1. TURE or FALSE, explain.

(1) $y' + y^2 = e^x$ is an ordinary differential equation of the 2 order.

(2) The initial value problem $y' = y^{1/4}$, $y(0) = 0$ has a unique solution.

(3) One can always transfer a high order ODE into a systems of first-Order equations equivalently.

(4) If $y_1, ..., y_n$ be the linear independent solutions of the homogeneous n-th order linear differential equation $y^n + p^1(t)y^{n-1} + ... + p^{n-1}(t)y' + p^n(t)y = 0$, then they form a fundamental set of solutions.

(5) Suppose $f(x, y)$ is continuous, then all the solutions of the first order ODE $y' = f(x, y)$ can be extended to the interval $(-\infty, \infty)$.

2. Solve the following ODEs

(1) $y' + y = x + \cos(x) + e^x$, find the general solution and the solution with $y(0) = 0$.

(2) $y' + y^3e^x = 0$.

(3) $(2x+3)+(2y-2)y' = 0$, determine whether it is exact. If it is exact, find all the solutions.

(4) $y'' + y' - 6y = 0$, $y(0) = 1$, $y'(0) = 1$.

(5) $y' = \frac{x+y}{x-y}$

(6)

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{x}$$

3. Suppose that $f(t)$ satisfies $f''(t) + af'(t) + f(t) = 0$ $(a > 0)$, and $f'(0) = n$, $f(0) = m$. Find

$$\int_0^{\infty} f(t)dt.$$

4. For the system of the first order ODEs below,

$$x_1' = F_1(t, x_1, x_2, ..., x_n),$$

$$x_2' = F_2(t, x_1, x_2, ..., x_n),$$

$$...$$

$$x_n' = F_n(t, x_1, x_2, ..., x_n).$$

Let each of the n functions $F_1, ..., F_n$ and the n^2 first partial derivatives $\frac{\partial F_i}{\partial x_j}$ be continuous in a region K of $\alpha < t < \beta, \alpha_1 < x_1 < \beta_1, ..., \alpha_n < x_n < \beta_n$, and let the point $t_0, x_1^0, ..., x_n^0$ be in K. Then there is an interval $|t - t_0| < h$ in which there exists a unique solution that also satisfies the initial conditions $x_i(t_0) = x_i^0$.

Describe the main steps to prove this Theorem using Picard's iteration method.

5. Let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ be solutions of following equation,

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2,$$

$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2.$$

And let W be the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. Prove that

$$\frac{dW}{dt} = (p_{11} + p_{22})W,$$

generalize this conclusion to the systems with arbitrary n.

6. Consider the following system:

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}.$$

(1) Find the critical point, and determine whether it is stable, asymptotically stable, or unstable.

(2) Find the fundamental matrix Ψ such that $\Psi(0) = I$.

(3) Then find the general solutions of

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ \cos(t) \end{bmatrix}.$$