计算固体力学 HW 2

1. Solution

$$\int_0^1 w_{,x} u_{,x}\, dx = \int_0^1 wf\, dx + w(0)h \qquad (W)$$

$$w \in \mathcal{V} = \{w : w \in H^1,\ w(1)=0\}$$

$$u \in \mathcal{S} = \{u : u \in H^1,\ u(1)=g\}$$

w and u are smooth on (x_A, x_{A+1})

$A = 1, 2, 3, \cdots, n$

slope discontinuities across element boundaries

IBP:

$$\int_0^1 w_{,x} u_{,x}\, dx = -\int_0^1 w u_{,xx}\, dx + \underbrace{w(1)}_{0} u_{,x}(1) - w(0)u_{,x}(0)$$

$$= -\int_0^1 w u_{,xx}\, dx - w(0)u_{,x}(0)$$

$$(W) \Rightarrow -\int_0^1 w u_{,xx}\, dx - w(0)u_{,x}(0) = \int_0^1 wf\, dx + w(0)h$$

$$\int_0^1 w(f + u_{,xx})\, dx + w(0)\left[u_{,x}(0) + h\right] = 0$$

$\Rightarrow$ $\Omega = (0,1)$

$$\sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w(f + u_{,xx})\, dx + w(0)\left[u_{,x}(0^+) + h\right] = 0$$

Since w and u are smooth on (x_A, x_{A+1}) and $\Omega = (0,1)$

$$\sum_{A=2}^{n} w(x_A)\left[u_{,x}(x_A^+) - u_{,x}(x_A^-)\right] \quad ?$$

The above process made one mistake, which is that the slop discontinulties across element boundaries are not considered.

Correction:

IBP:

$$\int_0^1 w_{,x} u_{,x}\, dx = \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w_{,x} u_{,x}\, dx$$

$$= \sum_{A=1}^{n} \left\{ -\int_{x_A}^{x_{A+1}} w u_{,xx}\, dx + (w u_{,x})\Big|_{x_A}^{x_{A+1}} \right\}$$

$$= \sum_{A=1}^{n} \left(-\int_{x_A}^{x_{A+1}} w u_{,xx}\, dx \right) + (w u_{,x})\Big|_{x_1=0}^{x_2} + \sum_{A=2}^{n} \left(w u_{,x} \Big|_{x_A}^{x_{A+1}} \right)$$

$$= \sum_{A=1}^{n} \left(-\int_{x_A}^{x_{A+1}} w u_{,xx}\, dx \right) + w(x_2)u_{,x}(x_2) - w(0)u_{,x}(0^+) + \sum_{A=2}^{n} \left(w u_{,x} \Big|_{x_A}^{x_{A+1}} \right)$$

$$= \sum_{A=1}^{n} \left(-\int_{x_A}^{x_{A+1}} w u_{,xx}\, dx \right) - w(0)u_{,x}(0^+) + \sum_{A=2}^{n} \left[w(x_A)u_{,x}(x_A^+) - w(x_A)u_{,x}(x_A^-) \right] \qquad (*)$$

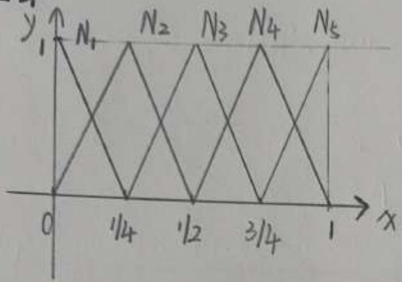
$$\text{Now, } (W) \Rightarrow (*) = \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} wf\, dx + w(0)h$$

$$\Rightarrow \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w(f + u_{,xx})\, dx + w(0)\left[h + u_{,x}(0^+)\right] + \sum_{A=2}^{n} w(x_A)\left[u_{,x}(x_A^+) - u_{,x}(x_A^-)\right] = 0$$

2. Solution

a) $\begin{cases} u_{,xx}(x) + ax = 0 \\ u(1) = 0 \\ u_{,x}(0) = 0 \end{cases}$

$n=4$



$N_1(x) = \begin{cases} -4x+1 & 0 \le x \le 1/4 \\ 0 & 1/4 < x < 1 \end{cases}$ $\quad N_{1,x}(x) = \begin{cases} -4 & 0 \le x \le 1/4 \\ 0 & 1/4 < x < 1 \end{cases}$

