Q3.1 (i) proof: 12012127 鄒催駒

$$\frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} - \cancel{c^2}\nabla^2\right) p' = \frac{A}{c^2} e^{-\sigma x_1^2} \delta(x_2)\delta(x_3) e^{i(\omega t - \alpha x_1)} = q(\underline{x}, t)$$

$$= \int_V q(\underline{y}, t)\, \delta(\underline{x} - \underline{y})\, d^3\underline{y}$$ (decompose $q(\underline{x},t)$ into a superposition of point sources)

for inhomogeneous wave equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla_x^2\right) p' = Q(\underline{y}, t)\, \delta(\underline{x} - \underline{y})$$

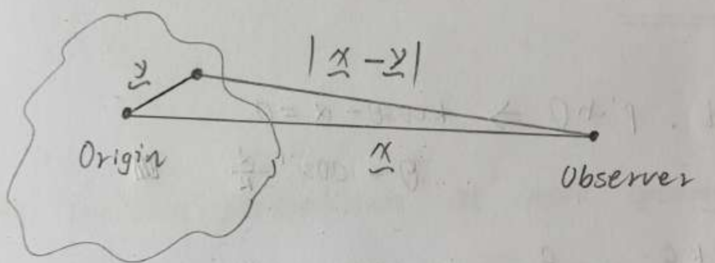
96'

solution $$p' = \frac{Q(\underline{y}, t - |\underline{x} - \underline{y}|/c)}{4\pi |\underline{x} - \underline{y}|}$$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla_x^2\right) \frac{q(\underline{y}, t - |\underline{x} - \underline{y}|/c)}{4\pi |\underline{x} - \underline{y}|} = q(\underline{y}, t)\, \delta(\underline{x} - \underline{y})$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla_x^2\right) \int_V \frac{q(\underline{y}, t - |\underline{x} - \underline{y}|/c)}{4\pi |\underline{x} - \underline{y}|} d^3y = \int_V q(\underline{y}, t)\, \delta(\underline{x} - \underline{y})\, dy^3 = q(\underline{x}, t)$$

where $$p'(\underline{x}, t) = \int_V \frac{q(\underline{y}, t - |\underline{x} - \underline{y}|/c)}{4\pi |\underline{x} - \underline{y}|} d^3y$$



from hint:

$$|\underline{x} - \underline{y}| = x\left(1 - \frac{\underline{x}\cdot\underline{y}}{x^2}\right) \quad \text{for } x \gg y$$

$$\frac{1}{|\underline{x} - \underline{y}|} = \frac{1}{|\underline{x}|} + O\left(\frac{|\underline{y}|}{|\underline{x}|^2}\right)$$

$$\Rightarrow p'(\underline{x}, t) = \frac{1}{4\pi x} \int_V q\left(\underline{y}, t - \frac{x}{c} + \frac{\underline{x}\cdot\underline{y}}{xc}\right) d^3y$$

$$= \frac{1}{4\pi x} \int_{-\infty}^{\infty} \frac{A}{c^2} e^{-\sigma y^2} \cdot e^{i\omega\left(t - \frac{x}{c} + \frac{\underline{x}\cdot\underline{y}}{xc}\right)} e^{i(-\alpha)y}\, dy$$

$\underline{x}\cdot\underline{y} = |\underline{x}|\cdot|\underline{y}|\cdot\cos\theta = xy\cos\theta$

$$= \frac{A}{c^2 4\pi x} \int_{-\infty}^{\infty} e^{-\sigma y^2} \cdot e^{i(\omega t - kx)} \cdot e^{i\cdot ky\cos\theta} \cdot e^{i(-\alpha)y}\, dy$$

