FEM HW8

Exercise 1 on Page 87

Solution.

$$a(N_A e_i, N_B e_j) = \int_\Omega \varepsilon(N_A e_i)^T D\, \varepsilon(N_B e_j)\, d\Omega$$

for $n_{sd} = 2$

$$e_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad e_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$B_A e_1 = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} N_{A,1} \\ 0 \\ N_{A,2} \end{Bmatrix}$$

$$B_A e_2 = \begin{Bmatrix} 0 \\ N_{A,2} \\ N_{A,1} \end{Bmatrix}$$

from (2.7.21): $\vec{\varepsilon}(\vec{u}) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix}$

$$N_A e_1 = N_A \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} N_A \\ 0 \end{Bmatrix}$$

$$\vec{\varepsilon}(N_A e_1) = \begin{Bmatrix} N_{A,1} \\ 0 \\ N_{A,2} \end{Bmatrix} = B_A e_1$$

$$\vec{\varepsilon}(N_A e_2) = B_A e_2$$

$$\therefore \vec{\varepsilon}(N_A \vec{e}_i) = B_A \vec{e}_i$$

for $n_{sd} = 3$

$$\vec{\varepsilon}(\vec{u}) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3} + u_{3,2} \\ u_{1,3} + u_{3,1} \\ u_{1,2} + u_{2,1} \end{Bmatrix} \quad (2.7.27)$$

$$e_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad e_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad e_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$\vec{\varepsilon}(N_A e_1) = \vec{\varepsilon}\left(\begin{Bmatrix} N_A \\ 0 \\ 0 \end{Bmatrix}\right) = \begin{Bmatrix} N_{A,1} \\ 0 \\ 0 \\ 0 \\ N_{A,3} \\ N_{A,2} \end{Bmatrix}$$

$$B_A e_1 = \begin{bmatrix} N_{A,1} & 0 & 0 \\ 0 & N_{A,2} & 0 \\ 0 & 0 & N_{A,3} \\ 0 & N_{A,3} & N_{A,2} \\ N_{A,3} & 0 & N_{A,1} \\ N_{A,2} & N_{A,1} & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} N_{A,1} \\ 0 \\ 0 \\ 0 \\ N_{A,3} \\ N_{A,2} \end{Bmatrix} = \vec{\varepsilon}(N_A e_1)$$

similarly, $B_A e_2 = \vec{\varepsilon}(N_A e_2)$, $B_A e_3 = \vec{\varepsilon}(N_A e_3)$

$$\therefore \vec{\varepsilon}(N_A \vec{e}_i) = B_A \vec{e}_i$$

Exercise 1 on Page 91

proof:

$$n_{sd}=2 \quad \therefore B_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix}$$

$$B_a d_a^e = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix} \begin{bmatrix} d_{1a}^e \\ d_{2a}^e \end{bmatrix} = \begin{bmatrix} N_{a,1} d_{1a}^e \\ N_{a,2} d_{2a}^e \\ N_{a,2} d_{1a}^e + N_{a,1} d_{2a}^e \end{bmatrix}$$

$$\sum_{a=1}^{nen} B_a d_a^e = \begin{Bmatrix} \sum_{a=1}^{nen} N_{a,1} d_{1a}^e \\ \sum_{a=1}^{nen} N_{a,2} d_{2a}^e \\ \sum_{a=1}^{nen} (N_{a,2} d_{1a}^e + N_{a,1} d_{2a}^e) \end{Bmatrix} = B d^e$$

