常微分方程B hw 11 week 13

1.

(1) 解:

(i) $X^{(1)\prime} = \begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} X^{(1)}$

$X^{(2)\prime} = \begin{pmatrix} -e^{-t} \\ -3e^{-t} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} X^{(2)}$

(ii) $W[X^{(1)}\ X^{(2)}] = e^t e^{-t}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2 \neq 0$

(iii) $X = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$X(0) = \begin{pmatrix} C_1 \\ C_1 \end{pmatrix} + \begin{pmatrix} C_2 \\ 3C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} C_1 = \frac{1}{2} \\ C_2 = \frac{1}{2} \end{cases}$

$X = e^t \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + e^{-t}\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$

(2) 解:

(i) $X^{(1)\prime} = \begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X^{(1)}$

$X^{(2)\prime} = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X^{(2)}$

(ii) $W[X^{(1)}\ X^{(2)}] = \begin{vmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & 2\sin t - \cos t \end{vmatrix}$

$= 10\sin t\cos t - 5\cos^2 t - 10\sin t\cos t - 5\sin^2 t$

$\neq 0$

(iii) $X(0) = C_1\begin{pmatrix} 5 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} C_1 = \frac{1}{5} \\ C_2 = -\frac{8}{5} \end{cases}$

$X = \frac{1}{5}\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} - \frac{8}{5}\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

$= \begin{pmatrix} \cos t - 8\sin t \\ 2\cos t - 3\sin t \end{pmatrix}$

2. 解:

proof:

$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x_1 = y$

$x_2 = y'$

for $y'' + py' + qy = 0$

$X = C_1 X^{(1)} + C_2 X^{(2)}$

$W[y_1\ y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} + C_2\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$

$W[X^{(1)}\ X^{(2)}] = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$

$\Rightarrow \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = C_3\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} + C_4\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$

$\begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = C_5\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} + C_6\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$

$\therefore W[y_1\ y_2] = \begin{vmatrix} C_3 x_{11} + C_4 x_{12} & C_5 x_{11} + C_6 x_{12} \\ C_3 x_{21} + C_4 x_{22} & C_5 x_{21} + C_6 x_{22} \end{vmatrix}$

$=$

$y_1 = a_{11} x_{11} + a_{12} x_{12}$

$y_2 = a_{21} x_{11} + a_{22} x_{12}$

$W[y_1\ y_2] = \begin{vmatrix} a_{11}x_{11} + a_{12}x_{12} & a_{21}x_{11} + a_{22}x_{12} \\ a_{11}x_{11}' + a_{12}x_{12}' & a_{21}x_{11}' + a_{22}x_{12}' \end{vmatrix}$

$= (a_{11}x_{11} + a_{12}x_{12})(a_{21}x_{11}' + a_{22}x_{12}') - (a_{21}x_{21} + a_{22}x_{22})(a_{11}x_{11}' + a_{12}x_{12}')$

$x_2 = y' = x_1'$

$\Rightarrow W[y_1\ y_2] = (a_{11}a_{22} - a_{12}a_{21})x_{11}x_{12}' - (a_{11}a_{22} - a_{12}a_{21})x_{12}x_{11}'$

$= (a_{11}a_{22} - a_{12}a_{21})x_{11}x_{22} - (a_{11}a_{22} - a_{12}a_{21})x_{12}x_{21}$

$= C \cdot \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$

$= C \cdot W[X^{(1)}\ X^{(2)}]$

3.解:

a) $x_1 = y$, $x_2 = y'$

$$\Rightarrow \begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = \dfrac{-cy-by'}{a} = -\dfrac{b}{a}x_2 - \dfrac{c}{a}x_1 \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

b) $A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$

$$|A - rI| = \begin{vmatrix} -r & 1 \\ -\frac{c}{a} & -\frac{b}{a} - r \end{vmatrix} = r\cdot\left(\frac{b}{a}+r\right) + \frac{c}{a} = 0$$

$$\Rightarrow \quad r(b+ar) + c = 0$$

$$ar^2 + br + c = 0$$

4.解:

(1) $X' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} X$

$$|A - rI| = \begin{vmatrix} 1-r & -2 \\ 3 & -4-r \end{vmatrix} = (1-r)(-4-r) + 6 = 0$$

$$r^2 + 3r + 2 = 0 \Rightarrow r_1 = -1,\ r_2 = -2$$

$r_1 = -1$时

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$r_2 = -2$时

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\xi^{(2)} = 0 \Rightarrow \xi^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$X = C_1\xi^{(1)}e^{r_1 t} + C_2\xi^{(2)}e^{r_2 t}$$

$$= C_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \boxed{\text{ANS}}$$

(2) $X' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} X$

$$|A - rI| = \begin{vmatrix} 1-r & 2 \\ -5 & -1-r \end{vmatrix} = (1-r)(-1-r) + 10 = 0$$

$$r^2 + 9 = 0 \Rightarrow r_1 = 0 + 3i,\ r_2 = 0 - 3i$$

$r_1 = 0 + 3i$时

$$\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix}\xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}$$

