力学与航空航天工程系
DEPARTMENT OF MECHANICS AND AEROSPACE ENGINEERING

MECHANICS OF MATERIALS

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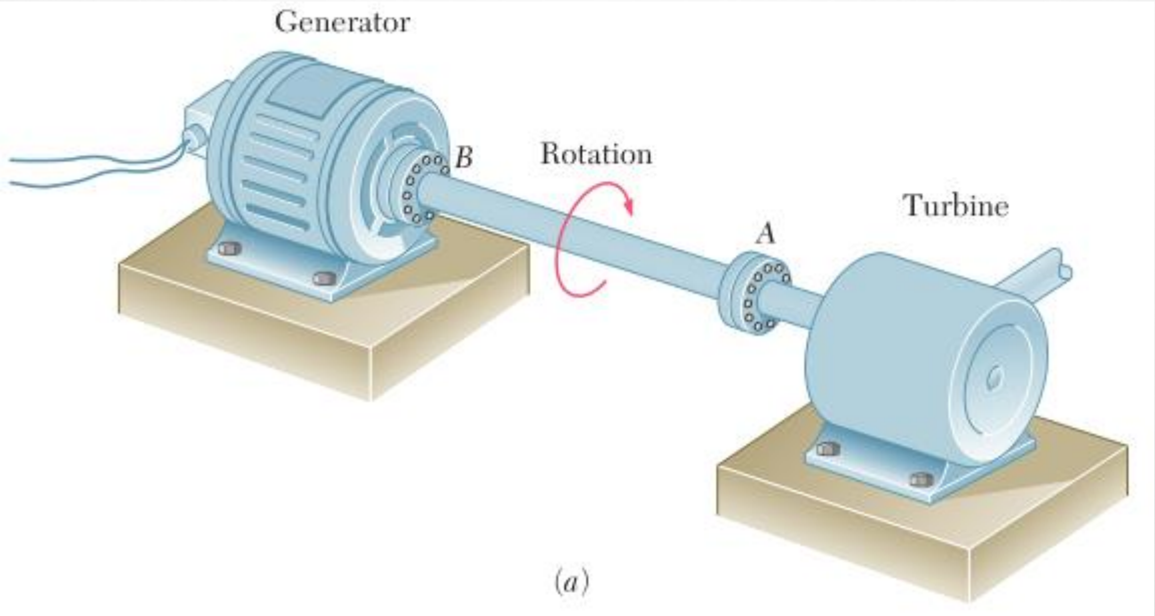
SPRING, 2022

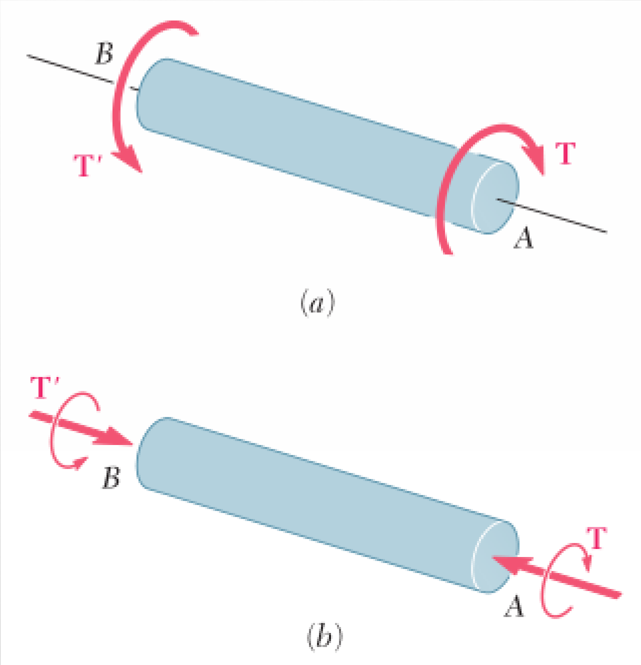
Lesson 4： Torsion

- Torsion, pure shear, shear strain
- Polar moment of inertia,
- Section modulus of torsion

§ 4.1 Introduction

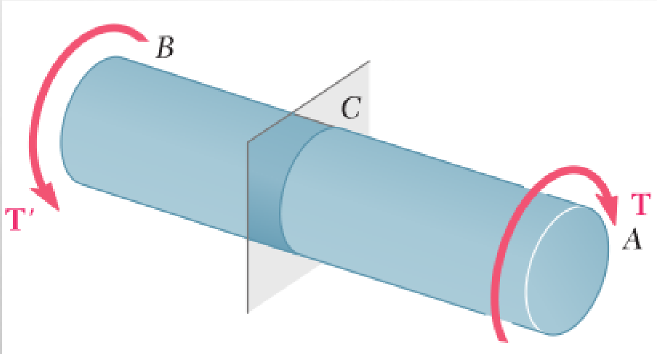
- Members of circular cross section subjected to twisting couples, or **torques**, **T** and **T′**

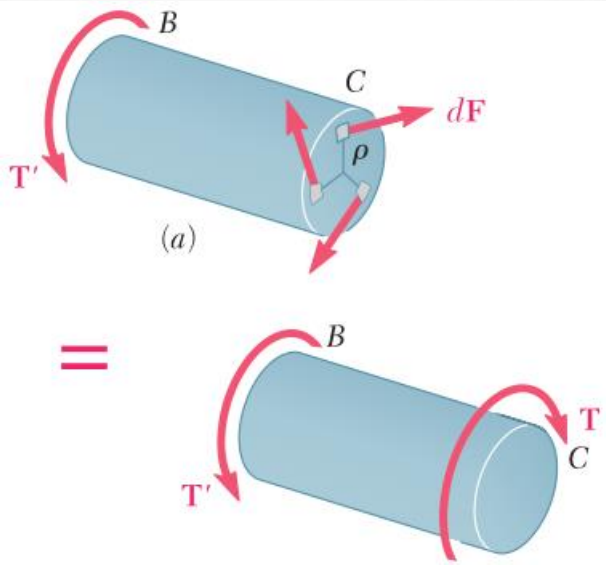




§ 4.2 Stress in a circular shaft

- Consider a circular shaft, applied with **torque, T**.



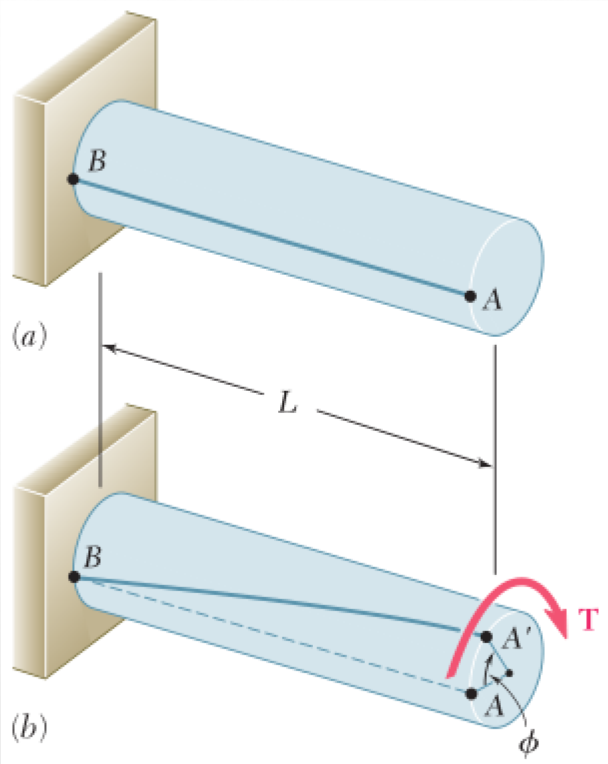


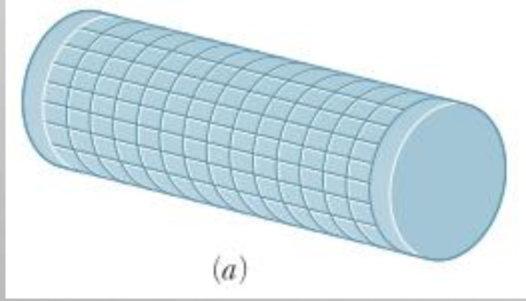
$$\int \rho \mathrm{d}F = \int \rho\tau \mathrm{dA} = T$$

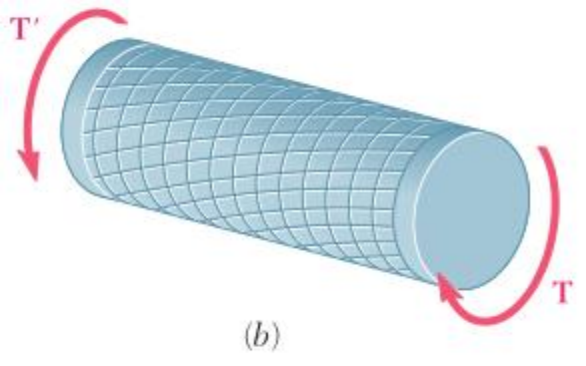
§ 4.3 Deformations in a circular shaft

- Consider a circular shaft that is attached to a fixed support at one end, applied with **torque**, **T**.

- When a circular shaft is subjected to torsion, every cross section remains plane and undistorted.