$N_2(x) = \begin{cases} 4x & 0 \le x \le 1/4 \\ -4x+2 & 1/4 < x \le 1/2 \\ 0 & 1/2 < x < 1 \end{cases}$ $\quad N_{2,x}(x) = \begin{cases} 4 & \sim \\ -4 & \sim \\ 0 & \sim \end{cases}$

$N_3(x) = \begin{cases} 0 & 0 \le x \le 1/4 \\ 4x-1 & 1/4 < x \le 1/2 \\ -4x+3 & 1/2 < x \le 3/4 \\ 0 & 3/4 < x \le 1 \end{cases}$ $\quad N_{3,x}(x) = \begin{cases} 0 & \sim \\ 4 & \sim \\ -4 & \sim \\ 0 & \sim \end{cases}$

$N_4(x) = \begin{cases} 0 & 0 \le x \le 1/2 \\ 4x-2 & 1/2 < x \le 3/4 \\ -4x+4 & 3/4 < x \le 1 \end{cases}$ $\quad N_{4,x}(x) = \begin{cases} 0 & \sim \\ 4 & \sim \\ -4 & \sim \end{cases}$

$N_5(x) = \begin{cases} 0 & 0 < x \le 3/4 \\ 4x-3 & 3/4 < x \le 1 \end{cases}$ $\quad N_{5,x}(x) = \begin{cases} 0 & \sim \\ 4 & \sim \end{cases}$

$K_{AB} = a(N_A, N_B)$

$$\Rightarrow K = \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & -4 & 8 \end{bmatrix}$$

$$F_A = (N_A, f) + N_A(0)h - a(N_A, N_{n+1})g$$

$$= (N_A, ax) + 0 - 0 = (N_A, ax)$$

$$F_1 = (N_1, ax) = \int_0^{1/4} (-4x+1)ax\,dx$$

$$= a \cdot \frac{1}{96}$$

$$F_2 = (N_2, ax) = \int_0^{1/4} 4x \cdot ax\,dx + \int_{1/4}^{1/2} (-4x+2) \cdot ax\,dx$$

$$= a \cdot \frac{1}{16}$$

$$F_3 = (N_3, ax) = \int_{1/4}^{1/2} (4x-1)ax\,dx + \int_{1/2}^{3/4} (-4x+3)ax\,dx$$

$$= a \cdot \frac{1}{8}$$

$$F_4 = (N_4, ax) = \int_{1/2}^{3/4} (4x-2)ax\,dx + \int_{3/4}^{1} (-4x+4)ax\,dx$$

$$= a \cdot \frac{3}{16}$$

$Kd = F$

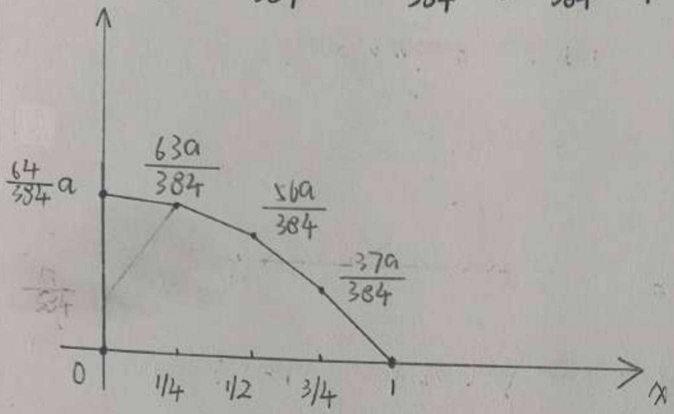
$$\begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{96}a \\ \frac{1}{16}a \\ \frac{1}{8}a \\ \frac{3}{16}a \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 4 & -4 & 0 & 0 & \frac{1}{96}a \\ -4 & 8 & -4 & 0 & \frac{1}{16}a \\ 0 & -4 & 8 & -4 & \frac{1}{8}a \\ 0 & 0 & -4 & 8 & \frac{3}{16}a \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 4 & 0 & 0 & 0 & \frac{64}{96}a \\ 0 & 4 & 0 & 0 & \frac{63}{96}a \\ 0 & 0 & 4 & 0 & \frac{56}{96}a \\ 0 & 0 & 0 & 4 & \frac{37}{96}a \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} \frac{64}{384}a \\ \frac{63}{384}a \\ \frac{56}{384}a \\ \frac{37}{384}a \end{bmatrix}$$

