

MECHANICS OF MATERIALS

YAHUI XUE (薛亚辉)

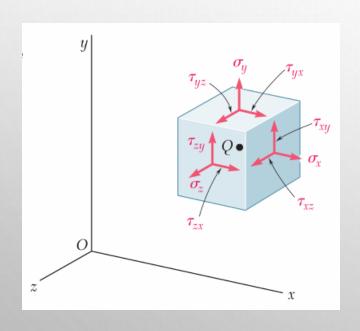
SPRING, 2022

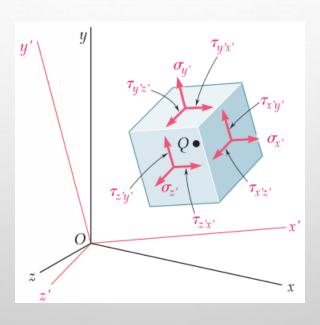
Lesson 9: Transformations of stress and strain

- Transformation of plane stress and strain
- Principal stresses and strains
- Mohr's circle for stress and strain
- Strain gages and strain rosette

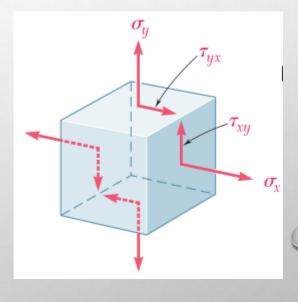
§ 9.1 Introduction

• The most general state of stress at a given point Q may be represented by six components. How will the components of stress be transformed under a rotation of the coordinate axes?



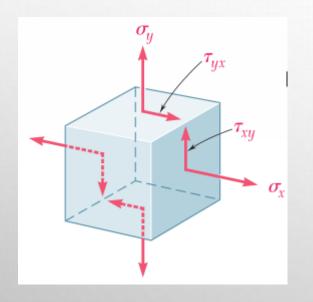


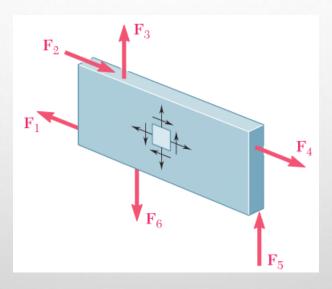
Plane stress

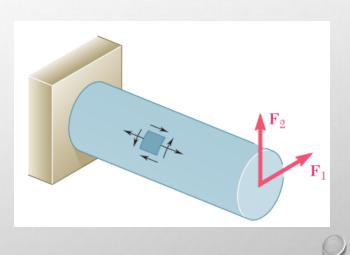


§ 9.1 Introduction

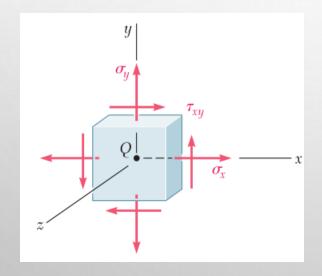
Plane stress: (1) a thin plate subjected to forces acting in the midplane of the plate; (2) the free surface of a structural element that is not subjected to an external force.

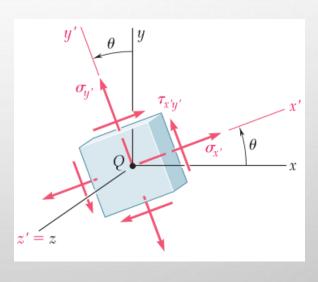






- Determine the components $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ associated with that element after it has been rotated through an angle θ about the z axis.
- Principal stresses θ_p : maximum and minimum of $\sigma_{x'}$, $\sigma_{y'}$
- Principal planes of stress: the faces of the corresponding element

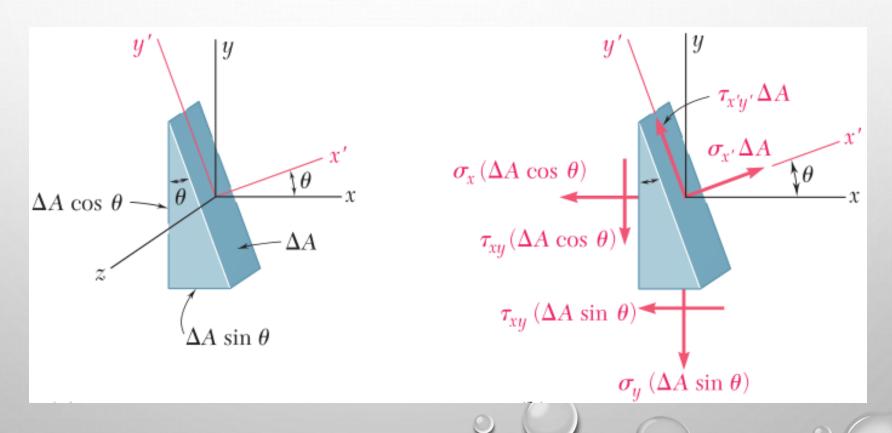




• Determine the normal stress $\sigma_{x'}$ and the shearing stress $\tau_{x'y'}$ exerted on the face perpendicular to the x' axis

$$\sum F_{x'} = 0$$

$$\sum F_{y'} = 0$$



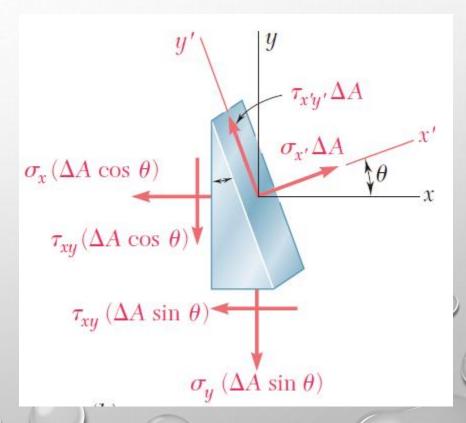
• Determine the normal stress $\sigma_{x'}$ and the shearing stress $\tau_{x'y'}$ exerted on the face perpendicular to the x' axis

$$\sum F_{x'} = 0$$

 $\sigma_{x'} \Delta A - \sigma_{x} (\Delta A \cos \theta) \cos \theta - \tau_{xy} (\Delta A \cos \theta) \sin \theta$ $-\sigma_{y} (\Delta A \sin \theta) \sin \theta - \tau_{xy} (\Delta A \sin \theta) \cos \theta = 0$

$$\sum F_{y'} = 0$$

 $\tau_{x'y'} \Delta A + \sigma_x (\Delta A \cos \theta) \sin \theta - \tau_{xy} (\Delta A \cos \theta) \cos \theta$ $-\sigma_y (\Delta A \sin \theta) \cos \theta + \tau_{xy} (\Delta A \sin \theta) \sin \theta = 0$



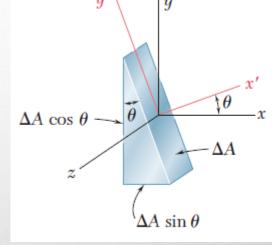
• Determine the normal stress $\sigma_{x'}$ and the shearing stress $\tau_{x'y'}$ exerted on the face perpendicular to the x' axis

$$\sigma_{x'} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$



Replace θ by $\theta + 90^{\circ}$:

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$$

• The equations obtained in the preceding section are the parametric equations of a circle.