10'

$$= \frac{A}{c^2 4\pi x} e^{i(\omega t - kx)} \int_{-\infty}^{\infty} e^{-\sigma y^2} e^{i(k\cos\theta - \alpha)y}\, dy$$

from hint: $$\int_{-\infty}^{\infty} e^{-\sigma y^2} e^{i(k\cos\theta - \alpha)y}\, dy = \sqrt{\frac{\pi}{\sigma}}\, e^{-(k\cos\theta - \alpha)^2/4\sigma}$$

$$= \frac{A}{c^2 4\pi x} e^{i(\omega t - kx)} \cdot \sqrt{\frac{\pi}{\sigma}}\, e^{-(k\cos\theta - \alpha)^2/4\sigma}$$

$$= \frac{A}{4c^2 x\sqrt{\pi\sigma}} e^{i(\omega t - kx)} e^{-(k\cos\theta - \alpha)^2/4\sigma}$$

(ii) phase speed $v_p = \frac{dx}{dt}$ when phase $wt - \beta x = \text{constant}$

far-field sound pressure: $p' = \frac{A}{4c^2 x \sqrt{\pi\sigma}} e^{i(wt-kx)} e^{-(k\cos\theta-\alpha)^2/4\sigma}$

from $\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) p' = A e^{-\sigma x_1^2} \delta(x_2)\delta(x_3) e^{i(wt-\alpha x_1)}$

phase $= wt - \alpha x_1 = \text{constant}$

$\downarrow \frac{d}{dt}$

$w - \alpha \frac{dx_1}{dt} = 0$

$\frac{dx_1}{dt} = \frac{w}{\alpha} = \text{phase speed } v_p$ ▨

4'

(iii) $\sigma \ll 1$

$p' = \frac{A}{4c^2 x \sqrt{\pi\sigma}} e^{i(wt-kx)} \underbrace{e^{-(k\cos\theta-\alpha)^2/4\underset{\ll 1}{\sigma}}}_{\sim 0}$

if we want the sound to be radiated. $p' \neq 0 \Rightarrow k\cos\theta - \alpha = 0$

$\theta = \cos^{-1}\frac{\alpha}{k}$ ▨

phase speed $v_p = \frac{w}{\alpha} = \frac{w}{k\cos\theta} = \frac{k\cdot c}{k\cos\theta} = \frac{c}{\cos\theta}$ supersonic ▨

6'

Q3.2

Proof:

the pressure perturbation at the neck opening : $p_t'(t)$

mass flow rate into the bulb : Q kg/s

in the bulb volume V : $V \cdot \frac{\partial \rho_b'}{\partial t} = Q$ ①

$$p_b' = c^2 \rho_b' \quad ②$$

assume harmonic disturbances with $w = \frac{\alpha U}{D}$ (characteristic frequency)

$$① \Rightarrow \rho_b' = \frac{Q}{iwV}$$

$$② \Rightarrow p_b' = \frac{c^2 Q}{iwV}$$

momentum balance in the neck : $p_t' - p_b' = \underbrace{\rho_0}_{\text{in the neck}} l \frac{\partial u_e}{\partial t}$ ③

where $u_e = \frac{Q}{\rho_0 S}$

$$\Rightarrow p_t' - p_b' = \rho_0 l \frac{\partial \frac{Q}{\rho_0 S}}{\partial t} = \frac{l}{S}\frac{\partial Q}{\partial t} = \frac{l}{S} Q (iw)$$

$$\Rightarrow p_t' = p_b' + \rho_0 l \frac{\partial u_e}{\partial t} = \frac{c^2 Q}{iwV} + \frac{iwlQ}{S}$$

$$= \left(w^2 - \frac{c^2 S}{Vl}\right)\frac{ilQ}{wS}$$

non-dimensional number

rms pressure perturbation at neck opening : $p_{t,rms}' = \frac{1}{2} k \rho_0 U^2$

$$\sim \left(w^2 - \frac{c^2 S}{Vl}\right)\frac{lQ}{wS}$$

$\left(\frac{\partial}{\partial t} \to iw \text{ , time scale is } \frac{D}{\alpha U} \quad \frac{\partial}{\partial t} \sim \frac{\alpha U}{D} = w \right)$

$$\Rightarrow Q \sim \frac{\rho_0 U^2 w S}{l}\left(w^2 - \frac{c^2 S}{Vl}\right)^{-1}$$

$$\sim \frac{\rho_0 S D U}{l \alpha}\left(1 - \frac{SD^2}{\alpha^2 M^2 V l}\right)^{-1}$$

