FEM HW7

Exercise 1 on Page 81

proof

$$a(w,u)=\int_\Omega w_{(i,j)}C_{ijkl}u_{(k,l)}\,d\Omega$$

$$w_{(i,j)}=\frac{w_{i,j}+w_{j,i}}{2}=w_{(j,i)}$$

$$u_{(k,l)}=\frac{u_{k,l}+u_{l,k}}{2}=u_{(l,k)}$$

$C_{ijkl}=C_{klij}$ major symmetry

$=C_{jikl}$
$=C_{ijlk}$ minor symmetries

$$a(u,w)=\int_\Omega u_{(k,l)}C_{klij}u_{(i,j)}\,d\Omega=a(w,u)\quad\blacksquare$$

$$(w,f)=\int_\Omega w_i f_i\,d\Omega$$
$$(f,w)=\int_\Omega f_i w_i\,d\Omega$$
$\Rightarrow (w,f)=(f,w)\quad\blacksquare$

$$(w,h)_\Gamma=\sum_{i=1}^{n_{sd}}\left(\int_{\Gamma_{h_i}}w_ih_i\,d\Gamma\right)$$
$$=\int_{\Gamma_{h_1}}w_1h_1\,d\Gamma+\cdots+\int_{\Gamma_{h_{nsd}}}w_{n_{sd}}h_{n_{sd}}\,d\Gamma$$
$$=\int_{\Gamma_{h_1}}h_1w_1\,d\Gamma+\cdots+\int_{\Gamma_{h_{nsd}}}h_{n_{sd}}w_{n_{sd}}\,d\Gamma$$
$$=\sum_{i=1}^{n_{sd}}\left(\int_{\Gamma_{h_i}}h_iw_i\,d\Gamma\right)=(h,w)_\Gamma\quad\blacksquare$$

$$a(c_1u+c_2v,w)=\int_\Omega(c_1u+c_2v)_{(i,j)}C_{ijkl}u_{(k,l)}\,d\Omega$$

$$(c_1u+c_2v)_{(i,j)}=\frac{(c_1u+c_2v)_{i,j}+(c_1u+c_2v)_{j,i}}{2}$$
$$=\frac{c_1u_{i,j}+c_2v_{i,j}+c_1u_{j,i}+c_2v_{j,i}}{2}$$
$$=\frac{c_1(u_{i,j}+u_{j,i})+c_2(v_{i,j}+v_{j,i})}{2}$$
$$=c_1u_{(i,j)}+c_2v_{(i,j)}$$

$$a(c_1u+c_2v,w)=c_1a(u,w)+c_2a(v,w)\quad\blacksquare$$

$$(c_1u+c_2v,w)=\int_\Omega(c_1u+c_2v)w\,d\Omega$$
$$=\int_\Omega c_1uw\,d\Omega+\int c_2vw\,d\Omega$$
$$=c_1(u,w)+c_2(v,w)\quad\blacksquare$$

$$(c_1u+c_2v,h)_\Gamma=\sum_{i=1}^{n_{sd}}\int_{\Gamma_{h_i}}\left[(c_1u+c_2v)_ih_i\,d\Gamma\right]$$
$$=\sum_{i=1}^{n_{sd}}\left[\int_{\Gamma_{h_i}}(c_1u)_ih_i\,d\Gamma+\int_{\Gamma_{h_i}}(c_2v)_ih_i\,d\Gamma\right]$$
$$=\sum_{i=1}^{n_{sd}}\left(c_1\int_{\Gamma_{h_i}}u_ih_i\,d\Gamma+c_2\int_{\Gamma_{h_i}}v_ih_i\,d\Gamma\right)$$
$$=c_1(u,h)_\Gamma+c_2(v,h)_\Gamma\quad\blacksquare$$

Exercise 2 on page 82

Solution

$$w_{(i,j)} C_{ijkl} u_{(k,l)} = \varepsilon(w)^T D \varepsilon(u)$$

$$n_{sd} = 2 \ , \quad 1 \le i,j,k,l \le 2$$

$$\varepsilon(u) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{Bmatrix} = \begin{Bmatrix} u_{(1,1)} \\ u_{(2,2)} \\ 2u_{(1,2)} \end{Bmatrix}$$

$$\varepsilon(w) = \begin{Bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2}+w_{2,1} \end{Bmatrix} = \begin{Bmatrix} w_{(1,1)} \\ w_{(2,2)} \\ 2w_{(1,2)} \end{Bmatrix}$$

$$D = [D_{IJ}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ \cdot & D_{22} & D_{23} \\ \cdot & \cdot & D_{33} \end{bmatrix}, \ D_{IJ} = C_{ijkl}$$