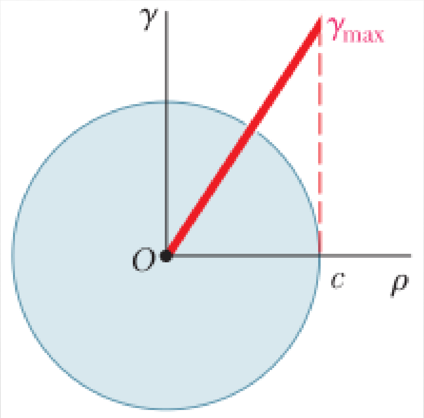


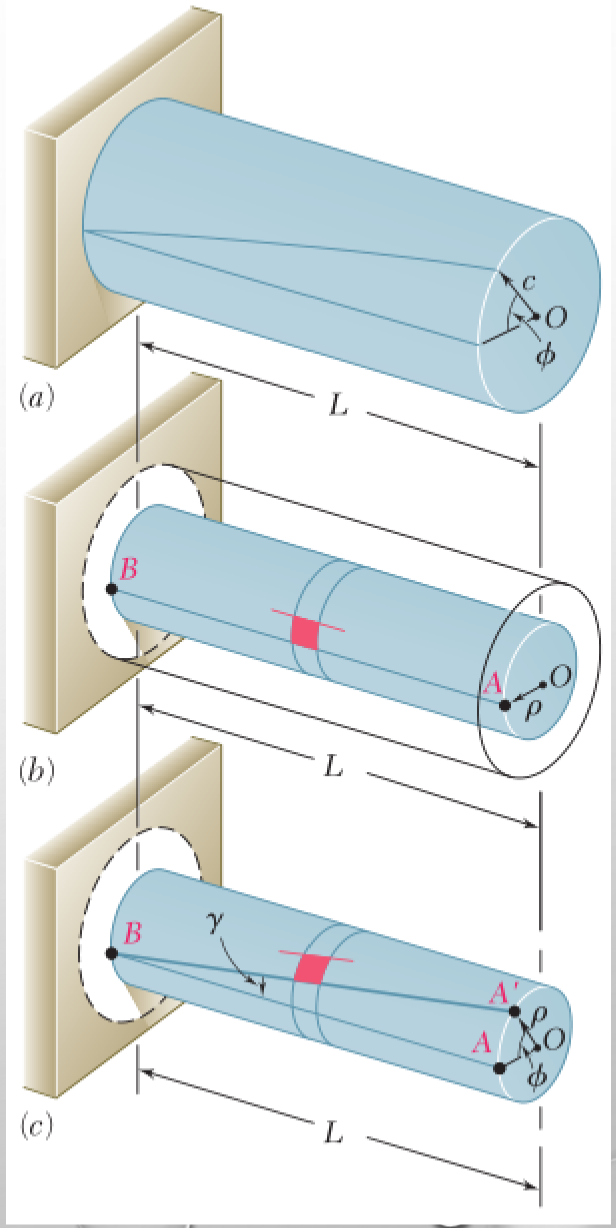
§ 4.4 Shear strain distribution

- Shear strain as a function of ρ,

$$\gamma = \frac{\rho\phi}{L} \qquad \gamma_{\max} = \frac{c\phi}{L}$$

$$\gamma = \frac{\rho}{c}\gamma_{\max}$$

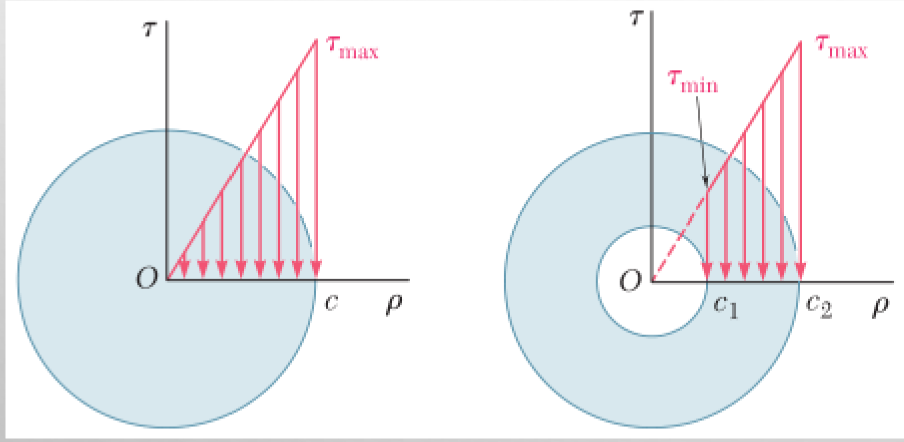




§ 4.5 Stresses in the elastic range

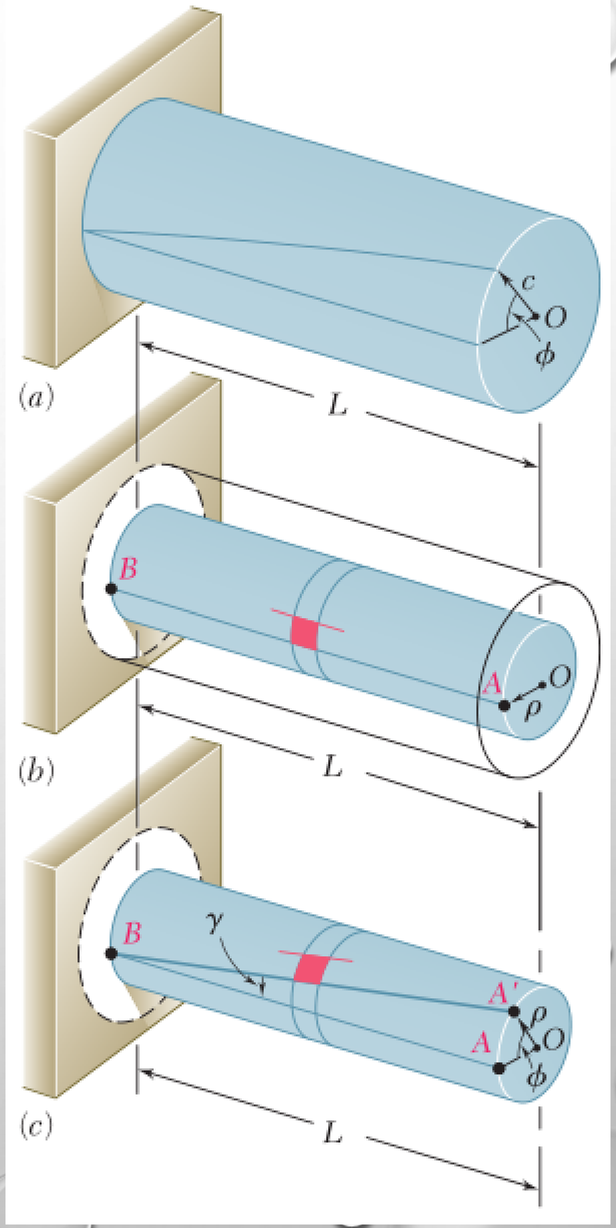
- Hooke's law of shear,

$$\tau = G\gamma \qquad \tau = G\frac{\rho\phi}{L} \qquad \tau_{max} = G\frac{c\phi}{L}$$

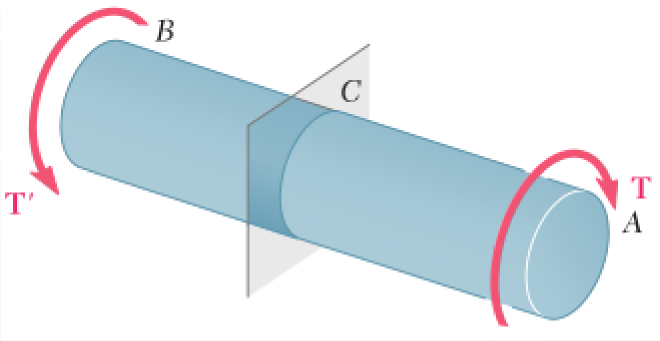


distribution of shearing stress

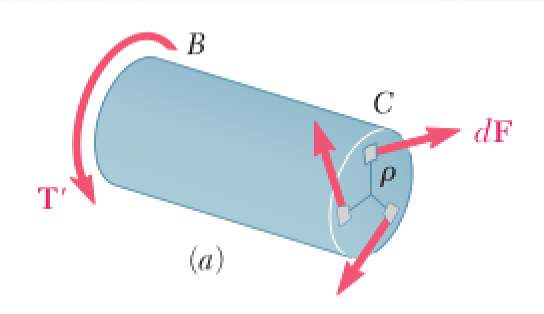
$$\tau = \frac{\rho}{c}\tau_{max}$$



§ 4.6 Elastic torsion formula



- The resultant of these elementary forces (d*F*) is equivalent to an internal torque T, equal and opposite to T′.



$$dF = \tau dA \qquad \tau = \frac{\rho}{c}\tau_{\max} \qquad \int \rho dF = \int \rho\tau dA = T$$

$$T = \frac{\tau_{\max}}{c} J \qquad J = \int \rho^2 dA \rightarrow \text{the polar moment of inertia}$$

$$\boxed{\tau_{\max} = \frac{Tc}{J}} \qquad \boxed{\tau = \frac{T\rho}{J}} \qquad \frac{J}{c} \rightarrow \text{section modulus of torsion}$$

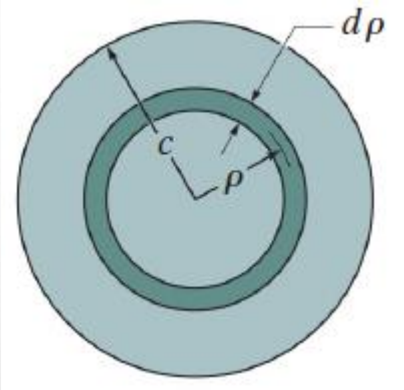
§ 4.7 Polar moment of inertia

- Geometric parameter

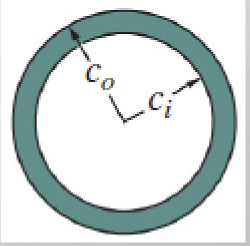
$$J = \int \rho^2 dA \rightarrow \text{the polar moment of inertia}$$

$$\frac{J}{c} \rightarrow \text{section modulus of torsion}$$

$$J = \frac{\pi}{2} c^4 \qquad\qquad J = \frac{\pi}{2}(c_o^4 - c_i^4)$$



circular

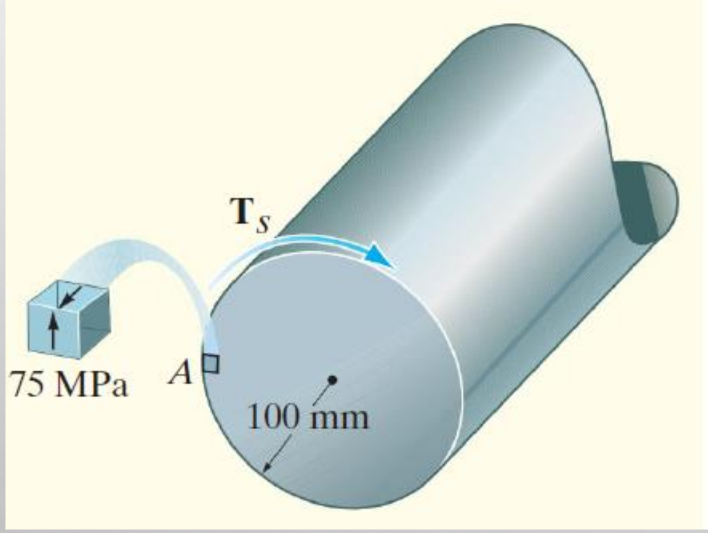


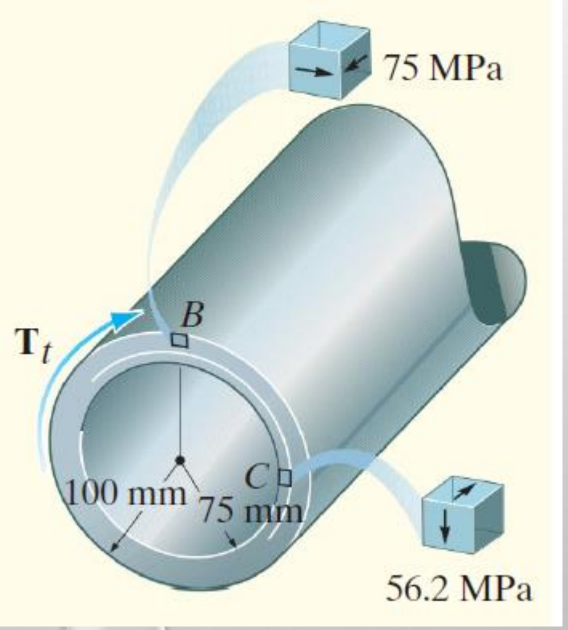
tube

Example 4.1

(Hibbeler, Page 191)

The solid shaft and tube as shown are made of a material having an allowable shear stress of 75 MPa. Determine the maximum torque that can be applied to each cross section, and show the stress acting on a small element of material at point A of the shaft, and points B and C of the tube.