$r_2 = -3i$时, $\xi^{(2)} = \begin{pmatrix} 2 \\ -1-3i \end{pmatrix}$

$$X^{(1)} = \xi^{(1)}e^{r_1 t} = \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}e^{3it} = \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}(\cos 3t + \sin 3t\, i)$$

$$= \begin{pmatrix} 2\cos 3t + 2\sin 3t\, i \\ -\cos 3t - 3\sin 3t - \sin 3t\, i + 3\cos 3t\, i \end{pmatrix} = \begin{pmatrix} 2\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix} + \begin{pmatrix} 2\sin 3t \\ -\sin 3t + 3\cos 3t \end{pmatrix} i$$

$$\therefore X = C_1\begin{pmatrix} 2\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix} + C_2\begin{pmatrix} 2\sin 3t \\ -\sin 3t + 3\cos 3t \end{pmatrix} \qquad \boxed{\text{ANS}}$$

(3) $X' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} X$

$$|A - rI| = \begin{vmatrix} 1-r & 1 & 1 \\ 1 & 1-r & 1 \\ 1 & 1 & 1-r \end{vmatrix} = \begin{vmatrix} -r & 0 & +r \\ 0 & -r & r \\ 1 & 1 & 1-r \end{vmatrix}$$

$$= -r\begin{vmatrix} -r & r \\ 1 & 1-r \end{vmatrix} + r\begin{vmatrix} 0 & -r \\ 1 & 1 \end{vmatrix}$$

$$= -r[(-r)(1-r) - r] + r(r)$$

$$= -r^3 + 3r^2 = 0$$

$r_1 = r_2 = 0$, $r_3 = 3$

for $r_3 = 3$,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\xi^{(3)} = 0, \quad \xi^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for $r_1 = r_2 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\xi = 0, \quad \xi^{(2)} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \xi^{(1)} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$X = C_1\xi^{(1)}e^{r_1 t} + C_2\xi^{(2)}e^{r_2 t} + C_3\xi^{(3)}e^{r_3 t}$$

$$= C_1\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + C_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{3t}$$

5.解:

(1) $X' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} X$

$|A - rI| = 0$

$\begin{vmatrix} -2-r & 1 \\ -5 & 4-r \end{vmatrix} = 0$

$(-2-r)(4-r)+5=0$

$r^2-2r-3=0$

$r_1=3,\ r_2=-1$

for $r_1=3$

$(A-r_1 I)\xi = 0$

$\begin{pmatrix} -5 & 1 \\ -5 & 1\end{pmatrix}\xi=0$, $\xi^{(1)} = \begin{pmatrix}1\\5\end{pmatrix}$

for $r_2=-1$

$\begin{pmatrix}-1&1\\-5&5\end{pmatrix}\xi=0$, $\xi^{(2)}=\begin{pmatrix}1\\1\end{pmatrix}$

$X = C_1\xi^{(1)}e^{r_1t}+C_2\xi^{(2)}e^{r_2t}$

$X(0)=\begin{pmatrix}1\\3\end{pmatrix} = C_1\begin{pmatrix}1\\5\end{pmatrix}+C_2\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \begin{cases}C_1=\frac12\\C_2=\frac12\end{cases}$

$\therefore X = \frac12\begin{pmatrix}1\\5\end{pmatrix}e^{3t}+\frac12\begin{pmatrix}1\\1\end{pmatrix}e^{-t}$

$=\begin{pmatrix}\frac12e^{3t}+\frac12e^{-t}\\ \frac52 e^{3t}+\frac12e^{-t}\end{pmatrix}$ $\boxed{\text{ANS}}$

(2) $\begin{vmatrix}-3-r & 2\\ -1 & -1-r\end{vmatrix}=0$

$(3+r)(1+r)+2=0$

$r^2+4r+5=0$

$\Rightarrow r_{1,2}=-2\pm i$

for $r_1=-2+i$

$\begin{pmatrix}-1-i & 2\\ -1 & 1-i\end{pmatrix}\xi=0 \Rightarrow \xi^{(1)}=\begin{pmatrix}1-i\\1\end{pmatrix}$

$\xi^{(2)}=\begin{pmatrix}1+i\\1\end{pmatrix}$

$X^{(1)}=\xi^{(1)}e^{r_1t}=\begin{pmatrix}1-i\\1\end{pmatrix}e^{(-2+i)t}$

$=\begin{pmatrix}1-i\\1\end{pmatrix}e^{-2t}\cdot(\cos t+\sin t\, i)$

$=e^{-2t}\cdot\begin{pmatrix}\sin t+\cos t+(\sin t-\cos t)i\\ \cos t+\sin t\, i\end{pmatrix}$

$=e^{-2t}\cdot\left[\begin{pmatrix}\sin t+\cos t\\ \cos t\end{pmatrix}+i\begin{pmatrix}\sin t-\cos t\\ \sin t\end{pmatrix}\right]$

$=\begin{pmatrix}\sin t+\cos t\\ \cos t\end{pmatrix}e^{-2t}+e^{-2t}\begin{pmatrix}\sin t-\cos t\\ \sin t\end{pmatrix}i$