$$\varepsilon(w)^T D \varepsilon(u)$$

$$= \begin{bmatrix} w_{1,1} & w_{2,2} & w_{1,2}+w_{2,1} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ \cdot & D_{22} & D_{23} \\ \cdot & \cdot & D_{33} \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{bmatrix}$$

$$= \begin{bmatrix} w_{1,1}D_{11} + w_{2,2}D_{12} + (w_{1,2}+w_{2,1})D_{13} \\ w_{1,1}D_{12} + w_{2,2}D_{22} + (w_{1,2}+w_{2,1})D_{23} \\ w_{1,1}D_{13} + w_{2,2}D_{23} + (w_{1,2}+w_{2,1})D_{33} \end{bmatrix}^T \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{bmatrix}$$

$$= w_{1,1}D_{11}u_{1,1} + w_{2,2}D_{12}u_{1,1} + (w_{1,2}+w_{2,1})D_{13}u_{1,1}$$
$$+ w_{1,1}D_{12}u_{2,2} + w_{2,2}D_{22}u_{2,2} + (w_{1,2}+w_{2,1})D_{23}u_{2,2}$$
$$+ w_{1,1}D_{13}(u_{1,2}+u_{2,1}) + w_{2,2}D_{23}(u_{1,2}+u_{2,1}) + (w_{1,2}+w_{2,1})D_{33}(u_{1,2}+u_{2,1})$$

$$D_{IJ} = C_{ijkl}$$

$$w_{(i,j)} = \frac{w_{i,j}+w_{j,i}}{2}$$

$$u_{(k,l)} = \frac{u_{k,l}+u_{l,k}}{2}$$

from Table 2.7.1 on textbook

$$w_{(1,1)}D_{11}u_{(1,1)} + w_{(2,2)}D_{12}u_{(1,1)} + 2w_{(2,1)}D_{13}u_{(1,1)}$$
$$+ w_{(1,1)}D_{12}u_{(2,2)} + w_{(2,2)}D_{22}u_{(2,2)} + 2w_{(1,2)}D_{23}u_{(2,2)}$$
$$+ w_{(1,1)}D_{13}\cdot 2u_{(1,2)} + w_{(2,2)}D_{23}u_{(1,2)}\cdot 2 + 4w_{(1,2)}D_{33}u_{(1,2)}$$

$$= \begin{bmatrix} w_{(1,1)} & w_{(2,2)} & 2w_{(1,2)} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ \cdot & D_{22} & D_{23} \\ \cdot & \cdot & D_{33} \end{bmatrix} \begin{bmatrix} u_{(1,1)} \\ u_{(2,2)} \\ 2u_{(1,2)} \end{bmatrix}$$

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1112} \\ \cdot & C_{2222} & C_{2212} \\ \cdot & \cdot & C_{1212} \end{bmatrix}$$

$$= \begin{bmatrix} w_{(1,1)}C_{1111} + w_{(2,2)}C_{1122} + 2w_{(1,2)}C_{1112} \\ w_{(1,1)}C_{1122} + w_{(2,2)}C_{2222} + 2w_{(1,2)}C_{2212} \\ w_{(1,1)}C_{1112} + w_{(2,2)}C_{2212} + 2w_{(1,2)}C_{1212} \end{bmatrix}^T \begin{bmatrix} u_{(1,1)} \\ u_{(2,2)} \\ 2u_{(1,2)} \end{bmatrix}$$

$$= w_{(1,1)}C_{1111}u_{(1,1)} + w_{(2,2)}C_{1122}u_{(1,1)} + 2w_{(1,2)}C_{1112}u_{(1,1)}$$
$$+ w_{(1,1)}C_{1122}u_{(2,2)} + w_{(2,2)}C_{2222}u_{(2,2)} + 2w_{(1,2)}C_{2212}u_{(2,2)}$$
$$+ 2w_{(1,1)}C_{1112}u_{(1,2)} + 2w_{(2,2)}C_{2212}u_{(1,2)} + 4w_{(1,2)}C_{1212}u_{(1,2)}$$

$$= w_{(i,j)}C_{ijkl}u_{(k,l)}$$ with $1 \le i,j,k,l \le 2$

i/k	j/l
1	1
1	2
2	1
2	2

Exercise 3 on Page 82

Solution

for $n_{sd} = 3$ $\quad \frac{3\times(3+1)}{2}$

$$\varepsilon(u) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{1,2}+u_{2,1} \end{Bmatrix}$$

