FEM HW 4.

Exercise 1.16

Solution

(a)

$(S) \Rightarrow (W)$: strong solution solves (W)

set u being the solution of (S):

$$\begin{cases} EI\,u_{,xxxx} = f \\ u(1) = 0 \\ u_{,x}(1) = 0 \\ EI\,u_{,xx}(0) = M \\ EI\,u_{,xxx}(0) = Q \end{cases}$$

$$w \cdot EI\,u_{,xxxx} = w \cdot f$$

$$\int_0^1 w \cdot EI\,u_{,xxxx}\,dx = \int_0^1 wf\,dx$$

$$-\int_0^1 w_{,x} EI\,u_{,xxx}\,dx + wEI\,u_{,xxx}\Big|_0^1 = \int_0^1 wf\,dx$$

$$-\left(-\int_0^1 w_{,xx} EI\,u_{,xx}\,dx + w_{,x}EI\,u_{,xx}\Big|_0^1\right) + wEI\,u_{,xxx}\Big|_0^1 = \int_0^1 wf\,dx$$

$$\int_0^1 w_{,xx} EI\,u_{,xx}\,dx - w_{,x}EI\,u_{,xx}\Big|_0^1 + wEI\,u_{,xxx}\Big|_0^1 = \int_0^1 wf\,dx$$

$$w_{,x}EI\,u_{,xx}\Big|_0^1 = \overset{0}{\overset{\parallel}{w_{,x}(1)}}EI\,u_{,xx}(1) - w_{,x}(0)\underbrace{EI\,u_{,xx}(0)}_{M}$$

$$= -w_{,x}(0)\cdot M$$

$$wEI\,u_{,xxx}\Big|_0^1 = \overset{0}{\overset{\parallel}{w(1)}}EI\,u_{,xxx}(1) - w(0)\underbrace{EI\,u_{,xxx}(0)}_{Q}$$

$$= -w(0)\cdot Q$$

$$\Rightarrow \int_0^1 w_{,xx}EI\,u_{,xx}\,dx = \int_0^1 wf\,dx - w_{,x}(0)M + w(0)\cdot Q$$

where $u \in \mathcal{S}$ and $w \in \mathcal{V}$

$$\mathcal{S} = \mathcal{V} = \{w \mid w \in H^2,\ w(1) = w_{,x}(1) = 0\}$$

$\therefore (S) \Rightarrow (W)$

$(W) \Rightarrow (S)$: weak solution solves (S)

$$\int_0^1 w_{,xx}EI\,u_{,xx}\,dx = \int_0^1 wf\,dx - w_{,x}(0)M + w(0)Q$$

$$\Downarrow$$

$$-\int_0^1 w_{,x}EI\,u_{,xxx}\,dx + \overset{0}{\overset{\parallel}{w_{,x}(1)}}EI\,u_{,xx}(1) - w_{,x}(0)EI\,u_{,xx}(0)$$

$$= \int_0^1 wf\,dx - w_{,x}(0)M + w(0)Q$$

$$-\int_0^1 w_{,x}EI\,u_{,xxx}\,dx - w_{,x}(0)EI\,u_{,xx}(0)$$

$$= \int_0^1 wf\,dx - w_{,x}(0)M + w(0)Q$$

$$-w_{,x}(0)EI\,u_{,xx}(0)$$

$$-\left[-\int_0^1 wEI\,u_{,xxxx}\,dx + \overset{0}{\overset{\parallel}{w(1)}}EI\,u_{,xxx}(1) - w(0)EI\,u_{,xxx}(0)\right]$$

$$= \int_0^1 wf\,dx - w_{,x}(0)M + w(0)Q$$

$$-w_{,x}(0)EI\,u_{,xx}(0)$$

$$\int_0^1 wEI\,u_{,xxxx}\,dx + w(0)EI\,u_{,xxx}(0) = \int_0^1 wf\,dx - w_{,x}(0)M + w(0)Q$$

$$\int_0^1 w(EI\,u_{,xxxx} - f)\,dx + w(0)[EI\,u_{,xxx}(0) - Q]$$

$$-w_{,x}(0)[EI\,u_{,xx}(0) - M] = 0$$

pick $w = \phi\cdot(EI\,u_{,xxxx} - f)$

with $\phi > 0$, $\phi(0) = \phi(1) = \phi_{,x}(0) = 0$

$$\Rightarrow \int_0^1 \phi(EI\,u_{,xxxx} - f)^2\,dx + 0 - 0 = 0$$

$$\Rightarrow EI\,u_{,xxxx} = f$$

$$w(0)[EI\,u_{,xxx}(0) - Q] - w_{,x}(0)[EI\,u_{,xx}(0) - M] = 0$$

$$w \in \mathcal{V} = \{w \mid w \in H^2,\ w(1) = w_{,x}(1) = 0\}$$

no restriction on $w(0)$ and $w_{,x}(0)$

$$\therefore EI\,u_{,xxx}(0) - Q = EI\,u_{,xx}(0) - M = 0$$

$$\Rightarrow EI\,u_{,xxx}(0) = Q,\quad EI\,u_{,xx}(0) = M$$

Since $u \in \mathcal{S} = \mathcal{V}$

$$\therefore u(1) = u_{,x}(1) = 0$$

$\therefore (W) \Rightarrow (S)$

Natural Boundary Conditions:

