力学与航空航天工程系
DEPARTMENT OF MECHANICS AND AEROSPACE ENGINEERING

MECHANICS OF MATERIALS

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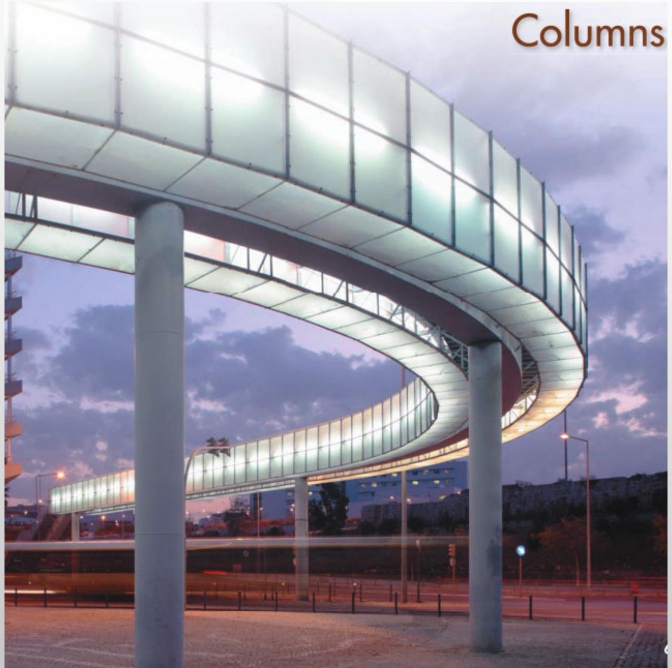
SPRING, 2022

Lesson 11： Columns

- Critical load for structure instability or buckling
- Euler’s formula
- Eccentric axial load

§ 11.1 Introduction

- Our concern will be with the stability of the structure, i.e., with its ability to support a given load without experiencing a sudden change in its configuration.

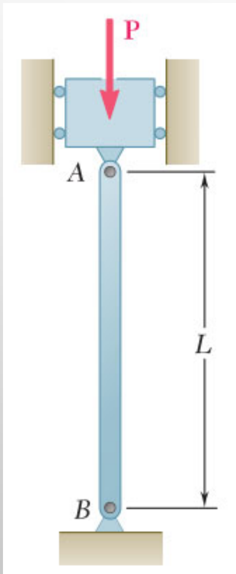


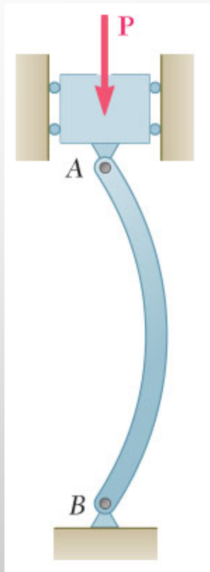
The curved pedestrian bridge is supported by a series of columns.

§ 11.2 Stability of structures

- Suppose we are to design a column AB of length L to support a given load P. The stress on a transverse section is less than the allowable stress for the material used. However, it may happen that, as the load is applied, the column will buckle; instead of remaining straight, it will suddenly become sharply curved.

$$P < P_{cr}$$





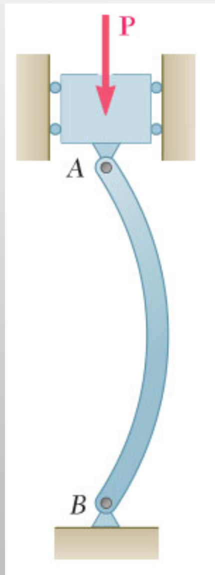
Buckled

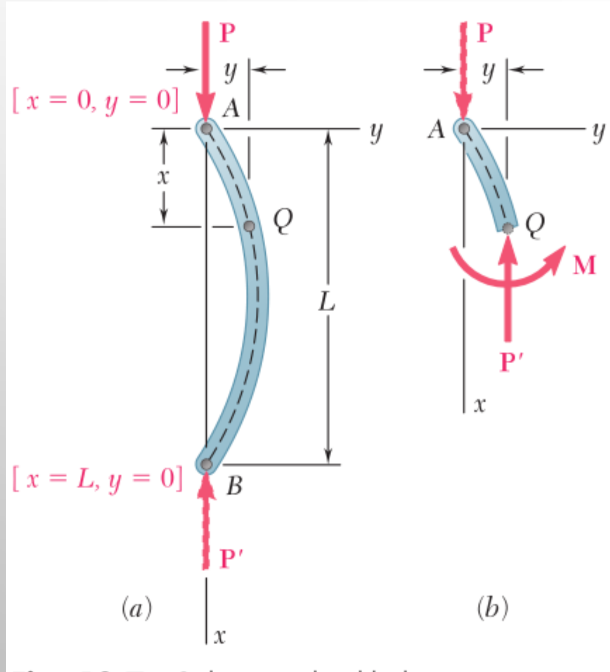
$$P > P_{cr}$$

The slightest misalignment or disturbance will cause the column to buckle

§ 11.3 Euler’s formula for pin-ended columns

- we propose to determine the critical value of the load P, i.e., the value P_{cr} of the load for which the position ceases to be stable. Our approach will be to determine the conditions under which the buckled configuration is possible.





