常微分方程 B. HW8

1.解:

(1) $y''+4y'+4y=0$

$r^2+4r+4=0$

$(r+2)^2=0$

$r=-2$

$y_0=(C_1+C_2t)e^{-2t}$

$y=[C_1(t)+C_2(t)\cdot t]e^{-2t}$

$$\begin{cases} C_1'(t)e^{-2t}+C_2'(t)te^{-2t}=0 \\ C_1'(t)e^{-2t}(-2)+C_2'(t)e^{-2t}(1-2t)=t^{-2}e^{-2t} \end{cases},\ t>0$$

$$\Rightarrow \begin{cases} C_1'(t)=-\frac{1}{t} \\ C_2'(t)=\frac{1}{t^2} \end{cases} \Rightarrow \begin{cases} C_1(t)=-\ln t+C_1 \\ C_2(t)=-\frac{1}{t}+C_2 \end{cases}$$

$\therefore y_p=(C_1-\ln t-1+t\cdot C_2)e^{-2t}$

$=(C_2\cdot t-\ln t+C_3)e^{-2t}$

(2) $y''-2y'+y=0$

$r^2-2r+1=0$

$(r-1)^2=0$

$r=1$

$y_0=(C_1+C_2t)e^t=C_1e^t+C_2t\cdot e^t$

$y=[C_1(t)+C_2(t)\cdot t]e^t$

$$\begin{cases} C_1'(t)e^t+C_2'(t)\cdot te^t=0 \\ C_1'(t)e^t+C_2'(t)e^t(1+t)=\frac{e^t}{1+t^2} \end{cases}$$

$$\Rightarrow \begin{cases} C_1'(t)=\frac{-t}{1+t^2} \\ C_2'(t)=\frac{1}{1+t^2} \end{cases} \Rightarrow \begin{cases} C_1(t)=-\frac{1}{2}\ln(1+t^2)+C_1 \\ C_2(t)=\tan^{-1}t+C_2 \end{cases}$$

$\therefore y=[-\frac{1}{2}\ln(1+t^2)+C_1+\tan^{-1}t\cdot t+C_2t]e^t$

2.解:

$ty''-(t+1)y'+y=t^2e^{2t}$, $t>0$

As $y_1(t)=t+1$, $y_2(t)=e^t$

$y_1'(t)=1$, $y_2'(t)=e^t$

$y_1''(t)=0$, $y_2''(t)=e^t$

$t\cdot y_1''(t)-(t+1)y_1'(t)+y_1(t)$

$=0-(t+1)\cdot 1+(t+1)=0$

$t\cdot y_2''(t)-(t+1)y_2'(t)+y_2(t)$

$=t\cdot e^t-(t+1)e^t+e^t=0$

let $y=C_1(t)y_1(t)+C_2(t)y_2(t)$

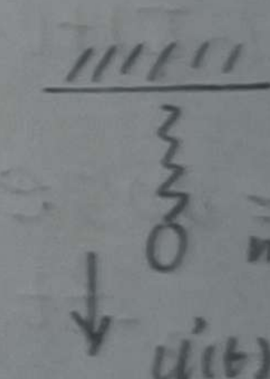
$$\begin{cases} C_1'(t)\cdot(t+1)+C_2'(t)\cdot e^t=0 \\ C_1'(t)\cdot 1+C_2'(t)\cdot e^t=t^2\cdot e^{2t} \end{cases}$$

$$\Rightarrow \begin{cases} C_1'(t)=-e^{2t} \\ C_2'(t)=(t^2+1)e^t \end{cases} \quad \begin{cases} C_1(t)=-\frac{1}{2}e^{2t}+C_1 \\ C_2(t)=e^t(t^2-2t+3)+C_2 \end{cases}$$

$\therefore y=(-\frac{1}{2}e^{2t}+C_1)(t+1)+$

$[e^t(t^2-2t+3)+C_2]e^t$

3.解: $k=\frac{mg}{\Delta x}=\frac{0.1\times 9.81}{0.05}=19.62\ N/m$



$mu''(t)+0u'(t)+ku(t)=0$

$mu''(t)+ku(t)=0$

$mr^2+k=0$

$r^2=\frac{-k}{m}$, $r=\pm\sqrt{\frac{k}{m}}\,i$

$u=A\cos\sqrt{\frac{k}{m}}t+B\sin\sqrt{\frac{k}{m}}t$

$u'=A(-\sin\sqrt{\frac{k}{m}}t)\cdot\sqrt{\frac{k}{m}}+B\cos\sqrt{\frac{k}{m}}t\cdot\sqrt{\frac{k}{m}}$

As $\begin{cases} u(0)=0 \\ u'(0)=10\ cm/s=0.01\ m/s \end{cases}$

$$\begin{cases} A+0=0 \\ 0+B\cdot\sqrt{\frac{k}{m}}=0.01 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=0.01\times\sqrt{\frac{0.1}{19.62}}=7.139\times 10^{-4} \end{cases}$$