$$EI\, u_{,xx}(0) = M$$

$$EI\, u_{,xxx}(0) = Q$$

b) $0 = x_1 < x_2 < \cdots < x_{n+1} = 1$

$$\mathcal{V}^h = \{ w^h \mid w^h \in C^1,\ w^h(1) = w^h_{,x}(1) = 0,$$

$$w^h \text{ restricted to } [x_n, x_{n+1}] \text{ cubic polynomial} \}$$

Let $w^h(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3$

$$w^h(x_1) = C_1 + C_2 x_1 + C_3 x_1^2 + C_4 x_1^3$$

$$w^h(x_2) = C_1 + C_2 x_2 + C_3 x_2^2 + C_4 x_2^3$$

$$w^h_{,x}(x) = C_2 + 2C_3 x + 3C_4 x^2$$

$$w^h_{,x}(x_1) = C_2 + 2C_3 x_1 + 3C_4 x_1^2$$

$$w^h_{,x}(x_2) = C_2 + 2C_3 x_2 + 3C_4 x_2^2$$

$$\Rightarrow \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} w^h(x_1) \\ w^h(x_2) \\ w^h_{,x}(x_1) \\ w^h_{,x}(x_2) \end{bmatrix}$$

$$\Rightarrow C_1 = -\frac{\left[2w^h_{,x}(x_2) - 3w^h_{,x}(x_1)x_1 - 3w^h_{,x}(x_1)\cdot x_2 + w^h(x_1)\cdot x_2^3 - 3w^h(x_1)x_1x_2^2 + 6w^h(x_2)x_1x_2 \right]}{(x_1 - x_2)^3}$$

$(h = x_2 - x_1)$

Similarly, obtain: C_2, C_3, C_4

Substitute them into $w^h(x)$:

$$w^h(x) = N_1(x)\, w^h(x_1) + N_3(x)\, w^h(x_2) + N_2(x)\, w^h_{,x}(x_1) + N_4(x)\, w^h_{,x}(x_2)$$

where $N_1(x)$、$N_2(x)$、$N_3(x)$、$N_4(x)$ are listed on the text book.

(g) (w) problem:

$a(w, u) = (w, f) - w_{,x}(0) M + w(0) Q$

find $u \in \mathcal{S}$ that for all $w \in \mathcal{V}$

note: $\mathcal{V} = \mathcal{S} = \{ w \mid w \in H^2, w(1) = w_{,x}(1) = 0 \}$

Green's function

pick $w = g \in \mathcal{V}^h \subset \mathcal{V}$ with y at x_A.

$a(g, u) = (g, f) - g_{,x}(0) M + g(0) Q$

$= (g, f) + g(0) Q$ (*)

Green's function problem.

$$(S) \begin{cases} g_{,xx} + \delta_y = 0 \\ g(1) = 0 \\ -g_{,x}(0) = 0 \end{cases}$$

for (G) problem:

(W) $a(w, g) = (w, \delta_y) + \underbrace{w(0) h}_{0} = w(y)$

$a(w^h, u^h) = (w^h, f) - w^h_{,x}(0) M + w^h(0) Q$

$a(g, u^h) = (g, f) - g_{,x}(0) M + g(0) Q$

$= (g, f) + g(0) Q.$ (Δ)

(*) − (Δ) $= 0 = a(g, u - u^h)$

$= a(u - u^h, g) = (u - u^h, \delta_y) = u(x_A) - u^h(x_A)$ ▨

(with y at x_A)

$u^h_{,x}(x_A) = u_{,x}(x_A)$. ?

(i) the functions in $\mathcal{V}^h$ have continuous first derivatives.

?

u^h describes the displacement of the thin beam

and $u^h \in \mathcal{V}^h = \mathcal{S}^h$

$u^h_{,x}$ means the slope of the beam under the boundary conditions (M, Q, f)

which must be continuous under the conditions of mechanics of materials.