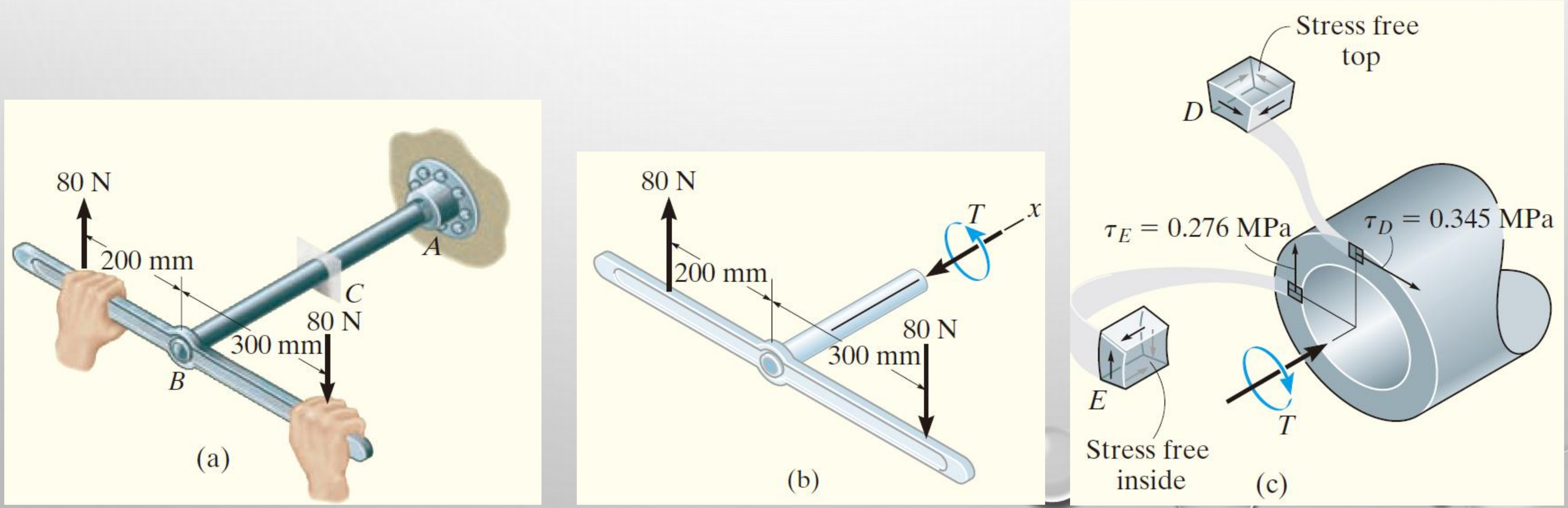




Example 4.2

(Hibbeler, Page 193)

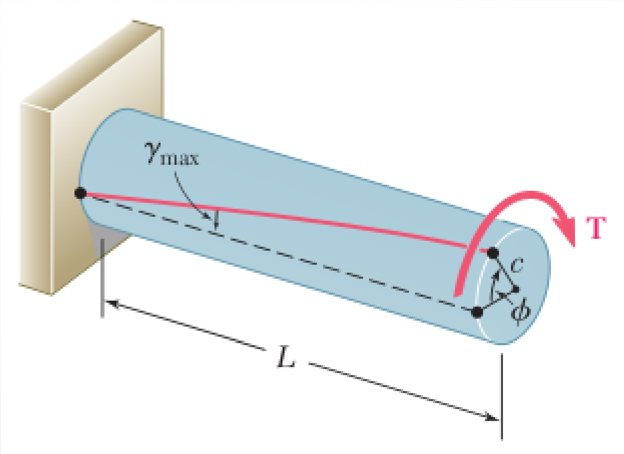
The pipe as shown has an inner radius of 40 mm and an outer radius of 50 mm. If its end is tightened against the support at A using the torque wrench, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe.



§ 4.8 Angle of twist in the elastic range

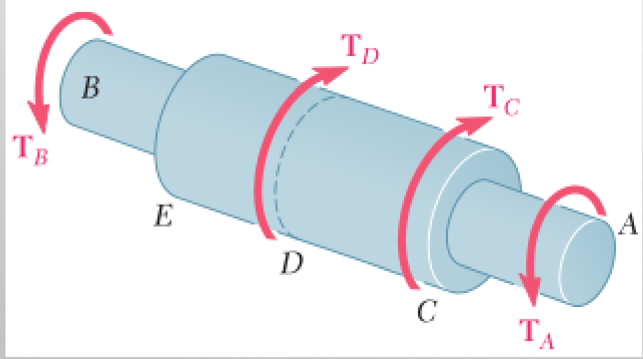
- The angle of twist ϕ is proportional to the torque T applied to the shaft,

$$\gamma_{max} = \frac{c\phi}{L} \qquad \gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

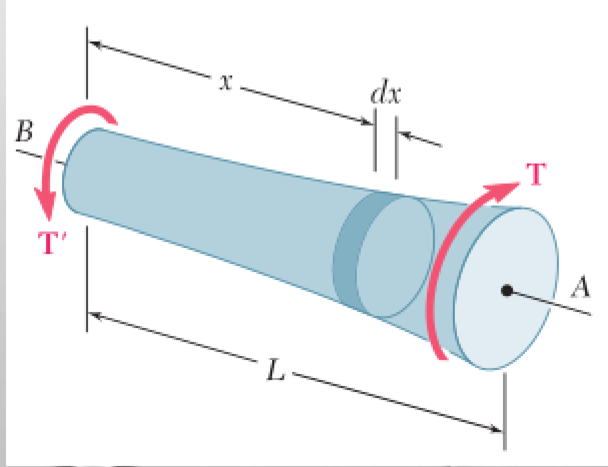


$$\phi = \frac{TL}{JG}$$

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

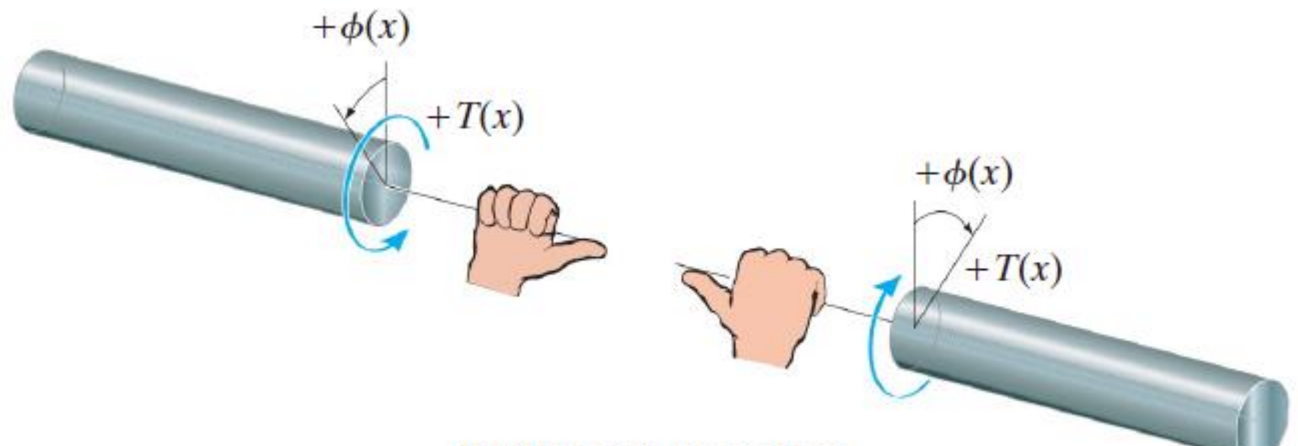


$$d\phi = \frac{T\,dx}{JG}$$



§ 4.9 Sign convention

- The right-hand rule

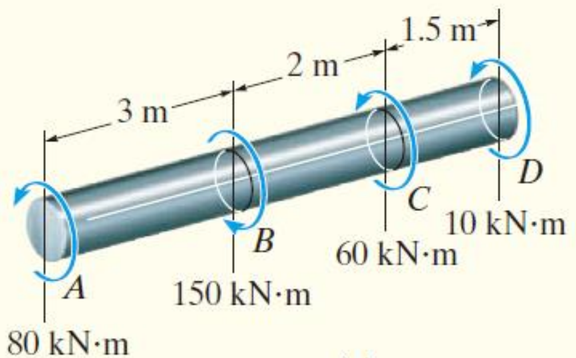


Positive sign convention for T and ϕ.

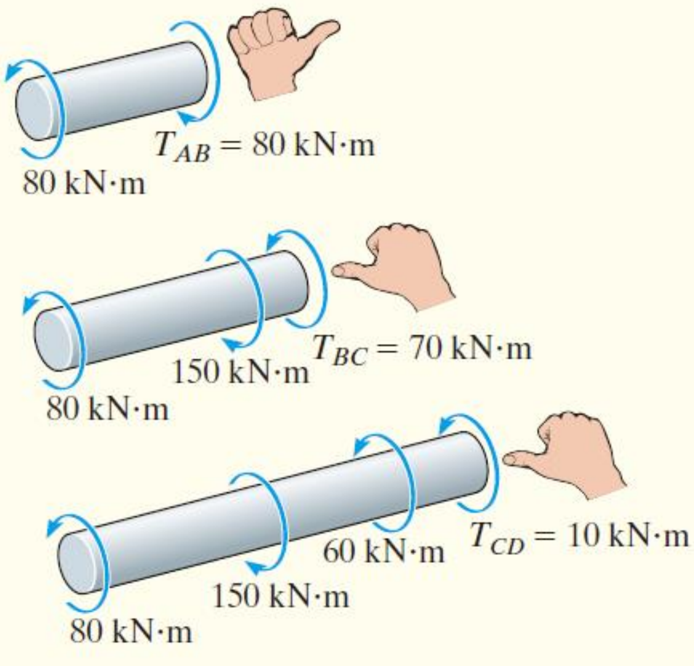
Example 4.3

(Hibbeler, Page 211)

Determine the angle of twist of the end A of the A-36 steel shaft as shown. Also, what is the angle of twist of A relative to C? The shaft has a diameter of 200 mm.