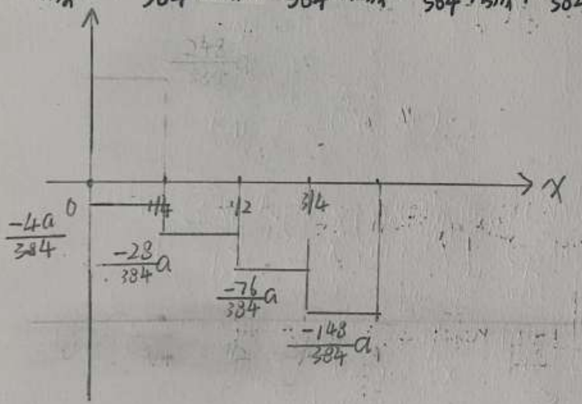
$$u^h = d_1 N_1 + d_2 N_2 + d_3 N_3 + d_4 N_4 + \overset{110}{9} N_5$$
$$= d_1 N_1 + d_2 N_2 + d_3 N_3 + d_4 N_4$$
$$= \frac{64a}{384} N_1 + \frac{63}{384} a N_2 + \frac{56}{384} a N_3 + \frac{37}{384} a N_4$$



$u^h(0) = \frac{64a}{384}$ $\quad u^h(1/2) = \frac{56a}{384}$

$u^h(1/4) = \frac{63a}{384}$ $\quad u^h(3/4) = \frac{37}{384}a$

$$u^h_{,x} = \frac{64a}{384} N_{1,x} + \frac{63a}{384} N_{2,x} + \frac{56a}{384} N_{3,x} + \frac{37a}{384} N_{4,x}$$



$$\frac{64a}{384}(-4) + \frac{63a}{384} \cdot 4 = \frac{-4}{384}a$$

$$\frac{63a}{384}(-4) + \frac{56a}{384} \cdot 4 = \frac{-28}{384}a$$

$$-4 \times \frac{56a}{384} + 4 \times \frac{37a}{384} = -\frac{76}{384}a$$

$$\frac{37a}{384} \times (-4) = \frac{-148}{384}a$$

b) $re_{,x} = \dfrac{|u^h_{,x} - u_{,x}|}{a/2}$

for $\begin{cases} u_{,xx} + ax = 0 \\ u(1) = 0 \\ u_{,x}(0) = 0 \end{cases}$

$$\Rightarrow u(x) = \frac{1}{6}a(1 - x^3)$$
$$u_{,x}(x) = -\frac{1}{2}ax^2$$

At midpoints of the four elements

$x = 1/8,\ 3/8,\ 5/8,\ 7/8$

$$re_{,x}(1/8) = \frac{\left|\frac{-4}{384}a - \left[-\frac{1}{2}a \times (\frac{1}{8})^2\right]\right|}{a/2} = \frac{1}{192}$$

$$re_{,x}(3/8) = \frac{\left|\frac{-28a}{384} + \frac{1}{2}a \times \frac{9}{64}\right|}{a/2} = \frac{1}{192}$$

$$re_{,x}(5/8) = \frac{\left|\frac{-76a}{384} + \frac{1}{2}a \times \frac{25}{64}\right|}{a/2} = \frac{1}{192}$$

$$re_{,x}(7/8) = \frac{\left|-\frac{148}{384}a + \frac{1}{2}a \times \frac{49}{64}\right|}{a/2} = \frac{1}{192}$$

They are all equal. $(\frac{1}{192})$

~~c) for h = 1~~

$\begin{cases} \cancel{u_{,xx} + ax = 0} \\ \cancel{u(1) = 0} \\ \cancel{u_{,x}(0) = h = 1} \end{cases} \Rightarrow \cancel{u_{,x}(x) = -\frac{1}{2}ax^2 - 1}$

