1. Solution.

a) Dirichlet BC: $u(0)=g$

Neumann BC: $u_{,x}(1)=h$

b) essential BC: $u(0)=g$ (in the $\mathcal{S}$)

natural BC: $u_{,x}(1)=h$ (in the (S) problem

(W) problem equation.)

trial function space: $\mathcal{S}=\{u: u\in H^1, u(0)=g\}$

test function space: $\mathcal{V}=\{w: w\in H^1, w(0)=0\}$

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c) (S) $\Rightarrow$ (W)

$$u_{,xx}+f=0$$

$$w\cdot(u_{,xx}+f)=0 \qquad w\in\mathcal{V}$$

$$w\cdot u_{,xx}+wf=0$$

$$\int_\Omega w\cdot u_{,xx}\,dx+\int_\Omega wf\,dx=0$$

$$-\int_\Omega w_{,x}u_{,x}\,dx+w\cdot u_{,x}\Big|_0^1+\int_\Omega wf\,dx=0$$

$$-\int_\Omega w_{,x}u_{,x}\,dx+w(1)\,u_{,x}(1)-\overset{w(0)=0}{0}+\int_\Omega wf\,dx=0$$

$$\int_\Omega w_{,x}u_{,x}\,dx=\int_\Omega wf\,dx+w(1)\underbrace{u_{,x}(1)}_{\overset{\text{(S)}}{=}\,h} \qquad ■$$

where $u\in\mathcal{S}$ and $w\in\mathcal{V}$. ■

$u(0)=g$.

(W) $\Rightarrow$ (S)

$$\int_0^1 w_{,x}u_{,x}\,dx=\int_0^1 wf\,dx+w(1)h$$

$$-\int_0^1 wu_{,xx}+wu_{,x}\Big|_0^1=\int_0^1 wf\,dx+w(1)h$$

$$-\int_0^1 wu_{,xx}+w(1)u_{,x}(1)-0=\int_0^1 wf\,dx+w(1)h$$

$$\int_0^1 w(u_{,xx}+f)\,dx+w(1)[h-u_{,x}(1)]=0$$

set $w=\phi\cdot(u_{,xx}+f)$

where $\phi(0)=\phi(1)=0 \Rightarrow w(1)=0$

$\phi(x)>0$.

so that: $\int_0^1\phi\cdot(u_{,xx}+f)^2\,dx+0=0$

fundamental lemma.

$\Rightarrow u_{,xx}+f=0.$ ■

Euler-Lagrange eq $\Rightarrow w(1)[h-u_{,x}(1)]=0$.

$h-u_{,x}(1)=0$ ■

$u\in\mathcal{V}\Rightarrow u(0)=g$ ■

d) stiffness matrix K

(G) problem: $a(w^h, u^h) = (w^h, f) + w^h(1)h$ where $u^h = v^h + \underbrace{g^h}_{= g \cdot N_{n+1}}$, $N_{n+1}(1)=1$.

$$w^h = \sum_{A=1}^{n} C_A N_A.$$

$$v^h = \sum_{A=1}^{n} d_B N_B$$

$$a\left(\sum_{A=1}^{n} C_A N_A, \sum_{B=1}^{n} d_B N_B\right) = \left(\sum_{A=1}^{n} C_A N_A, f\right) + \sum_{A=1}^{n} C_A N_A(1)\cdot h - \left(\sum_{A=1}^{n} C_A N_A, g\cdot N_{n+1}\right).$$

$$\Rightarrow \sum_{B=1}^{n} \underbrace{a(N_A, N_B)}_{K_{AB}} d_B = (N_A, f) + N_A(1)h - (N_A, N_{n+1})\cdot g$$

$$[K\cdot d = F]$$

① symmetric: $K^T = K$

since $K_{AB} = a(N_A, N_B) = \int_\Omega N_{A,x}\cdot N_{B,x}\, dx$
$= \int_\Omega N_{B,x}\cdot N_{A,x}\, dx = a(N_B, N_A) = K_{BA}$