(a)

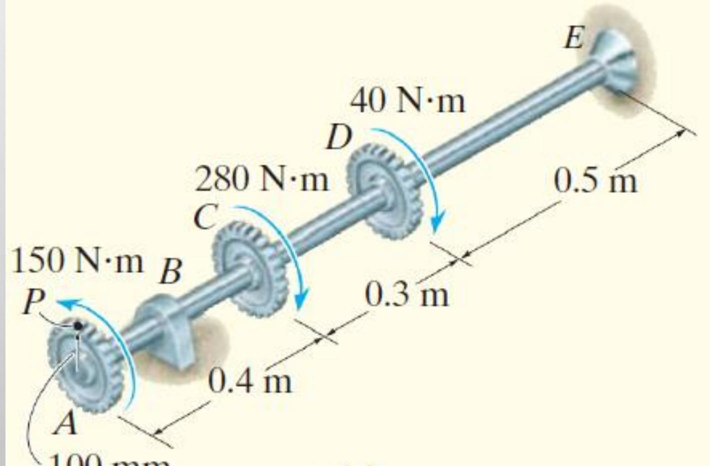


(b)

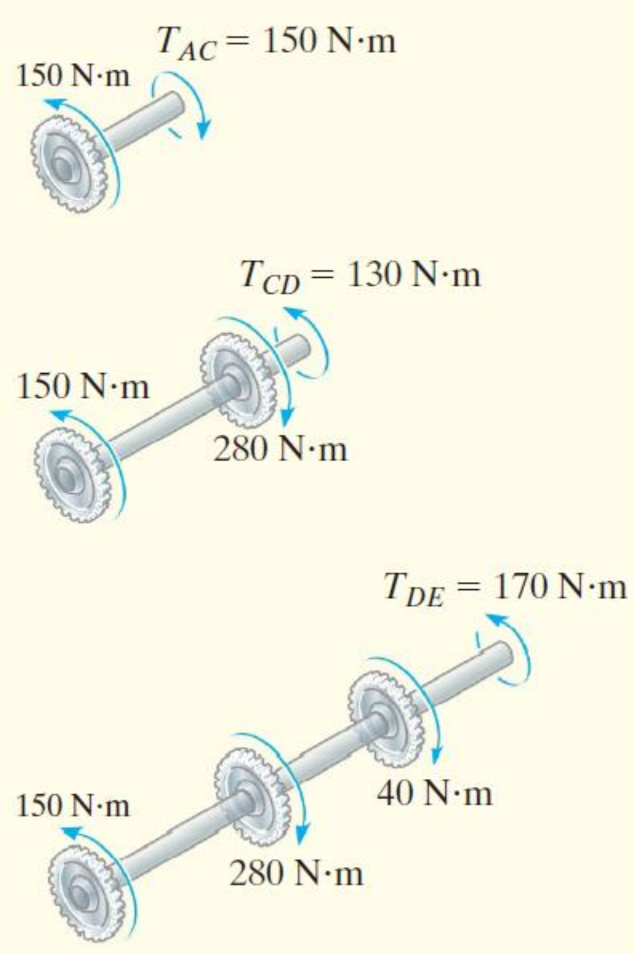
Example 4.4

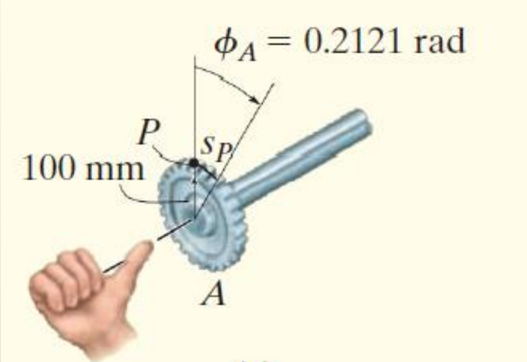
(Hibbeler, Page 212)

The gears attached to the fixed-end steel shaft are subjected to the torques as shown. If the shaft has a diameter of 14 mm, determine the displacement of the tooth P on gear A. G = 80 GPa.



(a)



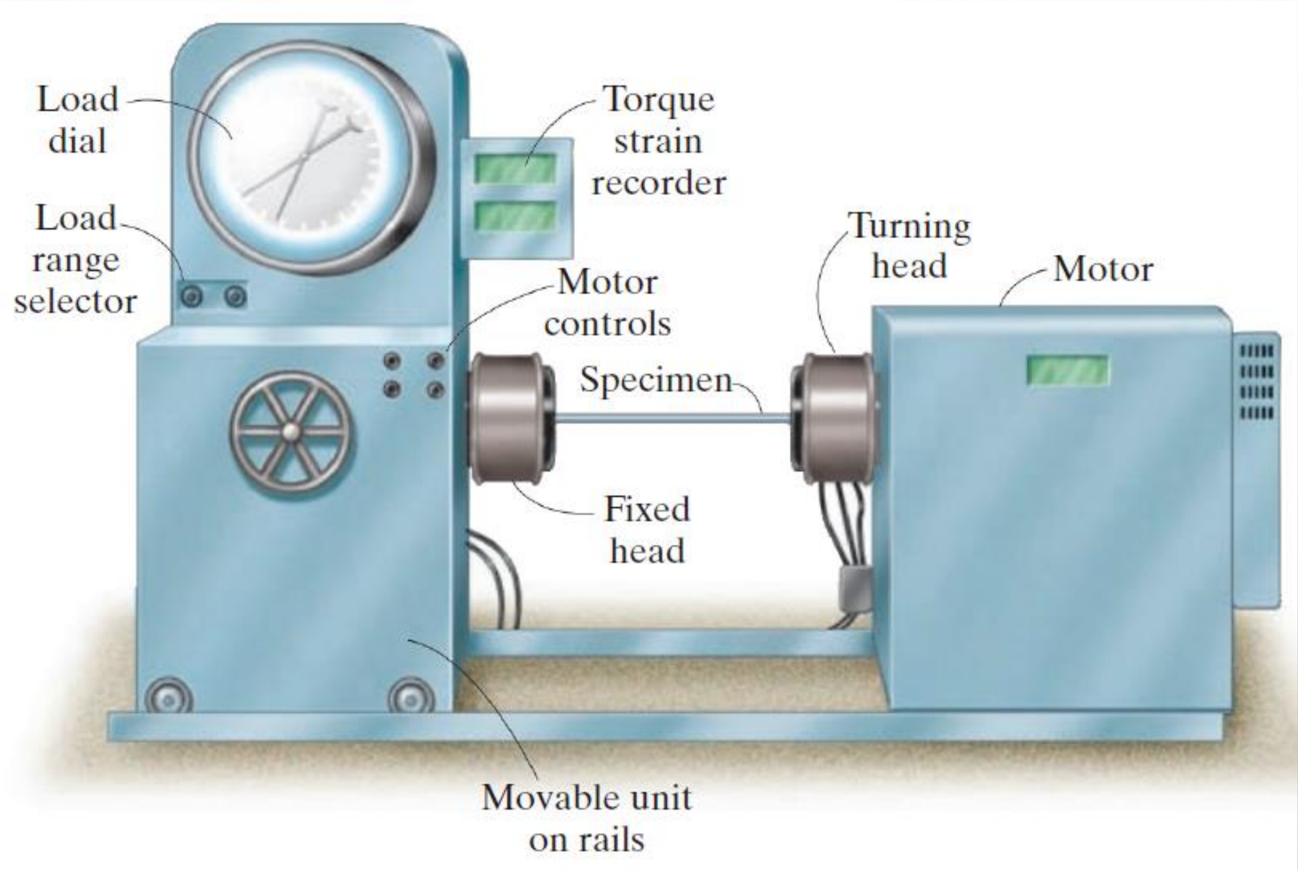


(c)

§ 4.10 Torsion tests

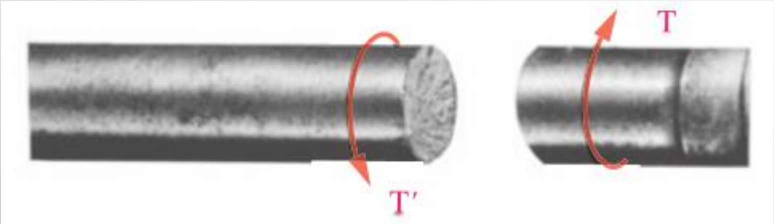
- A torsion testing machine to determine the shear modulus of G.

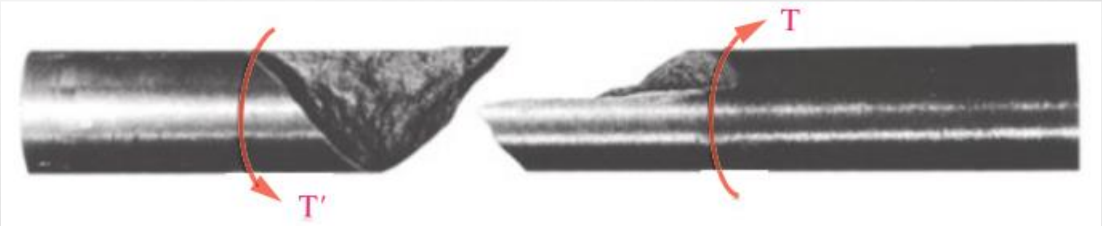
$$\phi = \frac{TL}{JG}$$



§ 4.10 Torsion tests

- Ductile materials generally fail in shear.
- Brittle materials are weaker in tension than in shear.





engineers
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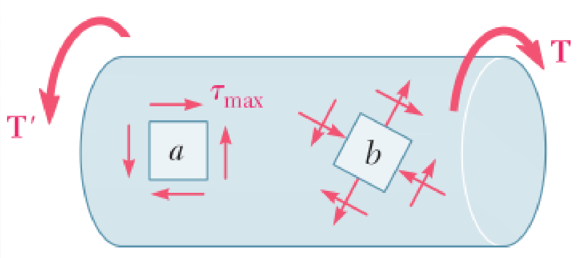
ϕ

