$

(i) $u^h = v^h + g^h$

Galerkin counterpart of (W):

$$a(w^h,u^h) + (w^h,\lambda u^h) = (w^h,f) + w^h(0)h$$

$$a(w^h,v^h) + a(w^h,g^h) + (w^h,\lambda v^h) + (w^h,\lambda g^h) = (w^h,f) + w^h(0)h$$

$$a(w^h,v^h) + \boxed{(w^h,\lambda v^h)} = (w^h,f) + w^h(0)h - a(w^h,g^h) - \boxed{(w^h,\lambda g^h)}$$

(ii) Remind that:

$$\mathcal{V}^h \ni w^h(x) = \sum_{A=1}^n c_A N_A(x)$$

$$\mathcal{V}^h \ni v^h(x) = \sum_{B=1}^n d_B N_B(x)$$

the basis of $\mathcal{V}^h$ is $\{N_A\}_{A=1}^n$

$\mathcal{V}^h \subset \mathcal{V} = \{w : w \in H^1, w(1) = 0\}$

$\therefore N_A(1) = 0$ for all A's

Set $N_{n+1}(1) = 1$, then $g^h(x) = gN_{n+1}(x)$

the equation in (i) can be written as:

$$a\Big(\sum_{A=1}^n c_AN_A, \sum_{B=1}^n d_BN_B\Big) + \Big(\sum_{A=1}^n c_AN_A, \lambda\sum_{B=1}^n d_BN_B\Big) = \Big(\sum_{A=1}^n c_AN_A, f\Big) + \sum_{A=1}^n c_AN_A(0)\cdot h - a\Big(\sum_{A=1}^n c_AN_A, gN_{n+1}\Big) - \Big(\sum_{A=1}^n c_AN_A, \lambda gN_{n+1}\Big)$$

$$a\Big(\sum_{A=1}^n c_AN_A, \sum_{B=1}^n d_BN_B\Big) + \Big(\sum_{A=1}^n c_AN_A, \lambda\sum_{B=1}^n d_BN_B\Big) - \Big(\sum_{A=1}^n c_AN_A, f\Big) - \sum_{A=1}^n c_AN_A(0)\cdot h + a\Big(\sum_{A=1}^n c_AN_A, gN_{n+1}\Big) + \Big(\sum_{A=1}^n c_AN_A, \lambda gN_{n+1}\Big) = 0$$

$$\sum_{A=1}^{n} C_A \left\{ a(N_A, \sum_{B=1}^{n} d_B N_B) + (N_A, \lambda \sum_{B=1}^{n} d_B N_B) - (N_A, f) - N_A(0)\cdot h + a(N_A, gN_{n+1}) + (N_A, \lambda g N_{n+1}) \right\} = 0$$

$$a(N_A, \sum_{B=1}^{n} d_B N_B) + (N_A, \lambda \sum_{B=1}^{n} d_B N_B) - (N_A, f) - N_A(0)\cdot h + a(N_A, gN_{n+1}) + (N_A, \lambda g N_{n+1}) = 0$$

$$a(N_A, \sum_{B=1}^{n} d_B N_B) + (N_A, \lambda \sum_{B=1}^{n} d_B N_B) = (N_A, f) + N_A(0)\cdot h - a(N_A, gN_{n+1}) + (N_A, \lambda g N_{n+1})$$

$$\sum_{B=1}^{n} a(N_A, N_B) d_B + \sum_{B=1}^{n} (N_A, N_B)\lambda d_B = (N_A, f) + N_A(0)\cdot h - a(N_A, N_{n+1})\cdot g + (N_A, N_{n+1})\cdot \lambda g$$

$$\sum_{B=1}^{n} \left[a(N_A, N_B) + (N_A, \lambda N_B) \right] d_B = F_A$$

$$\sum_{B=1}^{n} k_{AB}\, d_B = F_A$$

$$\therefore\ k_{AB} = a(N_A, N_B) + \boxed{(N_A, \lambda N_B)}$$

in element point of view

$$k^e_{AB} = a(N_A, N_B)^e + (N_A, \lambda N_B)^e$$

$k^e_{AB} \neq 0$ only if $A = e, e+1$
$B = e, e+1$

So that $k^e = [k_{ab}]_{2\times 2}$

$$k^e_{ab} = a(N_a, N_b)^e + \boxed{(N_a, \lambda N_b)^e} \qquad 1 \le a, b \le 2$$