$\therefore K^T = K$

② positive: $\forall \vec{C} \in R^n$, $\vec{C}^T K \vec{C} \geq 0$

set $\vec{C} = \{C_A\}_{A=1}^{n}$

so that: $\vec{C}^T K \vec{C} = \sum_{A=1}^{n}\sum_{B=1}^{n} C_A K_{AB} C_B = \sum_{A=1}^{n}\sum_{B=1}^{n} C_A \int_\Omega N_{A,x} N_{B,x}\, dx\, C_B$

$$= \int_\Omega \sum_{A=1}^{n} C_A N_{A,x} \sum_{B=1}^{n} C_B N_{B,x}\, dx$$

set $w^h = \sum_{A=1}^{n} C_A N_A \Rightarrow \int_\Omega (w^h_{,x})^2\, dx \geq 0$

③ definite: for $\vec{C} \in R^n$, if $\vec{C}^T K \vec{C} = 0$, then $\vec{C} = 0$

$\vec{C}^T K \vec{C} = 0 = \int_\Omega (w^h_{,x})^2\, dx \Rightarrow w^h_{,x} = 0$

$w^h(x) = \text{Constant}.$

since $w \in \mathcal{V}$, $w(0) = 0 = \text{constant}$.

$\therefore w^h(x) = 0 = \sum_{A=1}^{n} C_A N_A \Rightarrow \vec{C} = 0.$

线性无关

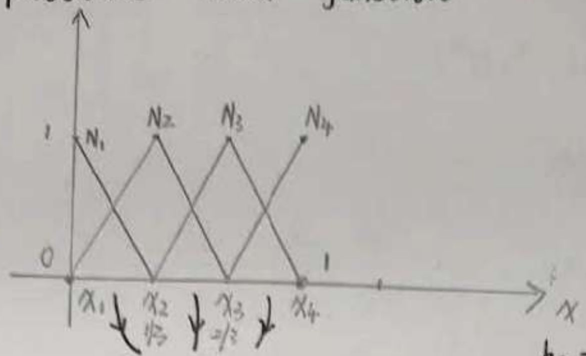
Why these properties important:

1) symmetric is the condition of positive and definite.
2) positive and definite make sure the matrix invertable (K^{-1})
3) $d = K^{-1}\cdot F$ can be obtained through 1) and 2).

+3

e) $x_1 = 0$, $x_2 = 1/3$, $x_3 = 2/3$, $x_4 = 1$.

piecewise linear function



3 linear elements

4 nodes.

$h_A = x_{A+1} - x_A$

$h_1 = h_2 = h_3 = h_4 = 1/3$

for $A = 1$

$$N_1(x) = \begin{cases} \dfrac{-x + x_2}{x_2 - x_1} = \dfrac{-x + x_2}{h_1} & 0 \le x \le x_2 \\ 0 & \text{else} \end{cases}$$

for $A = 4$

$$N_4(x) = \begin{cases} \dfrac{x - x_3}{x_4 - x_3} = \dfrac{x - x_3}{h_3} & x_3 \le x \le x_4 \\ 0 & \text{else} \end{cases}$$

for $A = 2, 3$

$$N_A(x) = \begin{cases} \dfrac{x - x_{A-1}}{x_A - x_{A-1}} = \dfrac{x - x_{A-1}}{h_{A-1}} & x_{A-1} \le x \le x_A \\ \dfrac{-x + x_{A+1}}{x_{A+1} - x_A} = \dfrac{-x + x_{A+1}}{h_A} & x_A \le x \le x_{A+1} \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \quad N_1(x) = \begin{cases} 3(-x + 1/3) = 1 - 3x & 0 \le x \le 1/3 \\ 0 & \text{else} \end{cases}$$