$p'(t) = \frac{\dot{Q}(t - r/c)}{4\pi r}$ for monopole source

Sound power radiated from the neck : $\text{Power} = \frac{\overline{p'^2}}{\rho_0 c} \cdot S = \frac{4\pi r^2}{\rho_0 c} \cdot \overline{\left[\frac{\dot{Q}(t-r/c)}{4\pi r}\right]^2}$

$$= \frac{1}{\rho_0 c} \cdot \frac{1}{4\pi} \cdot \overline{(iwQ)^2} \sim \frac{1}{\rho_0 c} \cdot w^2 Q^2$$

$$= \frac{1}{\rho_0 c} \cdot \frac{\alpha^2 U^2}{D^2} \cdot \frac{\rho_0^2 S^2 D^2 U^2}{l^2 \alpha^2} \cdot \left(1 - \frac{SD^2}{\alpha^2 M^2 V l}\right)^{-1}$$

$$= \frac{M U^3 S^2 \rho_0}{l^2}\left(1 - \frac{SD^2}{\alpha^2 M^2 V l}\right)^{-2} \quad ▨$$

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Q3.3

(i) an unbounded fluid.

piston vibrating surface displacement : $\varepsilon e^{i\omega t}$

$\backsim$ velocity : $\frac{\partial(\varepsilon e^{i\omega t})}{\partial t} = i\omega\varepsilon \cdot e^{i\omega t} = u_0 e^{i\omega t}$

linearised momentum equation on the surface :

$$\rho_0 \frac{\partial u_s}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

$$\rho_0 \frac{\partial u_s}{\partial t} = \rho_0 \cdot i\omega \cdot u_0 e^{i\omega t} = -\frac{\partial p'}{\partial x}$$

An impenetrable compact body changing in volume :

$$p'(\underline{x}, t) = \frac{\rho_0 \ddot{V}(t - |\underline{x}|/c)}{4\pi |\underline{x}|}$$

$$\dot{V}(t) = \int_S \underline{n} \cdot \underline{u} \, ds = l^2 \cdot u_0 e^{i\omega t}$$

$$\ddot{V}(t) = l^2 \cdot u_0 i\omega e^{i\omega t}$$

$$\Rightarrow p'(\underline{x}, t) = \frac{\rho_0 l^2 u_0 i\omega e^{i\omega(t - |\underline{x}|/c)}}{4\pi |\underline{x}|}$$

Since $kr \gg 1$, far field acoustic pressure :

$$p'(r, t) = \frac{\rho_0 l^2 u_0 i\omega e^{i\omega(t - r/c)}}{4\pi r} \quad \text{where } u_0 = i\omega\varepsilon$$

$$= -\frac{\rho_0 l^2 \omega^2 \varepsilon e^{i\omega(t-r/c)}}{4\pi r}$$

(ii) Cube in long tube

inside the tube: $p'(x,t) = I e^{i(wt-kx)} + R e^{i(wt+kx)}$

At $x=0$, surface displacement $= \varepsilon e^{iwt}$

$$u_s(t) = iw\varepsilon e^{iwt} = \frac{1}{\rho_0 c}\cdot(I-R)e^{iwt}$$

At $x=r$, $p' = I e^{i(wt-kx)} + R e^{i(wt+kx)} = (I e^{-ikr} + R e^{ikr})e^{iwt}$