$\cancel{F_A = (N_A, ax) + N_A(0)h = (N_A, ax) + N_A(0)}$

$\cancel{F_1 = (N_1, ax) + N_1(0) = \frac{1}{96}a + 1}$

$\cancel{F_2 = \frac{1}{16}a,\ F_3 = \frac{1}{8}a,\ F_4 = \frac{3}{16}a}$

$$\cancel{\Rightarrow \left[\begin{array}{cccc|c} 4 & -4 & 0 & 0 & \frac{1}{96}a+1 \\ -4 & 8 & -4 & 0 & \frac{1}{16}a \\ 0 & -4 & 8 & -4 & \frac{1}{8}a \\ 0 & 0 & -4 & 8 & \frac{3}{16}a \end{array}\right] \Rightarrow \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} \frac{64a}{384}+1 \\ \frac{63a}{384}+\frac{3}{4} \\ \frac{56a}{384}+\frac{2}{4} \\ \frac{37a}{384}+\frac{1}{4} \end{bmatrix}}$$

$$\cancel{u^h_{,x} = (\frac{64a}{384}+1)N_{1,x} + (\frac{63a}{384}+3/4)N_{2,x}}$$
$$\cancel{+ (\frac{56a}{384}+2/4)N_{3,x} + (\frac{37a}{384}+1/4)N_{4,x}}$$

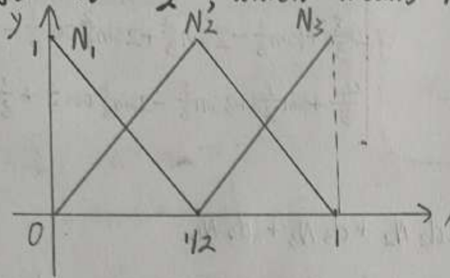
$$\cancel{\Rightarrow u^h_{,x} = \begin{cases} (\frac{64a}{384}+1)\times(-4) + (\frac{63a}{384}+3/4)\times 4 = \frac{-4a}{384} - 1, (0, 1/4) \\ (\frac{63a}{384}+3/4)\times(-4) + (\frac{56a}{384}+2/4)\times 4 = \frac{-28a}{384} - 1, (1/4, 1/2) \\ \cdots \quad = \frac{-76a}{384} - 1, (1/2, 3/4) \\ \cdots \quad = \frac{-148a}{384} - 1, (3/4, 1) \end{cases}}$$

c) for h = mesh parameter $= \frac{1}{4}$

$re_{,x}$ at the midpoints $= \frac{1}{192}$

as calculated in part (b)

for $h = \frac{1}{2}$, which means $n = 2$



we calculated this case in the class :

$$K = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

$$F_A = (N_A, ax) + N_A(0)\cdot 0 - a(N_A, N_{n+1})\cdot 0$$

$$= (N_A, ax)$$

$$\Rightarrow F_1 = \frac{a}{24}, \quad F_2 = \frac{a}{4}$$

$$Kd = F \Rightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6}a \\ \frac{7}{48}a \end{bmatrix}$$

$$u^h = d_1 N_1 + d_2 N_2 + g N_3$$

$$= \frac{1}{6}a N_1 + \frac{7}{48}a N_2$$

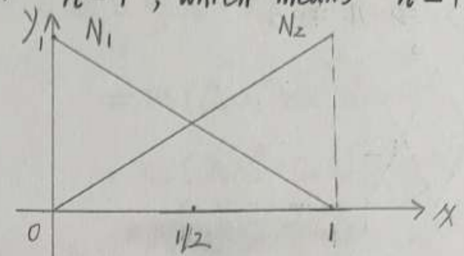
$$u^h_{,x} = \frac{1}{6}a N_{1,x} + \frac{7}{48}a N_{2,x}$$

$$= \begin{cases} -\frac{a}{24} & (0, 1/2) \\ -\frac{7a}{24} & (1/2, 1) \end{cases}$$

$$re_{,x}(1/4) = \frac{\left|-\frac{a}{24} + \frac{1}{2}a \times \frac{1}{16}\right|}{a/2} = \frac{1}{48}$$

$$re_{,x}(3/4) = \frac{\left|-\frac{7a}{24} + \frac{1}{2}a \times \frac{9}{16}\right|}{a/2} = \frac{1}{48}$$