pin-connected at both ends

$$M(x) = -Py$$

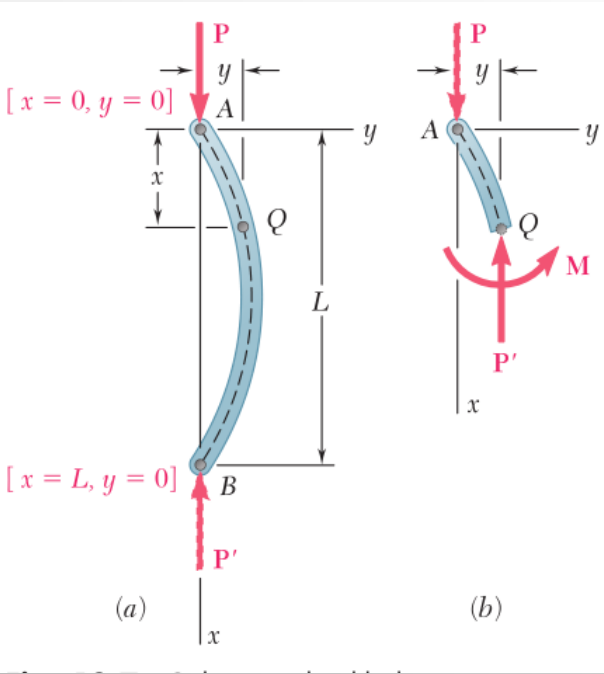
$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} = -\frac{Py}{EI}$$

$$\frac{d^2y}{dx^2} + p^2 y = 0 \qquad p^2 = \frac{P}{EI}$$

a linear, homogeneous differential equation of the second order with constant coefficients

§ 11.3 Euler's formula for pin-ended columns

- Euler's formula, after the Swiss mathematician Leonhard Euler (1707–1783).



$$\frac{d^2y}{dx^2} + p^2 y = 0 \qquad p^2 = \frac{P}{EI}$$

$$y = A\sin px + B\cos px \qquad \begin{cases} x = 0,\ y = 0 \\ x = L,\ y = 0 \end{cases}$$ general solution

$$B = 0 \qquad A\sin pL = 0$$

$$pL = n\pi \qquad P = \frac{n^2\pi^2 EI}{L^2} \xrightarrow{n=1} \boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$$

stable if $P < P_{cr}$

§ 11.3 Euler's formula for pin-ended columns

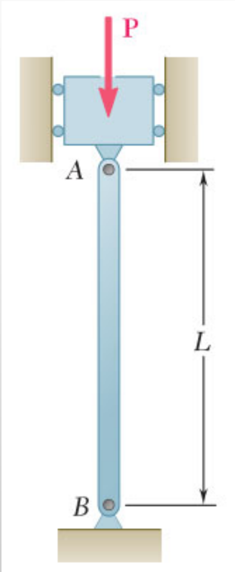
- The critical stress is proportional to the modulus of elasticity of the material, and inversely proportional to the square of the slenderness ratio of the column.

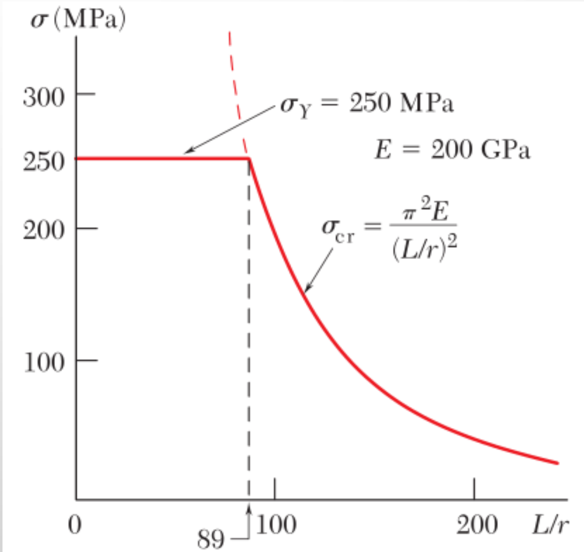
$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$I = Ar^2$$ gyration radius

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

L/r, slenderness ratio



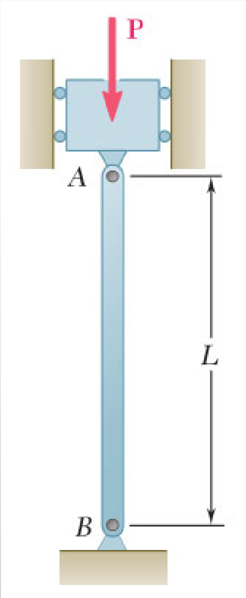


For structural steel, assuming E = 200 GPa and σ_Y = 250 Mpa.

