常微分方程B Week 10 邹佳驹 12012127

1.解：

(1) $f(t)=\sinh(at)=\frac{1}{2}(e^{at}-e^{-at})$

$$\mathcal{L}\{f(t)\}(s)=\int_0^\infty e^{-st}\cdot\frac{1}{2}(e^{at}-e^{-at})dt$$

$$=\frac{1}{2}\int_0^\infty\left(e^{(a-s)t}-e^{(-s-a)t}\right)dt$$

$$=\frac{1}{2}\lim_{A\to\infty}\left(\frac{1}{a-s}e^{(a-s)t}\Big|_0^A-\frac{1}{-s-a}e^{(-s-a)t}\Big|_0^A\right)$$

$$=\frac{1}{2}\lim_{A\to\infty}\left(\frac{1}{a-s}e^{(a-s)A}-\frac{1}{a-s}-\frac{1}{-s-a}e^{(-s-a)A}+\frac{1}{-s-a}\right)$$

$$\begin{cases}a-s<0\\-s-a<0\end{cases}\Rightarrow -s<a<s \text{ 时}$$

$$\text{原式}=\frac{1}{2}\left(\frac{1}{-s-a}-\frac{1}{a-s}\right)$$

$$=\frac{a}{s^2-a^2}\quad\boxed{\text{ANS}}$$

(2) $f(t)=e^{at}\cos(bt)$

$$\mathcal{L}\{f(t)\}(s)=\int_0^\infty e^{at}\cos bt\cdot e^{-st}dt$$

$$=\int_0^\infty e^{(a-s)t}\cos bt\, dt$$

$$=\int_0^\infty \cos bt\cdot\frac{1}{a-s}de^{(a-s)t}$$

$$=\lim_{A\to\infty}\left(\frac{\cos bt}{a-s}e^{(a-s)t}\Big|_0^A-\int_0^\infty\frac{e^{(a-s)t}}{a-s}d\cos bt\right)$$

$$=\lim_{A\to\infty}\left(C_1+\int_0^\infty\frac{e^{(a-s)t}}{a-s}\cdot\sin bt\cdot b\,dt\right)$$

$$=\lim_{A\to\infty}\left(C_1+\int_0^\infty\frac{b\cdot\sin bt}{(a-s)^2}\cdot de^{(a-s)t}\right)$$

$$=\lim_{A\to\infty}\left(C_1+\frac{b\cdot\sin bt}{(a-s)^2}\cdot e^{(a-s)t}\Big|_0^A-\int_0^\infty\frac{b^2\cdot e^{(a-s)t}}{(a-s)^2}\cdot\cos bt\,dt\right)$$

$$=\lim_{A\to\infty}\left(C_1+C_2-\frac{b^2}{(a-s)^2}\int_0^\infty e^{(a-s)t}\cdot\cos bt\cdot dt\right)$$

$$\left(1+\frac{b^2}{(a-s)^2}\right)\cdot\mathcal{L}\{f(t)\}(s)=\lim_{A\to\infty}(C_1+C_2)$$

$a-s<0$ ，$a<s$ 时

$$\mathcal{L}\{f(t)\}(s)=\frac{(a-s)^2\cdot\left(+\frac{1}{-a+s}+0\right)}{(a-s)^2+b^2}$$

$$=\frac{s-a}{(a-s)^2+b^2}\quad\boxed{\text{ANS}}$$

(3) $f(t)=t\sin at$

$$\mathcal{L}\{f(t)\}(s)=\int_0^\infty e^{-st}t\sin at\,dt$$

$$=\lim_{A\to\infty}\int_0^A t\sin at\cdot\frac{1}{-s}de^{-st}$$

$$=\lim_{A\to\infty}\left[\frac{t\sin at}{-s}\cdot e^{-st}\Big|_0^A+\int_0^A\frac{e^{-st}}{+s}(\sin at+t\cos at\cdot a)dt\right]$$

$$=\lim_{A\to\infty}\left[C_1+\frac{1}{s}\left(\int_0^A e^{-st}\sin at\,dt+\int_0^A e^{-st}t\cos at\cdot a\,dt\right)\right]$$

$$=\lim_{A\to\infty}\left[C_1+\frac{1}{s}\mathcal{L}\{\sin at\}(s)+\frac{a}{s}\cdot\int_0^A t\cos at\cdot\frac{1}{-s}de^{-st}\right]$$

$$=\lim_{A\to\infty}\left[C_1+C_2(s)+\frac{a}{s}\cdot\frac{t\cos at\cdot e^{-st}}{-s}\Big|_0^A+\right.$$
$$\left.\frac{a}{s^2}\cdot\int_0^A e^{-st}\cdot(\cos at+t\cdot a\cdot(-\sin at))dt\right]$$

$$=\lim_{A\to\infty}\left[C_1+C_2(s)+C_3+\frac{a}{s^2}\int_0^A e^{-st}\cos at\,dt\right.$$
$$\left.-\frac{a^2}{s^2}\int_0^A e^{-st}t\sin at\,dt\right]$$

$$=\lim_{A\to\infty}\left[C_1+C_2(s)+C_3+\frac{a}{s^2}\cdot\mathcal{L}\{\cos at\}(s)\right.$$
$$\left.-\frac{a^2}{s^2}\cdot\mathcal{L}\{f(t)\}(s)\right\}$$

$s>0$,

$$\left(\frac{a^2}{s^2}+1\right)\mathcal{L}\{f(t)\}(s)=0+\frac{1}{s}\cdot\frac{a}{s^2+a^2}+0+\frac{a}{s^2}\cdot\frac{s}{s^2+a^2}$$

$$\mathcal{L}\{f(t)\}(s)=\frac{2a\cdot s^{\cancel{2}}}{\cancel{s}(s^2+a^2)(a^2+s^2)}$$

$$=\frac{2as}{(s^2+a^2)^2}\quad\boxed{\text{ANS}}$$

2.解:

(1) proof:

$$\Gamma(p+1) = \int_0^{+\infty} x^p \cdot e^{-x} dx$$

$$= \int_0^{+\infty} (-1) x^p de^{-x}$$

$$= \lim_{A\to\infty} -x^p \cdot e^{-x}\Big|_0^A + \int_0^{+\infty} e^{-x} dx^p$$

$$= \int_0^{\infty} e^{-x} \cdot p \cdot x^{p-1} dx$$

$$= p\int_0^{\infty} e^{-x} x^{p-1} dx = p\Gamma(p)$$

(2) proof:

$$\Gamma(1) = \int_0^{+\infty} e^{-x} dx = \lim_{A\to\infty} (-1) e^{-x}\Big|_0^A = 1$$

$$\Gamma(n+1) = \int_0^{+\infty} x^n e^{-x} dx = n\cdot\Gamma(n)$$

$$= n\cdot(n-1)\cdot\Gamma(n-2) = n(n-1)(n-2)\Gamma(n-3)$$

$$= \cdots$$

$$= n(n-1)(n-2)\cdots\Gamma(1)$$

$$= n(n-1)(n-2)\cdots 1$$

$$= n!$$

(3) proof:

$$\mathcal{L}\{t^p\}(s) = \int_0^{\infty} e^{-st} t^p dt$$

$$= \int_0^{\infty} t^p \cdot e^{-st} dt$$

$$= \int_0^{\infty} (st)^p e^{-st} d(+st) \cdot \frac{1}{+s} \cdot \frac{1}{s^p}$$

$$= \frac{\Gamma(p+1)}{s^{p+1}} \quad , s>0 \; , p>-1.$$

(4) $\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} t^n e^{at} \cdot e^{-st} dt$

$$= \int_0^{\infty} t^n e^{-(s-a)t} dt = \int_0^{\infty} [(s-a)t]^n e^{-(s-a)t} d(s-a)t \cdot \frac{1}{(s-a)^{n+1}}$$

$$= \frac{\Gamma(n+1)}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}} \quad (n: \text{positive int})$$

3.解:

(1) $F(s) = \frac{3}{s^2+4} = \frac{2}{s^2+2^2} \times \frac{3}{2}$

$$= \mathcal{L}\{\sin 2t\} \times \frac{3}{2}$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{3}{2}\sin 2t$$

(2) $F(s) = \frac{4}{(s-1)^3} = \frac{2\times 1}{(s-1)^3} \times 2$

$$= \mathcal{L}\{t^2 \cdot e^t\} \times 2$$

$$\mathcal{L}^{-1}\{F(s)\} = 2t^2 e^t$$

(3) $F(s) = \frac{1-2s}{s^2+4s+5} = \frac{1-2s}{(s+2)^2+1}$

$$= \frac{1-2(s+2)+4}{(s+2)^2+1} = \frac{5-2(s+2)}{(s+2)^2+1}$$

$$= \frac{5}{(s+2)^2+1} - 2\frac{(s+2)}{(s+2)^2+1}$$

$$= 5\cdot\frac{1}{(s+2)^2+1} - 2\cdot\frac{(s+2)}{(s+2)^2+1}$$

$$= 5\mathcal{L}\{e^{-2t}\sin t\} - 2\mathcal{L}\{e^{-2t}\cos t\}$$

$$\therefore \mathcal{L}^{-1}\{F(s)\} = 5e^{-2t}\sin t - 2e^{-2t}\cos t$$

(4) $F(s) = \frac{8s^2-4s+12}{s(s^2+4)}$

$$= \frac{3(s^2+4)+5s^2-4s}{s(s^2+4)}$$

$$= \frac{3}{s} + \frac{5s}{s^2+4} - \frac{4}{s^2+4}$$

$$= 3\frac{1}{s-0} + 5\frac{s}{s^2+2^2} - 2\frac{2}{s^2+2^2}$$

$$= 3\mathcal{L}\{1\} + 5\mathcal{L}\{\cos 2t\} - 2\mathcal{L}\{\sin 2t\}$$

$$\therefore \mathcal{L}^{-1}\{F(s)\} = 3 + 5\cos 2t - 2\sin 2t$$

4.解:

(1) $y''+2y'+2y=0$

$\mathcal{L}\{y''+2y'+2y\}(s)=0$

$\mathcal{L}\{y''\}+2\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=0$

$s^2\cdot\mathcal{L}\{y\}-sy(0)-y'(0)+2[s\mathcal{L}\{y\}-y(0)]+2\mathcal{L}\{y\}=0$

$$\mathcal{L}\{y\}=\frac{s+2}{s^2+2s+2}=\frac{s+1}{(s+1)^2+1}+\frac{1}{(s+1)^2+1}$$

$$=\mathcal{L}\{e^{-t}\cos t\}+\mathcal{L}\{e^{-t}\sin t\}$$

$\therefore y=e^{-t}(\cos t+\sin t)$

(2) $y^{(4)}-y=0$

$\mathcal{L}\{y^{(4)}\}-\mathcal{L}\{y\}=0$

$s^4\mathcal{L}\{y\}-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)-\mathcal{L}\{y\}=0$

$$\mathcal{L}\{y\}=\frac{s^3+s}{s^4-1}=\frac{s(s^2+1)}{(s^2+1)(s^2-1)}=\frac{s}{s^2-1}$$

$$=\mathcal{L}\{\cosh t\}$$

$\therefore y=\cosh t$