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\left(\sigma_{x'} - \sigma_{avg}\right)^2 + \tau_{x'y'}^2 = R^2$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$R^{2} = \left(\frac{\sigma_{x} - \sigma_{y}}{2}\right)^{2} + \tau_{xy}^{2}$$

$$\sigma_{avg} = \frac{\sigma_{x} + \sigma_{y}}{2}$$

The equations obtained in the preceding section are the parametric equations of a circle.

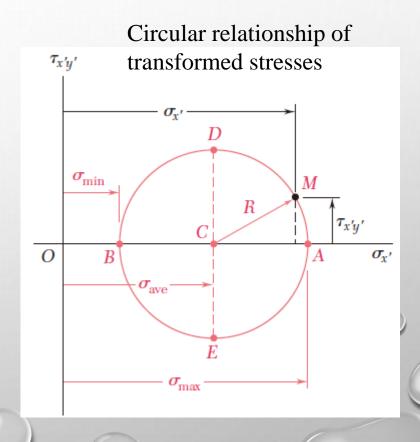
$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$R^{2} = \left(\frac{\sigma_{x} - \sigma_{y}}{2}\right)^{2} + \tau_{xy}^{2}$$

$$\sigma_{avg} = \frac{\sigma_{x} + \sigma_{y}}{2}$$

$$\left(\sigma_{x'} - \sigma_{avg}\right)^{2} + \tau_{x'y'}^{2} = R^{2}$$

$$\left(\sigma_{x'} - \sigma_{avg}\right)^2 + \tau_{x'y'}^2 = R^2$$

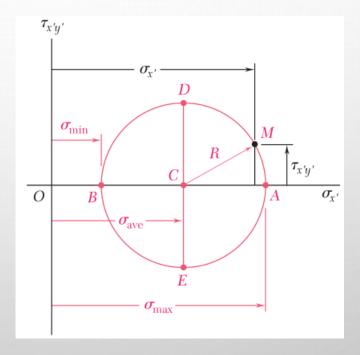


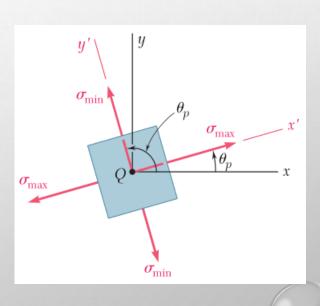
• Maximum and minimum stresses and principal planes of stress

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$\tau_{x'y'} = 0 \qquad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\sigma_{\text{max,min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$





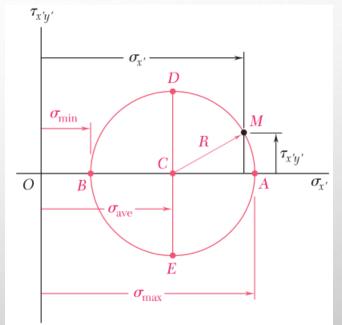
• Maximum and minimum shear stresses and the planes of maximum shearing stress

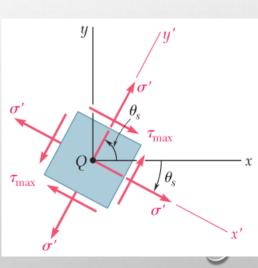
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\tau_{x'y'} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$





$$\sigma_{x'} = \sigma_{y'} = \sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

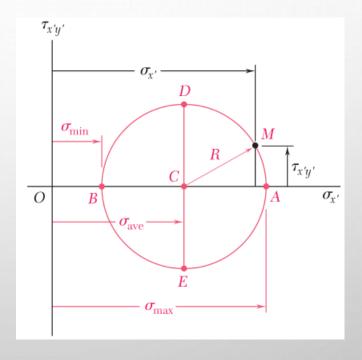
• The planes of maximum shearing stress are at 45° to the principal planes

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\tan 2\theta_p \cdot \tan 2\theta_s = -1$$

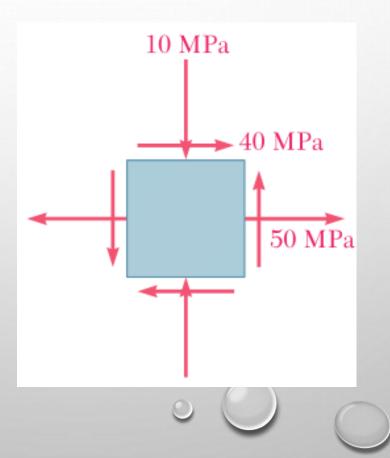
$$\left|\theta_{s}-\theta_{p}\right|=\frac{\pi}{4}$$



Example 9.1

(Beer, Page 446)

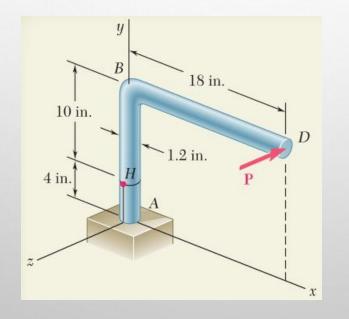
For the state of plane stress, determine (a) the principal planes, (b) the principal stresses, (c) the maximum shearing stress and the corresponding normal stress.

