常微分方程B week 14 hw12

1.

(1) 解: $X' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} X$

$|A - rI| = 0$

$\begin{vmatrix} 3-r & -4 \\ 1 & -1-r \end{vmatrix} = 0 \Rightarrow (r-3)(1+r)+4=0$

$r^2 - 2r + 1 = 0$

$r_{1,2} = 1$

for $r=1$

$(A-rI)\xi = 0$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\xi = 0$, $\xi = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$(A-rI)\eta = \xi$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\eta = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\eta = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$X = c_1 \xi e^{rt} + c_2(\xi t + \eta)e^{rt}$

$= c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 2t+3 \\ t+1 \end{pmatrix}$

(2) 解: $X' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} X$

$|A-rI|=0$

$\begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{vmatrix} = (1-r)[(1-r)^2-1] - 2(1-r) + (-2) = 0$

令 $1-r = m$

$m^3 - 3m - 2 = 0$

$(m+1)^2(m-2) = 0 \Rightarrow \begin{cases} 1-r=-1 \\ 1-r=2 \end{cases} \Rightarrow \begin{cases} r_{1,2}=2 \\ r_3=-1 \end{cases}$

for $r=2$

$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\xi = 0 \Rightarrow \xi = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\eta = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \eta = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

for $r=-1$

$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\xi_3 = 0 \Rightarrow \xi_3 = \begin{pmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{pmatrix}$

$\therefore X = c_1 \xi e^{r_1 t} + c_2(\xi t + \eta)e^{r_2 t} + c_3 \xi_3 e^{r_3 t}$

$= c_1 e^{2t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -1 \\ -t-1 \\ t \end{pmatrix} + c_3\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}e^{-t}$

(3) 解: $X' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} X + \begin{pmatrix} e^t \\ t \end{pmatrix}$

$|A-rI| = \begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = (2-r)(-2-r)+3$

$= r^2 - 1 = 0$, $r_{1,2} = \pm 1$

for $r_1 = 1$

$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for $r_2 = -1$

$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\xi^{(2)} = 0 \Rightarrow \xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

set $T = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$, $T^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$

令 $X = Ty$, $D = T^{-1}AT = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$\therefore Y' = Dy + T^{-1}g(t)$

$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 \\ -y_2 \end{pmatrix} + \begin{pmatrix} \frac{3}{2}e^t - \frac{1}{2}t \\ -\frac{1}{2}e^t + \frac{1}{2}t \end{pmatrix}$

① $y_1' - y_1 = \frac{3}{2}e^t - \frac{1}{2}t$

var of const

$y_0 = Ce^t$

$y_1 = c(t)e^t$

$c'(t)e^t + c(t)e^t - c(t)e^t = \frac{3}{2}e^t - \frac{1}{2}t$

$\therefore c'(t) = \frac{3}{2} - \frac{1}{2}t \cdot e^{-t}$

$c(t) = \frac{3}{2}t + \frac{1}{2}e^{-t}(1+t) + C_1$

$\therefore y_1 = \frac{3}{2}t \cdot e^t + \frac{1}{2}(t+1) + c_1 e^t$

② $y_2' + y_2 = -\frac{1}{2}e^t + \frac{1}{2}t$

$y_0 = Ce^{-t}$

$y_2 = c(t)e^{-t}$

$c'(t) \cdot e^{-t} = -\frac{1}{2}e^t + \frac{1}{2}t$

$c'(t) = -\frac{1}{2}e^{2t} + \frac{1}{2}te^t$

$c(t) = -\frac{1}{4}e^{2t} + \frac{1}{2}e^t(t-1) + C_2$

$y_2 = -\frac{1}{4}e^t + \frac{1}{2}(t-1) + C_2 e^{-t}$

$X = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ y_1 + 3y_2 \end{pmatrix}$

$= \begin{pmatrix} e^t\left(\frac{3}{2}t - \frac{1}{4}\right) + t + c_1 e^t + c_2 e^{-t} \\ e^t\left(\frac{3}{2}t - \frac{3}{4}\right) + 2t - 1 + c_1 e^t + 3c_2 e^{-t} \end{pmatrix}$

(4) $X' = \begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}X + \begin{pmatrix}-\cos t\\ \sin t\end{pmatrix}$

$|A-rI| = \begin{vmatrix}2-r & -5\\ 1 & -2-r\end{vmatrix} = (r+2)(r-2)+5=0$

$r_{1,2} = \pm i$

for $r=i$

$\begin{pmatrix}2-i & -5\\ 1 & -2-i\end{pmatrix}\xi^{(1)}=0 \Rightarrow \xi^{(1)}=\begin{pmatrix}5\\ 2-i\end{pmatrix}$

