Quiz 12

Date: 2022-05-06 Name: SID:

Use the Laplace transform to solve the given initial value problems:

1. $y'' + 4y = u_0(t), y(0) = y'(0) = 0;$
2. $y'' + 2y' + 5y = g(t), y(0) = 0, y'(0) = -1.$

2. $s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 5Y(s) = G(t).$

$$Y(s) = \frac{-1}{s^2+2s+5} + \frac{G(t)}{s^2+2s+5}$$

$$= \frac{-\frac{1}{2}\cdot 2}{(s+1)^2+4} + \frac{G(t)}{(s+1)^2+4}$$

$$y(t) = -\frac{1}{2}e^{-t}\sin 2t + \frac{1}{2}\int_0^t e^{(\tau-t)}\sin 2(t-\tau)\, g(\tau)\, d\tau.$$

Thus,

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{2-e^{-2s}}{(s+1)^2+1}\right\}(t)\\
&= 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}(t)\\
&\quad - \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{(s+1)^2+1}\right\}(t)\\
&= 2e^{-t}\sin t - H(t-2)e^{-(t-2)}\sin(t-2),
\end{aligned}$$

or, equivalently,

$$\begin{cases} 2e^{-t}\sin t, & 0 \le t < 2,\\ 2e^{-t}\sin t - e^{-(t-2)}\sin(t-2), & 2 \le t < \infty. \end{cases}$$

1. ✓

26. The forcing function f is described by

$$f(t) = \begin{cases} 0, & \text{if } t < 0,\\ 1, & \text{if } t \ge 0,\end{cases}$$

or equivalently

$$f(t) = H(t).$$

Hence,

$$F(s) = \mathcal{L}\{f(t)\}(s) = \frac{1}{s}.$$

Take the laplace transform of both sides of the given equation. Let $Y(s) = \mathcal{L}\{y(t)\}(s)$.

$$\begin{aligned}
y'' + 4y &= f(t)\\
\mathcal{L}(y'')(s) + 4\mathcal{L}(y)(s) &= \mathcal{L}\{f(t)\}(s)\\
s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= F(s)
\end{aligned}$$

Use the initial conditions $y(0) = y'(0) = 0$ and the laplace transform of f found earlier.

$$\begin{aligned}
s^2Y(s) + 4Y(s) &= \frac{1}{s}\\
Y(s) &= \frac{1}{s(s^2+4)}
\end{aligned}$$

A partial fraction decomposition is needed.

$$\begin{aligned}
\frac{1}{s(s^2+4)} &= \frac{A}{s} + \frac{Bs+C}{s^2+4}\\
1 &= A(s^2+4) + (Bs+C)s\\
1 &= (A+B)s^2 + Cs + 4A
\end{aligned}$$

Then,

$$\begin{aligned}
s = 0 \Rightarrow A &= \frac{1}{4}\\
A + B = 0 \Rightarrow B &= -\frac{1}{4}\\
C &= 0.
\end{aligned}$$

Therefore,

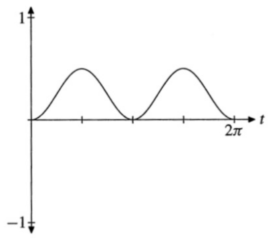
$$\frac{1}{s(s^2+4)} = \frac{1}{4s} - \frac{s}{4(s^2+4)}$$

which has transform

$$\frac{1}{4} - \frac{1}{4}\cos 2t.$$

Thus,

$$\boxed{\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t)\\ &= \frac{1}{4} - \frac{1}{4}\cos 2t. \end{aligned}}$$



27. The forcing function f is described by

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t < 1,\\ 0, & \text{otherwise,}\end{cases}$$

or equivalently,

$$\begin{aligned}
f(t) &= H_{01}(t)\\
&= H_0(t) - H_1(t)\\
&= H(t) - H(t-1).
\end{aligned}$$