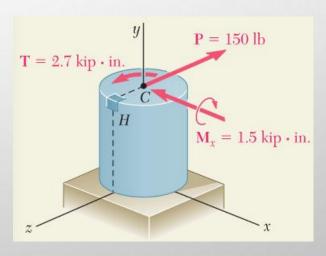


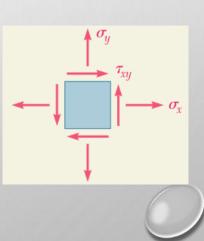
Example 9.2

(Beer, Page 447)

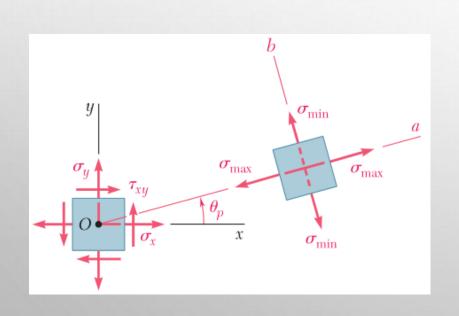
A single horizontal force P of magnitude 150 lb is applied to end D of lever ABD. Knowing that portion AB of the lever has a diameter of 1.2 in., determine (a) the normal and shearing stresses on an element located at point H and having sides parallel to the x and y axes, (b) the principal planes and the principal stresses at point H.

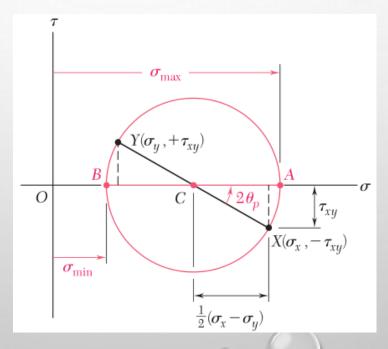






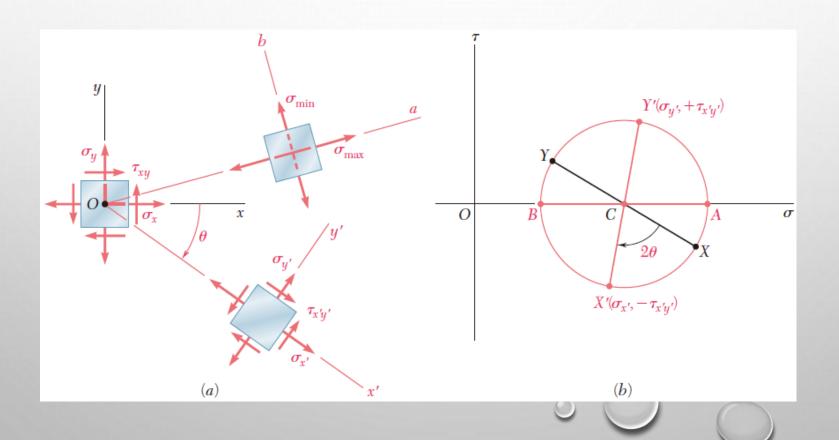
• The circle used in the preceding section to derive some of the basic formulas relating to the transformation of plane stress was first introduced by the German engineer Otto Mohr (1835–1918) and is known as Mohr's circle for plane stress.





$$\tan(XCA) = \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

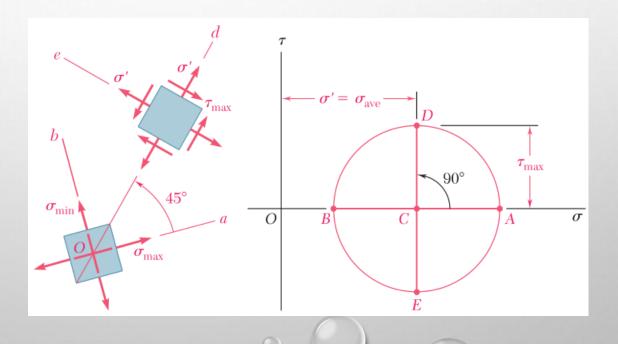
• Since Mohr's circle is uniquely defined, the same circle can be obtained by considering the stress components $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$,



• Points D and E on Mohr's circle correspond to the planes of maximum shearing stress, while A and B correspond to the principal planes. Since the diameters AB and DE of Mohr's circle are at 90° to each other, it follows that the faces of the corresponding elements are at 45° to each other.

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

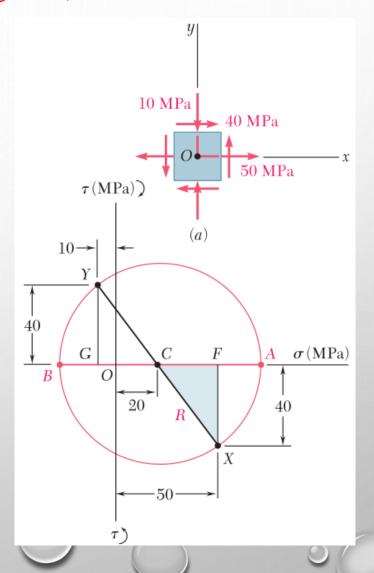
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$



Example 9.3

(Beer, Page 454)

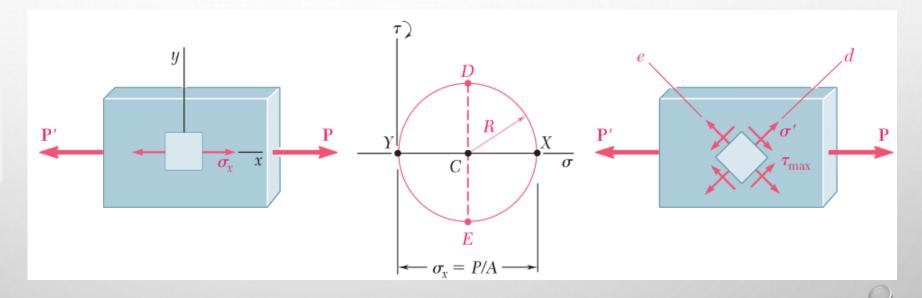
For the state of plane stress already considered, (a) construct Mohr's circle, (b) determine the principal stresses, (c) determine the maximum shearing stress and the corresponding normal stress.