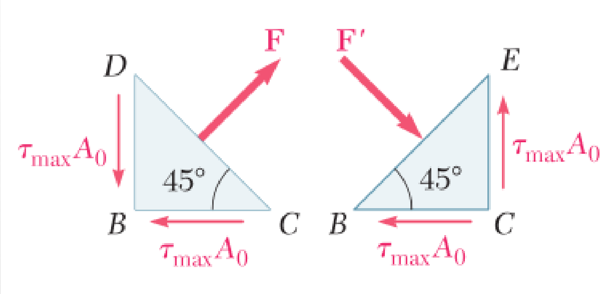
$\phi=$

§ 4.10 Torsion tests

- The angle of twist ϕ is proportional to the torque T applied to the shaft,

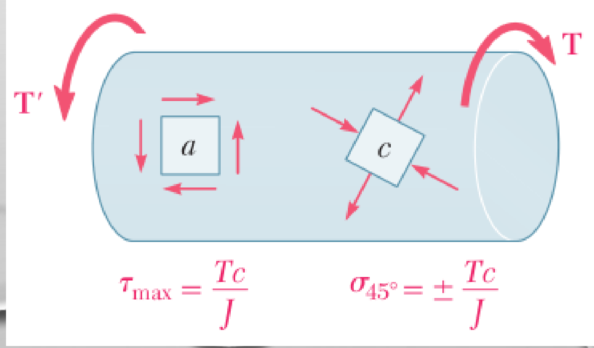


$$\tau_{max} = \frac{Tc}{J}$$



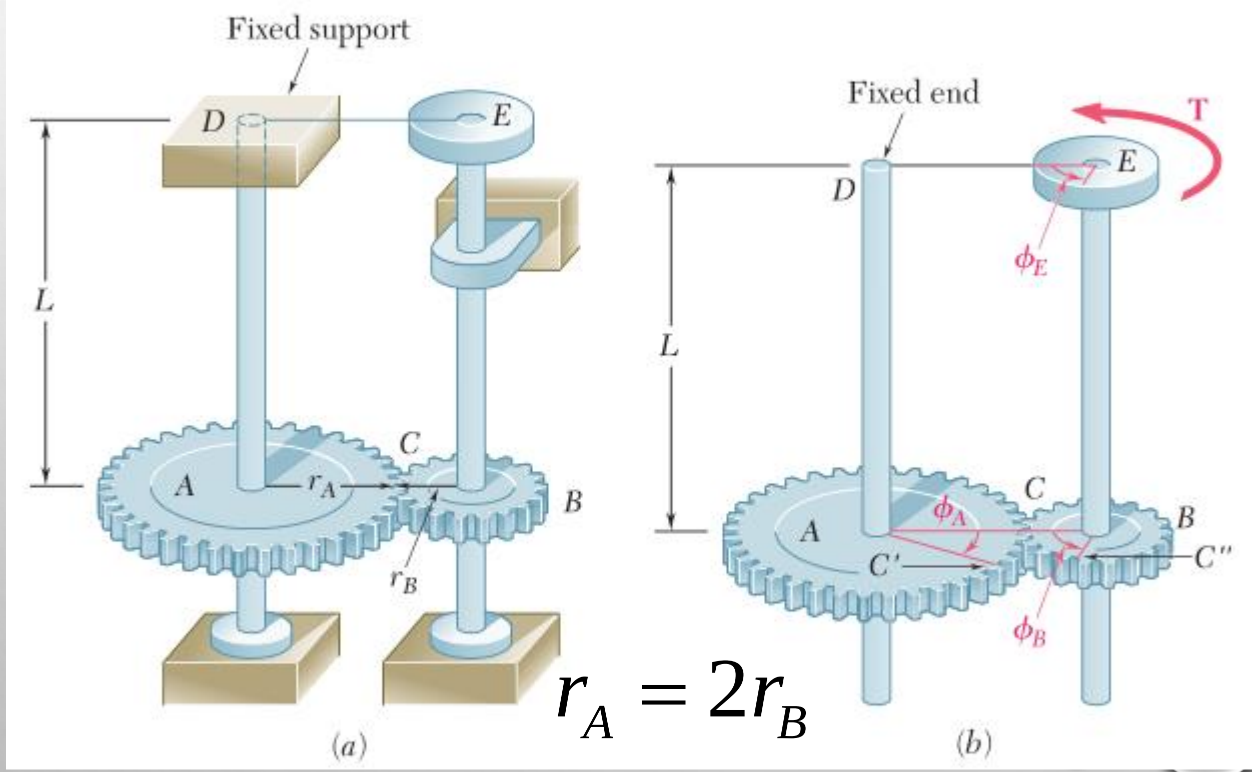
$$F = 2(\tau_{max} A_0)\cos 45^\circ = \tau_{max} A_0 \sqrt{2}$$

$$\sigma_{DC} = \frac{F}{A} = \tau_{max} \qquad \sigma_{BE} = \frac{F'}{A} = -\tau_{max}$$



§ 4.11 Gear assembly

- When both ends of a shaft rotate, the angle of twist of the shaft is equal to the angle through which one end of the shaft rotates with respect to the other.



$$r_A = 2r_B$$

$$\phi_{E/B} = \phi_E - \phi_B = \frac{TL}{JG}$$

$$\phi_A = \frac{2TL}{JG}$$

$$\phi_B = \frac{4TL}{JG}$$

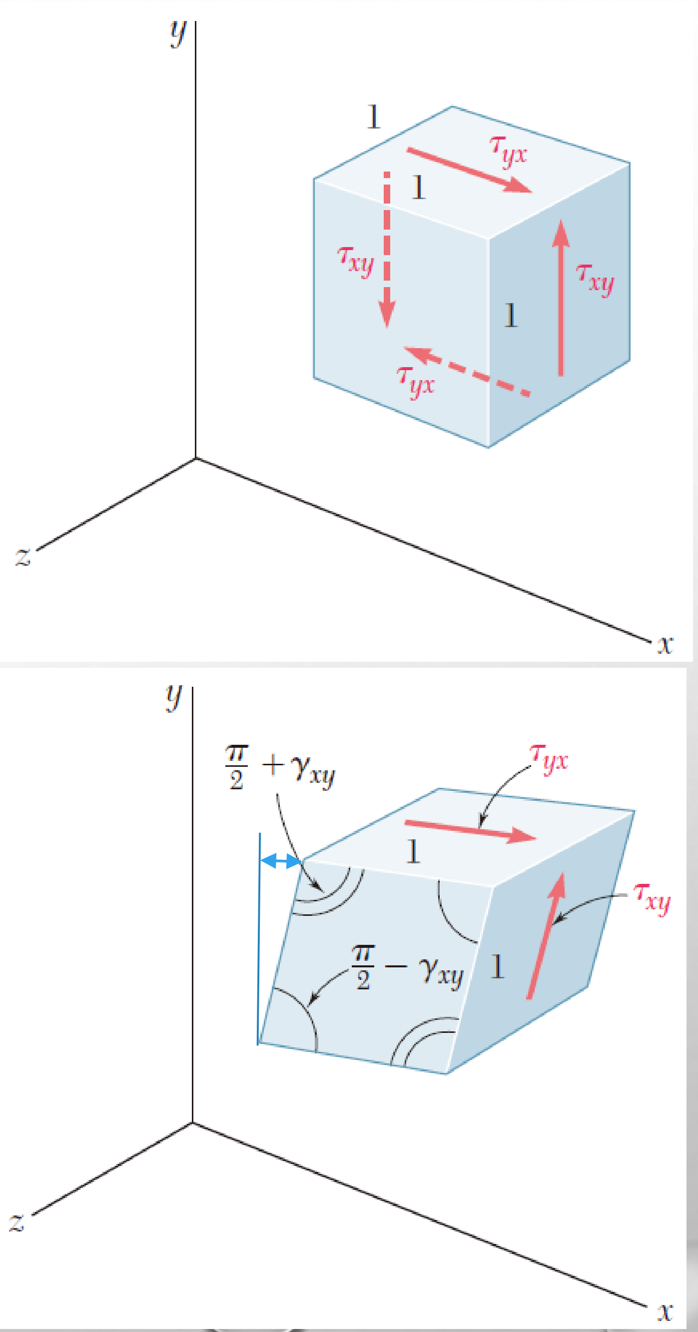
§ 4.12 Shear strain energy

- **Shear strain energy**: Internal energy stored in the material as deformed by an external load.

$$\Delta U = \frac{1}{2}\Delta F\,\Delta D = \frac{1}{2}\tau\gamma\Delta x\Delta y\Delta z = \frac{1}{2}\tau\gamma\Delta V$$

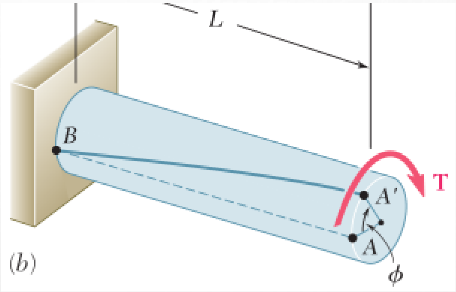
- **Shear energy density**: the strain energy per unit volume of material

$$u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\tau\gamma$$



§ 4.13 Power transmission

- The work transmitted by a rotating shaft equals the torque applied times the angle of rotation.



The instantaneous power

$$P = \frac{Td\theta}{dt} = T\omega = 2\pi fT$$

Shaft design

$$\frac{J}{c} = \frac{T}{\tau_{\max}}$$

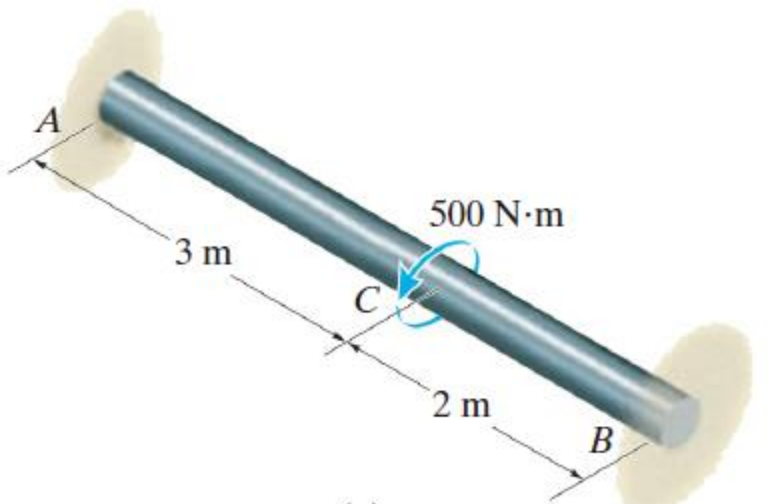
The belt drive transmits the torque developed by an electric motor to the shaft at A.

§ 4.14 Statically indeterminate problem

- A torsionally loaded shaft will be statically indeterminate if the moment equation of equilibrium, applied about the axis of the shaft, is not adequate to determine the unknown torques acting on the shaft.

$$500N \cdot m - T_A - T_B = 0$$

$$\phi_{A/B} = 0$$

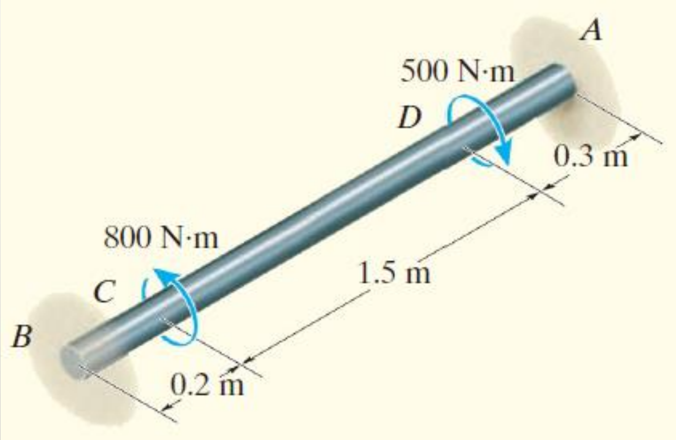


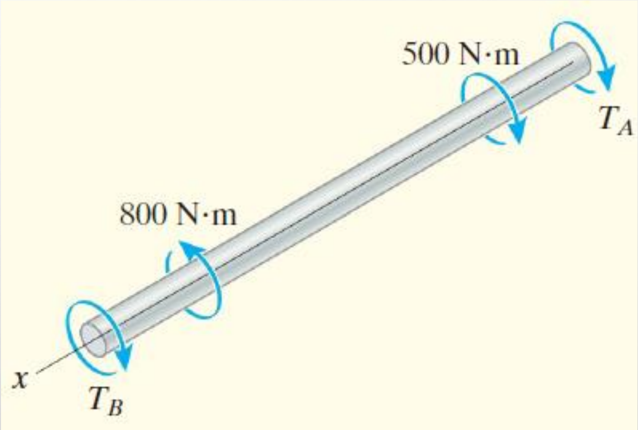
(a)

Example 4.5

(Hibbeler, Page 224)

The solid steel shaft as shown has a diameter of 20 mm. If it is subjected to the two torques, determine the reactions at the fixed supports
A and B.