(iii) $k^e = [k_{ab}]_{2\times 2}$

$$k^e_{ab} = a(N_a, N_b)^e + (N_a, \lambda N_b)^e = \int_{x_1^e}^{x_2^e} N_{a,x} N_{b,x}\, dx + \lambda \int_{x_1^e}^{x_2^e} N_a N_b\, dx$$

in text book: $\int_{-1}^{1} N_a N_b\, d\xi = \frac{1+\delta_{ab}}{3}$

$$= \int_{\xi_1}^{\xi_2} N_{a,\xi}\cdot \xi_{,x} \cdot N_{b,\xi} \cdot \underbrace{\xi_{,x}\cdot x_{,\xi}}_{+1\ (反函数)}\, d\xi + \lambda \int_{\xi_1}^{\xi_2} N_a(\xi) N_b(\xi)\, x_{,\xi}\, d\xi$$

$$= \int_{-1}^{1} \frac{1}{2}(-1)^a \cdot \frac{2}{h^e} \cdot \frac{1}{2}(-1)^b\, d\xi + \lambda \int_{-1}^{1} N_a(\xi) N_b(\xi)\, d\xi \cdot \frac{h_e}{2}$$

$$= \frac{(-1)^{a+b}}{h^e} + \frac{\lambda h_e}{6}(1+\delta_{ab}) \qquad 1 \le a, b \le 2$$

So that
$$k^e = [k_{ab}] = \begin{bmatrix} \frac{1}{h^e} + \frac{\lambda h_e}{3} & \frac{-1}{h^e} + \frac{\lambda h_e}{6} \\ \frac{-1}{h^e} + \frac{\lambda h_e}{6} & \frac{1}{h_e} + \frac{\lambda h_e}{3} \end{bmatrix}$$

(iv) $K = [k_{AB}]_{n\times n}$

where $k_{AB} = a(N_A, N_B) + (N_A, \lambda N_B)$

Since $a(m,n) = a(n,m)$

$(m,n) = (n,m)$

$\therefore\ a(N_A, N_B) = a(N_B, N_A)$

$(N_A, \lambda N_B) = (\lambda N_B, N_A) = \lambda(N_B, N_A)$

$= (N_B, \lambda N_A)$

$\Rightarrow\ k_{AB} = k_{BA}$.

$\therefore\ K = K^T$, K is symmetric

(v.)

positive: $\forall \vec{c} \in R^n$, $\vec{c}^T K \vec{c} \geq 0$.

Set $\vec{c} = \{C_A\}_{A=1}^n$

$$\vec{c}^T K \vec{c} = \sum_{A=1}^n \sum_{B=1}^n C_A k_{AB} C_B$$

$$= \sum_{A=1}^n \sum_{B=1}^n C_A \left(\int_0^1 N_{A,x} N_{B,x} dx + \int_0^1 N_A \lambda N_B dx\right) C_B$$

$$= \sum_{A=1}^n \sum_{B=1}^n C_A \cdot \int_0^1 N_{A,x} N_{B,x} dx \cdot C_B$$

$$+ \sum_{A=1}^n \sum_{B=1}^n C_A \cdot \int_0^1 N_A N_B dx \cdot C_B \lambda$$

$$= \int_0^1 \sum_{A=1}^n C_A N_{A,x} \sum_{B=1}^n C_B N_{B,x} dx$$

$$+ \lambda \int_0^1 \sum_{A=1}^n C_A N_A \sum_{B=1}^n C_B N_B dx$$

Set $w^h = \sum_{A=1}^n C_A N_A$

原式 $= \int_0^1 (w^h_{,x})^2 dx + \lambda \int_0^1 (w^h)^2 dx \geq 0$

where λ is positive.