Since $d_{ia}^e = u_i^h(x_a^e)$

$$\sigma = D\varepsilon(\vec{u}) = D \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{Bmatrix}$$

$$\begin{Bmatrix} \sum_{a=1}^{nen} N_{a,1} u_1^h(x_a^e) \\ \sum_{a=1}^{nen} N_{a,2} u_2^h(x_a^e) \\ \sum_{a=1}^{nen} [N_{a,2} u_1^h(x_a^e) + N_{a,1} u_2^h(x_a^e)] \end{Bmatrix}$$

$$u_{1,1} = \sum_{a=1}^{nen} N_{a,1} u_1^h(x_a^e)$$

$$u_{2,2} = \sum_{a=1}^{nen} N_{a,2} u_2^h(x_a^e)$$

$$u_{1,2}+u_{2,1} = \sum_{a=1}^{nen} (N_{a,2} u_1^h + N_{a,1} u_2^h)$$

$$\Rightarrow \sigma = D\varepsilon = DBd^e = D\sum_{a=1}^{nen} B_a d_a^e$$

$$n_{sd}=3$$

$$B_a d_a^e = \begin{bmatrix} N_{a,1} & 0 & 0 \\ 0 & N_{a,2} & 0 \\ 0 & 0 & N_{a,3} \\ 0 & N_{a,3} & N_{a,2} \\ N_{a,3} & 0 & N_{a,1} \\ N_{a,2} & N_{a,1} & 0 \end{bmatrix} \begin{Bmatrix} d_{1a}^e \\ d_{2a}^e \\ d_{3a}^e \end{Bmatrix}$$

$$= \begin{Bmatrix} N_{a,1} d_{1a}^e \\ N_{a,2} d_{2a}^e \\ N_{a,3} d_{3a}^e \\ N_{a,3} d_{2a}^e + N_{a,2} d_{3a}^e \\ N_{a,3} d_{1a}^e + N_{a,1} d_{3a}^e \\ N_{a,2} d_{1a}^e + N_{a,1} d_{2a}^e \end{Bmatrix} \quad (\#)$$

$$\sum_{a=1}^{nen} (\#) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{1,2}+u_{2,1} \end{Bmatrix} = \varepsilon(u)$$

$$\Rightarrow \sigma = D\varepsilon = DBd^e = D\sum_{a=1}^{nen} B_a d_a^e$$

Exercise 1 on Page 98

Solution

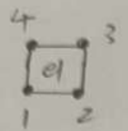
ID:

	1	2	3	4	5	6	7	8
1	1	3	0	5	7	9	0	12
2	2	4	0	6	8	10	11	13

$n_{np} = 8$

$n_{dof} = 2$

$n_{eq} = 13$



IEN:

	1	2	3	4
1	1	1	4	8
2	2	3	3	6
3	8	4	5	5
4	7	2	6	7

$n_{el} = 4$

$n_{en} = 4$

LM:

			1	2	3	4
1	1	1	1	1	5	12
2	1	2	2	2	6	13
3	2	1	3	0	0	9
4	2	2	4	0	0	10
5	3	1	12	5	7	7
6	3	2	13	6	8	8
7	4	1	0	3	9	0
8	4	2	11	4	10	11

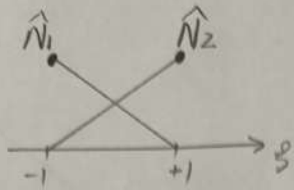
$n_{el} = 4$

$n_{ee} = 8$

Piecewise linears:

$\hat{N}_1 = \frac{1}{2}(1-\xi)$

$\hat{N}_2 = \frac{1}{2}(1+\xi)$



$$m^e_{ab} = \int_{\Omega^e} N_a \rho N_b \, d\Omega$$

$$= \int_{x_1^e}^{x_2^e} N_a \rho N_b \, dx \quad , \ 1 \le a, b \le 2$$

$$= \rho \int_{x_1^e}^{x_2^e} N_a N_b \, dx \quad (\text{constant } \rho)$$

$$= \rho \int_{x_1^e}^{x_2^e} \hat{N}_a(\xi(x)) \hat{N}_b(\xi(x)) \, dx$$

$$= \rho \int_{\xi_1}^{\xi_2} \hat{N}_a(\xi) \hat{N}_b(\xi) \, x_{,\xi} \, d\xi$$

$$= \rho \int_{-1}^{1} \frac{1}{2}(1+\xi_a \xi) \cdot \frac{1}{2}(1+\xi_b \xi) \cdot \frac{h}{2} \, d\xi$$

$$= \frac{\rho h}{4}\left(1 + \frac{1}{3}\xi_a \xi_b\right)$$

$$\Rightarrow m^e = \frac{\rho h}{4}\begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

$$= \frac{\rho h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Piecewise quadratics

