第 14 周习题
常微分方程 B

May 17, 2022

1. Find the general solution for each of the following systems.

   (1) $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

   7.8.1.4

   (2) $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$

   (3) $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

   (4) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

2. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x}.$$

7.8.17

   (a) Show that $r = 2$ is an eigenvalue of algebraic multiplicity 3 of $\mathbf{A}$, and that there is only one independent eigenvector, namely,

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

   Therefore, $\mathbf{x}^{(1)} = \boldsymbol{\xi}^{(1)} e^{2t}$ is the only solution of the purely exponential form $\boldsymbol{\xi} e^{rt}$.

(b) To find a second solution, assume that $\mathbf{x} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$. Show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Note that $\boldsymbol{\xi}$ is an eigenvector. Let us take $\boldsymbol{\xi} = \boldsymbol{\xi}^{(1)}$ and solve the second equation for $\boldsymbol{\eta}$. The second equation has infinitely many solutions, take one $\boldsymbol{\eta}$ and write down a second solution $\boldsymbol{\xi}^{(2)}$ of the given system. (You may safely neglect the multiple of $\boldsymbol{\xi}^{(1)}$ that appears in $\boldsymbol{\eta}$.)

(c) To find a third solution, assume that

$$\mathbf{x} = \boldsymbol{\xi} \frac{t^2}{2} e^{2t} + \boldsymbol{\eta} t e^{2t} + \boldsymbol{\zeta} e^{2t}.$$

Show that $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}.$$

Take $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as in part (a) and (b), then solve the third equation for $\boldsymbol{\zeta}$. Write down a third solution $\mathbf{x}^{(3)}$ of the given system.

(d) Write down a fundamental matrix $\Psi(t)$ for the system.

Remark. Let $\mathbf{T}$ be the matrix with the eigenvector $\boldsymbol{\xi}^{(1)}$ in the first column and the generalized eigenvectors $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ in the second and third columns. Then $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is the Jordan form of $\mathbf{A}$.

3. Let $\mathbf{J} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where λ is a real number.

(a) Find $\mathbf{J}^2, \mathbf{J}^3$ and show inductively that $\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

(b) Determine $\exp(\mathbf{J}t)$.

(c) Use $\exp(\mathbf{J}t)$ as a fundamental matrix to solve the initial value problem $\mathbf{x}' = \mathbf{J}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$.

4. With each matrix $\mathbf{A}$ given below, consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) Classify the equilibrium point of the system to type (node, saddle, spiral point, center), and determine whether it is stable, asymptotically stable, or unstable.

(b) Sketch the phase portrait by hand.

(1) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

(2) $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

(3) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$

(4) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$