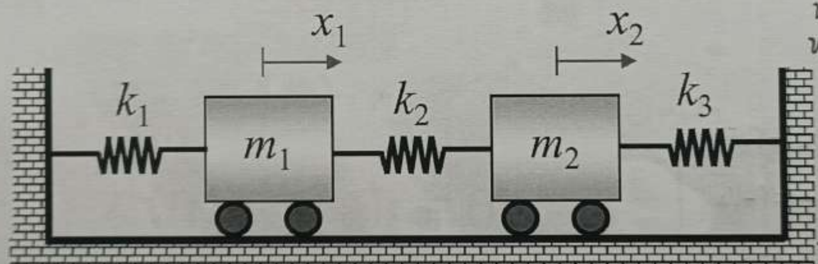
Homework

For the undamped linear system with two degree-of-freedom system as shown in the figure, assume that

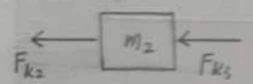
$$m_1 = m,\ m_2 = 2m,\quad k_1 = k_2 = k,\ k_3 = 2k$$

Derive the differential equation governing the two degree-of-freedom system. Then determine the frequency and mode of the motion.



解:

FBD of mass 2

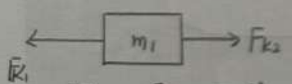


$-F_{k2} - F_{k3} = m_2\ddot{x}_2$

$\Rightarrow -k_2(x_2 - x_1) - k_3 x_2 = m_2\ddot{x}_2$

$\Rightarrow m_2\ddot{x}_2 - k_2 x_1 + (k_2+k_3)x_2 = 0$ ①

FBD of mass 1



$F_{k2} - F_{k1} = m_1\ddot{x}_1$

$\Rightarrow k_2(x_2 - x_1) - k_1 x_1 = m_1\ddot{x}_1$

$\Rightarrow m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = 0$ ②

DGE: $\begin{cases} 2m\ddot{x}_2 - kx_1 + 3kx_2 = 0 \\ m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \end{cases}$ $\boxed{\text{ANS}}$

由①②:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Assume solution

$\vec{x}(t) = \vec{X}\cos(wt+\varphi)$, $\ddot{\vec{x}}(t) = -\vec{X}w^2\cos(wt+\varphi)$

$\Rightarrow [\vec{k} - \vec{m}w^2]\vec{X} = 0$

$$|\vec{k} - \vec{m}w^2| = \left| \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} - \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} w^2 \right| = 0$$

$\Rightarrow m_1 m_2 w^4 - [(k_2+k_3)m_1 + (k_1+k_2)m_2]w^2 + [(k_1+k_2)(k_2+k_3) - k_2^2] = 0$

natural frequency

$$\begin{matrix} w_1^2 \\ w_2^2 \end{matrix} = \frac{1}{2}\left[\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right] \mp \frac{1}{2}\sqrt{\left[\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right]^2 - 4\left[\frac{(k_1+k_2)(k_2+k_3) - k_2^2}{m_1 m_2}\right]}$$