• Stresses under a centric axial loading

$$\sigma_x = P/A, \sigma_y = 0$$

$$\tau_{xy} = 0$$

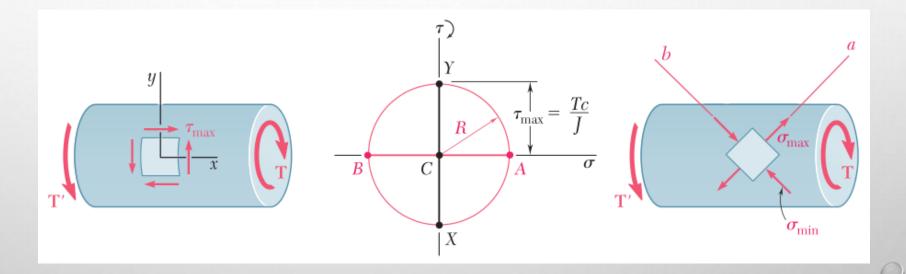


$$\tau_{\rm max} = P/2A$$

• Stresses under a torsional loading

$$\sigma_x = 0, \sigma_y = 0$$

$$\tau_{xy} = \frac{Tc}{J}$$

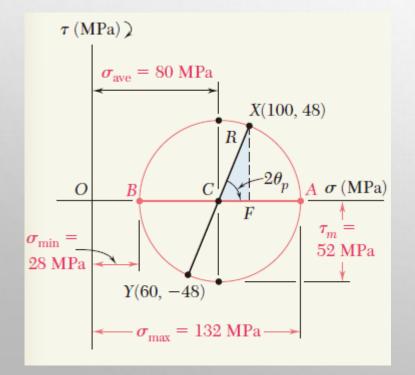


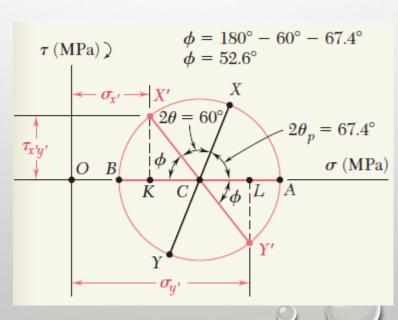
$$\sigma_{\rm max,min} = \pm \frac{Tc}{J}$$

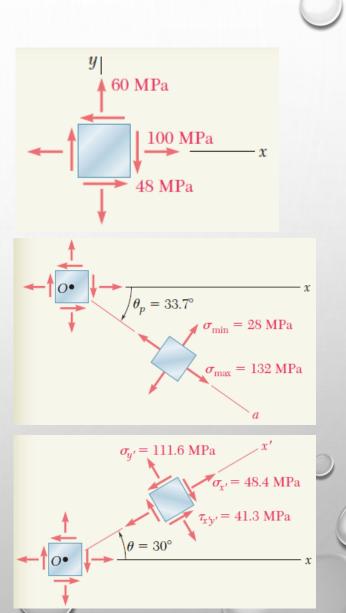


(Beer, Page 457)

For the state of plane stress shown, determine (a) the principal planes and the principal stresses, (b) the stress components exerted on the element obtained by rotating the given element counterclockwise through 30°.

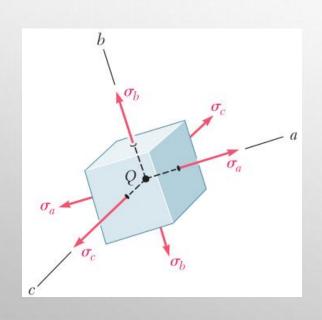


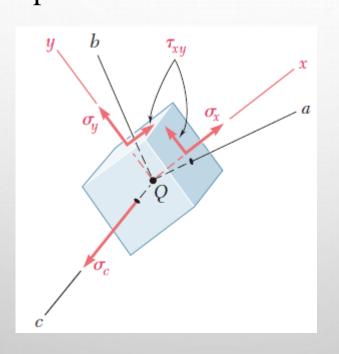


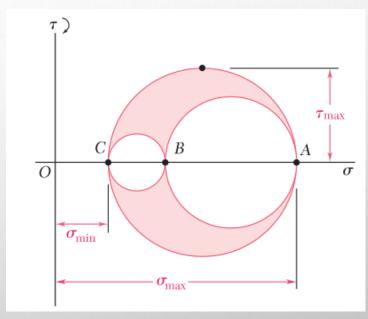


§ 9.5 Application of Mohr's circle in 3D

• If the element is rotated about one of the principal axes at Q, say the c axis, the corresponding transformation of stress can be analyzed by means of Mohr's circle as if it were a transformation of plane stress.



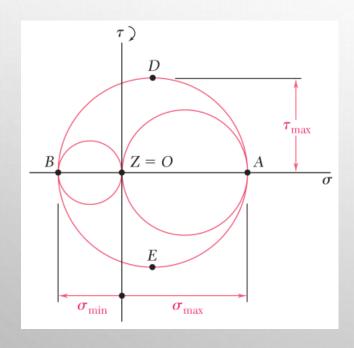




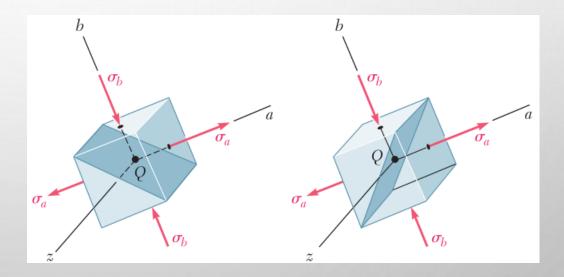
$$\tau_{\max} = \frac{1}{2} |\sigma_{\max} - \sigma_{\min}|$$

§ 9.5 Application of Mohr's circle in 3D

• For the particular case of plane stress, if A and B are located on opposite sides of the origin O,

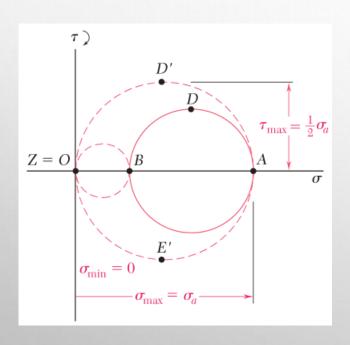


the planes of maximum shearing stress and principal planes

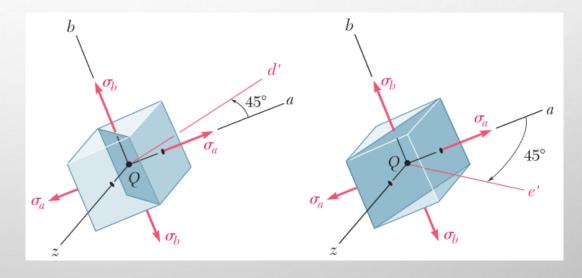


§ 9.5 Application of Mohr's circle in 3D

• For the particular case of plane stress, if, on the other hand, A and B are on the same side of O,