$\hat{N}_1 = \frac{\xi^2 - \xi}{2}$

$\hat{N}_2 = 1 - \xi^2$ Lagrange polynomial

$\hat{N}_3 = \frac{\xi^2 + \xi}{2}$

$$\int_{-1}^{1} \hat{N}_1 \hat{N}_2 \, d\xi = \int_{-1}^{1} \frac{\xi^2-\xi}{2} \cdot (1-\xi^2) \, d\xi = \frac{2}{15}$$

$$\int_{-1}^{1} \hat{N}_1 \hat{N}_1 \, d\xi = \int_{-1}^{1} \left(\frac{\xi^2-\xi}{2}\right)^2 d\xi = \frac{4}{15}$$

$$\int_{-1}^{1} \hat{N}_1 \hat{N}_3 \, d\xi = \int_{-1}^{1} \frac{\xi^2-\xi}{2} \cdot \frac{\xi^2+\xi}{2} \, d\xi = -\frac{1}{15}$$

$$\int_{-1}^{1} \hat{N}_2 \hat{N}_2 \, d\xi = \int_{-1}^{1} (1-\xi^2)^2 \, d\xi = \frac{16}{15}$$

$$\int_{-1}^{1} \hat{N}_2 \hat{N}_3 \, d\xi = \int_{-1}^{1} (1-\xi^2) \frac{\xi^2+\xi}{2} \, d\xi = \frac{2}{15}$$

$$\int_{-1}^{1} \hat{N}_3 \hat{N}_3 \, d\xi = \int_{-1}^{1} \left(\frac{\xi^2+\xi}{2}\right)^2 d\xi = \frac{4}{15}$$

$$x^e(\xi) = x_1^e \hat{N}_1(\xi) + x_2^e \hat{N}_2(\xi) + x_3^e \hat{N}_3(\xi)$$

$$x^e(\xi)_{,\xi} = x_1^e \cdot \frac{2\xi - 1}{2} + x_2^e \cdot (-2\xi) + x_3^e \cdot \frac{2\xi+1}{2}$$

$$= x_1^e \xi - \frac{1}{2}x_1^e - 2\xi x_2^e + x_3^e \cdot \xi + \frac{1}{2}x_3^e$$

$$= \xi(-h_1) + \xi \cdot h_2 + \frac{1}{2}\left(x_3^e - x_2^e + x_2^e - x_1^e\right)$$

$$= \frac{1}{2} \cdot h$$

$$\therefore m^e = \frac{\rho h}{2} \cdot \begin{bmatrix} \frac{4}{15} & \frac{2}{15} & \frac{-1}{15} \\ \cdot & \frac{16}{15} & \frac{2}{15} \\ \cdot & \cdot & \frac{4}{15} \end{bmatrix}$$

$$= \frac{\rho h}{30}\begin{bmatrix} 4 & 2 & -1 \\ \cdot & 16 & 2 \\ \cdot & \cdot & 4 \end{bmatrix}$$

Symmetric.