$\therefore X = C_1 e^{-2t}\begin{pmatrix}\sin t+\cos t\\ \cos t\end{pmatrix}+C_2e^{-2t}\begin{pmatrix}\sin t-\cos t\\ \sin t\end{pmatrix}$

$X(0)=\begin{pmatrix}1\\-2\end{pmatrix}=C_1\begin{pmatrix}1\\1\end{pmatrix}+C_2\begin{pmatrix}-1\\0\end{pmatrix}$

$C_1=-2,\ C_2=-3$

$\therefore X = -2e^{-2t}\begin{pmatrix}\sin t+\cos t\\ \cos t\end{pmatrix}+3e^{-2t}\begin{pmatrix}\cos t-\sin t\\ -\sin t\end{pmatrix}$

6.解:

$X'=\begin{pmatrix}-1&-4\\1&-1\end{pmatrix}X$

$\begin{vmatrix}-1-r & -4\\ 1 & -1-r\end{vmatrix}=0$

$(1+r)^2+4=0$

$r_{1,2}=-1\pm2i$

for $r_1=-1+2i$

$\begin{pmatrix}-2i&-4\\1&-2i\end{pmatrix}\xi=0\Rightarrow\xi^{(1)}=\begin{pmatrix}2i\\1\end{pmatrix}$

$X^{(1)}=\xi^{(1)}e^{r_1t}=\begin{pmatrix}2i\\1\end{pmatrix}e^{(-1+2i)t}$

$=\begin{pmatrix}2i\\1\end{pmatrix}\cdot e^{-t}(\cos2t+\sin2t\cdot i)$

$=e^{-t}\begin{pmatrix}-2\sin2t+2\cos2t\cdot i\\ \cos2t+\sin2t\cdot i\end{pmatrix}$

$=e^{-t}\left[\begin{pmatrix}-2\sin2t\\ \cos2t\end{pmatrix}+i\begin{pmatrix}2\cos2t\\ \sin2t\end{pmatrix}\right]$

$=\begin{pmatrix}-2\sin2t\\ \cos2t\end{pmatrix}e^{-t}+\begin{pmatrix}2\cos2t\\ \sin2t\end{pmatrix}e^{-t}\cdot i$

$X=C_1\begin{pmatrix}-2\sin2t\\ \cos2t\end{pmatrix}e^{-t}+C_2\begin{pmatrix}2\cos2t\\ \sin2t\end{pmatrix}e^{-t}$

$\psi(t)=\begin{pmatrix}-2\sin2t\cdot e^{-t} & 2\cos2t\cdot e^{-t}\\ \cos2t\cdot e^{-t} & \sin2t\cdot e^{-t}\end{pmatrix}$

$\Phi(0)=\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}C_1&C_2\\C_3&C_4\end{bmatrix}\Rightarrow\begin{bmatrix}C_1&C_2\\C_3&C_4\end{bmatrix}=\begin{bmatrix}0&1\\ \frac12&0\end{bmatrix}$

$\therefore\Phi(t)=\begin{bmatrix}\cos2t\,e^{-t} & -2\sin2t\,e^{-t}\\ \frac12\sin2t\,e^{-t} & \cos2t\,e^{-t}\end{bmatrix}$

$X=\Phi(t)\cdot X^0=\begin{pmatrix}3\cos2t-2\sin2t\\ \frac32\sin2t+\cos2t\end{pmatrix}e^{-t}$

7.解：

a) $\begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = 0 \Rightarrow (3-r)(-2-r)+4=0$

$r^2-r-2=0$

$r_1=2 \qquad r_2=-1$

for $r_1=2$

$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

for $r_2=-1$

$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \xi^{(2)} = 0 \Rightarrow \xi^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b) proof

$S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad S^{-1} = \frac{1}{-1+4}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} +\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & +\frac{2}{3} \end{pmatrix}$

$S^{-1}AS = \begin{pmatrix} +\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & +\frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} +\frac{4}{3} & -\frac{2}{3} \\ +\frac{1}{3} & -\frac{2}{2} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} r_1 & \\ & r_2 \end{pmatrix}$

c) proof: $y = S^{-1}X$

$y' = S^{-1}X' = S^{-1}AX = S^{-1}ASS^{-1}X$

$= DS^{-1}X$

$= Dy$

d) $y' = Dy = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} y$

$\Rightarrow \begin{pmatrix} e^{2t} & \\ & e^{-t} \end{pmatrix} = y$

$X = Sy = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^{2t} & \\ & e^{-t} \end{pmatrix}$

$= \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix} = [X^{(1)} \quad X^{(2)}]$

e) $A = SDS^{-1}$

$e^{At} = e^{SDS^{-1}t} = Se^{Dt}S^{-1}$

$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^{2t} & \\ & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & +\frac{2}{3} \end{pmatrix}$

$= \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

$= \begin{pmatrix} \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} \end{pmatrix}$

(f) $\psi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \psi^{-1}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$X = \psi(t)\psi^{-1}(0)X^0$

$= \begin{pmatrix} \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{t} \end{pmatrix}$