I, J indices take on values: $1, 2, \cdots, 6$

$1 \le i, j, k, l \le n_{sd} = 3$

$$\varepsilon(w) = \begin{Bmatrix} w_{1,1} \\ w_{2,2} \\ w_{3,3} \\ w_{2,3}+w_{3,2} \\ w_{1,3}+w_{3,1} \\ w_{1,2}+w_{2,1} \end{Bmatrix}$$

$$D = [D_{IJ}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ & & D_{33} & D_{34} & D_{35} & D_{36} \\ & \text{Symmetric} & & D_{44} & D_{45} & D_{46} \\ & & & & D_{55} & D_{56} \\ & & & & & D_{66} \end{bmatrix}$$

$w_{(i,j)} C_{ijkl} u_{(k,l)} = \varepsilon(w)^T D \, \varepsilon(u)$

$D_{IJ} = C_{ijkl}$

⇒

I/J	i/k	j/l
1	1	1
2	2	2
3	3	3
4	2	3
4	3	2
5	1	3
5	3	1
6	1	2
6	2	1

Exercise 4 on Page 82

Proof

$n_{sd} = 2$

$$D\varepsilon(u) = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ \cdot & D_{22} & D_{23} \\ \cdot & \cdot & D_{33} \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{Bmatrix}$$

$$= \begin{bmatrix} D_{11}u_{1,1} + D_{12}u_{2,2} + D_{13}(u_{1,2}+u_{2,1}) \\ D_{12}u_{1,1} + D_{22}u_{2,2} + D_{23}(u_{1,2}+u_{2,1}) \\ D_{13}u_{1,1} + D_{23}u_{2,2} + D_{33}(u_{1,2}+u_{2,1}) \end{bmatrix}$$

through generalized Hooke's law:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} = D_{IJ} \cdot \varepsilon_{kl}$$

$\sigma_{11} = D_{IJ}\varepsilon_{kl} = D_{11}\varepsilon_{11} + D_{12}\varepsilon_{22} + D_{13}(\varepsilon_{12}+\varepsilon_{21})$

where $\varepsilon_{kl} = u_{(k,l)} = \frac{u_{k,l}+u_{l,k}}{2}$

$\varepsilon_{11} = u_{1,1}$, $\varepsilon_{22} = u_{2,2}$, $\varepsilon_{12}+\varepsilon_{21} = u_{1,2}+u_{2,1}$

$\therefore \sigma_{11} = D_{11}u_{1,1} + D_{12}u_{2,2} + D_{13}(u_{1,2}+u_{2,1})$

Similarly, $\sigma_{12} = D_{13}u_{1,1} + D_{23}u_{3,2} + D_{33}(u_{1,2}+u_{2,1})$

$\sigma_{22} = D_{12}u_{1,1} + D_{22}u_{2,2} + D_{23}(u_{1,2}+u_{2,1})$

$$\Rightarrow \sigma = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = D\varepsilon(u) \quad \blacksquare$$

$n_{sd} = 3$,

$1 \le I, J \le 6$; $1 \le i, j, k, l \le 3$

$$D\varepsilon(u) = [D_{IJ}] \cdot \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{1,2}+u_{2,1} \end{Bmatrix} = [D_{IJ}] \cdot u_{(k,l)}$$

$u_{(k,l)} = \varepsilon_{kl} = \frac{u_{k,l}+u_{l,k}}{2}$

generalized Hooke's law

$D_{IJ} = C_{ijkl}$

$\Rightarrow D_{IJ} \cdot \varepsilon_{kl} = \boxed{\sigma_{ij} = C_{ijkl} \cdot \varepsilon_{kl}}$

$1 \le i, j \le 3$, $1 \le k, l \le 3$, $1 \le I, J \le 6$

$\Rightarrow \sigma = [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{23}\ \sigma_{13}\ \sigma_{12}]^T = [D_{IJ}] \cdot [\varepsilon_{kl}]$

$= D \cdot \varepsilon(u)$

Exercise 5 on Page 83

Solution

$n_{sd} = 2.$

$$D = [D_{IJ}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ \cdot & D_{22} & D_{23} \\ \cdot & \cdot & D_{33} \end{bmatrix} = [C_{ijkl}]$$

$$D_{11} = C_{1111} = \mu(\delta_{11}\delta_{11} + \delta_{11}\delta_{11}) + \lambda\delta_{11}\delta_{11}$$
$$= 2\mu + \lambda$$

$$D_{12} = C_{1122} = \mu(\delta_{12}\delta_{12} + \delta_{12}\delta_{12}) + \lambda\delta_{11}\delta_{22}$$
$$= \lambda$$

$$D_{13} = C_{1112} = \mu(\delta_{11}\delta_{12} + \delta_{12}\delta_{11}) + \lambda\delta_{11}\delta_{12}$$
$$= C_{1121} = 0$$

$$D_{22} = C_{2222} = \mu(\delta_{22}\delta_{22} + \delta_{22}\delta_{22}) + \lambda\delta_{22}\delta_{22}$$
$$= 2\mu + \lambda$$

$$D_{23} = C_{2212} = \mu(\delta_{21}\delta_{22} + \delta_{22}\delta_{21}) + \lambda\delta_{22}\delta_{12}$$
$$= C_{2221} = 0$$

$$D_{33} = C_{1212} = C_{2112} = \mu(\delta_{11}\delta_{22} + \delta_{12}\delta_{21}) + \lambda\delta_{12}\delta_{12}$$
$$= C_{1221} = C_{2121} = \mu$$

$$\therefore D = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ & \lambda+2\mu & 0 \\ \text{symmetric} & & \mu \end{bmatrix}$$