$$= \frac{7k}{4m} \mp \frac{1}{2} \times \frac{3k}{2m}$$

$\Rightarrow w_1 = \sqrt{\frac{4k}{4m}} = \sqrt{\frac{k}{m}}$

$w_2 = \sqrt{\frac{10k}{4m}} = \sqrt{\frac{5k}{2m}}$ $\boxed{\text{ANS}}$

Mode 1: $r_1 = \dfrac{k_2}{-m_2 w_1^2 + (k_2+k_3)} = 1$

Mode 2: $r_2 = \dfrac{k_2}{-m_2 w_2^2 + (k_2+k_3)} = -\frac{1}{2}$

x_1, x_2 same direction in the first mode

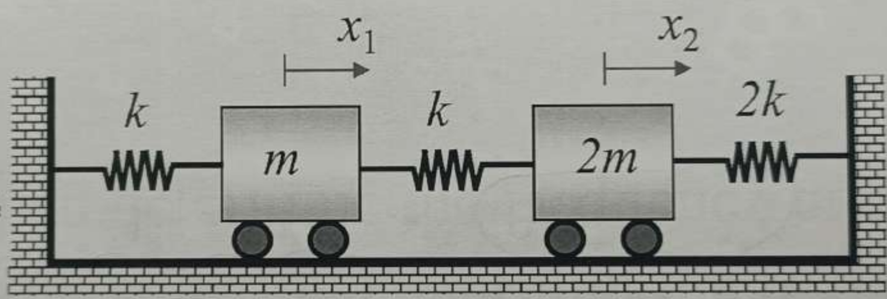
oppsite ······· second ··

$\boxed{\text{ANS}}$

Homework

For the undamped linear system with two degree-of-freedom system, the initial conditions are shown as below. Please determine the mode of the motion.

$$x_1(0) = x_2(0) = 1, \qquad \dot{x}_1(0) = \dot{x}_2(0) = 0,$$



解:

$$X_1^{(1)} = \frac{1}{|r_2 - r_1|}\sqrt{[r_2 x_1(0) - x_2(0)]^2 + \frac{[-r_2\dot{x}_1(0) + \dot{x}_2(0)]^2}{w_1^2}}$$

$$= \frac{1}{|-\frac{1}{2}-1|}\sqrt{(-\frac{1}{2}\times 1 - 1)^2 + \frac{+\frac{1}{2}\times 0 + 0}{\frac{k}{m}}}$$

$$= +1$$

$$X_1^{(2)} = \frac{1}{|r_2 - r_1|}\sqrt{[-r_1 x_1(0) + x_2(0)]^2 + \frac{[r_1\dot{x}_1(0) - \dot{x}_2(0)]^2}{w_2^2}}$$

$$= \frac{1}{|-\frac{1}{2}-1|}\sqrt{(-1\times 1 + 1)^2 + 0}$$

$$= 0$$

$$\phi_1 = \tan^{-1}\left\{\frac{-r_2\dot{x}_1(0) + \dot{x}_2(0)}{w_1[r_2 x_1(0) - x_2(0)]}\right\} = 0$$

$$\phi_2 = \tan^{-1}\left\{\frac{r_1\dot{x}_1(0) - \dot{x}_2(0)}{w_2[-r_1 x_1(0) + x_2(0)]}\right\} = 0$$

mass 1: $x_1(t) = X_1^{(1)}\cos(w_1 t + \phi_1) + X_1^{(2)}\cos(w_2 t + \phi_2) = +\cos(\sqrt{\frac{k}{m}}t)$

mass 2: $x_2(t) = r_1 X_1^{(1)}\cos(w_1 t + \phi_1) + r_2 X_1^{(2)}\cos(w_2 t + \phi_2) = +\cos(\sqrt{\frac{k}{m}}t)$ $\boxed{\text{ANS}}$

the two block vibrate together.

Another way:

$$\begin{cases} x_1(t) = C_1\cos(w_1 t + \phi_1) + C_2\cos(w_2 t + \phi_2) \\ x_2(t) = C_1\cos(w_1 t + \phi_1) - \frac{1}{2}C_2\cos(w_2 t + \phi_2) \\ \dot{x}_1(t) = -C_1 w_1\sin(w_1 t + \phi_1) - C_2 w_2\sin(w_2 t + \phi_2) \\ \dot{x}_2(t) = -C_1 w_1\sin(w_1 t + \phi_1) + \frac{1}{2}C_2 w_2\sin(w_2 t + \phi_2) \end{cases}$$

$$\begin{cases} C_1\cos\phi_1 + C_2\cos\phi_2 = 1 \\ C_1\cos\phi_1 - \frac{1}{2}C_2\cos\phi_2 = 1 \\ -C_1 w_1\sin\phi_1 - C_2 w_2\sin\phi_2 = 0 \\ -C_1 w_1\sin\phi_1 + \frac{1}{2}C_2 w_2\sin\phi_2 = 0 \end{cases}$$

$\Rightarrow \phi_1 = \phi_2 = 0$

$C_1 = 1, \; C_2 = 0$

$x_1(t) = \cos(\sqrt{\frac{k}{m}}t)$

$x_2(t) = \cos(\sqrt{\frac{k}{m}}t)$