第 15 周习题
常微分方程 B

May 24, 2022

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1. Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and suppose that $\mathbf{A}$ has one zero eigenvalue, *i.e.* $\det A = 0$.

   (a) Show that in addition to the critical point $\mathbf{x} = \mathbf{0}$, every point on a certain line through the origin is also a critical point. (*Hint*. A critical point of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$.)

   (b) Let $r_1 = 0$ and $r_2 \neq 0$, and let $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ be the corresponding eigenvectors. Show that the trajectories are as indicated in Figure 1.

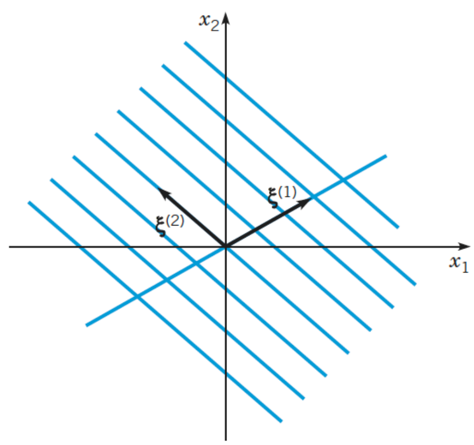

Figure 1: Trajectories for a linear system with nonisolated critical points. Every point on the line through $\boldsymbol{\xi}^{(1)}$ is a critical point.

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2. The equation of motion of a spring-mass system with damping is

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0,$$

where m, c, and k are positive. Write this second-order equation as a system of two first-order equations for $x = u$, $y = du/dt$. Show that $(x, y) = (0, 0)$ is a critical point, and analyze the nature and stability of the critical point as a function of the parameters m, c, and k.

Remark. The same analysis can be applied to the electric circuit equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0.$$

3. Consider the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \tag{1}$$

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(a) Determine the critical point (x_0, y_0) for the system.

(b) Make the transformation $u = x - x_0$, $v = y - y_0$, and show that

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{2}$$

(c) For the system (2), classify the type and stability of the critical point $(u, v) = (0, 0)$. Note that the transformation $(u, v) = (x, y) - (x_0, y_0)$ is only a translation (平移), this indicates that the critical point (x_0, y_0) for the system (1) is of the same nature as the critical point $(0, 0)$ for (2).

(d) Let r_1 and r_2 be the two eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, let $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ be the corresponding eigenvectors. Let $\mathbf{T} = [\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}]$, *i.e.* $\mathbf{T}$ is the matrix with $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ being its first and second column respectively. Consider the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{T} \begin{pmatrix} w \\ z \end{pmatrix}.$$

Show that

$$\begin{pmatrix} w \\ z \end{pmatrix}' = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}. \tag{3}$$

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Note that the transformation $(u, v)^T = \mathbf{T}(w, z)^T$ is invertible, the critical points for the systems (2) and (3) are of the same type.

4. Consider the system

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$$\mathbf{x}' = \begin{pmatrix} 2 & -2.5 \\ 1.8 & -1 \end{pmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}. \tag{4}$$

(a) Find the eigenvalues r_1 and r_2 for $\mathbf{A}$. They are a pair of complex conjugates.

(b) Find a (complex) eigenvector $\boldsymbol{\xi}^{(1)}$ corresponding to r_1. Let $\boldsymbol{\xi}^{(1)} = \mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are real vectors. Let $\mathbf{P} = [\mathbf{u}\ \mathbf{v}]$, *i.e.* the first and second columns of $\mathbf{P}$ are $\mathbf{u}$ and $\mathbf{v}$ respectively. Let $\mathbf{x} = \mathbf{P}\mathbf{y}$ and substitute for $\mathbf{x}$ in equation (4). Show that

$$\mathbf{y}' = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\mathbf{y}. \tag{5}$$

(c) Suppose $r_1 = \lambda + i\mu$. Find $\mathbf{P}^{-1}$ and show that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}.$$

Remark. This exercise shows that how a 2×2 system with eigenvalues $\lambda \pm i\mu$ can be transformed into the system (5)

$$\mathbf{y}' = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \mathbf{y}.$$

Note that the transformation $\mathbf{x} = \mathbf{P}\mathbf{y}$ is invertible, hence the critical points for the systems (4) and (5) are of the same type. Remember that the stability of the critical point relies on λ. For the system (5), the type and stability of the critical point are known at a glance.

5. For each of the following systems,

a. Find all the critical points (equilibrium solutions).

   Remark. If you find the following parts (b. c. and d.) rather difficult, you may just ignore them.

b. Use an appropriate graphing tool to draw a direction field (or vector field) for the system. For this purpose, you are free to use the tools at this link or this link. You are encouraged to draw a phase portrait of the system.

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c. From the plot(s) in the last part, determine whether each critical point is asymptotically stable, stable, or unstable, and classify it as to type (saddle, node, spiral point, center).

9.2.10 d. Describe the basin of attraction for each asymptotically stable critical point.

(1) $dx/dt = x^2 - y^2 - 4, \quad dy/dt = 2xy$

(2) $dx/dt = (2+x)(y-x), \quad dy/dt = (4-x)(y+x)$

6. For each of the following systems, find an equation of the form $H(x, y) = c$ satisfied by the trajectories.

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(2) $dx/dt = 2y, \quad dy/dt = -8x$

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(3) $dx/dt = -x + y, \quad dy/dt = -x - y$

7. For each of the following systems, verify that $(0, 0)$ is a critical point, show that the system is locally linear, and discuss the type and stability of the critical point $(0, 0)$ by examining the corresponding linear system.

9.3 (1) $dx/dt = x - y^2, \quad dy/dt = x - 2y + x^2$

1.2 (2) $dx/dt = (1+x)\sin y, \quad dy/dt = 1 - x - \cos y$

8. For the following system,

$$dx/dt = (2+x)(y-x), \quad dy/dt = (4-x)(y+x),$$

(1) Determine all critical points and find the corresponding linear system near each critical point.

9.3.4 (2) Find the eigenvalues of each linear system. What conclusions can you draw about the nonlinear system?