$$N_2(x) = \begin{cases} 3(x - 0) \\ 3(-x + 2/3) \\ 0 \end{cases} = \begin{cases} 3x & 0 \le x \le 1/3 \\ -3x + 2 & 1/3 \le x \le 2/3 \\ 0 & \text{else.} \end{cases}$$

$$N_3(x) = \begin{cases} 3(x - 1/3) \\ 3(-x + 1) \\ 0 \end{cases} = \begin{cases} 3x - 1 & 1/3 \le x \le 2/3 \\ -3x + 3 & 2/3 \le x \le 1 \\ 0 & \text{else.} \end{cases}$$

$$N_4(x) = \begin{cases} 3x - 2 & 2/3 \le x \le 1 \\ 0 & \text{else.} \end{cases}$$

f) (G) problem

9 $a(w^h, u^h) = (w^h, f) + w^h(1)\cdot h$

$a(w^h, v^h) = (w^h, f) + w^h(1)\cdot h - a(w^h, g^h)$

$\mathcal{V}^h \ni w^h = \sum_{A=2}^{4} c_A N_A = c_2 N_2 + c_3 N_3 + c_4 N_4.$

$\mathcal{V}^h \ni v^h = \sum_{B=2}^{4} d_B N_B = d_2 N_2 + d_3 N_3 + d_4 N_4$

and $g^h = g N_1$ $\quad (N_1(0) = 1)$

$f = \text{Constant}.$

$$\Rightarrow a\left(\sum_{A=2}^{4} c_A N_A, \sum_{B=2}^{4} d_B N_B\right) = \left(\sum_{A=2}^{4} c_A N_A, f\right) + \sum_{A=2}^{4} c_A N_A(1) h - a\left(\sum_{A=2}^{4} c_A N_A, g N_1\right)$$

$$a\left(N_A, \sum_{B=2}^{4} d_B N_B\right) = (N_A, f) + N_A(1)\cdot h - a(N_A, g N_1)$$

$$\sum_{B=2}^{4} a(N_A, N_B) d_B = (N_A, f) + N_A(1)\cdot h - a(N_A, g N_1).$$

$k_{AB} = a(N_A, N_B)$. where $A, B = 2, 3, 4$.

$$\left.\begin{aligned} a(N_2, N_2) &= \frac{1}{h_1} + \frac{1}{h_2} = 6 \\ a(N_2, N_3) &= \frac{-1}{h_2} = -3 \\ a(N_3, N_3) &= \frac{1}{h_2} + \frac{1}{h_3} = 6 \\ a(N_3, N_4) &= \frac{-1}{h_3} = -3 \\ a(N_4, N_4) &= 3 \end{aligned}\right\} \Rightarrow k = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$a(N_A, N_B) = \begin{cases} \frac{-1}{h_{A-1}} & B = A-1 \\ \frac{1}{h_{A-1}} + \frac{1}{h_A} & B = A \\ \frac{-1}{h_A} & B = A+1 \end{cases}$$

Load vector $F_A = (N_A, f) + N_A(1)\cdot h - a(N_A, g N_1)$

$$\begin{cases} F_2 = (N_2, c) + 0 - a(N_2, N_1) g \\ F_3 = (N_3, c) + 0 - \underset{0}{\underline{a(N_3, N_1)}} g \\ F_4 = (N_4, c) + h - \underset{0}{\underline{a(N_4, N_1)}} g \end{cases}$$

$a(N_2, N_1) = -3.$

$(N_3, c) = \int_0^1 c N_3 \, dx = c \cdot \frac{1}{3}$

$(N_4, c) = \int_0^1 c N_4 \, dx = c \cdot \frac{1}{6}$

$(N_2, c) = \int_0^1 c N_2 \, dx = c \cdot 1/3$

$$\Rightarrow F_2 = \frac{1}{3}c + 3g$$

$$F_3 = \frac{1}{3}c$$

$$F_4 = \frac{1}{6}c + h.$$

g) IEN.

