常微分方程B Week 16

1.解：
$$\begin{cases}\dfrac{dx}{dt} = -x^3+xy^2\\ \dfrac{dy}{dt} = -2x^2y-y^3\end{cases}$$

$V(x,y) = ax^2+cy^2$

$\dot{V}(x,y) = V_x F + V_y G$

$= 2ax\cdot(-x^3+xy^2)+2cy\cdot(-2x^2y-y^3)$

$= -2ax^4+2ax^2y^2-4cx^2y^2-2cy^4$

$= (2a-4c)x^2y^2-2ax^4-2cy^4$

令 $a>0$，$c>0$，$2a-4c=0$ $\Rightarrow$ $V(0,0)=0$，$V(x,y)>0$ if $(x,y)\neq 0$，正定；$\dot{V}(0,0)=0$，$\dot{V}(x,y)<0$ if $(x,y)\neq 0$，负定

$\therefore (0,0)$ is asymptotically stable

3.解：
$$\begin{cases}\dfrac{dx}{dt} = x^3-y^3\\ \dfrac{dy}{dt} = 2xy^2+4x^2y+2y^3\end{cases}$$

$V(x,y) = ax^2+cy^2$

$\dot{V}(x,y) = V_x F+V_y\cdot G$

$= 2ax\cdot(x^3-y^3)+2cy\cdot(2xy^2+4x^2y+2y^3)$

$= 2ax^4-2axy^3+4cxy^3+8cx^2y^2+4cy^4$

$= 2ax^4+8cx^2y^2+4cy^4+xy^3(4c-2a)$

令 $a>0$，$c>0$，$4c-2a=0$ $\Rightarrow$ $V(0,0)=0$，$V(x,y)>0$ if $(x,y)\neq 0$，正定；$\dot{V}(0,0)=0$，$\dot{V}(x,y)>0$ if $(x,y)\neq 0$，正定

$\therefore (0,0)$ is unstable

4.解：

$V(x) = cx^2+cy^2$

$\dot{V}(x) = V_x F+V_y\cdot G$

$= 2cx\cdot(y-xf)+2cy\cdot(-x-yf)$

$= 2cxy-2cx^2f+(-2c)xy-2cy^2f$

$= -2cf\cdot(x^2+y^2)$

$= -2c\cdot f(x,y)\cdot(x^2+y^2)$

令 $C>0$；$V(0,0)=0$，$V(x,y)>0$ if $(x,y)\neq 0$，正定

if $f(x,y)>0$ near $(0,0)$,

$\dot{V}(x,y)$ 负定，$(0,0)$ is asymptotically stable

if $f(x,y)<0$ near $(0,0)$,

$\dot{V}(x,y)$ 正定，$(0,0)$ is unstable

6.解：

a. proof：
$$\begin{cases}y=0\\ -y-\sin x=0\end{cases}$$
$\Rightarrow (0,0)$ is a critical point

b. proof：$V = x^2+y^2$

$\dot{V}(x,y) = V_x F+V_y G$

$= 2x\cdot y+2y\cdot(-y-\sin x)$

$= 2xy-2y^2-2y\sin x$

$= 2y(x-\sin x)-2y^2$

ignore $2y^2$

set $y>0$，$\dot{V}(x,y) = 2y(x-\sin x)$

$x>0$，$x-\sin x>0$

$x<0$，$x-\sin x<0$

$\therefore \dot{V}(x,y)$ is not a Liapunov function

c. proof: near $(0,0)$

$V(x,y) = \frac{1}{2}y^2+(1-\cos x)>0$ if $(x,y)\neq 0$，正定

$\dot{V}(x,y) = \sin x\cdot y + y\cdot(-y-\sin x) = -y^2$，半负定 $(4ac-b^2=0)$

$\therefore (0,0)$ is stable (at least)

d. $V(x,y)$ 正定

$\dot{V}(x,y) = [(x+y)+2x]\cdot y+[(x+y)+y]\cdot(-y-\sin x)$

$= 2xy-y^2-\sin x(x+2y)$

$= 2xy-y^2-(x+2y)\cdot\left(x-\frac{x^3}{3!}\cdot x\right)$

$= -(x^2+y^2) + \frac{\alpha}{3!}x^4 + 2y\cdot\alpha\cdot\frac{x^3}{3!}$

令 $x = r\cos\theta$

$y = r\sin\theta$

$\dot{V}(r\cos\theta, r\sin\theta) = -r^2 + \frac{\alpha r^4}{6}(\cos\theta + 2\sin\theta)\cos^3\theta$

$= -r^2\left[1 - \frac{\alpha r^2}{6}(\cos\theta+2\sin\theta)\cos^3\theta\right]$

$1-\frac{\alpha r^2}{6}(\cos\theta+2\sin\theta)\cos^3\theta > 0$, as r small, $\dot{V}(x,y)$ 负定.