(j) $k^e_{pq} = \int_{x_1^e}^{x_2^e} N_{p,xx} EI N_{q,xx} dx \qquad 1 \le p, q \le 4$

$= EI \cdot \int_{x_1^e}^{x_2^e} N_{p,xx} N_{q,xx} dx$

N_1, N_2, N_3, N_4 are listed in part (b)

$$N_{1,xx} = \frac{4(2x-2x_2)}{h^3} + \frac{2(h+2x-2x_1)}{h^3}$$

$$N_{2,xx} = \frac{2(2x-2x_2)}{h^2} + \frac{2(x-x_1)}{h^2}$$

$$N_{3,xx} = \frac{2(h-2x+2x_2)}{h^3} - \frac{4(2x-2x_1)}{h^3}$$

$$N_{4,xx} = \frac{2(2x-2x_1)}{h^2} + \frac{2(x-x_2)}{h^2}$$

(solving with matlab)

$$\Rightarrow \quad k^e_{pq} = \begin{bmatrix} \frac{12EI}{h^3} & \frac{6EI}{h^2} & \frac{-12EI}{h^3} & \frac{6EI}{h^2} \\ \frac{6EI}{h^2} & \frac{4EI}{h} & \frac{-6EI}{h^2} & \frac{4EI}{h} \\ \frac{-12EI}{h^3} & \frac{-6EI}{h^2} & \frac{12EI}{h^3} & \frac{-6EI}{h^2} \\ \frac{6EI}{h^2} & \frac{4EI}{h} & \frac{-6EI}{h^2} & \frac{4EI}{h} \end{bmatrix}$$

(k)

(w) problem:

$$a(w,u) = (w,l) - w_{,x}(0)M + w(0)Q$$

$$\int_0^1 w_{,xx} EI\, u_{,xx}\, dx = \int_0^1 w f\, dx - w_{,x}(0)M + w(0)Q$$

$$\int_0^1 w_{,xx} EI\, u_{,xx}\, dx = \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w_{,xx} EI\, u_{,xx}\, dx$$

$$= \sum_{A=1}^{n} \left\{ -\int_{x_A}^{x_{A+1}} w_{,x} EI\, u_{,xxx}\, dx + \underline{w_{,x} EI\, u_{,xx} \Big|_{x_A}^{x_{A+1}}} \right\}$$

$$= \sum_{A=1}^{n} \left(-\int_{x_A}^{x_{A+1}} w_{,x} EI\, u_{,xxx}\, dx \right) + w_{,x} EI\, u_{,xx} \Big|_{x_1=0}^{x_2} + \sum_{A=2}^{n} \left(w_{,x} EI\, u_{,xx} \Big|_{x_A}^{x_{A+1}} \right)$$

$$= \sum_{A=1}^{n} \left(\int_{x_A}^{x_{A+1}} w EI\, u_{,xxxx}\, dx - w EI\, u_{,xxx} \Big|_{x_A}^{x_{A+1}} \right) + \underline{w_{,x}(x_2) EI\, u_{,xx}(x_2^-)}$$

$$- w_{,x}(0) EI\, u_{,xx}(0^+) + \underline{\sum_{A=2}^{n} \left(w_{,x} EI\, u_{,xx} \Big|_{x_A}^{x_{A+1}} \right)}$$

add.

$$w_{,x}(x_3) EI\, u_{,xx}(x_3^-) - w_{,x}(x_2) EI\, u_{,xx}(x_2^+)$$

$$+ \quad x_4 \quad x_4^- \quad x_3 \quad x_3^+$$

$$= \left\{ \sum_{A=1}^{n} \left(\int_{x_A}^{x_{A+1}} w EI\, u_{,xxxx}\, dx \right) - w EI\, u_{,xxx} \Big|_{x_1=0}^{x_2} - \sum_{A=2}^{n} \left(w EI\, u_{,xxx} \Big|_{x_A}^{x_{A+1}} \right) \right\}$$

$$- w_{,x}(0) EI\, u_{,xx}(0^+) + \sum_{A=2}^{n} w_{,x}(x_A) EI \left[u_{,xx}(x_A^-) - u_{,xx}(x_A^+) \right]$$

$$= \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w EI\, u_{,xxxx}\, dx + w(0) EI\, u_{,xxx}(0^+) - \sum_{A=2}^{n} w(x_A) EI \left[u_{,xxx}(x_A^-) - u_{,xxx}(x_A^+) \right]$$

$$- w_{,x}(0) EI\, u_{,xx}(0^+) + \sum_{A=2}^{n} w_{,x}(x_A) EI \left[u_{,xx}(x_A^-) - u_{,xx}(x_A^+) \right]$$

$$= \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w f\, dx - w_{,x}(0) M + w(0) Q.$$

$$\Longrightarrow \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w \left(EI\, u_{,xxxx} - f \right) dx - w_{,x}(0) \left[EI\, u_{,xx}(0^+) - M \right] + w(0) \left[EI\, u_{,xxx}(0^+) - Q \right]$$

$$+ \sum_{A=2}^{n} w(x_A) EI \left[u_{,xxx}(x_A^+) - u_{,xxx}(x_A^-) \right] - \sum_{A=2}^{n} w_{,x}(x_A) EI \left[u_{,xx}(x_A^+) - u_{,xx}(x_A^-) \right]$$

$$= 0.$$