常微分方程B HW4 邹佳驹 12012127

1.

(1) 解: $\frac{dy}{dt} - ry = -ky^2$, $r, k > 0$

令 $z = y^{1-2} = \frac{1}{y}$, $y = \frac{1}{z}$, check: $y \neq 0$

$$\frac{dz}{dt} = -1 \cdot \frac{1}{y^2} \cdot \frac{dy}{dt} = \frac{-1}{y^2} \cdot (ry - ky^2) = \frac{-r}{y} + k$$

$\Rightarrow \frac{dz}{dt} = -r \cdot z + k$, check: $-rz + k = 0$

$$\int \frac{1}{-r(-r \cdot z + k)} d(-rz + k) = \int dt$$

$$\frac{1}{-r} \cdot \ln|-rz + k| = t + C$$

$$\frac{1}{-r} \cdot \ln\left|-r \cdot \frac{1}{y} + k\right| = t + C$$

由 check: $y = 0$, $y = \frac{r}{k}$ 也是解

(2) 解: $\frac{dy}{dx} + \frac{1}{x} \cdot y = x \cdot y^2$, $x \neq 0$

令 $z = y^{1-2} = \frac{1}{y}$, $y = \frac{1}{z}$ check: $y \neq 0$

$$\frac{dz}{dx} = -1 \cdot \frac{1}{y^2} \cdot \left(xy^2 - \frac{y}{x}\right) = \frac{1}{xy} - x = \frac{z}{x} - x$$

$\Rightarrow \frac{dz}{dx} + \frac{-1}{x} z = -x$

$$u(x) = e^{\int \frac{-1}{x} dx} = \frac{1}{|x|}$$

$$z = \frac{1}{u(x)} \left(\int u(x)(-x) dx + C \right)$$

$x > 0$: $z = x\left(\int (-1) dx + C\right) = x(-x + C_1) = -x^2 + C_1 x$

$x < 0$: $z = -x\left(\int 1 dx + C\right) = -x(x + C_1) = -x^2 - C_1 x$

$\Rightarrow z = -x^2 + Ax$, $\frac{1}{y} = -x^2 + Ax$

由 check: $y = 0$ 也是解

(3) 解: $\frac{dy}{dx} = \frac{x+1}{x-y}$

令 $\begin{cases} x = X + a \\ y = Y + b \end{cases}$, $\frac{dy}{dx} = \frac{dY}{dX} = \frac{X+a+1}{X+a-Y-b}$

$\begin{cases} a+1=0 \\ a-b=0 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=-1 \end{cases}$

$$\frac{dY}{dX} = \frac{X}{X-Y} = \frac{1}{1-\frac{Y}{X}}$$

令 $u = \frac{Y}{X}$, $Y = uX$

$$\frac{dY}{dX} = u' \cdot X + u$$

$$\frac{du}{dX} = \frac{\left(\frac{1}{1-u} - u\right)}{X}$$

$$\int \frac{1-u}{1-u(1-u)} du = \int \frac{1}{X} dX$$

?

(4) 解: $\frac{dy}{dx} = (4x - y + 1)^2$

特解: ~~$4x-y+1=0$~~

$y = 4x+1$ ×

令 $y = u + (4x+1)$

$$\frac{dy}{dx} = \frac{du}{dx} + 4 = (4x+1-u-4x-1)^2 = u^2$$

$\Rightarrow \frac{du}{dx} = u^2 - 4$, check: $u \neq \pm 2$

$$\int \frac{1}{u^2-4} du = \int dx$$

$$\cancel{\frac{1}{4} \cdot 2 \cdot \frac{1}{\left(\frac{u}{2}\right)^2 - 1} d\frac{u}{2} = \int dx}$$

$$\frac{1}{2 \times 2} \ln\left|\frac{u-2}{u+2}\right| + C = x$$

$$\frac{1}{4} \ln\left|\frac{y-4x-3}{y-4x+1}\right| + C = x$$

特解: $u = \pm 2 \Rightarrow y_2 = 4x+3$

$y_3 = 4x - 1$

特解: $y_1 = 4x - 1$

令 $y = u + 4x - 1$, $u = y - 4x + 1$

$$\frac{dy}{dx} = u' + 4$$

$$= (4x - u - 4x + 2)^2$$

$\Rightarrow u' = (2-u)^2 - 4$

check: $2 - u \neq \pm 2$

$$\int \frac{1}{(u-2)^2 - 4} du = \int dx$$

$$\int \frac{1}{(u-2)^2 - 2^2} d(u-2) = \int dx$$

$$\frac{1}{2 \times 2} \ln\left|\frac{u-2-2}{u-2+2}\right| + C = x$$

$$\frac{1}{4} \ln\left|\frac{y-4x-3}{y-4x+1}\right| + C = x$$

由 check, $u = 0$, $y = 4x - 1$

$u = 4$, $y = 4x + 3$ 也为解

2.解:

$$\frac{dy}{dx} = +y^2 - \frac{1}{x}y - \frac{4}{x^2}$$

令 $y_1 = Cx^a$

$$C\cdot a\cdot x^{a-1} = c^2\cdot x^{2a} - C\cdot x^{a-1} - \frac{4}{x^2}$$

$$\Rightarrow \left.\begin{matrix} a-1=2a=-2 \\ ca = c^2 - c - 4 \end{matrix}\right\} \Rightarrow \begin{cases} a=-1 \\ c=\pm 2 \end{cases}，特解 \ y = \pm\frac{2}{x}$$