Linearised momentum equation: $\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p'}{\partial x} = 0$

$$\frac{\partial u}{\partial t} = \frac{-1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$u = \frac{-1}{\rho_0 iw}\frac{\partial p'}{\partial x}$$

$$= \frac{-1}{\rho_0 \cancel{i}\cancel{w}}\cdot I e^{iwt}\cdot e^{-ikx}(-\cancel{i}k) + R e^{iwt}\cdot e^{ikx}\cdot(\cancel{i}k)$$

$$= \frac{1}{\rho_0 c}(I e^{-ikx} - R e^{ikx})e^{iwt}$$

At the open end of the tube, $p' = 0 = (I e^{-ik x_\infty} + R e^{ik x_\infty})e^{iwt}$

$\Rightarrow R = 0$, $I = iw\varepsilon\cdot\rho_0 c$ (infinitely long tube)

$$p' = iw\varepsilon\rho_0 c\, e^{i(wt-kr)}$$

10'

$$\text{Power 1} = \frac{1}{\rho_0 c}\cdot\overline{p'^2}\cdot 4\pi r^2 = \frac{1}{\cancel{\rho_0 c}}\cdot\frac{\rho_0^2 L^4 w^2\varepsilon^2\cdot w^2\cdot\frac{1}{2}}{\cancel{16\pi^2\cdot r^2}\ 4\pi\cdot c}\ \cancel{4\pi r^2}$$

$$= \frac{\rho_0 L^4 w^4 \varepsilon^2}{8\pi c}$$

$$\frac{4\,8\pi\cancel{c}\cdot\cancel{\rho_0}\cancel{w^2}\cancel{\varepsilon^2}c\cancel{L^2}}{\cancel{2}\cdot\cancel{\rho_0}\cancel{L^4}\cancel{w^4}\cancel{\varepsilon^2}w^2 L^2}$$

$$\text{Power 2} = \frac{1}{\rho_0 c}\overline{p'^2}\cdot L^2 = \frac{1}{\rho_0 c}\cdot\frac{1}{2}w^2\varepsilon^2\rho_0^2 c^2\cdot L^2$$

12'

$$= \frac{1}{2}\rho_0 w^2\varepsilon^2 c L^2$$

$$\frac{\text{Power 2}}{\text{Power 1}} = \frac{\cancel{\rho_0}\cancel{w^2}\cancel{\varepsilon^2}c\cancel{L^2}\cdot 4\pi c}{\cancel{\rho_0}\cancel{L^4}\cancel{w^4}\cancel{\varepsilon^2}L^2 w^2} = 4\pi\cdot L^{-2}\cdot k^{-2} = 2\pi(kL)^{-2}$$ ▨

Q3.4

i) Proof :

flame with the rate of heat addition per unit volume $w(\underset{\sim}{x}, t)$

→ for the acoustic source : $\rho = \rho(p, s)$

$$d\rho = \frac{\partial \rho}{\partial p}\Big|_s dp + \frac{\partial \rho}{\partial s}\Big|_p ds$$
$$= \frac{1}{c^2} dp + \frac{\partial \rho}{\partial s}\Big|_p ds$$

for perfect gas : $s - s_0 = C_v \ln \frac{p}{p_0} - C_p \ln \frac{\rho}{\rho_0}$

$$\Rightarrow \frac{\partial \rho}{\partial s}\Big|_p = -\frac{\rho}{C_p}$$

$$\Rightarrow d\rho = \frac{1}{c^2} dp - \frac{\rho}{C_p} ds$$

$$\frac{d\rho}{dt} = \frac{1}{c^2}\frac{dp}{dt} - \frac{\rho}{C_p}\frac{ds}{dt} \underset{\uparrow}{=} \frac{1}{c^2}\frac{dp}{dt} - \frac{\gamma - 1}{c^2} w$$

$$\left(\rho T \frac{ds}{dt} = w\right)$$
$$\left(\frac{1}{C_p T} = \frac{\gamma - 1}{c^2}\right)$$