$\therefore (0,0)$ is asymptotically stable

9.解:

a. $\begin{cases} \frac{dx}{dt} = a_{11}x + a_{12}y \\ \frac{dy}{dt} = a_{21}x + a_{22}y \end{cases}$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad T = a_{11}+a_{22}$, $D = a_{11}a_{22} - a_{21}a_{12}$

since $(0,0)$ is asymptotically stable

$\begin{matrix} T<0 \\ D>0 \end{matrix} \Rightarrow \begin{matrix} a_{11}+a_{22}<0 \\ a_{11}a_{22}-a_{21}a_{12}>0 \end{matrix}$

b. $V(x,y) = Ax^2+Bxy+Cy^2$

$\dot{V}(x,y) = (2Ax+By)\cdot(a_{11}x+a_{12}y) + (Bx+2Cy)\cdot(a_{21}x+a_{22}y)$

$= 2Aa_{11}x^2 + 2Aa_{12}xy + Ba_{11}xy + Ba_{12}y^2 + Ba_{21}x^2 + Ba_{22}xy + 2Ca_{21}xy + 2Ca_{22}y^2$

$= (2Aa_{11}+Ba_{21})x^2 + (Ba_{12}+2Ca_{22})y^2 + (2Aa_{12}+Ba_{11}+Ba_{22}+2Ca_{21})xy$

$\begin{cases} 2Aa_{12}+Ba_{11}+Ba_{22}+2Ca_{21} = 0 \\ 2Aa_{11}+Ba_{21} = Ba_{12}+2Ca_{22} = -1 \end{cases}$

$\Rightarrow A = -\frac{a_{21}^2+a_{22}^2+(a_{11}a_{22}-a_{12}a_{21})}{2\Delta}$

$B = \frac{a_{12}a_{22}+a_{11}a_{21}}{\Delta}$

$C = -\frac{a_{11}^2+a_{12}^2+(a_{11}a_{22}-a_{12}a_{21})}{2\Delta}$

c. $\Delta = (a_{11}+a_{22})(a_{11}a_{22}-a_{12}a_{21}) < 0$

$-2\Delta > 0$

$a_{21}^2+a_{22}^2+(a_{11}a_{22}-a_{12}a_{21}) > 0$

$\therefore A>0$

$\Delta^2 > 0$

$(a_{11}a_{22}-a_{12}a_{21})^2 > 0$

$\therefore 4AC-B^2 > 0$

$\therefore V$ 正定

10. 解:

a. proof:

$$\dot{V}(x,y) = (2Ax+By)\cdot[a_{11}x+a_{12}y+F_1(x,y)] +$$
$$(Bx+2Cy)\cdot[a_{21}x+a_{22}y+G_1(x,y)]$$
$$= (2Ax+By)\cdot F_1(x,y) + (Bx+2Cy)\cdot G_1(x,y) +$$
$$(2Ax+By)(a_{11}x+a_{12}y)+$$
$$(Bx+2Cy)\cdot(a_{12}x+a_{22}y)$$

$$(2Ax+By)(a_{11}x+a_{12}y)+(Bx+2Cy)\cdot(a_{12}x+a_{22}y)$$
$$= -x^2-y^2$$ 由 problem 9.

$$\therefore \dot{V}(x,y) = -(x^2+y^2) + (2Ax+By)F_1(x,y) +$$
$$(Bx+2Cy)\cdot G_1(x,y)$$

b. proof:

$$M = \max\{|2A|, |B|, |2C|\}$$
$$x = r\cos\theta,\ y = r\sin\theta$$
$$\varepsilon > 0$$
$$|F_1(x,y)| < \varepsilon r,$$
$$|G_1(x,y)| < \varepsilon r, \quad 0<r<R$$

$$\dot{V}(x,y) \le -x^2-y^2 + |(2Ax+yB)F_1(x,y)| + |(Bx+2Cy)G_1(x,y)|$$
$$\le -r^2\cos^2\theta - r^2\sin^2\theta + (|2A|r+|B|r)|F_1(x,y)|$$
$$+ (|B|r+|2C|r)|G_1(x,y)|$$
$$\le -r^2(\cos^2\theta+\sin^2\theta) + (Mr+Mr)|F_1(x,y)| +$$
$$(Mr+Mr)|G_1(x,y)|$$
$$\le -r^2 + 2Mr\cdot 2\varepsilon r = -r^2(1-4M\varepsilon)$$

$$\varepsilon \to 0 \Rightarrow \dot{V}(x,y) < 0$$

$\therefore$ $V(x,y)$ is a Liapunov function.