$x^{(1)}(t) = \xi^{(1)}e^{r_1 t} = \begin{pmatrix}5\\ 2-i\end{pmatrix}e^{it} = \begin{pmatrix}5\\ 2-i\end{pmatrix}(\cos t+\sin t\cdot i)$

$= \begin{pmatrix}5\cos t\\ 2\cos t+\sin t\end{pmatrix} + i\begin{pmatrix}5\sin t\\ 2\sin t-\cos t\end{pmatrix}$

for $X' = \begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}X$

$X_0(t) = \begin{pmatrix}5\cos t & 5\sin t\\ 2\cos t+\sin t & 2\sin t-\cos t\end{pmatrix}C.$

$X_p(t) = \begin{pmatrix}5\cos t & 5\sin t\\ 2\cos t+\sin t & 2\sin t-\cos t\end{pmatrix}c(t)$

$\begin{pmatrix}5\cos t & 5\sin t\\ 2\cos t+\sin t & 2\sin t-\cos t\end{pmatrix}c'(t) = \begin{pmatrix}-\cos t\\ \sin t\end{pmatrix}$

$c'(t) = \begin{pmatrix}\frac{6}{5}\sin^2 t - \frac{2}{5}\cos\sin t - \frac{1}{5}\\ -\frac{2}{5}\cos^2 t - \frac{6}{5}\cos t\sin t\end{pmatrix}$

$c(t) = \begin{pmatrix}-\frac{3}{5}\cos t\sin t + \frac{2}{5}t - \frac{1}{5}\sin^2 t\\ -\frac{1}{5}\cos t\sin t - \frac{1}{5}t - \frac{3}{5}\sin^2 t\end{pmatrix}$

$\therefore X_p(t) = \begin{pmatrix}t(2\cos t-\sin t) - 3\sin t - 2\cos t\sin^2 t\\ \frac{1}{5}\cos^2 t\sin t - \frac{1}{5}(4\cos t+\sin t)\sin^2 t + t\cos t - \frac{6}{5}\sin t\end{pmatrix}$

$X(t) = C_1\begin{pmatrix}5\cos t\\ 2\cos t+\sin t\end{pmatrix} + C_2\begin{pmatrix}5\sin t\\ 2\sin t-\cos t\end{pmatrix} + X_p(t).$

2.解:

(1) proof:

$|A-rI| = \begin{vmatrix}1-r & 1 & 1\\ 2 & 1-r & -1\\ -3 & 2 & 4-r\end{vmatrix}$

$= (1-r)[(1-r)(4-r)+2] - [2(4-r)-3] +$
$4 + 3(1-r)$

$= (1-r)^2(4-r) + 5(1-r) - 2(4-r) + 7 = 0$

$r^3 - 6r^2 + 12r - 8 = 0$

$(r-2)^3 = 0$

$r_{1,2,3} = 2$

$(A-rI)\xi = 0$

$\begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 2 & 2\end{pmatrix}\xi = 0 \Rightarrow \xi = \begin{pmatrix}0\\ 1\\ -1\end{pmatrix}$

b) proof:

$(A-2I)\xi = \begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ 3 & 2 & 2\end{pmatrix}\begin{pmatrix}0\\ 1\\ -1\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$, $\boxed{\text{check}}$

$X = \xi te^{2t} + \eta e^{2t}$

$X' = \xi e^{2t} + \xi t\cdot 2e^{2t} + 2\eta e^{2t} = AX$

$= A\xi te^{2t} + A\eta e^{2t}$

$\Rightarrow \xi + 2\eta = A\eta$

$(A-2I)\eta = \xi$ $\boxed{\text{check}}$

$\begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 2 & 2\end{pmatrix}\eta = \begin{pmatrix}0\\ 1\\ -1\end{pmatrix} \Rightarrow \eta = \begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$

$x^{(2)}(t) = (\xi t+\eta)e^{2t} = e^{2t}t\begin{pmatrix}0\\ 1\\ -1\end{pmatrix} + e^{2t}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$

c) proof:

$(A-2I)\xi=0$, $(A-2I)\eta=\xi$ 在(b)中已证.