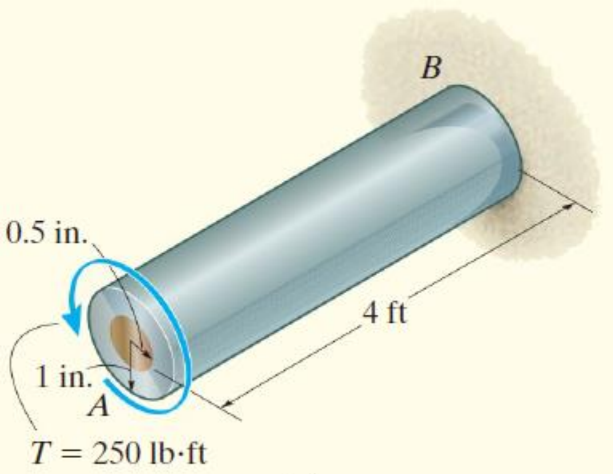




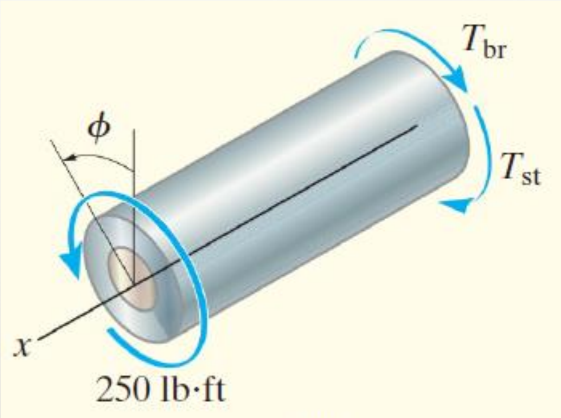
Example 4.6

(Hibbeler, Page 225)

The shaft as shown is made from a steel tube, which is bonded to a brass core. If a torque of T = 250 lb*ft is applied at its end, plot the shear-stress distribution along a radial line on its cross section. Take G_{st} = 11.411032 ksi, G_{br} = 5.2011032 ksi.



(a)



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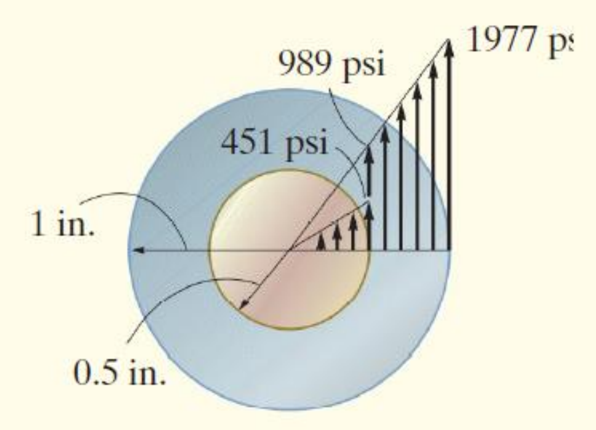


Shear–stress distribution

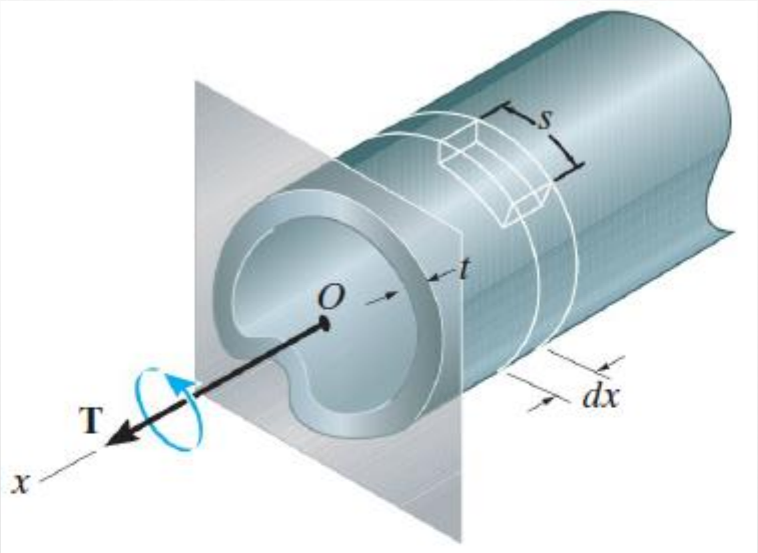
§ 4.15 Thin-walled tubes with closed cross sections

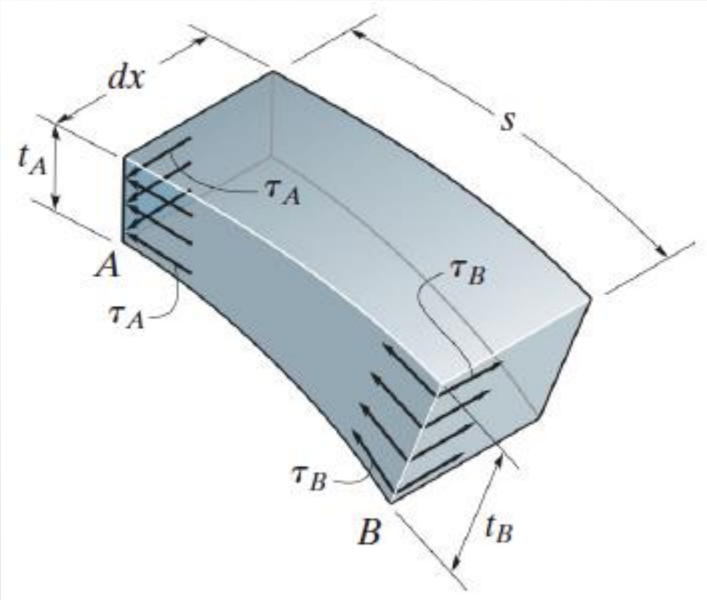
- The product of the average shear stress and the thickness of the tube is the same at each location on the cross section.

$$\tau_A t_A = \tau_B t_B$$

Shear flow

$$q = \tau t = \text{constant}$$

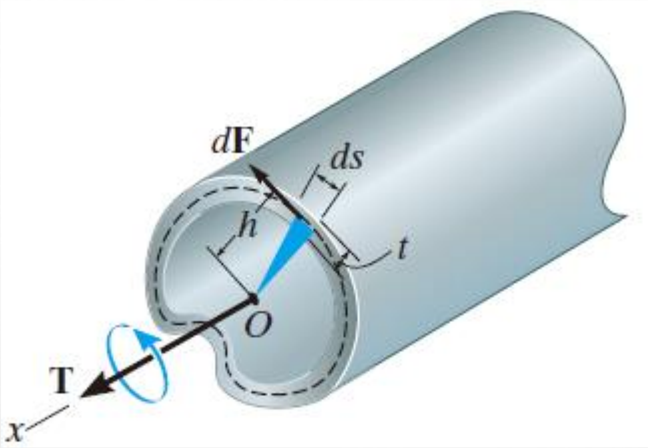




§ 4.15 Thin-walled tubes with closed cross sections

- Average Shear Stress

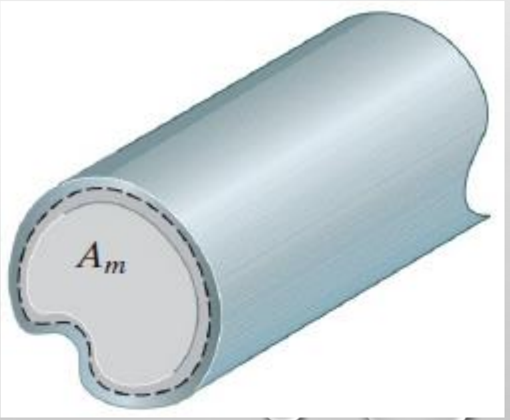
$$dF = \tau_{avg} t ds \quad dT = h\tau_{avg} t ds$$



$$T = \tau_{avg} t \oint h ds$$

$$T = 2\tau_{avg} t A_m$$

$$\tau_{avg} = \frac{T}{2tA_m}$$



- Angle of twist?

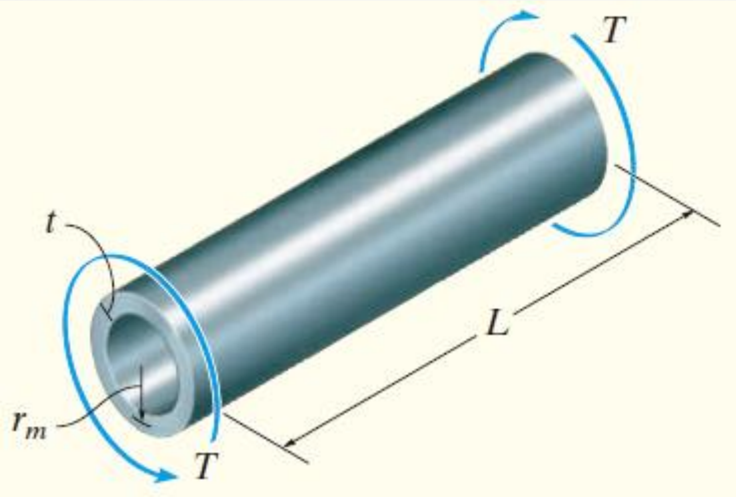
$$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t}$$

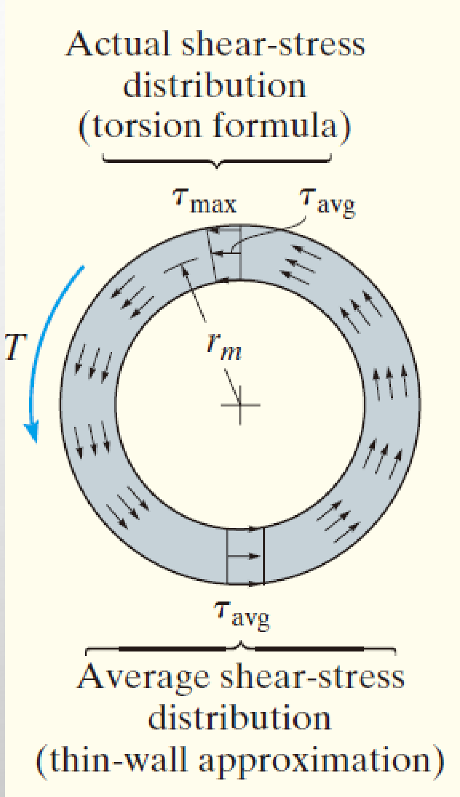
(elastic energy = external work)

Example 4.7

(Hibbeler, Page 235)

Calculate the average shear stress in a thin-walled tube having a circular cross section of mean radius r_m and thickness t, which is subjected to a torque T. Also, what is the relative angle of twist if the tube has a length L?