for $h = 1$, which means $n = 1$



$N_1 = -x + 1$, $N_2 = x$ at $[0, 1]$

$N_{1,x} = -1$, $N_{2,x} = 1$

$$K = a(N_1, N_1) = \int_0^1 (-1)(-1)\,dx = 1$$

$$F_1 = (N_1, ax) = \int_0^1 (-x+1)\,ax\,dx = \frac{1}{6}a$$

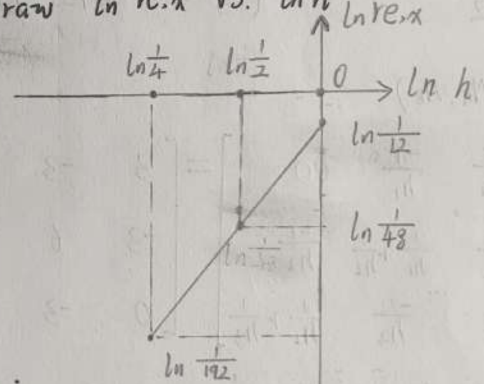
$$\Rightarrow d_1 = \frac{1}{6}a$$

$$u^h = d_1 N_1 + g N_2 = \frac{1}{6}a N_1$$

$$u^h_{,x} = \frac{1}{6}a N_{1,x} = -\frac{1}{6}a \quad [0, 1]$$

$$re_{,x}(1/2) = \frac{\left|-\frac{1}{6}a + \frac{1}{2}a \times \frac{1}{4}\right|}{a/2} = \frac{1}{12}$$

Draw ln $re_{,x}$ vs. ln h



d) i.

$$\frac{\ln\frac{1}{48} - \ln\frac{1}{192}}{\ln\frac{1}{2} - \ln\frac{1}{4}} = \frac{\ln 4}{\ln 2} = \frac{\ln\frac{1}{12} - \ln\frac{1}{48}}{\ln 1 - \ln\frac{1}{2}} = \text{slope}$$

significance : the relative error in $u_{,x}$

the slope ↑, the error ↑

ii.

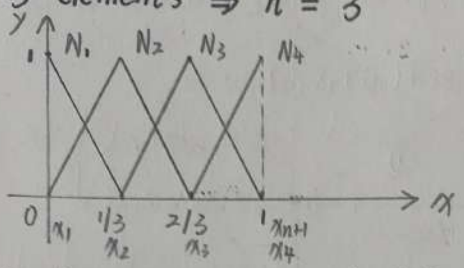
y-intercept significance :

when mesh parameter = 1 , the relative error in $u_{,x}$.

3. Solution

$$\begin{cases} u_{,xx} + \sin x = 0 \\ u(1) = g \\ -u_{,x}(0) = h \end{cases}$$

3 elements $\Rightarrow n = 3$



$$N_1(x) = \begin{cases} -3x+1 \\ 0 \end{cases} \qquad N_{1,x}(x) = \begin{cases} -3 & (0, 1/3) \\ 0 & (1/3, 1) \end{cases}$$

$$N_2 = \begin{cases} 3x \\ -3x+2 \\ 0 \end{cases} \qquad N_{2,x} = \begin{cases} 3 & (0, 1/3) \\ -3 & (1/3, 2/3) \\ 0 & (2/3, 1) \end{cases}$$

$$N_3 = \begin{cases} 0 \\ 3x-1 \\ -3x+3 \end{cases} \qquad N_{3,x} = \begin{cases} 0 & (0, 1/3) \\ 3 & (1/3, 2/3) \\ -3 & (2/3, 1) \end{cases}$$

$$N_4 = \begin{cases} 0 \\ 3x-2 \end{cases} \qquad N_{4,x} = \begin{cases} 0 & (0, 2/3) \\ 3 & (2/3, 1) \end{cases}$$

$$K_{AB} = a(N_A, N_B)$$

$$K = \begin{bmatrix} \frac{1}{h_1} & \frac{-1}{h_1} & 0 \\ \frac{-1}{h_1} & \frac{1}{h_1}+\frac{1}{h_2} & \frac{-1}{h_2} \\ 0 & \frac{-1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 6 \end{bmatrix}$$

