常微分方程B Week 11.

1.解:

$$f(t) = u_{0,2}(t)\cdot t + u_{2,5}(t)\cdot 2 + u_{5,7}(t)\cdot(7-t) + 0$$
$$= t[u_0(t)-u_2(t)] + 2[u_2(t)-u_5(t)] + (7-t)[u_5(t)-u_7(t)]$$
$$= t + u_2(t)(2-t) + u_5(t)\cdot(5-t) - u_7(t)(7-t)$$

2.解:

(1) $f(t) = (t-2)^2 u(t-2)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-2)\cdot(t-2)^2\} = e^{-2s}\cdot\frac{2}{s^3}$$

(2) $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

$$= (t-2)u(t-2) - u_2(t) - (t-3)u_3(t) - u_3(t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-2)\cdot(t-2)\} - \mathcal{L}\{u_2(t)\} - \mathcal{L}\{u(t-3)(t-3)\} - \mathcal{L}\{u_3(t)\}$$
$$= e^{-2s}\cdot\frac{1}{s^2} - \frac{e^{-2s}}{s} - e^{-3s}\cdot\frac{1}{s^2} - \frac{e^{-3s}}{s}$$
$$= e^{-2s}\left(\frac{1}{s^2}-\frac{1}{s}\right) - e^{-3s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

(3) ① $f(t) = \int_0^t (t-\tau)^2\cos(2\tau)\,d\tau$

$$= \int_0^t (t-\tau)^2\, d(\sin 2\tau)\cdot\frac{1}{2}$$
$$= \frac{1}{2}\cdot(t-\tau)^2\cdot\sin 2\tau\Big|_0^t + \int_0^t \cancel{\tfrac{1}{2}}\sin 2\tau\cdot\cancel{2}(t-\tau)\,d\tau$$
$$= \int_0^t \sin 2\tau\cdot t\,d\tau - \int_0^t \sin 2\tau\cdot\tau\,d\tau$$
$$= t\cdot(-\cos 2\tau)\cdot\frac{1}{2}\Big|_0^t + \int_0^t \tau\, d\cos 2\tau\cdot\frac{1}{2}$$
$$= -\frac{1}{2}t\cdot\cos 2t + \frac{1}{2}t + \frac{1}{2}\tau\cdot\cos 2\tau\Big|_0^t - \frac{1}{2}\int_0^t \cos 2\tau\,d\tau$$
$$= -\frac{1}{2}t\cdot\cos 2t + \frac{1}{2}t + \frac{1}{2}t\cdot\cos 2t - \frac{1}{2}\cdot\sin 2\tau\cdot\frac{1}{2}\Big|_0^t$$
$$= \frac{1}{2}t - \frac{1}{4}\sin 2t$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\mathcal{L}\{t\} - \frac{1}{4}\mathcal{L}\{\sin 2t\}$$
$$= \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{4}\cdot\frac{2}{s^2+4}$$
$$= \frac{1}{2s^2} - \frac{1}{2s^2+8}$$

② $f(t) = \int_0^t (t-\tau)^2\cos 2\tau\,d\tau = (g*h)(t)$, $g(t) = t^2$, $h(t) = \cos 2t$

$$\mathcal{L}\{(g*h)(t)\} = G(s)\cdot H(s)$$
$$= \frac{2}{s^3}\cdot\frac{s}{s^2+4} = \frac{2}{s^2(s^2+4)}$$

3.解:

proof:

令 $f(t) = g(t) + u_T(t)f(t-T)$; $g(t) = f(t)$, $0\le t<T$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} + \mathcal{L}\{u_T(t)f(t-T)\}$$
$$= \int_0^T e^{-st}f(t)\,dt + e^{-sT}\int_0^\infty e^{-st}f(t)\,dt$$
$$= \int_0^T e^{-st}f(t)\,dt + e^{-sT}\mathcal{L}\{f(t)\}$$
$$\therefore \mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st}f(t)\,dt}{(1-e^{-sT})}$$