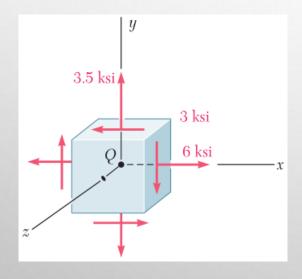
the planes of maximum shearing stress and principal planes

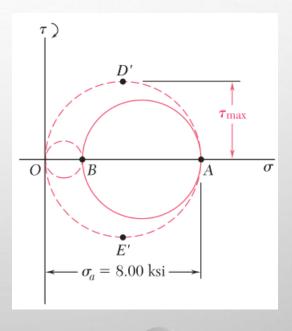


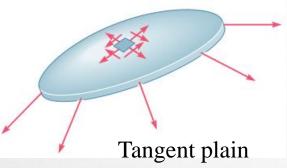
Example 9.5

(Beer, Page 466)

For the state of plane stress, determine (a) the three principal planes and principal stresses, (b) the maximum shearing stress.

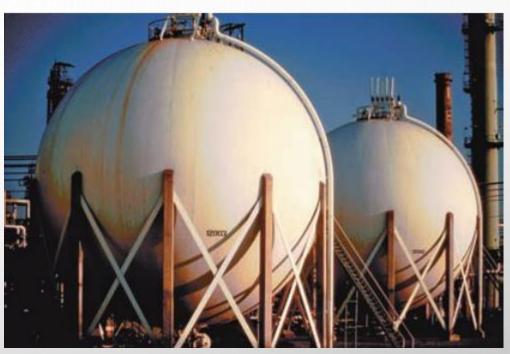








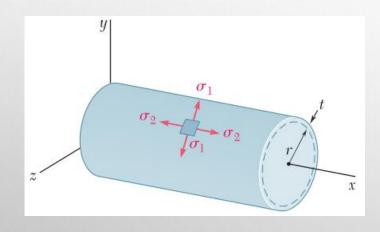
Cylindrical pressure vessels

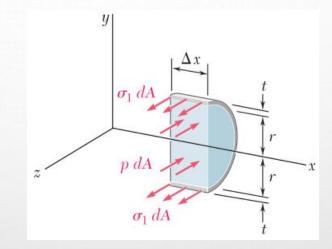


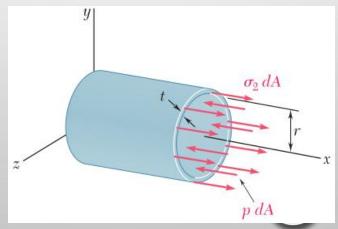
Spherical pressure vessels

• Consider a cylindrical vessel of inner radius r and wall thickness t containing a fluid

under pressure,







$$\sum F_z = 0$$

$$\sigma_1(2t\Delta x) - p(2r\Delta x) = 0$$

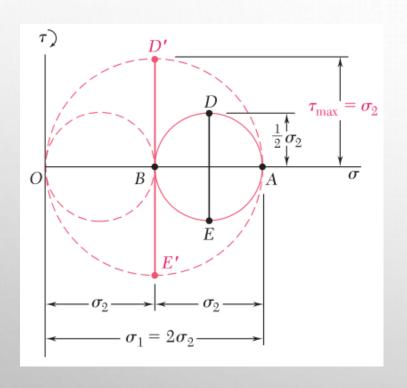
$$\sigma_1 = pr/t$$

$$\sum F_{x} = 0$$

$$\sigma_2(2\pi rt) - p(\pi r^2) = 0$$

$$\sigma_2 = pr/2t$$

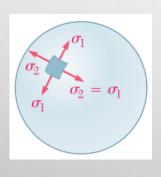
• Mohr's circle for element of cylindrical pressure vessel.

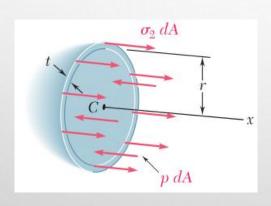


$$\tau_{\text{max(in plane)}} = \frac{\sigma_2}{2} = \frac{pr}{4t}$$

$$\tau_{\text{max (out of plain)}} = \sigma_2 = \frac{pr}{2t}$$

• Now consider a spherical vessel of inner radius *r* and wall thickness t, containing a fluid under a gage pressure *p*.



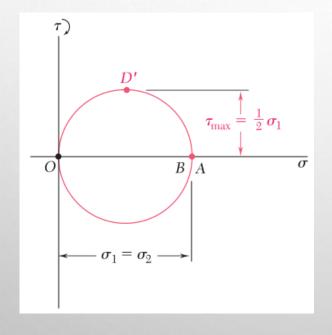


$$\sum F_{x} = 0$$

$$\sigma_2(2\pi rt) - p(\pi r^2) = 0$$

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

• Now consider a spherical vessel of inner radius *r* and wall thickness t, containing a fluid under a gage pressure *p*.

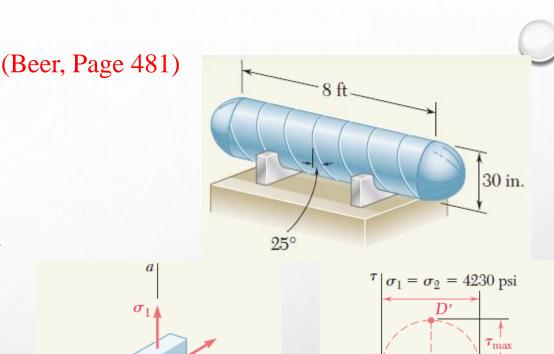


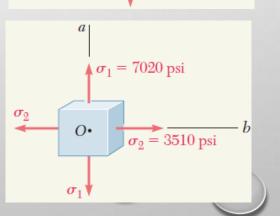
$$\tau_{\text{max(in plain)}} = 0$$

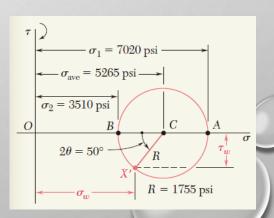
$$\tau_{\text{max(out of plain)}} = \frac{\sigma_1}{2} = \frac{pr}{4t}$$

Example 9.6

A compressed-air tank is supported by two cradles as shown; one of the cradles is designed so that it does not exert any longitudinal force on the tank. The cylindrical body of the tank has a 30-in. outer diameter and is fabricated from a 3/8-in. steel plate by butt welding along a helix that forms an angle of 25° with a transverse plane. The end caps are spherical and have a uniform wall thickness b of 5/16 in. For an internal gage pressure of 180 psi, determine (a) the normal stress and the maximum shearing stress in the spherical caps. (b) the stresses in directions perpendicular and parallel to the helical weld.