(b)

$$F_A = (N_A, \sin x) + N_A(0)h - a(N_A, N_4)g$$

$$F_1 = (N_1, \sin x) + N_1(0)h - a(N_1, N_4)g$$

$$= \int_0^{1/3} (-3x+1)\sin x\, dx + h - 0 = 1 - 3\sin\tfrac{1}{3} + h$$

$$F_2 = \int_0^{1/3} 3x \cdot \sin x\, dx + \int_{1/3}^{2/3} (-3x+2)\sin x\, dx + 0 - 0$$

$$= 3\left(2\sin\tfrac{1}{3} - \sin\tfrac{2}{3}\right)$$

$$F_3 = \int_{1/3}^{2/3} (3x-1)\sin x\, dx + \int_{2/3}^{1} (-3x+3)\sin x\, dx + 0$$

$$- \int_{2/3}^{1} (-3)\cdot 3\, dx \cdot g = 3\left(-2\sin\tfrac{2}{3}\cos(\tfrac{1}{3}) + 2\sin\tfrac{2}{3} + g\right)$$

$$K^{-1} = \frac{1}{27}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = K^{-1}F$$

$$u^h(x) = d_1 N_1 + d_2 N_2 + d_3 N_3 + \overset{g}{d_4} N_4$$

in the assembly of the load vector, where:

$$\begin{cases} d_1 = F_1 - \frac{2}{3}F_2 + \frac{1}{3}F_3 \\ d_2 = -\frac{2}{3}F_1 + \frac{2}{3}F_2 - \frac{1}{3}F_3 \\ d_3 = \frac{1}{3}F_1 - \frac{1}{3}F_2 + \frac{1}{3}F_3 \end{cases}$$

nodally exact solution:

$$u^h(0) = d_1$$

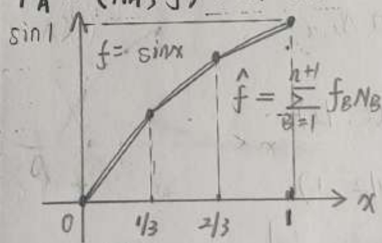
$$u^h(1/3) = d_2$$

$$u^h(2/3) = d_3$$

$$u^h(1) = g$$

(a) $\hat{f} \simeq \sum_{B=1}^{n+1} f_B N_B$, $(N_A, f) \to (N_A, \hat{f})$

$$\hat{F}_A = (N_A, \hat{f}) + N_A(0)h - a(N_A, N_4)g$$



$$\hat{f} \simeq \sum_{B=1}^{n+1} f_B N_B = 0\cdot N_1 + \sin\tfrac{1}{3} N_2 + \sin\tfrac{2}{3} N_3 + \sin 1\, N_4$$

$$= \begin{cases} \sin\frac{1}{3} N_2 & (0, 1/3) \\ \sin\frac{1}{3} N_2 + \sin\frac{2}{3} N_3 & (1/3, 2/3) \\ \sin\frac{2}{3} N_3 + \sin 1\, N_4 & (2/3, 1) \end{cases}$$

$$\hat{F}_1 = (N_1, \hat{f}) + N_1(0)h - 0$$

$$= \int_0^{1/3} N_1 \cdot \sin\tfrac{1}{3} N_2\, dx + h = h + \tfrac{1}{18}\sin\tfrac{1}{3}$$

$$\hat{F}_2 = (N_2, \hat{f}) + 0 - 0 = \int_0^{1/3} N_2 \cdot \sin\tfrac{1}{3} N_2\, dx + \int_{1/3}^{2/3} N_2 \cdot (\sin\tfrac{1}{3} N_2 + \sin\tfrac{2}{3} N_3)\, dx$$

$$= \tfrac{2}{9}\sin\tfrac{1}{3} + \tfrac{1}{18}\sin\tfrac{2}{3}$$

$$\hat{F}_3 = (N_3, \hat{f}) + 0 - a(N_3, N_4)g$$

$$= \int_{1/3}^{2/3} N_3 \cdot (\sin\tfrac{1}{3} N_2 + \sin\tfrac{2}{3} N_3)\,dx$$

