Quiz 13

Date: 2022-05-13 Name: SID:

Find the solutions of the following equations.

Q1. $x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x;$

Q2. $x' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} x;$

Q3. $x' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x, x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

1. $|A-\lambda I| = \lambda^2 - \lambda - 2 = 0$. $\lambda_1 = -1$. $\lambda_2 = 2$.

$\xi_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. $\xi_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$.

2. 5. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $(1-r)(r^2-2r+5)=0$, with roots $r_1 = 1$, $r_2 = 1+2i$ and $r_3 = 1-2i$. Setting $r=1$, the equations reduce to $\xi_1 - \xi_3 = 0$ and $3\xi_1 + 2\xi_2 = 0$. If we choose $\xi_2 = -3$, the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2,-3,2)^T$. With $r = 1+2i$, the system of equations is equivalent to $i\xi_1 = 0$ and $i\xi_2 + \xi_3 = 0$. An eigenvector is given by $\boldsymbol{\xi}^{(2)} = (0,1,-i)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^t(\cos 2t + i \sin 2t).$$

Taking the real and imaginary parts, we obtain

$$e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \quad \text{and} \quad e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}.$$

Thus the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t + c_2 e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix},$$

which spirals to ∞ about the x_1 axis in the $x_1x_2x_3$ space as $t \to \infty$ (for most initial conditions).

3. 7. Solution of the system of ODEs requires that

$$\begin{pmatrix} 1-r & -5 \\ 1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Substituting $r = -1+i$, the equations are equivalent to $\xi_1 = (2+i)\xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i, 1)^T$. One of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-t}(\cos t + i \sin t) =$$

$$= e^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i e^{-t} \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}.$$

Hence the general solution is

$$\mathbf{x} = c_1 e^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}.$$

$2c_1 + c_2 = 2$

$c_1 = 1$.

Invoking the initial conditions, we obtain the system of equations $2c_1 + c_2 = 1$ and $c_1 = 1$. Solving for the coefficients, the solution of the initial value problem is

$c_2 = 0$

$$\mathbf{x} = e^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - e^{-t} \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} = e^{-t} \begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix},$$

$x = e^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$

which spirals to zero as $t \to \infty$, due to the e^{-t} term.