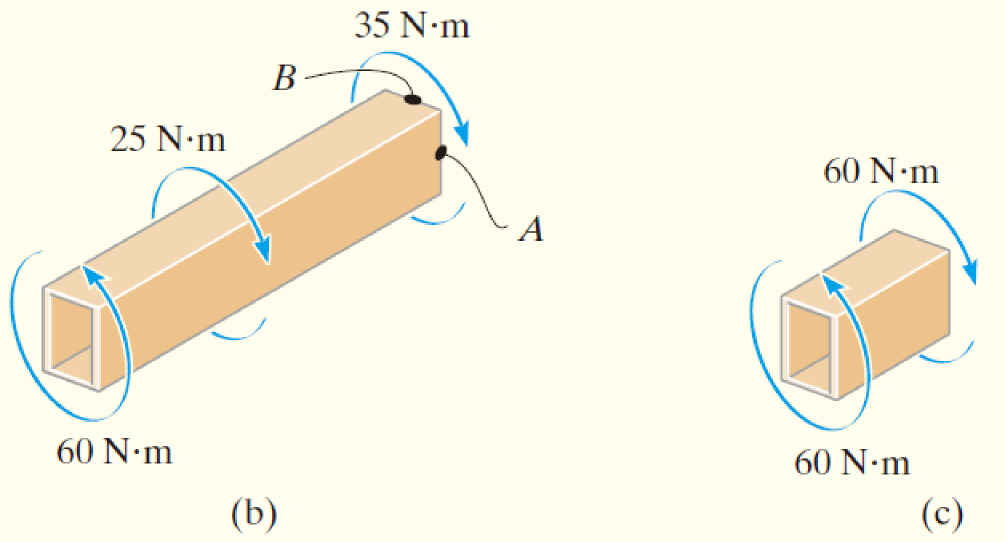


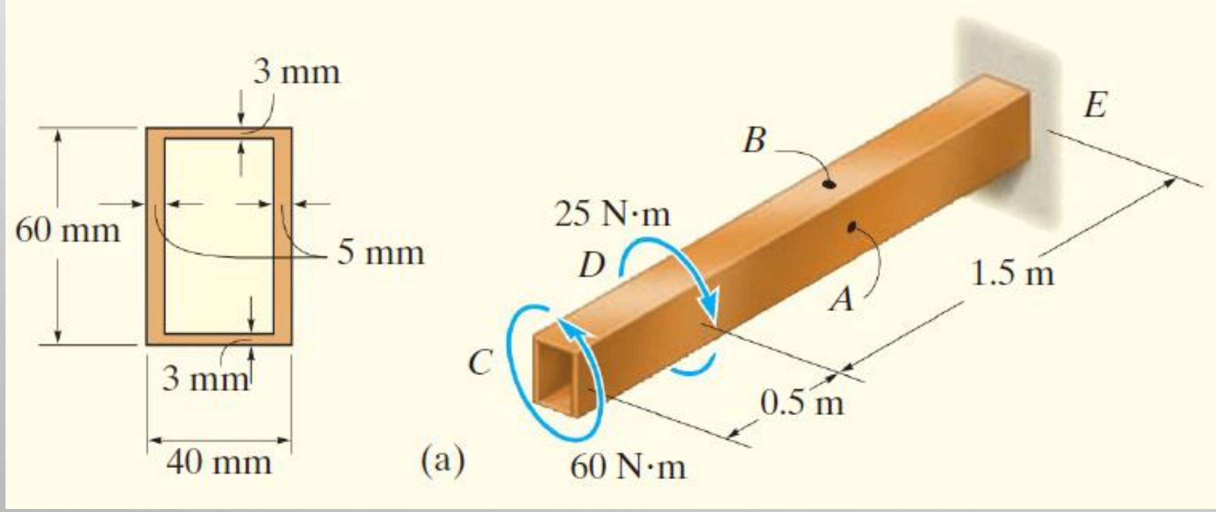


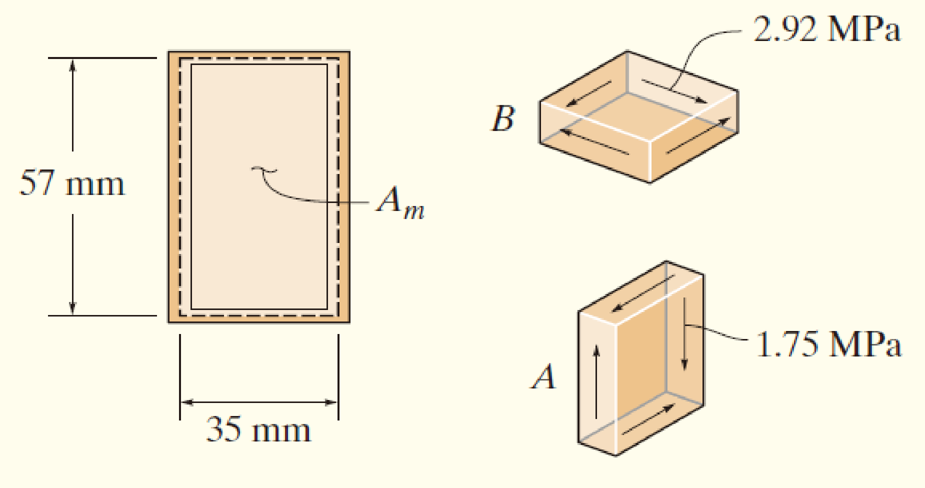
Example 4.8

(Hibbeler, Page 236)

The tube is made of C86100 bronze and has a rectangular cross section as shown. If it is subjected to the two torques, determine the average shear stress in the tube at points A and B. Also, what is the angle of twist of end C? The tube is fixed at E.



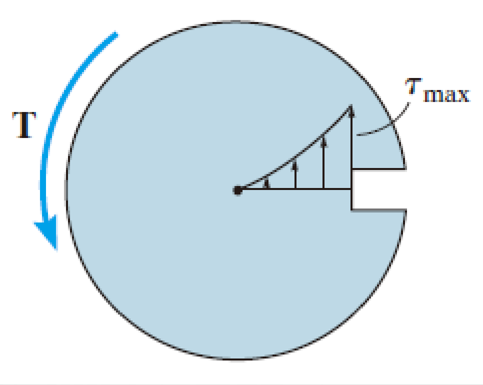




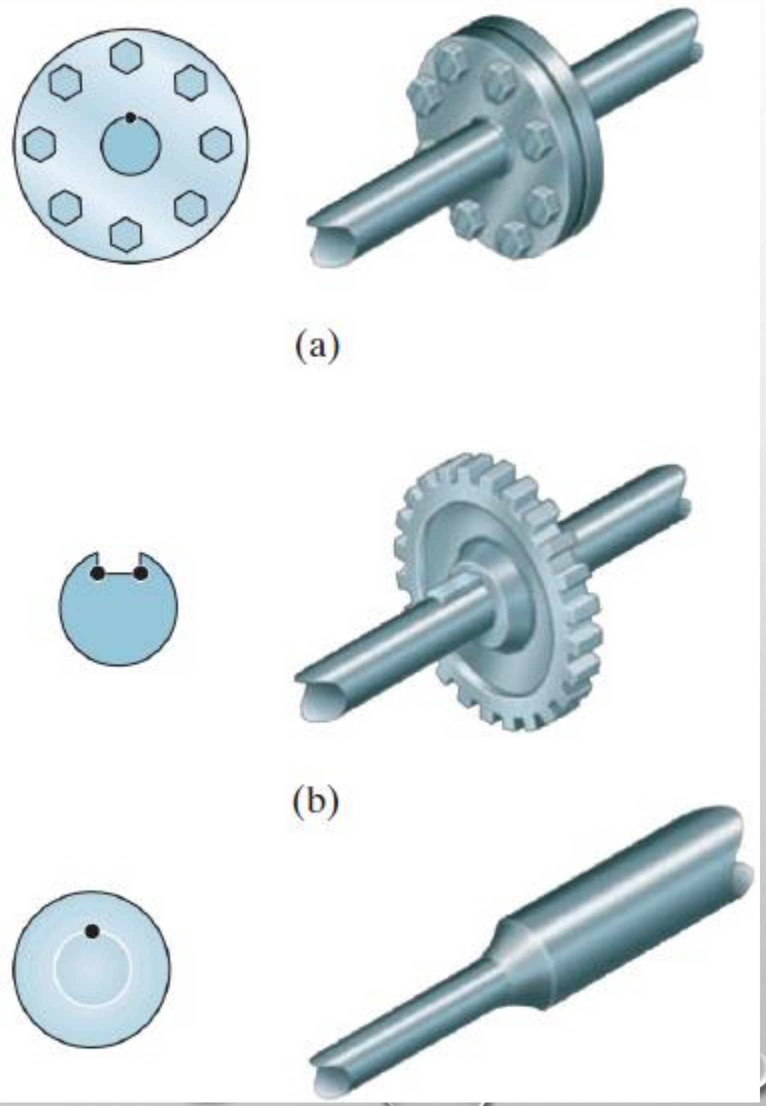
§ 4.16 Stress concentration

- Discontinuities of the cross section
- Torsional stress concentration factor, K.

$$\tau_{\max} = \frac{Tc}{J}$$

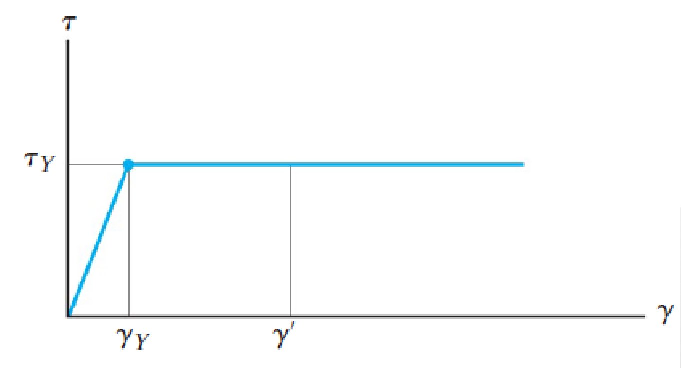


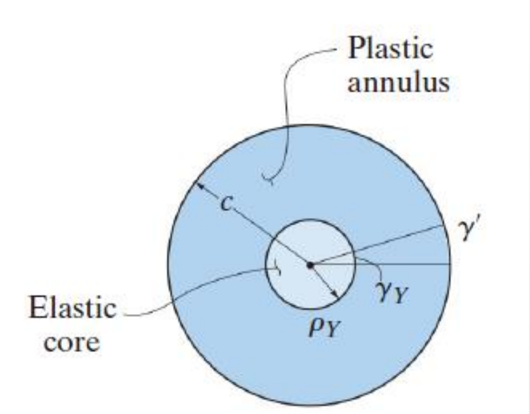
$$\tau_{\max} = K\frac{Tc}{J}$$

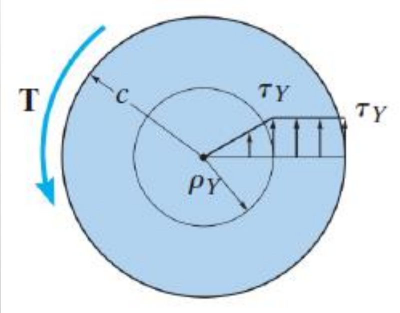


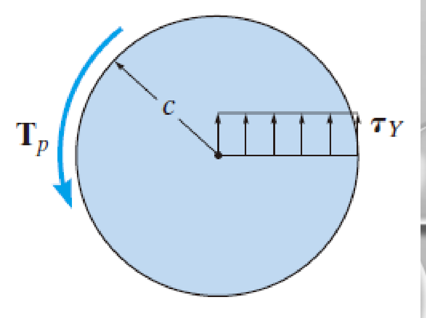
§ 4.17 Inelastic torsion

- Consider the material in the shaft to exhibit an elastic perfectly plastic behavior.









