Mid-term Test

Wednesday, November 2, 2022 (110 minutes)

Student Name: 邹佳驹

30

Problem 1. (a) The displacement components in a body are given by

$$u = Axy, \quad v = A(x^2 - y^2), \quad w = 0.$$

Obtain the strain components.

(b) Transform the components of the stress state

$$[\sigma] = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}$$

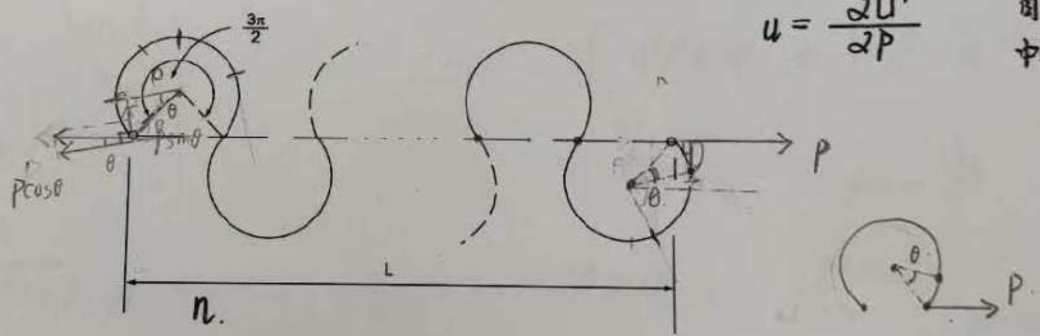
to a new system rotated counterclockwise by 30° from the old system.

35

Problem 2. The flat tension spring shown in figure consists of a length of wire of circular cross-section having a diameter d and Young's modulus E. The spring consists of n open loops each of which subtends an angle of $3\pi/2$ radians at its center; the total length between the ends of the spring is L. Considering bending and axial strains only, calculate the stiffness of the spring using the Castigliano's first theorem.

$u = \frac{\partial U^*}{\partial P}$

首尾两个.
中间部分.



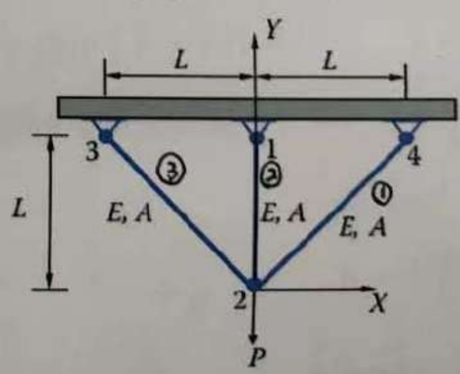
35

Problem 3. The plane truss is loaded with force P as shown below. Constants E and A for each bar are as shown in the diagram. Determine:

(a) The nodal displacements;

(b) The reaction forces;

(c) The stress in each bar element.



1.解:

a) $\varepsilon_{xx} = \frac{\partial u}{\partial x} = Ay$

$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -2Ay$

$\varepsilon_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cdot\frac{1}{2}$

$= (Ax + 2Ax)\cdot\frac{1}{2}$

$= \frac{3}{2}Ax$

15

b) counterclockwise $30^\circ \Rightarrow \theta = 30^\circ$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$

$$[Q] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$[\sigma'] = [Q][\sigma][Q]^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 8 + \frac{9}{4}\sqrt{3} & -2\sqrt{3} + \frac{3}{2} \\ -2\sqrt{3} + \frac{3}{2} & 4 - \frac{3}{2}\sqrt{3} \end{bmatrix}$$

15

装订线内不答题！

2. 解:

Assume load P subjected to the spring horizontally.

$$k \cdot u = P$$

$$k = \frac{P}{u}$$

$$2u = \frac{\partial U^*_{total}}{\partial P}$$

Consider one loop

due to bending: $U^* = \frac{1}{2}\int_0^{\frac{3}{2}\pi} \frac{M^2(\theta)}{EI} r \, d\theta$

$$\cancel{M(\theta) = P r \sin\theta}$$

$$U^* = \frac{1}{2}\frac{1}{EI}\int_0^{\frac{3}{2}\pi} P^2 r^3 \sin^2\theta \, d\theta$$

$$= \frac{P^2 r^3 \cdot 3\pi}{2EI}$$

$$U^*_{total} = \frac{3n\pi P^2 r^3}{2EI}$$

$$2u = \frac{3n\pi P r^3}{EI \cdot 2}$$

$$k = \frac{2EI}{3n\pi r^3}$$

$$M(\theta) = P r \left[1 - \sin\left(\frac{\pi}{4} - \theta\right)\right]$$

$$= P r \left(1 - \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)$$

$$U^* = \frac{1}{2EI}\int_0^{\frac{3}{2}\pi} M^2(\theta) \, r \, d\theta.$$

$$= \frac{P^2 r^3}{2EI}\int_0^{\frac{3}{2}\pi}\left(1 - \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)^2 d\theta$$