3//

a \ e	1	2	3
1	1	2	3
2	2	3	4

$ID(A) = P \longrightarrow$ eqn index $\quad 1 \le P \le n_{eq}$ — # of eqns

A: nodal index $\quad 1 \le A \le n_{np}$ # of nodal pts

ID

e	1	2	3	4
a	1	2	3	0

		1	2	3 — ID
	1	2	3	4
	$u=g$			flux.

$LM = ID[IEN(a,e)]$

a \ e	1	2	3
1	1	2	3
2	2	3	0

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h) 1) $u^h(x_A) = u(x_A)$ the solution is nodally exact.

8// 2) there exists a point c between x_A and x_{A+1}, which satisfies the derivative accuracy.

$$\frac{u^h(x_{A+1}) - u^h(x_A)}{h_A} = \frac{u(x_{A+1}) - u(x_A)}{h_A} = u_{,x}(c)$$

OK.

Barlow pt ?

i). $u_{,x}(1) = \beta(u - u_r)$

5// $\int_\Omega w\, u_{,xx}\, dx + \int_\Omega w f\, dx = 0$

$-\int_\Omega w_{,x} u_{,x}\, dx + w u_{,x}\Big|_0^1 + \int_\Omega w f\, dx = 0$

$-\int_\Omega w_{,x} u_{,x}\, dx + w(1) u_{,x}(1) - 0 + \int_\Omega w f\, dx = 0$

$\int_\Omega w_{,x} u_{,x}\, dx = \int_\Omega w f\, dx + w(1)\cdot \beta(u - u_r)$?

2. Solution

a) $n_{en} = 3$

$$a=1:\ l_1^2(\xi) = N_1(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{\xi\cdot(\xi-1)}{-1\cdot(-2)} = \frac{1}{2}(\xi^2-\xi)$$

$$a=2:\ l_2^2(\xi) = N_2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = \frac{(\xi+1)\cdot(\xi-1)}{-1} = \xi^2-1$$

$$a=3:\ l_3^2(\xi) = N_3(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{(\xi+1)\cdot\xi}{2\cdot 1} = \frac{1}{2}(\xi^2+\xi)$$

b) $$\tilde{f}(\xi) = f(-1)\,N_1(\xi) + f(0)\,N_2(\xi) + f(1)\,N_3(\xi)$$

$$= f(-1)\cdot\frac{1}{2}(\xi^2-\xi) + f(0)(\xi^2-1) + f(1)\cdot\frac{1}{2}(\xi^2+\xi)$$

integral of $\tilde{f}(\xi)$: $$\int_{-1}^{1}\left[f(-1)\,N_1(\xi) + f(0)\,N_2(\xi) + f(1)\,N_3(\xi)\right]d\xi$$

$$= \int_{-1}^{1}\left[f(-1)\cdot\frac{1}{2}(\xi^2-\xi) + f(0)\cdot(\xi^2-1) + f(1)\cdot\frac{1}{2}(\xi^2+\xi)\right]d\xi$$

$$= f(-1)\cdot\frac{1}{2}\left(\frac{1}{3}\xi^3-\frac{1}{2}\xi^2\right)\Big|_{-1}^{1} + f(0)\cdot\left(\frac{1}{3}\xi^3-\xi\right)\Big|_{-1}^{1} + f(+1)\cdot\frac{1}{2}\cdot\left(\frac{1}{3}\xi^3+\frac{1}{2}\xi^2\right)\Big|_{-1}^{1}$$

$$= \frac{1}{3}f(-1) + \left(-\frac{4}{3}\right)f(0) + \frac{1}{3}f(+1)$$

c) it can integrate ξ^4 exactly.
cannot integrate ξ^5 exactly

-4

d) the three-point Gaussian quadrature rule
$n_{int} = 3$,

$2n_{int} - 1 = 5$

so it can integrate up to ξ^5 exactly, which is better than Simpson's rule.

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