Example 11.1

(Beer, Page 638)

A 2-m-long pin-ended column of square cross section is to be made of wood. Assuming $E = 13$ GPa, $\sigma_{all} = 12$ MPa, and using a factor of safety of 2.5 in computing Euler's critical load for buckling, determine the size of the cross section if the column is to safely support (a) a 100-kN load, (b) a 200-kN load.



Gyration radius (回转半径)

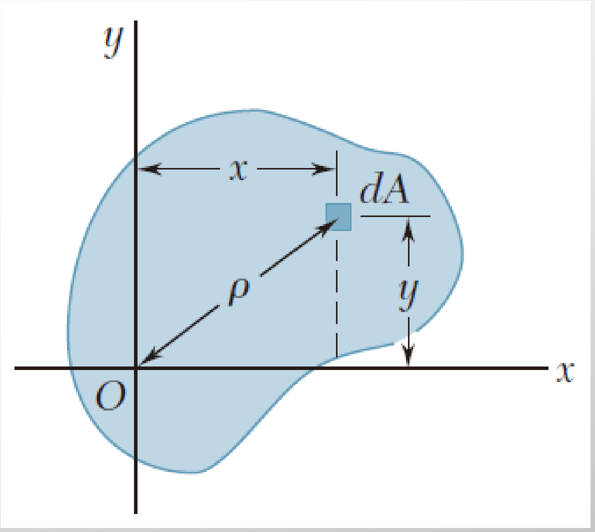
- The radius of gyration of an area A with respect to the x axis is defined as the quantity r_x, that satisfies the relation where I_x is the moment of inertia of A with respect to the x axis.

$$I_x = Ar_x^2$$ L/r_x, slenderness ratio

$$I_x = \int_A y^2\,dA \qquad I_y = \int_A x^2\,dA$$

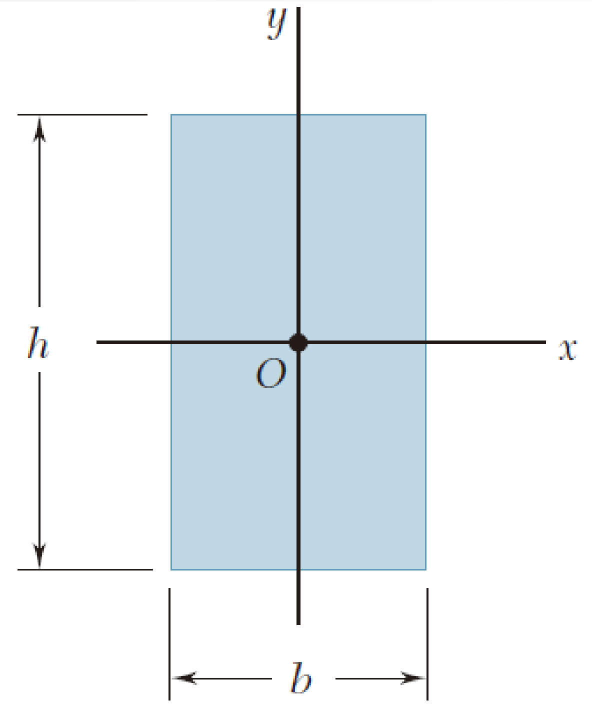
$$J_O = \int_A \rho^2\,dA$$

$$J_O = I_x + I_y$$



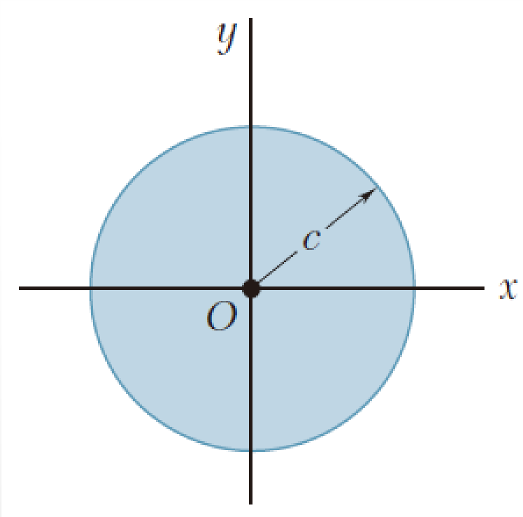
Gyration radius (回转半径)