$\therefore\ K$ is positive

definite: for $\vec{c} \in R^n$, if $\vec{c}^T K \vec{c} = 0$
then $\vec{c} = \vec{0}$

Set $\vec{c}^T K \vec{c} = \int_0^1 (w^h_{,x})^2 dx + \lambda \int_0^1 (w^h)^2 dx = 0$

$\Rightarrow\ (w^h_{,x})^2 = (w^h)^2 = 0 \qquad (*)$

$w^h_{,x} = w^h = 0$

namely $w^h(x) = 0 = \sum_{A=1}^n C_A N_A$

$\Rightarrow \vec{c} = \{C_A\}_{A=1}^n = \vec{0}$

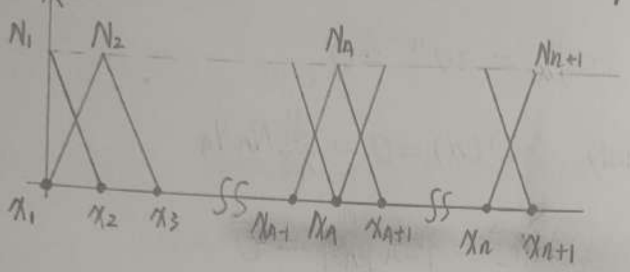
not necessary to employ $w^h(1) = 0$

since in $(*)$ we can get $w^h(x) = 0$

(vi) $g_{,xx} - \lambda g + \delta_y = 0$

$$g(x)=\begin{cases} c_1e^{px}+c_2e^{-px} & 0\le x\le y\\ c_3e^{px}+c_4e^{-px} & y\le x\le 1\end{cases}$$

piecewise linear finite element space



Case 1: if $y \neq x_A$, whice means

y is not on the nodes.

In this case, $g(x)$ can't be expressed by the space elements.

~~Case 2: if $y = x_A$, y on one of the nodes~~

$\cancel{g \in \mathcal{V}^h \subseteq \mathcal{V}}$

~~(S) ⇒ (W)~~

~~the progress has been shown before.~~

$\cancel{a(w,g)+(w,\lambda g) = (w,\delta_y) + w(0)\cdot h}$ $=0$

$\cancel{a(w,g)+(w,\lambda g) = (w,\delta_y) = w(y) \quad (W)}$

for (G) ~~problem~~

$\cancel{a(w^h,g^h)+(w^h,\lambda g^h) = (w^h,\delta_y) = w^h(y)}$

$\cancel{a(g,g^h)+(g,\lambda g^h) = (g,\delta_y)}$

$\cancel{a(g,g-g^h)+\lambda(g,g-g^h)=0}$

$\cancel{a(g-g^h,g)+(g-g^h,\lambda g)=0=(g-g^h,\delta_y)}$

$\cancel{=(g-g^h)(y)}$

$\cancel{g(x_A)-g^h(x_A)\neq 0}$?

$\cancel{u(x_A)-u^h(x_A)\neq 0}$?

$\cancel{w^h \in \mathcal{V}^h \subseteq \mathcal{V}}$

$\cancel{(W) \Rightarrow a(w^h,u)+(w^h,\lambda u)=(w^h,\delta_y)=w^h(y)}$

$\cancel{(G)\ a(w^h,u^h)+(w^h,\lambda u^h)=(w^h,\delta_y)=w^h(y)}$

$\cancel{\Rightarrow a(w^h,u-u^h)+(w^h,\lambda(u-u^h))=0}$ ①

~~By the defination of δ_y:~~

$\cancel{u(y)-u^h(y)=(u-u^h,\delta_y)}$

~~with (W): $u(y)-u^h(y)$~~

$\cancel{=(u-u^h,\delta_y)}$

$\cancel{=a(u-u^h,g)+(u-u^h,\lambda g)}$ ②

~~Since $y=x_A$,~~

$\cancel{g\in\mathcal{V}^h\subseteq\mathcal{V}}$

~~from ① and ②:~~

$\cancel{u(x_A)-u^h(x_A)=0}$?

Case 2: if $y = x_A$, y on one of the nodes

$a(w^h,u)+(w^h,\lambda u)=(w^h,\delta_y)$

$a(w^h,u^h)+(w^h,\lambda u^h)=(w^h,\delta_y)$

$\Rightarrow a(w^h,u-u^h)+\lambda(w^h,u-u^h)=0$

Although y is on one of the nodes $g(x)$ is curve and can't be expressed by the piecewise linear finite element space.

(g can't be picked to replace w^h)

(vii)

exponential element shape function $N_1(x), N_2(x)$

$$u^h(x) = d_1^e N_1(x) + d_2^e N_2(x)$$
$$= C_1 e^{px} + C_2 e^{-px}$$

$d_1^e = u^h(x_1^e)$

$d_2^e = u^h(x_2^e)$

$$u^h(x) = u^h(x_1^e) N_1(x) + u^h(x_2^e) N_2(x)$$

attribute : nodally exact solutions ?

can be attained by piecewise linear finite element space.