第 13 周习题
常微分方程 B

May 10, 2022

1. For the following problems,

   (i) show that the functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the given system;

   (ii) show that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions;

   (iii) find the solution of the given system that satisfies the initial condition $\mathbf{x}(0) = (1, 2)^T$.

   (1) $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}; \quad \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$

   (2) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}; \quad \mathbf{x}^{(1)} = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

2. For the second-order equation

$$y'' + p(t)y' + q(t)y = 0, \tag{1}$$

let $x_1 = y$ and $x_2 = y'$, then it corresponds to the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -q(t)x_1 - p(t)x_2. \end{aligned} \tag{2}$$

Show that if $\mathbf{x}^{(1)} = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}$ and $\mathbf{x}^{(2)} = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$ form a fundamental set of solutions of equation (2), and if y_1 and y_2 form a fundamental set of solutions of equation (1), then

$$W[y_1, y_2] = cW[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}],$$

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where c is a nonzero constant. *Hint:* $y_1(t)$ and $y_2(t)$ must be linear combinations of $x_{11}(t)$ and $x_{12}(t)$.

3. Consider the equation

$$ay'' + by' + cy = 0, \tag{3}$$

where a, b and c are constants with $a \neq 0$. We know that the general solution depends on the roots of the characteristic equation

$$ar^2 + br + c = 0. \tag{4}$$

(a) Transform equation (3) into a system of first-order equations by letting $x_1 = y,\ x_2 = y'$. Find the system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ satisfied by $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(b) Find the characteristic polynomial $\det(r\mathbf{I} - \mathbf{A})$ of $\mathbf{A}$. Verify that the characteristic equation for the matrix $\mathbf{A}$ is exactly the equation (4).

4. Find the general solutions for each of the following problems:

(1) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

(2) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

(3) $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{x}$

5. Solve the given initial value problems.

(1) $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(2) $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

6. Find a fundamental matrix $\Psi(t)$ for the following system:

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}.$$

Find also the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = \mathbf{I}$, and use this $\Phi(t)$ to find the solution that satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

7. Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{x} = (x_1, x_2)^T$ and

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$

(a) Find the eigenvalues r_1 and r_2 for $\mathbf{A}$, find also the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.

(b) Let $\mathbf{S}$ be the matrix whose columns are $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$, *i.e.* $\mathbf{S} = [\boldsymbol{\xi}^{(1)}\ \boldsymbol{\xi}^{(2)}]$. Show that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is the diagonal matrix $\mathbf{D} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$. Note that this step is called the diagonalization of $\mathbf{A}$.

(c) Let $\mathbf{y} = \mathbf{S}^{-1}\mathbf{x}$, show that $\mathbf{y}' = \mathbf{D}\mathbf{y}$.

(d) Solve the system $\mathbf{y}' = \mathbf{D}\mathbf{y}$, then use $\mathbf{x} = \mathbf{S}\mathbf{y}$ to find the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(e) Note that $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$. Use $\exp(\mathbf{D}t) = \begin{pmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{pmatrix}$ to find $\exp(\mathbf{A}t)$. You may use the formula $\exp(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}t) = \mathbf{S}\exp(\mathbf{D}t)\mathbf{S}^{-1}$.

(f) Use the fundamental matrix $\Psi(t) = \exp(\mathbf{A}t)$ to solve the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$