- For the rectangular and circular areas,



$$I_x = \frac{1}{12} b h^3$$

$$r_x = h/\sqrt{12}$$



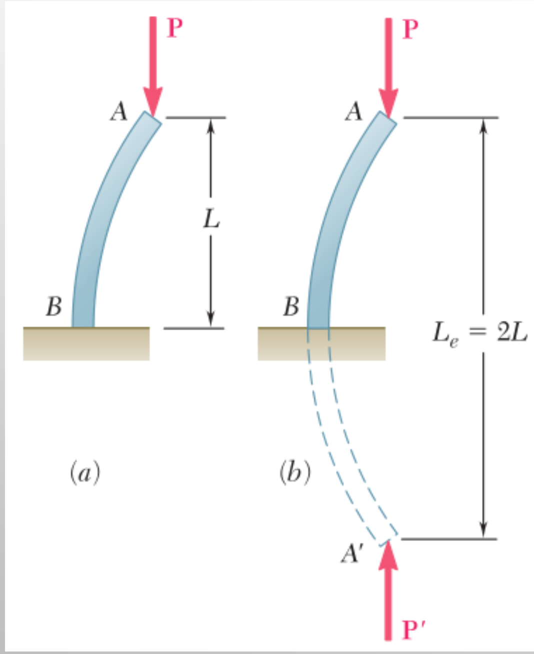
$$J_O = \frac{1}{2}\pi c^4$$

$$I_x = I_y = \frac{1}{4}\pi c^4$$

$$r_x = r_y = r_O/\sqrt{2} = c/2$$

§ 11.4 Columns with other end conditions

- In the case of a column with one free end A supporting a load P and one fixed end B, we observe that the column will behave as the upper half of a pin-connected column.



$$\frac{d^2y}{dx^2} + p^2 y = 0 \qquad p^2 = \frac{P}{EI}$$

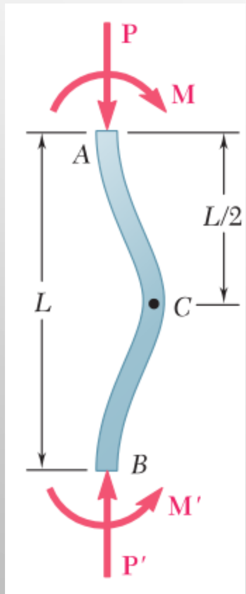
$$y = A\sin px + B\cos px \qquad \begin{cases} x = 0,\ y = 0 \\ x = L,\ dy/dx = 0 \end{cases}$$

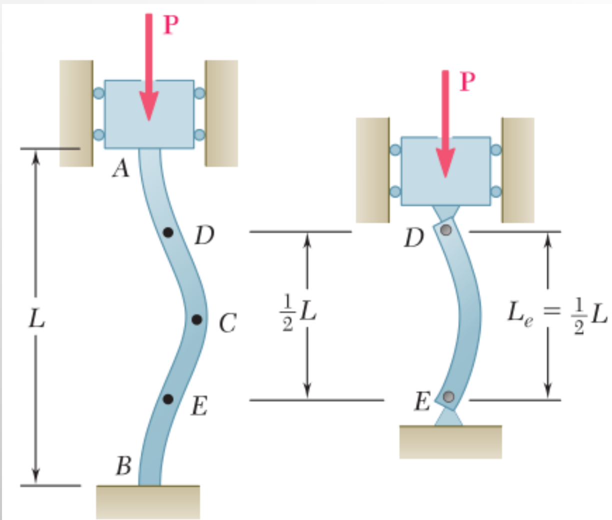
$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \qquad \sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}$$

$$L_e = 2L$$

§ 11.4 Columns with other end conditions

- Consider next a column with two fixed ends A and B supporting a load.





$$\frac{d^2y}{dx^2} + p^2 y = \frac{M}{EI} \qquad p^2 = \frac{P}{EI}$$

$$y = A\sin px + B\cos px + \frac{M}{p^2 EI}$$

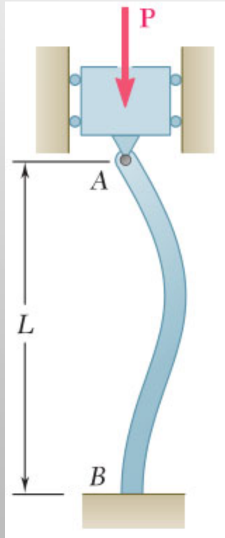
$$\begin{cases} x = 0,\ y = 0, dy/dx = 0 \\ x = L,\ y = 0, dy/dx = 0 \end{cases}$$