Fully plastic torque

$$T = \int \rho\tau \mathrm{dA} = 2\pi\int_0^c \rho^2\tau \mathrm{d}\rho$$

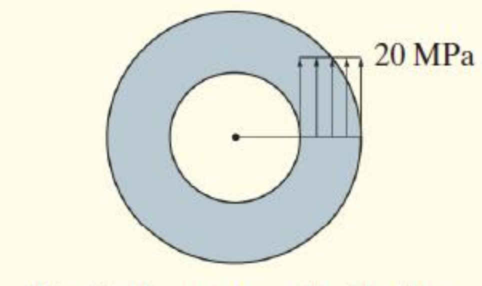
$$= 2\pi\int_0^{\rho_Y}\left(\tau_Y\frac{\rho}{\rho_Y}\right)\rho^2\mathrm{d}\rho + 2\pi\int_{\rho_Y}^{c}\tau_Y\rho^2\mathrm{d}\rho$$

$$= \frac{\pi\tau_Y}{6}\left(4c^3 - \rho_Y^3\right) \qquad T_Y = \frac{\pi\tau_Y}{2}c^3 \qquad T_p = \frac{2\pi\tau_Y}{3}c^3$$

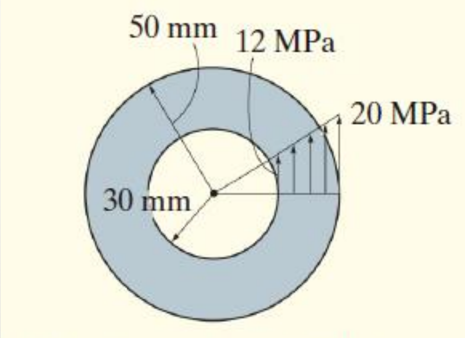
Example 4.9

(Hibbeler, Page 250)

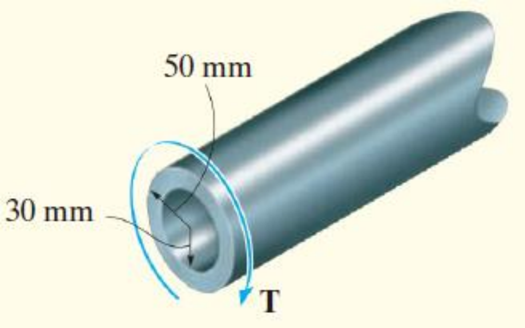
The tubular shaft is made of an aluminum alloy that is assumed to have an elastic perfectly plastic τ-γ diagram as shown. Determine the maximum torque that can be applied to the shaft without causing the material to yield, and the plastic torque that can be applied to the shaft. Also, what should the minimum shear strain at the outer wall be in order to develop a fully plastic torque?

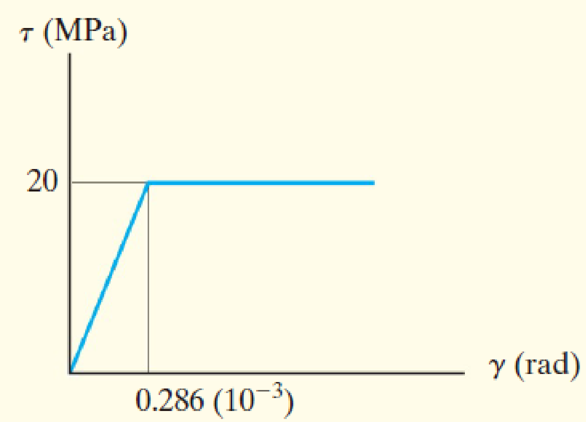


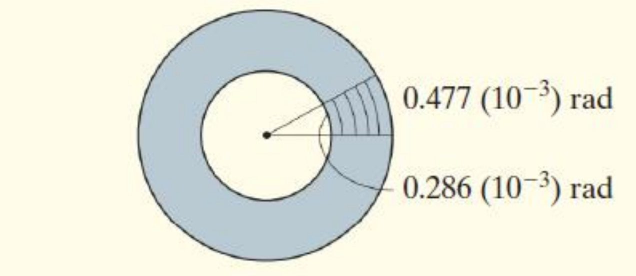
Plastic shear–stress distribution



Elastic shear–stress distribution



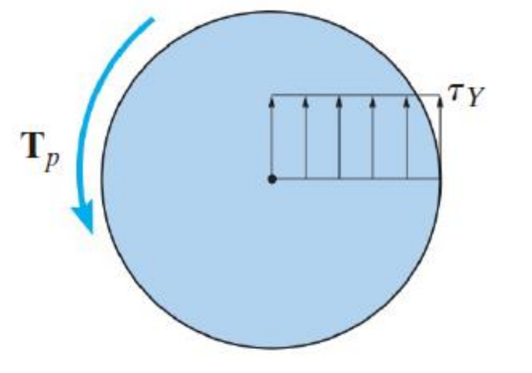




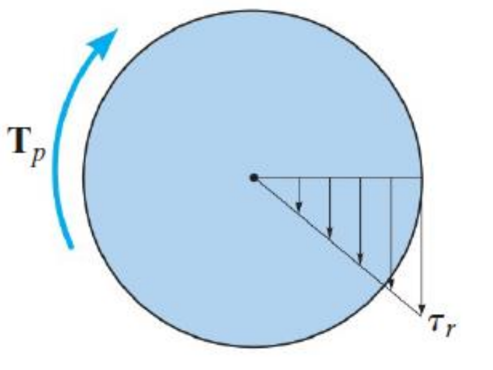
Initial plastic shear–strain distribution

§ 4.18 Residual Stress

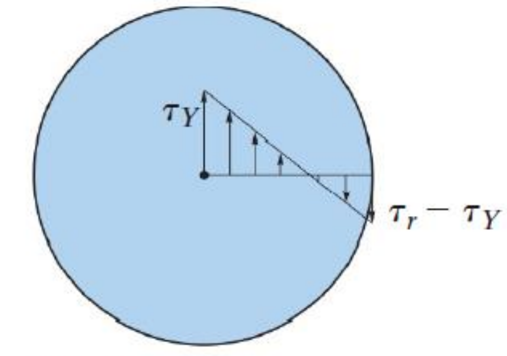
- When a shaft is subjected to plastic shear strains caused by torsion, removal of the torque will cause some shear stress to remain in the shaft. This stress is referred to as residual stress, and its distribution can be calculated using superposition.



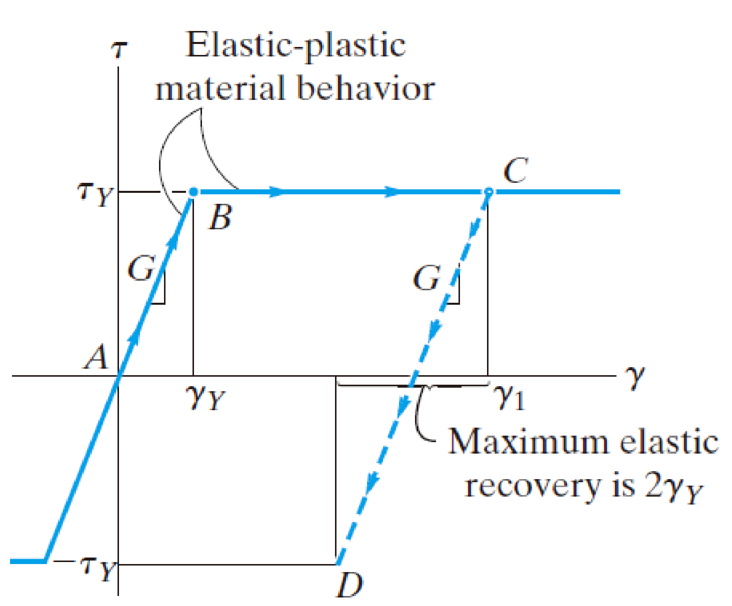
Plastic torque applied causing plastic shear strains throughout the shaft



Plastic torque reversed causing elastic shear strains throughout the shaft



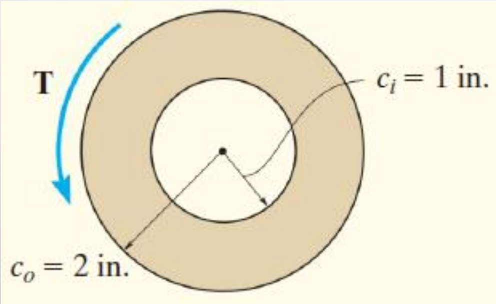
Residual shear–stress distribution in shaft

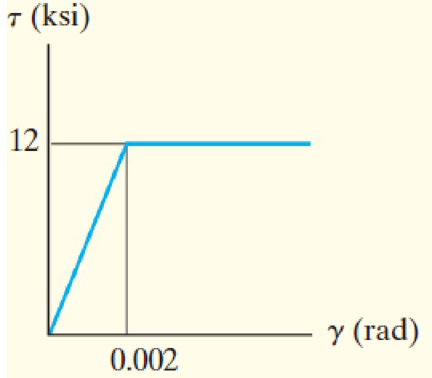


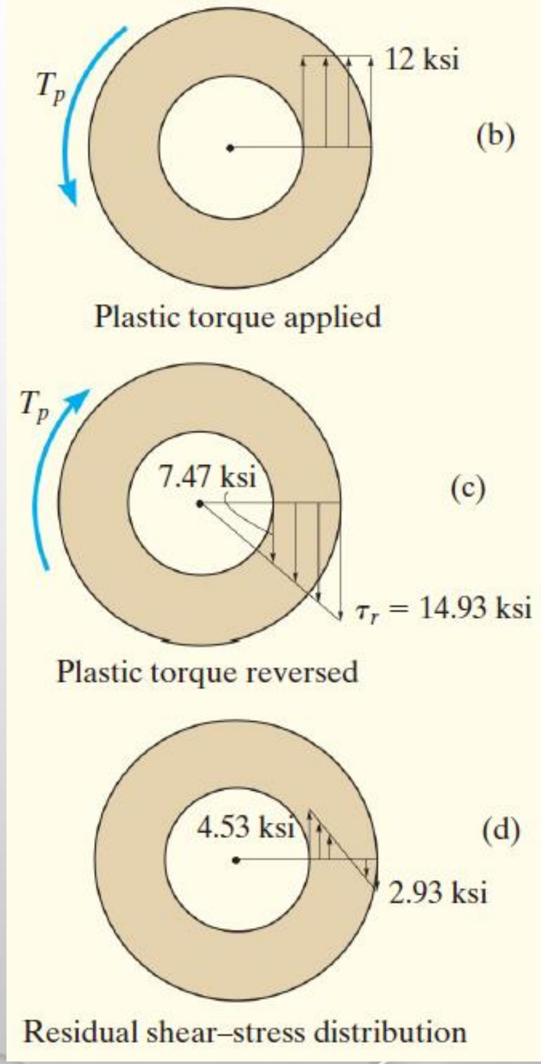
Example 4.10

(Hibbeler, Page 251)

The 5-ft-long tube and its elastic perfectly plastic τ-γ diagram are as shown. Determine the plastic torque T_p. What is the residual shear-stress distribution in the tube if T_p is removed just after the tube becomes fully plastic?







§ 4.17 Summary

- Deformation of a circular member under torsion
- Torsion formula
- Angle of twist
- Statically indeterminate problem
- Torsion of thin-walled tubes
- Stress concentration