取 $y_1 = \frac{2}{x}$

令 $y = u + \frac{2}{x}$，$u = y - \frac{2}{x}$

$$u' + 2\cdot\frac{(-1)}{x^2} = \left(u+\frac{2}{x}\right)^2 - \frac{1}{x}\left(u+\frac{2}{x}\right) - \frac{4}{x^2}$$

$$u' = \frac{3u}{x} + u^2$$

$$\frac{du}{dx} - \frac{3}{x}u = u^2$$

令 $z = u^{1-2} = \frac{1}{u}$ ，check: $u \neq 0$

$$\frac{dz}{dx} = -1\cdot\frac{1}{u^2}\cdot\left(u^2 + \frac{3}{x}u\right) = -1 - \frac{3}{xu}$$

$$\frac{dz}{dx} + \frac{3}{x}u = -1$$

$$u'(x) = e^{\int\frac{3}{x}dx} = e^{3\ln|x|} = |x|^3$$

$$z = \frac{1}{u'(x)}\left(\int u'(x)\cdot(-1)dx + C\right)$$

$$= \frac{1}{|x|^3}\left(\int |x|^3(-1)dx + C\right)$$

$$x>0:\ z = \frac{1}{x^3}\left(\int -x^3 dx + C\right) = -\frac{1}{4}x + \frac{C_1}{x^3}$$

$$x<0:\ z = \frac{1}{-x^3}\left(\int +x^3 dx + C\right) = -\frac{1}{4}x - \frac{C_1}{x^3}$$

$$\Rightarrow z = -\frac{1}{4}x + \frac{A}{x^3} = \frac{1}{u} = \frac{1}{y-\frac{2}{x}}$$

即: $\frac{1}{y-\frac{2}{x}} = -\frac{1}{4}x + \frac{A}{x^3}$

特解: $y = \pm\frac{2}{x}$

$$\frac{dz}{dx} = \frac{d\frac{1}{u}}{dx} = -1\cdot\frac{1}{u^2}\cdot u' = -1\cdot\frac{1}{u^2}\cdot\left(u^2 + \frac{3}{x}u\right)$$

$$= -1 - \frac{3}{x}\cdot\frac{1}{u} = -1 - \frac{3}{x}\cdot z$$

$$\frac{dz}{dx} + \frac{3}{x}\cdot z = -1,$$

$$u'(x) = e^{\int\frac{3}{x}dx} = |x|^3$$

$$z = \frac{1}{|x|^3}\left(\int -|x|^3 dx + c\right) \Rightarrow z = -\frac{1}{4}x + \frac{A}{x^3}$$

$\therefore\ -\frac{1}{4}x + \frac{A}{x^3} = \frac{1}{y-\frac{2}{x}}$ 是方程的解

特解 $y = \pm\frac{2}{x}$

3.解:

a) 见附图

b) $\frac{dy}{dt} - 3y = -ty^2$

令 $z = y^{1-2} = \frac{1}{y}$ ，check: $y \neq 0$

$$\frac{dz}{dt} = \frac{d\frac{1}{y}}{dt} = -1\cdot\frac{1}{y^2}\cdot y' = \frac{-1}{y^2}(3y - ty^2) = \frac{-3}{y} + t$$

即: $\frac{dz}{dt} = -3z + t$

$$\frac{dz}{dt} + 3z = t$$

$$u(t) = e^{\int 3dt} = e^{3t}$$

$$z = \frac{1}{e^{3t}}\cdot\left(\int e^{3t}\cdot t\,dt + C\right)$$

$$= \frac{1}{e^{3t}}\cdot\left(\frac{1}{3}t\cdot e^{3t} - \frac{1}{9}e^{3t} + C_1\right)$$

$$\frac{1}{y} = \frac{1}{3}t - \frac{1}{9} + \frac{C_1}{e^{3t}}$$

代入 $y(0) = 0.5$

$$2 = -\frac{1}{9} + \frac{C_1}{1} \Rightarrow C_1 = \frac{19}{9}$$

$$\therefore\ y = \frac{1}{\frac{1}{3}t - \frac{1}{9} + \frac{19}{9e^{3t}}}$$

$y(1) = 3.055037$

$y(1.5) = 2.42518$

$y(2) = 1.78320$

$y(2.5) = 1.38238$

$y(3.0) = 1.12467$

After comparing, the smaller h the more accurate.

h=0.1

n	t	y
10	1.0	3.0660498
15	1.5	2.4402971
20	2.0	1.7720355
25	2.5	1.3734809
30	3.0	1.1192479

h=0.05

n	t	y
20	1.0	3.0605109
30	1.5	2.432919
40	2.0	1.778066
50	2.5	1.3779508
60	3.0	1.1219106

h=0.025

n	t	y
40	1.0	3.057689
60	1.5	2.4290493
80	2.0	1.7807386
100	2.5	1.3801748
120	3.0	1.1232798

h=0.01

n	t	y
100	1.0	3.0560708
150	1.5	2.4267223
200	2.0	1.7822415
250	2.5	1.3815007
300	3.0	1.1241117