linearising : $\frac{1}{c^2}\frac{\partial p'}{\partial t} - \frac{\partial \rho'}{\partial t} = \frac{\gamma - 1}{c^2} w$

$$\left.\begin{array}{l} \frac{\partial}{\partial t} \rightarrow \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \underset{\sim}{v} = 0 \\ \nabla \cdot \rightarrow \rho_0 \frac{\partial \underset{\sim}{v}}{\partial t} + \nabla p' = 0 \end{array}\right\} \Rightarrow \frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = 0$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \frac{\gamma - 1}{c^2}\frac{\partial w}{\partial t} \quad \text{(wave equation)}.$$

where $p'(\underset{\sim}{x}, t) = \frac{\gamma - 1}{4\pi c^2}\frac{\partial}{\partial t}\int_V \frac{w(\underset{\sim}{y}, t - |\underset{\sim}{x} - \underset{\sim}{y}|/c)}{|\underset{\sim}{x} - \underset{\sim}{y}|} d^3\underset{\sim}{y}$

Given : $\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = \frac{\gamma - 1}{c^2}\frac{\partial w}{\partial t}$

$$\Rightarrow \rho'(\underset{\sim}{x}, t) = \frac{\gamma - 1}{4\pi c^4}\frac{\partial}{\partial t}\int_V \frac{w(\underset{\sim}{y}, t - |\underset{\sim}{x} - \underset{\sim}{y}|/c)}{|\underset{\sim}{x} - \underset{\sim}{y}|} d^3\underset{\sim}{y} \checkmark$$

Since the flame source distribution is compact, far-field : $|\underset{\sim}{x}| \gg |\underset{\sim}{y}|$

$$\rho'(\underset{\sim}{x}, t) = \frac{\gamma - 1}{4\pi c^4}\frac{\partial}{\partial t}\int_V \frac{w(\underset{\sim}{y}, t - |\underset{\sim}{x}|/c)}{|\underset{\sim}{x}|} d^3\underset{\sim}{y}$$

$$= \frac{\rho_0}{4\pi c^2 |\underset{\sim}{x}|}\frac{\partial^2}{\partial t^2}\int_V \beta(\underset{\sim}{y}, t - |\underset{\sim}{x}|/c)\, d^3\underset{\sim}{y} \quad \text{(from hint)}$$

[3]

$$\frac{\partial}{\partial t} \sim \frac{u}{L}, \quad \int_V d^3y \sim L^3, \quad \beta \sim 1$$

$$\Rightarrow \rho' \sim \frac{\rho_0}{c^2 |\underset{\sim}{x}|}\cdot\frac{u^2}{L^2}\cdot L^3 = \frac{\rho_0 L}{|\underset{\sim}{x}|}\cdot\left(\frac{u}{c}\right)^2 = \frac{\rho_0 L}{|\underset{\sim}{x}|}\cdot M^2 \quad ▨$$

(ii) $p' \sim \rho_0 \left(\frac{L}{x}\right) M^2 \quad \Rightarrow \quad \overline{p'^2} \sim \rho_0^2 \frac{L^2}{x^2} M^4$

$p' = c^2 \cdot \rho'$

far-field acoustic power

$$\text{Power} = \int_S \overline{p'u}\, dS = 4\pi x^2 \cdot \frac{\overline{p'^2}}{\rho_0 c}$$

$$\to 4\pi x^2 \cdot \frac{\overline{(c^2\rho')^2}}{\rho_0 c} = 4\pi x^2 \cdot \frac{c^4 \cdot \overline{\rho'^2}}{\rho_0 c}$$

$$p' = \frac{\gamma - 1}{4\pi x c^2} \cdot \frac{\partial}{\partial t} \int_V w(\underline{y}, t - x/c)\, d^3\underline{y}$$