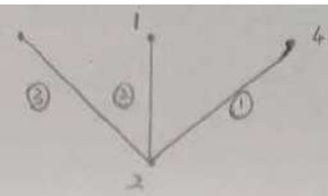
$$2u = \frac{\partial U^* \cdot n}{\partial P} = \frac{P r^3 n}{EI}\int_0^{\frac{3}{2}\pi}\left(1 - \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)^2 d\theta.$$

$$\therefore k = \frac{P}{u} = \frac{2EI}{r^3 \cdot \int_0^{\frac{3}{2}\pi}\left(1 - \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)^2 d\theta \cdot n}$$

20

装订线内不答题！

3. 解：



FEA

for Bar ①:

$\theta = 45^\circ$, $l = \cos\theta = \frac{\sqrt{2}}{2}$, $m = \sin\theta = \frac{\sqrt{2}}{2}$

$$[k_①] = \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \frac{AE}{\sqrt{2}L} = \frac{AE}{\sqrt{2}L} \begin{array}{c} \begin{matrix} u_2 & v_2 & u_4 & v_4 \end{matrix} \\ \left[\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \hline -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \end{array}$$

for Bar ②:

$\theta = 90^\circ$, $l = \cos\theta = 0$, $m = \sin\theta = 1$

$$[k_②] = \begin{array}{c} \begin{matrix} u_2 & v_2 & u_1 & v_1 \end{matrix} \\ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \end{array} \frac{AE}{L} = \frac{AE}{\sqrt{2}L} \begin{array}{c} \begin{matrix} u_2 & v_2 & u_1 & v_1 \end{matrix} \\ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{2} \\ \hline 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} \end{array}\right] \end{array} = \frac{AE}{\sqrt{2}L} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} \end{bmatrix} \end{array}$$

for Bar ③:

$\theta = 135^\circ$, $l = -\frac{\sqrt{2}}{2}$, $m = \frac{\sqrt{2}}{2}$

$$[k_③] = \frac{AE}{\sqrt{2}L} \begin{array}{c} \begin{matrix} u_2 & v_2 & u_3 & v_3 \end{matrix} \\ \left[\begin{array}{cc|cc} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \hline -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \end{array}$$

Combine above:

$$\frac{AE}{\sqrt{2}L} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & & & & \\ 0 & +\sqrt{2} & 0 & -\sqrt{2} & & & & \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\sqrt{2} & 0 & 1+\sqrt{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ & & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & & \\ & & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & & \\ & & -\frac{1}{2} & -\frac{1}{2} & & & \frac{1}{2} & \frac{1}{2} \\ & & -\frac{1}{2} & -\frac{1}{2} & & & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \begin{matrix} 0 \\ 0 \\ \\ \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} \begin{matrix} \\ \\ 0 \\ -P \\ \\ \\ \\ \\ \end{matrix}$$

BC: $u_1 = v_1 = 0$, $u_3 = v_3 = 0$, $u_4 = v_4 = 0$, $F_{2y} = -P$, $F_{2x} = 0$

We can solve and get: $\cancel{u_2 = 0}$ $\cancel{F_{1x} = 0}$ $\cancel{F_{2x} = 0}$ $\cancel{F_{3x} = \frac{\sqrt{2}+1}{2}P}$

$\cancel{v_2 = \frac{(2+\sqrt{2})PL}{AE}}$ $\cancel{F_{1y} = (2+\sqrt{2})P}$ $\cancel{F_{3y} =}$

$$\begin{cases} u_2 = 0 \\ v_2 = \frac{-\sqrt{2}PL}{(1+\sqrt{2})AE} \end{cases} \quad \begin{cases} F_{1x} = 0 \\ F_{1y} = \frac{\sqrt{2}P}{1+\sqrt{2}} \end{cases}$$

$$F_① = \frac{\sqrt{2}P}{2+2\sqrt{2}}, \quad F_② = \frac{\sqrt{2}P}{1+\sqrt{2}}, \quad F_③ = \frac{\sqrt{2}P}{2+2\sqrt{2}}$$

$$\begin{cases} F_{3x} = \frac{-P}{2+2\sqrt{2}} \\ F_{3y} = \frac{P}{2+2\sqrt{2}} \end{cases} \quad \begin{cases} F_{4x} = \frac{+P}{2+2\sqrt{2}} \\ F_{4y} = \frac{P}{2+2\sqrt{2}} \end{cases}$$

$$\sigma_① = \frac{P}{A(\sqrt{2}+2)}, \quad \sigma_② = \frac{\sqrt{2}P}{A(\sqrt{2}+1)}, \quad \sigma_③ = \frac{P}{A(\sqrt{2}+2)}$$

$\boxed{\text{ANS}}$

35