∎

$n_{sd} = 3$ ijkl kjil iljk ijkl

$$D_{11} = C_{1111} = 2\mu + \lambda$$

$$D_{12} = C_{1122} = \mu(0+0) + \lambda = \lambda$$

$$D_{13} = C_{1133} = \mu(0+0) + \lambda = \lambda$$

$$D_{14} = C_{1123} = \mu(0+0) + \lambda \cdot 0 = 0$$
$$= C_{1132}$$

$$D_{15} = C_{1113} = \mu(0+0) + \lambda \cdot 0 = 0$$
$$= C_{1131}$$

$$D_{16} = C_{1112} = \mu(0+0) + \lambda \cdot 0 = 0$$
$$= C_{1121}$$

$$D_{22} = C_{2222} = 2\mu + \lambda$$

$$D_{23} = C_{2233} = \lambda$$

$$D_{24} = C_{2223} = C_{2232} = \mu(0+0) + \lambda \cdot 0 = 0.$$

$$D_{25} = C_{2213} = C_{2231} = \mu(0+0) + \lambda \cdot 0 = 0$$

$$D_{26} = C_{2212} = C_{2221} = \mu(0+0) + \lambda \cdot 0 = 0.$$

$$D_{33} = C_{3333} = 2\mu + \lambda$$

$$D_{34} = C_{3323} = C_{3332} = 0.$$

$$D_{35} = C_{3313} = C_{3331} = 0.$$

$$D_{36} = C_{3312} = C_{3321} = 0$$

$$D_{44} = C_{2323} = C_{2332} = C_{3232} = C_{3223}$$
$$= \mu \cdot (1+0) + \lambda \cdot 0 = \mu$$

$$D_{45} = C_{3213} = C_{2313} = C_{3231} = C_{2331} = 0$$

$$D_{46} = C_{3212} = C_{3221} = C_{2312} = C_{2321} = 0$$

$$D_{56} = C_{1312} = C_{1321} = C_{3121} = C_{3112} = 0$$

$$D_{55} = C_{1331} = C_{1313} = C_{3113} = C_{3131} = \mu$$

$$D_{66} = C_{1212} = C_{1221} = C_{2121} = C_{2112} = \mu$$

$$D = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ & & 2\mu+\lambda & 0 & 0 & 0 \\ & \text{symmetric} & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}$$

∎

Exercise 1 on Page 462

Solution

implementation 1: v-form

$$(M + \alpha \Delta t K) v_{n+1} = F_{n+1} - K \tilde{d}_{n+1}$$

implementation 2: d-form

$$v_{n+1} = \frac{d_{n+1} - \tilde{d}_{n+1}}{\alpha \Delta t}$$

$$\Rightarrow (M + \alpha \Delta t K) \frac{d_{n+1} - \tilde{d}_{n+1}}{\alpha \Delta t} = F_{n+1} - K \tilde{d}_{n+1}$$

$$(M + \alpha \Delta t K) d_{n+1} = \alpha \Delta t F_{n+1} - \alpha \Delta t K \tilde{d}_{n+1} + (M + \alpha \Delta t K) \tilde{d}_{n+1}$$

Since $\tilde{d}_{n+1} = d_n + (1-\alpha) \Delta t\, v_n$

$$\Rightarrow$$

$$(M + \alpha \Delta t K) d_{n+1} = \alpha \Delta t F_{n+1} + M \cdot [d_n + (1-\alpha) \Delta t\, v_n]$$

$$= \alpha \Delta t F_{n+1} + M d_n + M \cdot (1-\alpha) \Delta t \cdot v_n \qquad (\#)$$

from $M v_{n+1} + K d_{n+1} = F_{n+1}$

$$\Rightarrow \quad v_n = \frac{F_n - K d_n}{M} \quad \text{at } t_n$$

$$(\#) = \alpha \Delta t F_{n+1} + M d_n + (1-\alpha) \Delta t (F_n - K d_n)$$

$$= [M - (1-\alpha) \Delta t K] d_n + \Delta t [\alpha F_{n+1} + (1-\alpha) F_n] = (M + \alpha \Delta t K) d_{n+1} \qquad \blacksquare$$