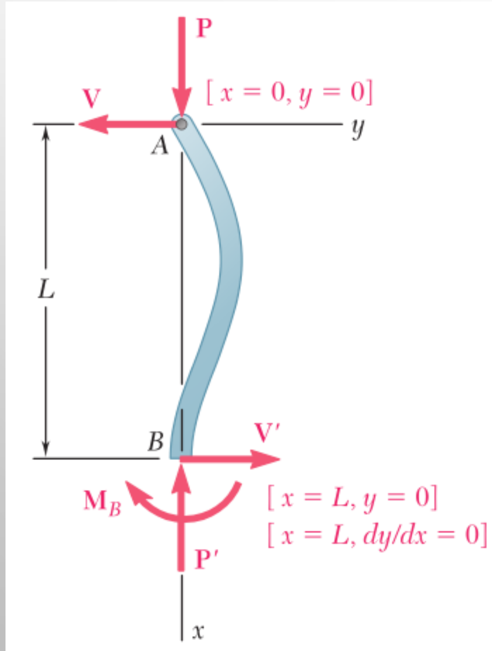
$$\boxed{P_{cr} = \frac{\pi^2 EI}{L_e^2}} \qquad \boxed{L_e = \frac{L}{2}}$$

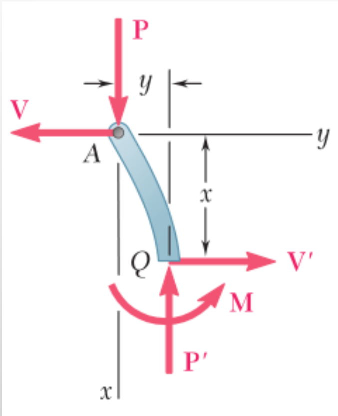
At D and E, $M = 0$ (inflection point)

§ 11.4 Columns with other end conditions

- In the case of a column with one fixed end B and one pin-connected end A supporting a load P, we must write and solve the differential equation of the elastic curve to determine the effective length of the column.







$$M(x) = -Py - Vx$$

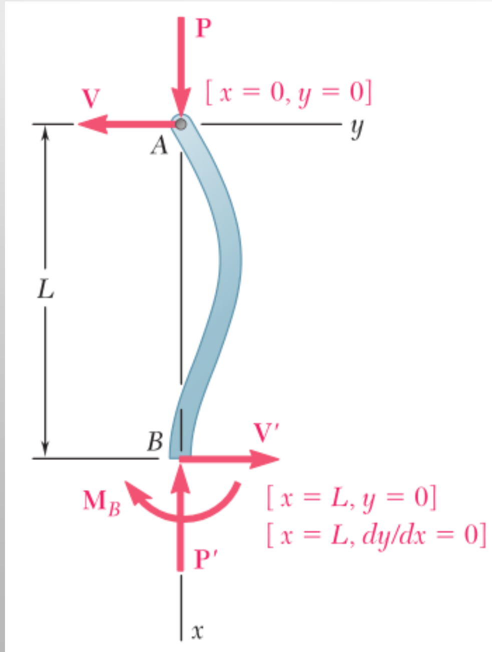
$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} = -\frac{Py}{EI} - \frac{Vx}{EI}$$

$$\frac{d^2y}{dx^2} + p^2 y = -\frac{Vx}{EI}$$

$$y = A\sin px + B\cos px - \frac{V}{P}x$$

§ 11.4 Columns with other end conditions

- In the case of a column with one fixed end B and one pin-connected end A supporting a load P, we must write and solve the differential equation of the elastic curve to determine the effective length of the column.