$$= \frac{\rho_0}{4\pi x} \frac{\partial^2}{\partial t^2} \int_V \beta(\underline{y}, t - x/c)\, d^3\underline{y} \sim \frac{\rho_0}{x} \cdot \frac{u^2}{L^2} \cdot L^3 = \frac{\rho_0 u^2 L}{x}$$

$$\sim \frac{\cancel{4\pi x^2}\, c^3 \cdot}{\cancel{\rho_0}} \rho_0^2 \frac{L^2}{\cancel{x^2}} M^4$$

$$= c^3 \rho_0 L^2 M^4$$

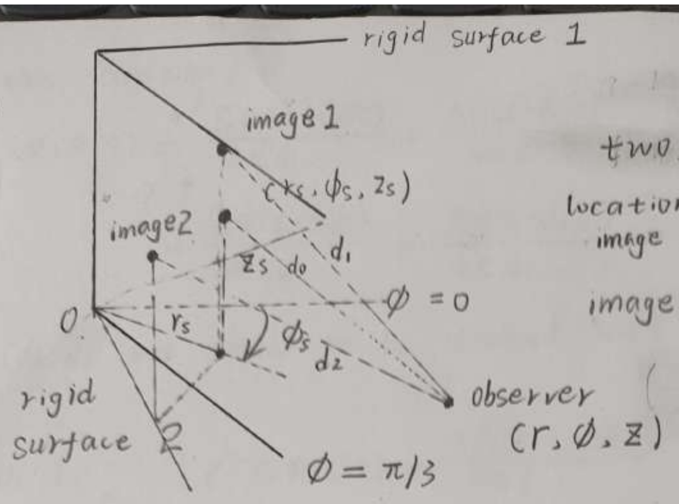
$$= u^3 \rho_0 L^2 M$$

$$\Rightarrow \text{Power} \sim \cancel{x^2} \cdot \frac{1}{\rho_0 c} \cdot \frac{\rho_0^2 u^4 L^2}{\cancel{x^2}} = \rho_0 L^2 u^3 \cdot M$$

7

Q3.5

Solution:



two basic necessary images of the loudspeaker

locations:

image 1 : $(r_s, -\phi_s, z_s)$

image 2 : $(r_s, \frac{2}{3}\pi - \phi_s, z_s)$

small loudspeaker with a rate of volume outflow $Q(t)$ $\rho_0 \dot{Q}(t)$.

image source is symmetric about the rigid surface

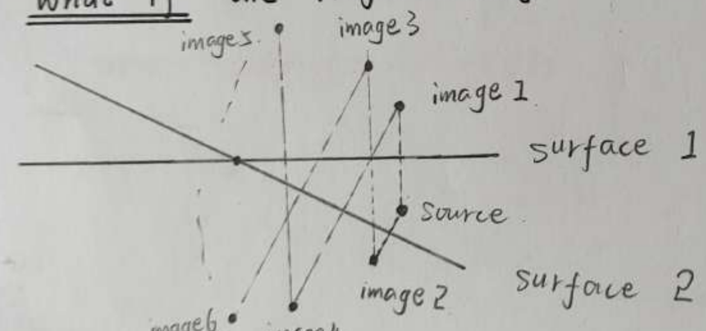
$$p'(r, \phi, z, t) = \frac{\rho_0 \dot{Q}(t - d_0/c)}{4\pi d_0} + \frac{\rho_0 \dot{Q}(t - d_1/c)}{4\pi d_1} + \frac{\rho_0 \dot{Q}(t - d_2/c)}{4\pi d_2}$$

$$d_0 = \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2} = \sqrt{(r\cos\phi - r_s\cos\phi_s)^2 + (r\sin\phi - r_s\sin\phi_s)^2 + (z - z_s)^2}$$

$$d_1 = \sqrt{(r\cos\phi - r_s\cos\phi_s)^2 + (r\sin\phi + r_s\sin\phi_s)^2 + (z - z_s)^2}$$