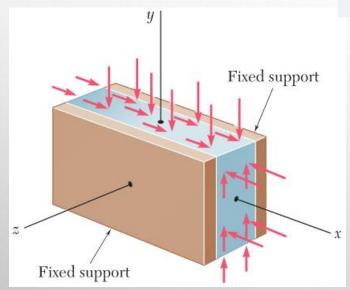




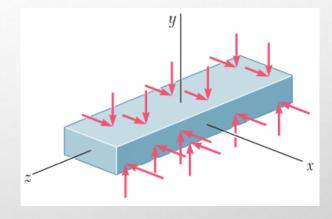


• Plain strain examples: the deformations of the material take place within parallel planes, and are the same in each of these planes.

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

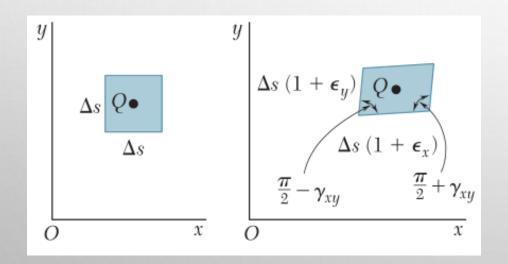


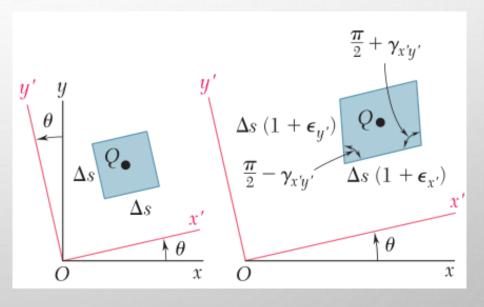
Laterally restrained plate: uniformly distributed loads along its edge



Bar of infinite length: uniformly distributed transverse loads

• Plane strain element deformation and Transformation of plane strain element: to determine in terms of ε_x , ε_y , γ_{xy} , and θ the strain components $\varepsilon_{x'}$, $\varepsilon_{y'}$, and $\gamma_{x'y'}$ associated with the frame of reference x'y' obtained by rotating the x and y axes through the angle θ .





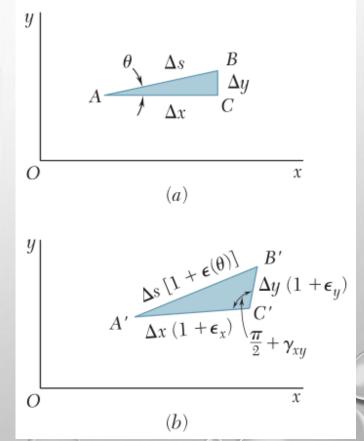
• First derive an expression for the normal strain $\varepsilon(\theta)$ along a line AB forming an arbitrary angle θ with the x axis.

The law of cosines to triangle A'B'C',

$$(A'B')^{2} = (A'C')^{2} + (C'B')^{2} - 2(A'C')(C'B')\cos(\frac{\pi}{2} + \gamma_{xy})$$

$$(\Delta s)^{2}[1+\varepsilon(\theta)]^{2} = (\Delta x)^{2}(1+\varepsilon_{x})^{2} + (\Delta y)^{2}(1+\varepsilon_{y})^{2}$$
$$-2(\Delta x)(1+\varepsilon_{x})(1+\varepsilon_{y})\cos(\frac{\pi}{2}+\gamma_{xy})$$

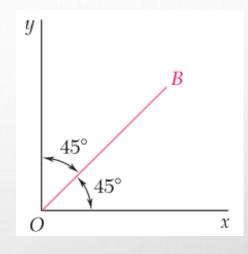
$$\Delta x = (\Delta s) \cos \theta$$
 $\Delta y = (\Delta s) \sin \theta$



• Small strain assumption

$$\cos(\frac{\pi}{2} + \gamma_{xy}) = -\sin\gamma_{xy} \approx -\gamma_{xy} \qquad (1 + \varepsilon)^2 \approx 1 + 2\varepsilon$$

$$(\Delta s)^{2}[1+\varepsilon(\theta)]^{2} = (\Delta x)^{2}(1+\varepsilon_{x})^{2} + (\Delta y)^{2}(1+\varepsilon_{y})^{2}$$
$$-2(\Delta x)(1+\varepsilon_{x})(1+\varepsilon_{y})\cos(\frac{\pi}{2}+\gamma_{xy})$$



$$\varepsilon(0^{\circ}) = \varepsilon_{x}$$
 $\varepsilon(90^{\circ}) = \varepsilon_{y}$

$$\varepsilon(45^{\circ}) = \frac{1}{2} (\varepsilon_x + \varepsilon_y + \gamma_{xy})$$

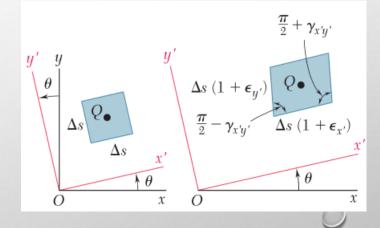
$$\gamma_{xy} = 2\varepsilon_{OB} - (\varepsilon_x + \varepsilon_y)$$

• The main purpose of this section is to express the strain components associated with the frame of reference x'y' in terms of the angle θ and the strain components associated with the x and y axes

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta - \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\varepsilon_{x'} + \varepsilon_{y'} = \varepsilon_x + \varepsilon_y$$



$$\varepsilon_{OB'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2(\theta + \frac{\pi}{4}) + \frac{\gamma_{xy}}{2} \sin 2(\theta + \frac{\pi}{4})$$

$$\gamma_{x'y'} = 2\varepsilon_{OB'} - (\varepsilon_{x'} + \varepsilon_{y'}) \longrightarrow \frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

§ 9.8 Mohr's circle for plain strain

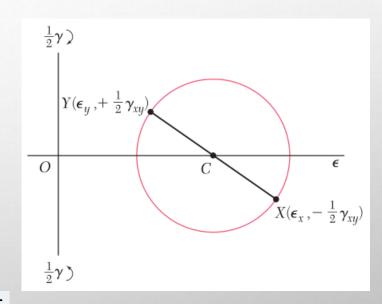
Since the equations for the transformation of plane strain are of the same form as the equations for the transformation of plane stress, the use of Mohr's circle can be extended to the analysis of plane strain.