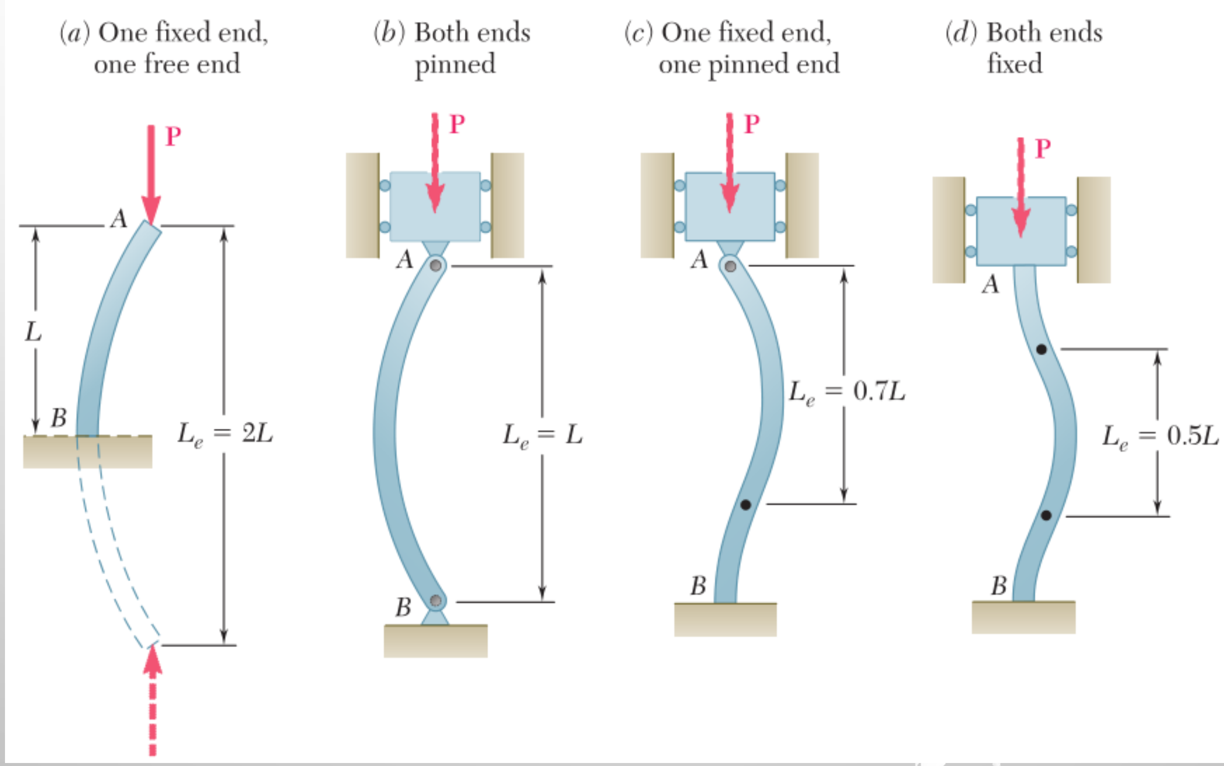
$$y = A\sin px + B\cos px - \frac{V}{P}x \qquad \begin{cases} x = 0,\ y = 0 \\ x = L,\ y = 0 \\ x = L,\ dy/dx = 0 \end{cases}$$

$$B = 0 \quad \tan pL = pL$$

$$pL = 4.49$$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{20.19EI}{L^2} \qquad L_e = 0.7L$$

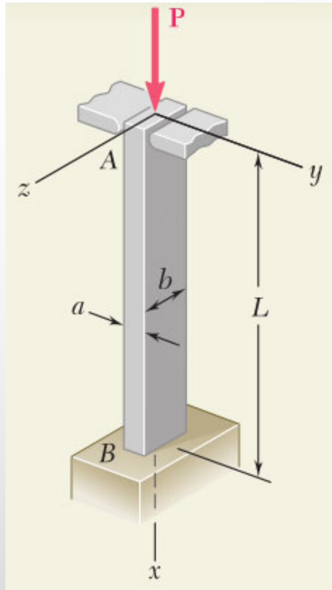
§ 11.4 Columns with other end conditions

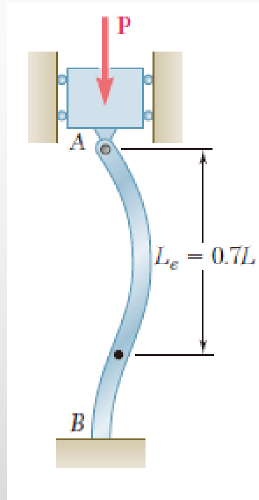


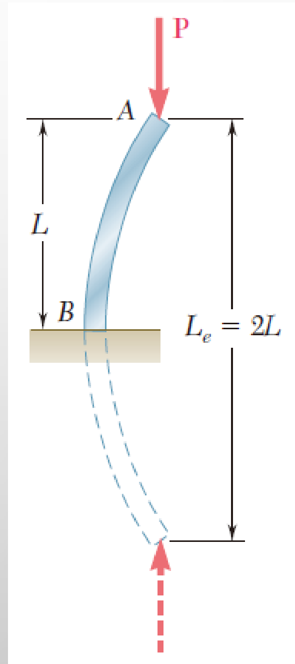
Example 11.2

(Beer, Page 643)

An aluminum column of length L and rectangular cross section has a fixed end B and supports a centric load at A. Two smooth and rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry of the column, but allow it to move in the other plane. (a) Determine the ratio a/b of the two sides of the cross section corresponding to the most efficient design against buckling. (b) Design the most efficient cross section for the column, knowing that L = 20 in., E = 10.1 × 10^6 psi, P = 5 kips, and that a factor of safety of 2.5 is required.