Exercise 1 on Page 492

Solution

eliminated.

$$(9.1.4): \quad M a_{n+1} + C v_{n+1} + K d_{n+1} = F_{n+1}$$

$$(9.1.9): \quad d_{n+1} = \tilde{d}_{n+1} + \beta \Delta t^2 \cdot a_{n+1}$$

$$(9.1.10): \quad v_{n+1} = \tilde{v}_{n+1} + \gamma \Delta t \cdot a_{n+1}$$

$$a_{n+1} = (d_{n+1} - \tilde{d}_{n+1}) / \beta \Delta t^2$$

$$M a_{n+1} + C \cdot (\tilde{v}_{n+1} + \gamma \Delta t \cdot a_{n+1}) + K d_{n+1} = F_{n+1}$$

$$(M + C \gamma \Delta t) a_{n+1} + C \cdot \tilde{v}_{n+1} + K d_{n+1} = F_{n+1}$$

$$(M + C \gamma \Delta t) \cdot (d_{n+1} - \tilde{d}_{n+1}) + C \cdot \beta \Delta t^2 \cdot \tilde{v}_{n+1} + \beta \Delta t^2 \cdot K d_{n+1} = \beta \Delta t^2 F_{n+1}$$

eliminated a_{n+1} and v_{n+1}

Exercise 3 on Page 495

Proof

$$M a_{n+1} + C v_{n+1} + K d_{n+1} = F_{n+1} \qquad (9.1.4)$$

$$\begin{cases} d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}\cdot\left(\frac{1}{2}a_n + \frac{1}{2}a_{n+1}\right) & \beta = 1/4 \qquad (9.1.5) \\ v_{n+1} = v_n + \Delta t\left(\frac{1}{2}a_n + \frac{1}{2}a_{n+1}\right) & \gamma = 1/2 \qquad (9.1.6) \end{cases}$$

$$\Rightarrow (\#) \begin{cases} d_{n+1} = d_n + \frac{\Delta t}{2}\cdot\left[v_n + \underbrace{v_n + \frac{\Delta t}{2}\cdot(a_n + a_{n+1})}_{v_{n+1}}\right] = d_n + \frac{\Delta t}{2}(v_n + v_{n+1}) & (1) \\ v_{n+1} = v_n + \frac{\Delta t}{2}(a_n + a_{n+1}) & (2) \end{cases}$$

$$\begin{cases} y_{n+1} = y_n + \frac{\Delta t}{2}(z_n + z_{n+1}) \qquad (9.1.27) \\ y_{n+1} = \begin{Bmatrix} d_{n+1} \\ \dot{d}_{n+1} \end{Bmatrix}, \quad y_n = \begin{Bmatrix} d_n \\ \dot{d}_n \end{Bmatrix} \end{cases}$$

$$\Rightarrow d_{n+1} = d_n + \frac{\Delta t}{2}(z_n + z_{n+1}) = d_n + \frac{\Delta t}{2}\cdot\left[f(y_n, t_n) + f(y_{n+1}, t_{n+1})\right] \qquad (9.1.25)$$

$$f(y_n, t_n) = \dot{y}_n = \begin{Bmatrix} \dot{d}_n \\ \ddot{d}_n \end{Bmatrix} \quad 取第1项\rightarrow$$

$$f(y_n, t_n) = \dot{y}_n = \dot{d}_n = v_n, \qquad f(y_{n+1}, t_{n+1}) = \dot{y}_{n+1} = \dot{d}_{n+1} = v_{n+1}$$

displacement

velocity

$$\therefore d_{n+1} = d_n + \frac{\Delta t}{2}(v_n + v_{n+1}) \quad \blacksquare$$

equivalent to (#)(1).

Similarly,
$$\dot{d}_{n+1} = \dot{d}_n + \frac{\Delta t}{2}\cdot\left[f(y_n, t_n) + f(y_{n+1}, t_{n+1})\right]$$
$$= \dot{d}_n + \frac{\Delta t}{2}\cdot\left(\ddot{d}_n + \ddot{d}_{n+1}\right) \leftarrow 取第2项$$

$$v_{n+1} = v_n + \frac{\Delta t}{2}\cdot(a_n + a_{n+1}) \quad \blacksquare$$

equivalent to (#)(2).