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

$$\left(\varepsilon_{x'} - \varepsilon_{avg}\right)^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2 = R^2$$

$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \varepsilon_{avg} = \frac{\varepsilon_x + \varepsilon_y}{2}$$



$$\varepsilon_{avg} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

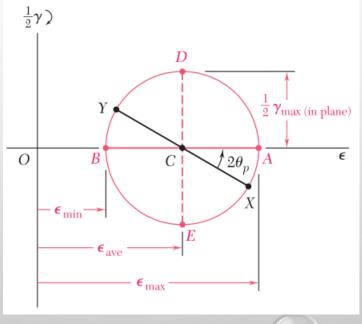
§ 9.8 Mohr's circle for plain strain

• If the shear deformation causes a given side to rotate clockwise, the corresponding point on Mohr's circle for plane strain is plotted above the horizontal axis, and if the deformation causes the side to rotate counterclockwise, the corresponding point is plotted below the horizontal axis.

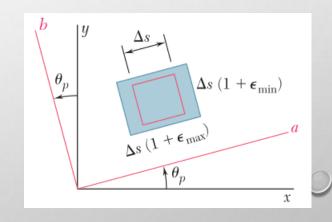
$$\varepsilon_{\rm max,min} = \varepsilon_{avg} \pm R$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\gamma_{\text{max}} = 2R = \sqrt{\left(\varepsilon_{x} - \varepsilon_{y}\right)^{2} + \gamma_{xy}^{2}}$$



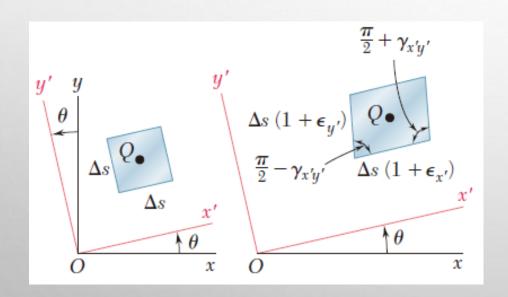
principal axes of strain/stress

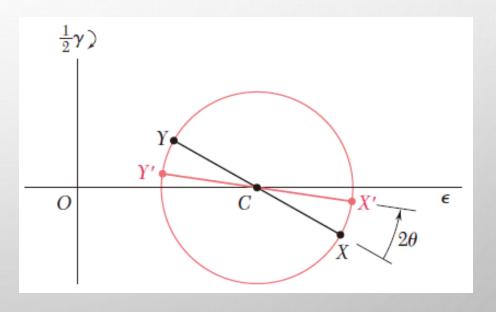


$$\tau_{xy} = G\gamma_{xy} = 0$$

§ 9.8 Mohr's circle for plain strain

• The points X' and Y' that define the components of strain corresponding to a rotation of the coordinate axes through an angle θ are obtained by rotating the diameter XY of Mohr's circle in the same sense through an angle 2θ .

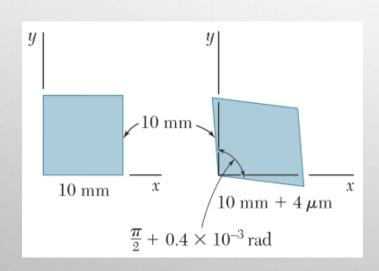


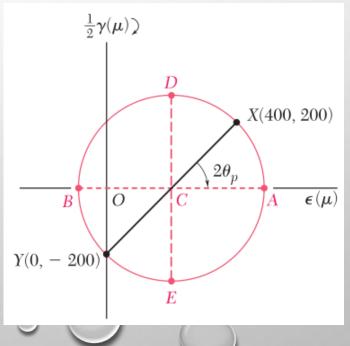


Example 9.7

(Beer, Page 481)

In a material in a state of plane strain, it is known that the horizontal side of a 10×10 -mm square elongates by 4 µm, while its vertical side remains unchanged, and that the angle at the lower left corner increases by 0.4×10^{-3} rad. Determine (a) the principal axes and principal strains, (b) the maximum shearing strain and the corresponding normal strain.

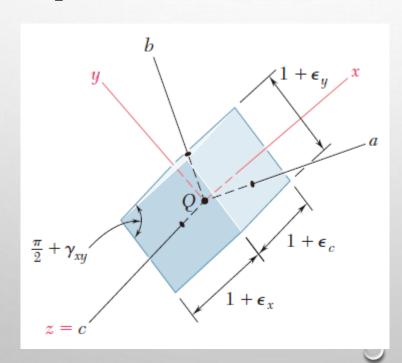


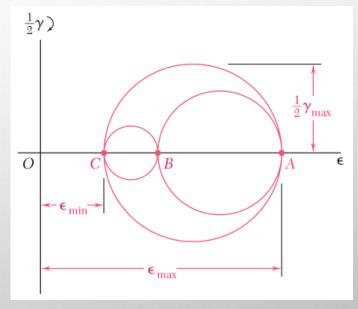


§ 9.9 Three-dimensional analysis of strain

• In the most general case of stress, we can determine three coordinate axes a, b, and c, called the principal axes of stress. If the element is rotated about one of the principal axes at Q, the method of analysis for the transformation of plane strain can be used to determine the strain components.