$\therefore u(t)=7.139\times 10^{-4}\sin\sqrt{\frac{19.62}{0.1}}t$

$=7.139\times 10^{-4}\sin 14.01t$ m/s

4.解

$k = \frac{3N}{10cm} = \frac{3}{0.1} = 30\ N/m$

$r = \frac{3N}{5m/s} = 3N\cdot s/m$

$u_0(0) = 0.05m$

$u_0'(0) = 0.1m/s$

$mu''(t) + ru'(t) + ku(t) = 0$

$m\lambda^2 + r\lambda + k = 0$

$\lambda = \frac{-r \pm \sqrt{r^2-4mk}}{2m}$

$r^2 - 4mk = 9 - 4\times2\times30 = -231$

$\lambda = \frac{-r}{2m} \pm \frac{\sqrt{-231}}{2m} = \frac{-3}{4} \pm \frac{\sqrt{-231}}{4}$

$= -\frac{3}{4} \pm 3.800i$

$u(t) = Ae^{-\frac{3}{4}t}\cos3.8t + Be^{-\frac{3}{4}t}\sin3.8t$

$u'(t) = A\cdot\cos3.8t\cdot(-\frac{3}{4})\cdot e^{-\frac{3}{4}t} + A\cdot e^{-\frac{3}{4}t}\cdot3.8\cdot(-\sin3.8t) +$

$B\sin3.8t\cdot(-\frac{3}{4})\cdot e^{-\frac{3}{4}t} + Be^{-\frac{3}{4}t}\cdot3.8\cos3.8t$

$\Rightarrow \begin{cases} A+0=0.05 \\ A(-\frac{3}{4})+0+0+B(3.8)=0.1 \end{cases} \quad \begin{cases} A=0.05 \\ B=0.03618 \end{cases}$

$\therefore u(t) = 0.05e^{-\frac{3}{4}t}\cos3.8t + 0.03618e^{-\frac{3}{4}t}\sin3.8t$

$= e^{-\frac{3}{4}t}(0.05\cos3.8t + 0.03618\sin3.8t)$

quasi-frequency $\mu = \frac{\sqrt{4km-r^2}}{2m} = \frac{\sqrt{8\times30-9}}{4} = 3.800$

$\frac{\mu}{w_0} = \frac{\frac{\sqrt{4km-r^2}}{2m}}{\sqrt{\frac{k}{m}}} = (1-\frac{r^2}{4km})^{\frac{1}{2}}$

$= \sqrt{1-\frac{9}{4\times30\times2}}$

$= 0.9811$

5.解

$Q(0) = 10^{-6}C$

$Q'(0) = 0$

$L\frac{di}{dt} + R\cdot i + \frac{Q}{C} = 0$

$L\cdot Q'' + RQ' + \frac{1}{C}Q = 0$

$Lr^2 + Rr + \frac{1}{C} = 0$

$r = \frac{-R \pm \sqrt{R^2-\frac{4L}{C}}}{2L}$

$R^2 - \frac{4L}{C} = 300^2 - \frac{4\times0.2}{10^{-5}} = 10000$

$r = \frac{-300\pm100}{0.4} = -750 \pm 250$

$r_1 = -1000,\ r_2 = -500$

$Q(t) = c_1e^{r_1t} + c_2e^{r_2t}$

$= c_1\cdot e^{-1000t} + c_2e^{-500t}$

$Q'(t) = -1000c_1\cdot e^{-1000t} - 500c_2e^{-500t}$

$\Rightarrow \begin{cases} c_1+c_2 = 10^{-6} \\ -1000c_1 - 500c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = -10^{-6} \\ c_2 = 2\times10^{-6} \end{cases}$

$\therefore Q(t) = -10^{-6}\times e^{-1000t} + 2\times10^{-6}e^{-500t}$

6.解:

proof: $mu'' + ru' + ku = 0$

$m\lambda^2 + r\lambda + k = 0$

$\lambda = \frac{-r\pm\sqrt{r^2-4mk}}{2m}$

① overdamped: $r^2-4mk>0$

$u(t) = c_1e^{\lambda_1t} + c_2e^{\lambda_2t}$

let $u(t)=0$,

$c_1e^{\lambda_1t} + c_2e^{\lambda_2t} = 0$, only $c_1=c_2=0$

no t can be found.

② critical: $r^2-4mk=0$

$u(t) = (c_1+c_2t)e^{\lambda t}$

let $u(t)=0$

$(c_1+c_2t)e^{\lambda t} = 0$

$c_1+c_2t=0$

$t = \frac{-c_1}{c_2}$, one t can be found

from ①②: mass can pass the EP for at most once.