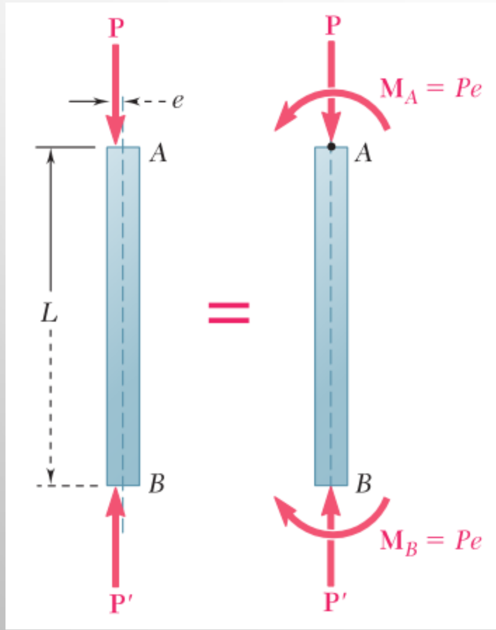


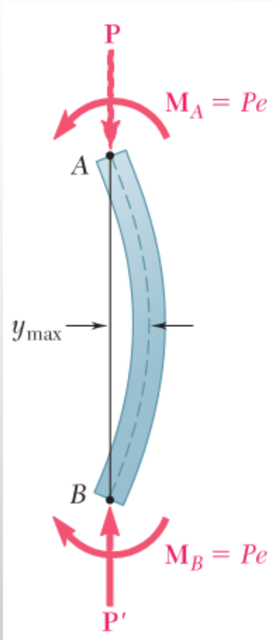




§ 11.5 Eccentric loading

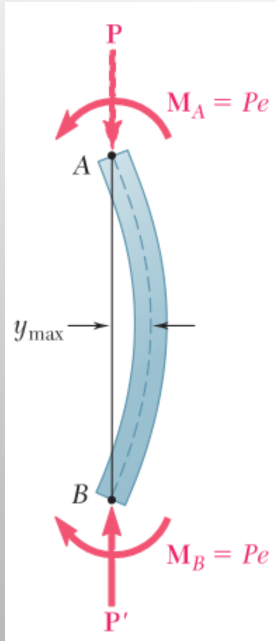
- The problem of column buckling will be approached in a different way, by observing that the load P applied to a column is never perfectly centric.

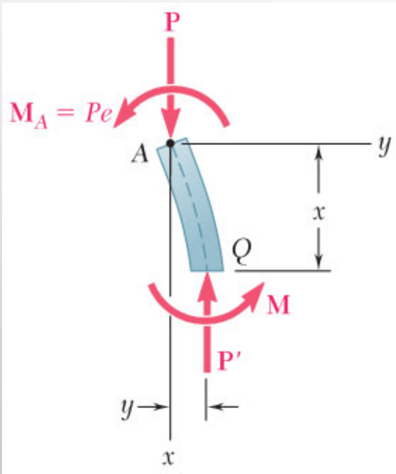




§ 11.5 Eccentric loading

- The problem of buckling is how much the column can be permitted to bend under the increasing load, if the allowable stress is not to be exceeded and if the deflection y_{max} is not to become excessive.





$$M(x) = -Py - M_A = -Py - Pe$$

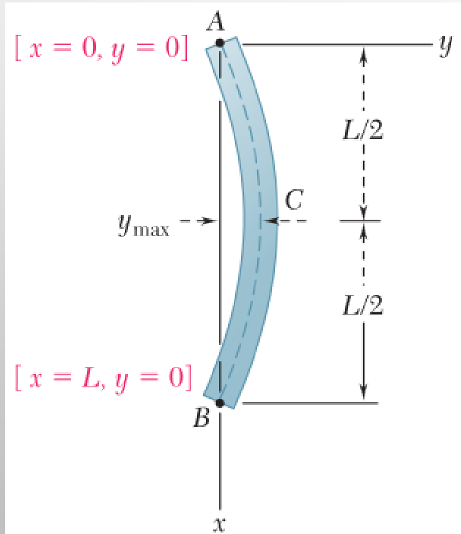
$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} = -\frac{Py}{EI} - \frac{Pe}{EI}$$

$$\frac{d^2y}{dx^2} + p^2y = -p^2e$$

$$y = A\sin px + B\cos px - e$$

§ 11.5 Eccentric loading

- While the deflection does not actually become infinite, it nevertheless becomes unacceptably large, and P should not be allowed to reach the critical value



$$y = A\sin px + B\cos px - e \qquad \begin{cases} x = 0,\ y = 0 \\ x = L,\ y = 0 \end{cases} \qquad \text{Pin-connected}$$