$X = \frac{1}{2}\xi t^2e^{2t} + \eta te^{2t} + \varepsilon e^{2t}$

$X' = \xi te^{2t} + \xi t^2e^{2t} + \underline{\eta e^{2t}} + 2\eta te^{2t} + \underline{2\varepsilon e^{2t}} = AX$

$= A(\frac{1}{2}\xi t^2e^{2t} + \eta te^{2t} + \underline{\varepsilon e^{2t}})$

$\Rightarrow \eta + 2\varepsilon = A\varepsilon$

$(A-2I)\varepsilon = \eta$ $\boxed{\text{check}}$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \xi = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\xi = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

$$X^{(3)}(t) = \frac{t^2}{2} e^{2t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

d) $X = C_1 X^{(1)}(t) + C_2 X^{(2)}(t) + C_3 X^{(3)}(t)$

$= \psi(t) \cdot C$

$$\psi(t) = \begin{bmatrix} 0 & 1 & t+2 \\ 1 & t+1 & t+\frac{t^2}{2} \\ -1 & -t & 3-\frac{t^2}{2} \end{bmatrix} e^{2t}$$

Remark.

$$T = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$J = T^{-1} A T = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

3.解

a) $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

$$J^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$$

$$J^3 = J^2 \cdot J = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$$

for $n=1$, $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, for $n=k$, $J^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}$

for $n=k+1$, $J^{k+1} = J^k \cdot J = \begin{pmatrix} \lambda^{k+1} & (k+1)\lambda^k \\ 0 & \lambda^{k+1} \end{pmatrix}$

$$\therefore J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \quad n \geq 1.$$

b) $$e^{Jt} = 1 + Jt + \frac{(Jt)^2}{2!} + \frac{(Jt)^3}{3!} + \cdots$$

$$= 1 + Jt + \frac{J^2 t^2}{2!} + \frac{J^3 t^3}{3!} + \cdots$$

$$= 1 + \begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} \lambda^2 t^2 & 2\lambda t^2 \\ 0 & \lambda^2 t^2 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots & 0 + t + \frac{2\lambda t^2}{2!} + \cdots \\ 0 & 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

c) $X' = JX$, $X(0) = X^0$

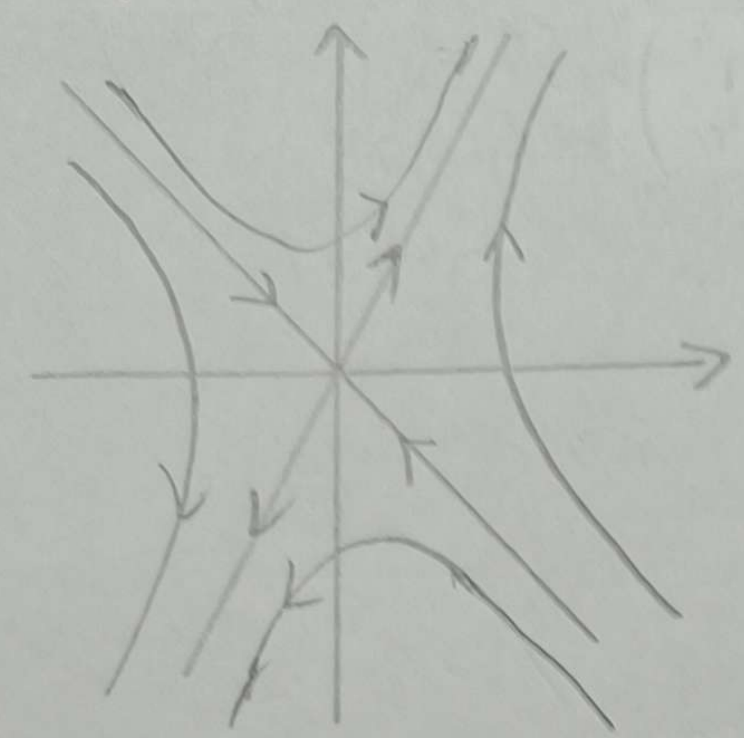
$$X = e^{Jt} \cdot C = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \cdot X^0$$

4. 解:

(1) $A=\begin{pmatrix}3&-2\\2&-2\end{pmatrix}$

$|A-rI|=\begin{vmatrix}3-r&-2\\2&-2-r\end{vmatrix}=(2+r)(r-3)+4=0$

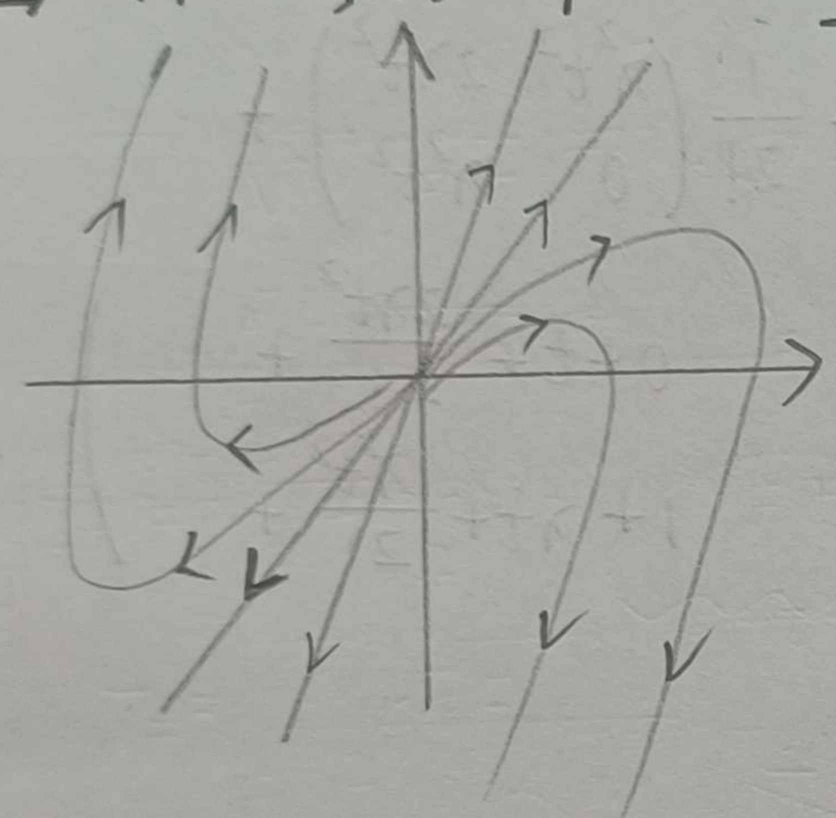
$\Rightarrow r_1=-1,\ r_2=2$ saddle, unstable



(2) $A=\begin{pmatrix}5&-1\\3&1\end{pmatrix}$

$|A-rI|=\begin{vmatrix}5-r&-1\\3&1-r\end{vmatrix}=(1-r)(5-r)+3=0$

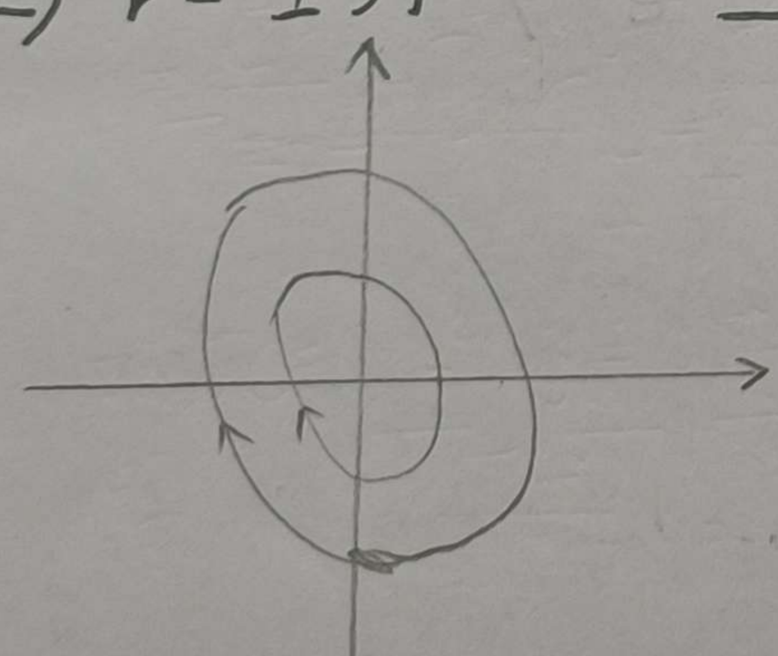
$\Rightarrow r_1=2,\ r_2=4$ node, unstable



(3) $A=\begin{pmatrix}1&2\\-5&-1\end{pmatrix}$

$|A-rI|=\begin{vmatrix}1-r&2\\-5&-1-r\end{vmatrix}=(r-1)(r+1)+10=0$

$\Rightarrow r=\pm 3i$ center, stable



(4) $A=\begin{pmatrix}3&-2\\4&1\end{pmatrix}$

$|A-rI|=\begin{vmatrix}3-r&-2\\4&-1-r\end{vmatrix}=(3-r)(1-r)+8=0$

$\Rightarrow r=\dfrac{+2\pm\sqrt{-16}}{2}=+1\pm 2i$, spiral point unstable