b $1 + \epsilon_b$ a $1 + \epsilon_c$

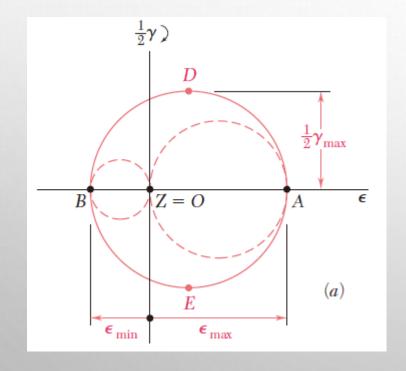


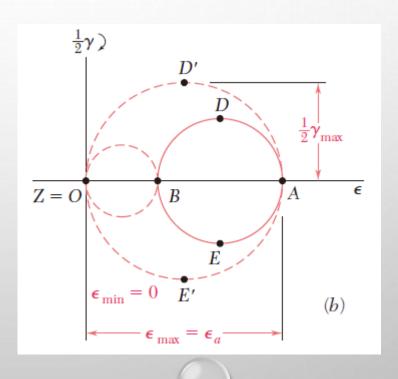


$$\gamma_{\max} = \left| \varepsilon_{\max} - \varepsilon_{\min} \right|$$

§ 9.9 Three-dimensional analysis of strain

• For the particular case of plane strain, the points A and B that define the principal axes within the plane of strain fall on opposite sides of O, and A and B are on the same side of O.

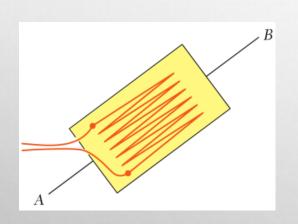


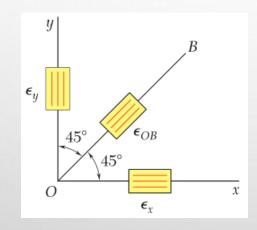


§ 9.10 Measurement of strain

• A more convenient and more accurate method for the measurement of normal strains is provided by electrical strain gages: the elongation causes the electrical resistance of the gage to increase.

$$\varepsilon(\theta) = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta$$





$$L_2$$
 ϵ_2
 ϵ_3
 θ_2
 ϵ_1
 ϵ_1
 ϵ_3
 ϵ_3
 ϵ_4
 ϵ_3
 ϵ_4
 ϵ_5

$$\gamma_{xy} = 2\varepsilon_{OB} - (\varepsilon_x + \varepsilon_y)$$

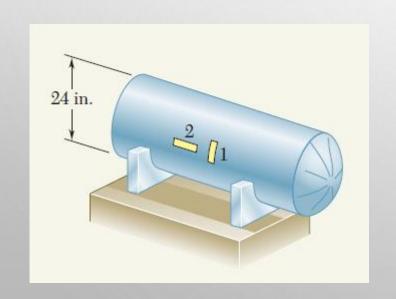
A third measurement of normal strain, made along the bisector OB enables to determine the shearing strain.

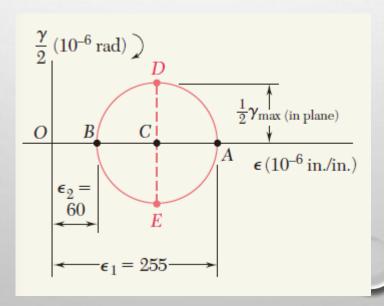
The strain components could be obtained from normal strain measurements made along any three lines drawn through that point.

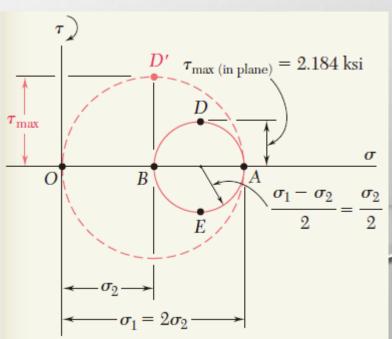


(Beer, Page 496)

A cylindrical storage tank used to transport gas under pressure has an inner diameter of 24 in. and a wall thickness of $\frac{3}{4}$ in. Strain gages attached to the surface of the tank in transverse and longitudinal directions indicate strains of 255×10^{-6} and 60×10^{-6} in./in. respectively. Knowing that a torsion test has shown that the modulus of rigidity of the material used in the tank is $G = 11.2 \times 10^6$ psi, determine (a) the gage pressure inside the tank, (b) the principal stresses and the maximum shearing stress in the wall of the tank.







Example 9.9

(Beer, Page 497)

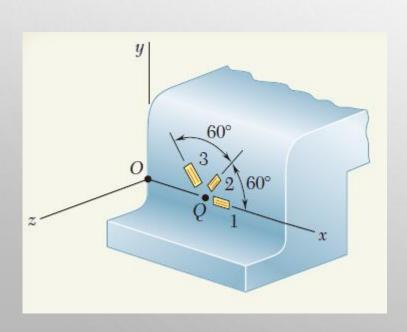
Using a 60° rosette, the following strains have been determined at point Q on the surface of a steel machine base:

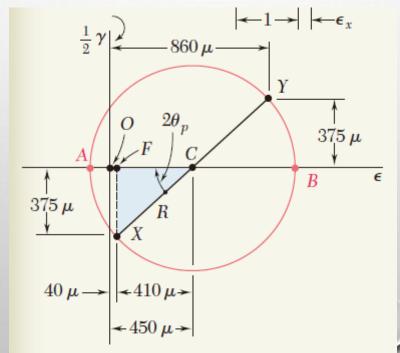
$$\varepsilon_1 = 40 \mu$$

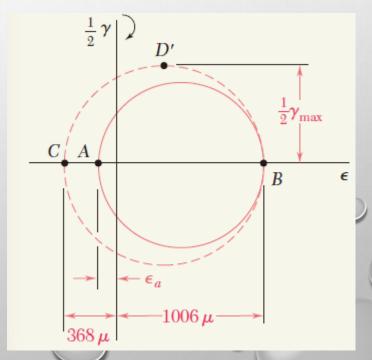
$$\varepsilon_2 = 980 \,\mu$$

$$\varepsilon_1 = 40 \,\mu$$
 $\varepsilon_2 = 980 \,\mu$ $\varepsilon_3 = 330 \,\mu$

Using the coordinate axes shown, determine at point Q, (a) the strain components ε_x , ε_y , and γ_{xy} , (b) the principal strains, (c) the maximum shearing strain. (Use v = 0.29.)







§ 9.10 Summary

- Transformation of plane stress: Principal planes. Principal stresses
- Mohr's circle for stress: 2D, 3D
- Thin-walled pressure vessels
- Transformation of plane strain
- Mohr's circle for strain: 2D, 3D, Strain gages. Strain rosette