$$+ \int_{2/3}^{1} N_3 \cdot (\sin\tfrac{2}{3} N_3 + \sin 1\, N_4)\,dx$$

$$- \int_{2/3}^{1} (-3)\cdot 3\,dx \cdot g$$

$$= 3g + \tfrac{1}{18}\sin\tfrac{1}{3} + \tfrac{2}{9}\sin\tfrac{2}{3} + \tfrac{1}{18}\sin 1$$

$$\hat{d} = K^{-1}\hat{F} \Rightarrow \begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{bmatrix} = \begin{bmatrix} F_1 - \frac{2}{3}F_2 + \frac{1}{3}F_3 \\ -\frac{2}{3}F_1 + \frac{2}{3}F_2 - \frac{1}{3}F_3 \\ \frac{1}{3}F_1 - \frac{1}{3}F_2 + \frac{1}{3}F_3 \end{bmatrix}$$

$$\hat{u}^h = \hat{d}_1 N_1 + \hat{d}_2 N_2 + \hat{d}_3 N_3 + g N_4$$

$\hat{u}^h(0) = \hat{d}_1$

$\hat{u}^h(1/3) = \hat{d}_2$

$\hat{u}^h(2/3) = \hat{d}_3$

$\hat{u}^h(1) = g$

it's not nodally exact.

Comparing (a) and (b).

(a) can't get the exact nodally solution while (b) can.

4. Solution

two-point Gaussian rule

for $g(x) = a_1 x + a_0$

$$\int_{-1}^{1} g(x)\,dx = \frac{1}{2}a_1 x^2 + a_0 x \Big|_{-1}^{1} = 2a_0$$

$$= w_1(a_1 x_1 + a_0) + w_2(a_1 x_2 + a_0)$$

$$\Rightarrow \begin{cases} w_1 + w_2 = 2 \\ w_1 x_1 + w_2 x_2 = 0 \end{cases} \quad ①$$

$$\text{where} \begin{cases} w_1 = w_2 = 1 \\ x_1 = -\frac{1}{\sqrt{3}},\ x_2 = \frac{1}{\sqrt{3}} \end{cases}$$

satisfies ①

for $g(x) = a_2 x^2 + a_1 x + a_0$

$$\int_{-1}^{1} g(x)\,dx = \frac{2}{3}a_2 + 2a_0$$

$$= w_1(a_2 x_1^2 + a_1 x_1 + a_0) + w_2(a_2 x_2^2 + a_1 x_2 + a_0)$$

$$\Rightarrow \begin{cases} w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3} \\ w_1 x_1 + w_2 x_2 = 0 \\ w_1 + w_2 = 2 \end{cases} \quad ②$$

$$\text{where} \begin{cases} w_1 = w_2 = 1 \\ x_1 = -\frac{1}{\sqrt{3}},\ x_2 = \frac{1}{\sqrt{3}} \end{cases} \quad (*)$$

satisfies ②

for $g(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

$$\int_{-1}^{1} g(x)\,dx = \frac{2}{3}a_2 + 2a_0$$

$$= w_1(a_3 x_1^3 + a_2 x_1^2 + a_1 x_1 + a_0) + w_2(a_3 x_2^3 + a_2 x_2^2 + a_1 x_2 + a_0)$$

$$\Rightarrow \begin{cases} w_1 x_1^3 + w_2 x_2^3 = 0 \\ w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3} \\ w_1 x_1 + w_2 x_2 = 0 \\ w_1 + w_2 = 2 \end{cases} \quad ③$$

(*) satisfies ③

for $g(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$

$$\int_{-1}^{1} g(x)\,dx = \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0$$

$$= w_1(a_4 x_1^4 + a_3 x_1^3 + a_2 x_1^2 + a_1 x_1 + a_0) + w_2(a_4 x_2^4 + a_3 x_2^3 + a_2 x_2^2 + a_1 x_1 + a_0)$$

$$\Rightarrow \begin{cases} w_1 x_1^4 + w_2 x_2^4 = \frac{2}{5} \\ w_1 x_1^3 + w_2 x_2^3 = 0 \\ w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3} \\ w_1 x_1 + w_2 x_2 = 0 \\ w_1 + w_2 = 2 \end{cases} \quad ④$$

(*) can't satisfy ④

So the two-point Gaussian rule can exactly integrate the monomials $1, \xi^2, \xi^3$ but not ξ^4.