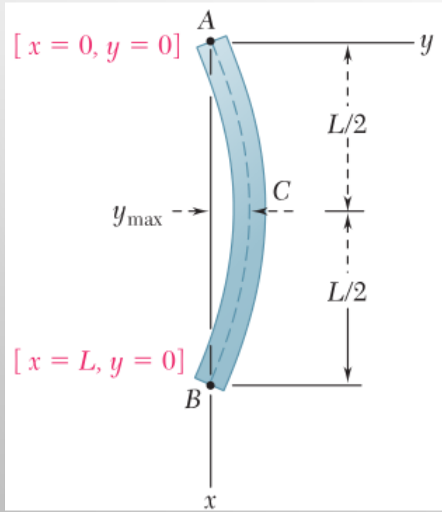
$$B = e \qquad A = e\tan\frac{pL}{2}$$

$$y = e\left(\tan\frac{pL}{2}\sin px + \cos px - 1\right)$$

$$y_{\max} = e\left(\sec\frac{pL}{2} - 1\right) \to \infty \quad \frac{pL}{2} = \frac{\pi}{2} \quad \longrightarrow \quad \boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$$

§ 11.5 Eccentric loading

- The maximum stress $\sigma_{\max}$ occurs in the section of the column where the bending moment is maximum, i.e., in the transverse section through the midpoint C.



$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$y_{\max} = e\left(\sec\sqrt{\frac{P}{EI}}\frac{L}{2} - 1\right)$$

$$y_{\max} = e\left(\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} - 1\right)$$

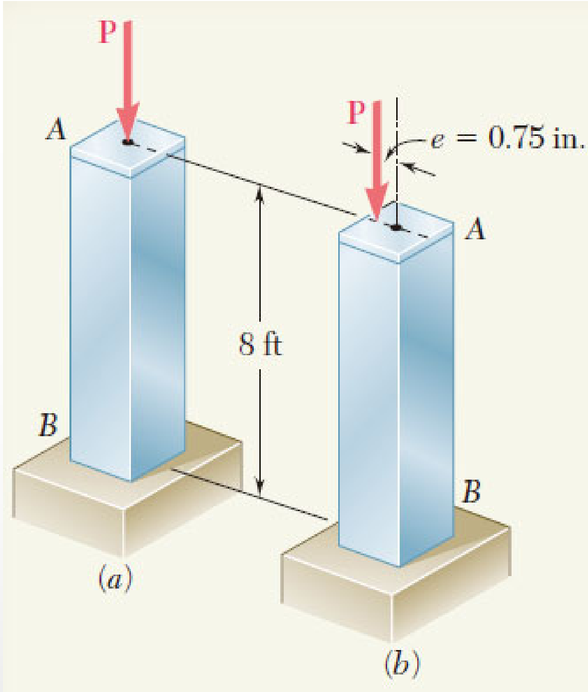
$$\sigma_{\max} = \frac{P}{A} + \frac{M_{\max}c}{I} \qquad M_{\max} = P(y_{\max} + e)$$

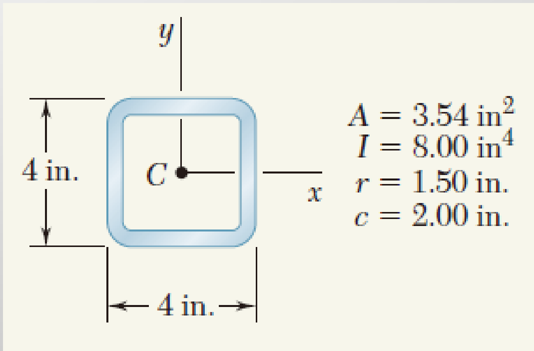
$$\sigma_{\max} = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

Example 11.3

(Beer, Page 654)

The uniform column AB consists of an 8-ft section of structural tubing having the cross section shown. (a) Using Euler's formula and a factor of safety of two, determine the allowable centric load for the column and the corresponding normal stress. (b) Assuming that the allowable load, found in part a, is applied as shown at a point 0.75 in. from the geometric axis of the column, determine the horizontal deflection of the top of the column and the maximum normal stress in the column. Use E $=29 \times 10^6$ psi.





§ 11.6 Summary

- **Critical load for structure instability**
- **Euler’s formula**
- **Effective length**
- **Eccentric axial load**