7.解: $y''' + 2y'' - y' - 2y = 0$

① $(e^t)' = e^t$

$(e^t)'' = e^t$

$(e^t)''' = e^t$

$e^t + 2e^t - e^t - 2e^t = 0$

② $(e^{-t})' = -e^{-t}$

$(e^{-t})'' = e^{-t}$

$(e^{-t})''' = -e^{-t}$

$-e^{-t} + 2e^{-t} + e^{-t} - 2e^{-t} = 0$

③ $(e^{-2t})' = -2e^{-2t}$

$(e^{-2t})'' = 4e^{-2t}$

$(e^{-2t})''' = -8e^{-2t}$

$-8e^{-2t} + 8e^{-2t} + 2e^{-2t} - 2e^{-2t} = 0$

$$W[e^t, e^{-t}, e^{-2t}] = \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix}$$

$$= e^t \cdot e^{-t} \cdot e^{-2t} \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} \end{array}$$

$$= e^{-2t}\left(\begin{vmatrix} -1 & -2 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \right)$$

$$= e^{-2t}(6 - 6 + 2)$$

$$= 2e^{-2t} \neq 0$$

so they form a fundamental set of solutions

8.解:

(1) $y''' - 3y'' + 3y' - y = 0$

$r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0$

$r = 1$, repeat 3

$\therefore y = (C_1 + C_2 t + C_2 t^2) e^t$

(2) $y^{(4)} - 4y''' + 4y'' = 0$

set $m = y''$

$m'' - 4m' + 4m = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$, repeat 2

$m(t) = (C_1 + C_2 t)e^{2t} = y''$

$y' = \int (C_1 e^{2t} + C_2 t e^{2t})\, dt$

$= C_1 \cdot \frac{1}{2} e^{2t} + C_2 \frac{1}{2} e^{2t}(t - \frac{1}{2}) + C_3$

$y = \int (\frac{1}{2} C_1 e^{2t} + \frac{1}{2} C_2 t e^{2t} - \frac{1}{4} C_2 e^{2t} + C_3)\, dt$

$= \int (A_1 e^{2t} + A_2 t e^{2t} + A_3)\, dt$

$= \frac{1}{2} A_1 e^{2t} + \frac{A_2}{2} e^{2t}(t - \frac{1}{2}) + A_3 t + A_4$

$= B_1 e^{2t} + B_2 e^{2t}(t - \frac{1}{2}) + A_3 t + A_4$

(3) $y^{(4)} + 2y'' + y = 3 + \cos 2t$

$y^{(4)} + 2y'' + y = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2 + 1)^2 = 0$

$r^2 = -1$, $r = \pm i$, repeat 2

$y_0(t) = A\cos t + B\sin t + Ct\cos t + Dt\sin t$

$y(t) = A(t)\cos t + B(t)\sin t + C(t)\cdot t\cos t + D(t)\, t\sin t$

$$\begin{cases} A'(t)\cos t + B'(t)\sin t + C'(t)\, t\cos t + D'(t)\, t\sin t = 0 \\ A'(t)(-\sin t) + B'(t)\cos t + C'(t)\cdot(\cos t - t\cdot\sin t) + D'(t)\cdot(\sin t + t\cos t) = 0 \\ A'(t)(-\cos t) + B'(t)(-\sin t) + C'(t)(-2\sin t - t\cos t) + D'(t)(2\cos t - t\sin t) = 0 \\ A'(t)\sin t + B'(t)(-\cos t) + C'(t)(-3\cos t + t\sin t) + D'(t)(-3\sin t - t\cos t) = 3 + \cos 2t \end{cases}$$

$$\Rightarrow \begin{cases} A'(t) = \\ B'(t) = \\ C'(t) = \\ D'(t) = \end{cases}$$

(3) $y^{(4)} + 2y'' + y = 3 + \cos 2t$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)^2 = 0$

$r^2 = -1$, $r = \pm i$ ~~repeat 2~~

$\cancel{Y(t) = C_1 + (A_1 \cos 2t + A_2 \sin 2t)\, t^2}$

$\cancel{Y'(t) = 0 + [A_1 \cdot (-\sin 2t) \cdot 2 + A_2 \cdot \cos 2t \cdot 2] \cdot t^2 +}$

$\cancel{(A_1 \cos 2t + A_2 \sin 2t) \cdot 2t}$

$\cancel{Y''(t) = [2A_1 (-\cos 2t) \cdot 2 + 2A_2 (-\sin 2t) \cdot 2] \cdot t^2 +}$

$\cancel{[2A_1 (-\sin 2t) + 2A_2 \cdot \cos 2t] \cdot 2t +}$

$\cancel{[A_1 \cdot (-\sin 2t) \cdot 2 + A_2 \cos 2t \cdot 2] \cdot 2t +}$

$\cancel{(A_1 \cos 2t + A_2 \sin 2t) \cdot 2}$

$Y(t) = A + C \cos 2t + D \sin 2t$

$Y''(t) = -4C \cos 2t - 4D \sin 2t$

$Y^{(4)}(t) = 16C \cos 2t + 16D \sin 2t$

$16C \cos 2t + 16D \sin 2t + 2(-4C \cos 2t - 4D \sin 2t)$

$+ A + C \cos 2t + D \sin 2t = 3 + \cos 2t$

$$\Rightarrow \begin{cases} A = 3 \\ C = \frac{1}{9} \\ D = 0 \end{cases}$$

$\therefore Y(t) = 3 + \frac{1}{9} \cos 2t$ particular solution

$y(t) = C_1 \cos t + C_2 \sin t + C_3 t \cos t + C_4 t \sin t$

$+ 3 + \frac{1}{9} \cos 2t$ general solution.