第 10 周习题
常微分方程 B

April 19, 2022

1. Find the Laplace transform of each of the following functions (where a, b are real constants):

(1) $f(t) = \sinh(at) = \dfrac{1}{2}(e^{at} - e^{-at})$.

(2) $f(t) = e^{at}\cos(bt)$.

(3) $f(t) = t\sin(at)$.

2. The gamma function $\Gamma(p)$ is defined by the integral

$$\Gamma(p) = \int_0^{+\infty} x^{p-1}e^{-x}dx.$$

(1) Show that for $p > 0$,

$$\Gamma(p+1) = p\Gamma(p).$$

(2) Show that $\Gamma(1) = 1$ and $\Gamma(n+1) = n!$, where $n \geq 0$ is an integer. Recall that $0! = 1$.

(3) Show that for any $p > -1$,

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0.$$

(4) Find the Laplace transform of $f(t) = t^n e^{at}$, where a is a real constant.

3. Find the inverse Laplace transform of the given functions:

(1) $F(s) = \dfrac{3}{s^2+4}$

(2) $F(s) = \dfrac{4}{(s-1)^3}$

(3) $F(s) = \dfrac{1-2s}{s^2+4s+5}$

(4) $F(s) = \dfrac{8s^2-4s+12}{s(s^2+4)}$

4. Use the Laplace transform to solve the given initial value problem:

(1) $y'' + 2y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 0$

(2) $y^{(4)} - y = 0; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$