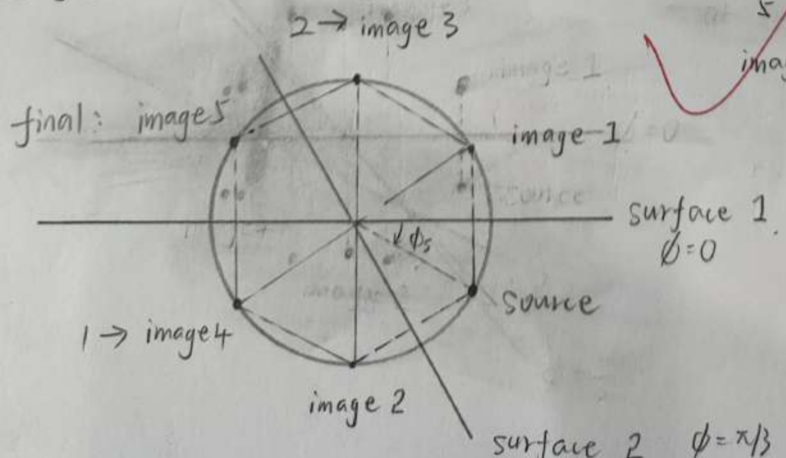
$$d_2 = \sqrt{\left[r\cos\phi - r_s\cos(\tfrac{2}{3}\pi - \phi_s)\right]^2 + \left[r\sin\phi - r_s\sin(\tfrac{2}{3}\pi - \phi_s)\right]^2 + (z - z_s)^2}$$

<u>what if</u> the image → image in another surface.



the number of the images is dependent of the location of source

To find the necessary images, consider:



5 images in total are necessary

image 3 $(r_s, -\frac{2}{3}\pi + \phi_s, z_s)$

4 $(r_s, \frac{2}{3}\pi + \phi_s, z_s)$

5 $(r_s, \phi_s + \pi, z_s)$

9'

in this condition :

$$p'(r,\phi,z,t) = \frac{\rho_0\dot{Q}(t-d_0/c)}{4\pi d_0} + \frac{\rho_0\dot{Q}(t-d_1/c)}{4\pi d_1} + \frac{\rho_0\dot{Q}(t-d_2/c)}{4\pi d_2} + \frac{\rho_0\dot{Q}(t-d_3/c)}{4\pi d_3} + \frac{\rho_0\dot{Q}(t-d_4/c)}{4\pi d_4} + \frac{\rho_0\dot{Q}(t-d_5/c)}{4\pi d_5}$$ ▨

d_3, d_4, d_5 are similar to be obtained with d_0, d_1, d_2.

small r_s : $p'(r,\phi,z,t) = \dfrac{3\rho_0\dot{Q}(t-r/c)}{4\pi r}$

unbounded : $\dfrac{\rho_0\dot{Q}}{4\pi r}$

(if image → image : $p' = \dfrac{6\rho_0\dot{Q}(t-r/c)}{4\pi r}$) ✓

acoustic power ~ $\overline{p'^2}$ ⇒ a factor of 9 ✗

(of 36) ✓ ▨ 5'

~~hemisphere~~

Because the sound source is mirrored by the rigid surfaces, creating increased sound pressure.

$p' = 6p_0'$

$\overline{p'^2} = 36\,\overline{p_0'^2}$

~~area $4\pi r^2 \cdot \frac{1}{2}$ $r = \frac{1}{2} r_0$~~

$\dfrac{\overline{p_0'^2}}{\rho_0 c} \cdot 4\pi r_0^2$

$\frac{\pi}{3}$扇形角面积.

$\dfrac{\overline{p'^2}}{\rho_0 c} \cdot \left(4\pi r_0^2 \cdot \frac{1}{6}\right)$

$\dfrac{36\,\overline{p_0'^2}}{\rho_0 c} \cdot \frac{1}{6} \cdot 4\pi r_0^2 \rightarrow 6$倍 Power.