$f(t) = |\sin t|$, $T = \pi$

$$\mathcal{L}\{f(t)\} = \frac{\int_0^\pi e^{-st}\sin t\,dt}{1-e^{-s\pi}} \quad ①$$
$$= \frac{1}{1-e^{-s\pi}}\cdot\frac{1}{-s}\cdot\int_0^\pi \sin t\cdot de^{-st}$$
$$= \frac{1}{s\cdot e^{-s\pi}-s}\cdot\left(\sin t\cdot e^{-st}\Big|_0^\pi - \int_0^\pi e^{-st}\cos t\,dt\right)$$
$$= \frac{1}{s\cdot e^{-s\pi}-s}\left(\frac{1}{s}\int_0^\pi \cos t\cdot de^{-st}\right)$$
$$= \frac{1}{s^2\cdot e^{-s\pi}-s^2}\cdot\left(\cos t\cdot e^{-st}\Big|_0^\pi + \int_0^\pi e^{-st}\sin t\,dt\right)$$
$$= \frac{1}{s^2\cdot e^{-s\pi}-s^2}\left([(-1)\cdot e^{-s\pi}-1] + \int_0^\pi e^{-st}\sin t\,dt\right) \quad ②$$

$$① = ② \Rightarrow \int_0^\pi e^{-st}\sin t\,dt = \frac{1+e^{-s\pi}}{1+s^2}$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1+e^{-s\pi}}{(1+s^2)(1-e^{-s\pi})} \quad s>0.$$

4.解:

(1) $F(s)=\dfrac{2s+1}{4s^2+4s+5}=\dfrac{2s+1}{4(s^2+s+\frac{1}{4})+4}$

$=\dfrac{\cancel{2}}{\cancel{4}}\cdot\dfrac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+1}\cdot\dfrac{1}{2}$

$\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2}e^{-\frac{1}{2}t}\cos t$

(2) $F(s)=\dfrac{2(s-1)e^{-2s}}{s^2-2s+2}=\dfrac{2(s-1)\cdot e^{-2s}}{(s-1)^2+1}$

$=e^{-2s}\cdot\dfrac{s-1}{(s-1)^2+1}\cdot 2$

$\mathcal{L}\{u(t-c)\cdot f(t-c)\}=e^{-cs}\cdot F(s)$

$\mathcal{L}^{-1}\{e^{-2s}\cdot\dfrac{s-1}{(s-1)^2+1}\cdot 2\}=u(t-2)\cdot e^{\tau}\cdot\cos\tau\cdot 2$

$=2u(t-2)\cdot e^{t-2}\cos(t-2)$

(3) $H(s)=\dfrac{1}{(s+1)^2\cdot(s^2+4)}=\dfrac{1}{(s+1)^2}\cdot\dfrac{2}{(s^2+2^2)}\cdot\dfrac{1}{2}$

$\mathcal{L}\{e^{ct}\cdot f(t)\}=F(s-c)$

$H(s)=F(s)\cdot G(s)$

$F(s)=\dfrac{1}{(s+1)^2}$, $\mathcal{L}^{-1}\{F(s)\}=e^{-t}\cdot t$

$G(s)=\frac{1}{2}\cdot\dfrac{2}{s^2+2^2}$, $\mathcal{L}^{-1}\{G(s)\}=\frac{1}{2}\cdot\sin 2t$

$\therefore \mathcal{L}^{-1}\{H(s)\}=\int_0^t \frac{1}{2}\sin 2u\cdot e^{-(t-u)}\cdot(t-u)\,dt$

5.解:

(1) $\mathcal{L}\{y\}=\dfrac{1}{(s^2+1)(s^2+4)}-\dfrac{e^{-2\pi s}}{(s^2+1)(s^2+4)}$

$=(1-e^{-2\pi s})\cdot\dfrac{1}{(s^2+1)(s^2+4)}$

$=(1-e^{-2\pi s})\cdot\left[\dfrac{1}{s^2+1}\cdot\dfrac{1}{3}+\dfrac{2}{s^2+2^2}\cdot(-\dfrac{1}{6})\right]$

$=\dfrac{1}{s^2+1}\cdot\dfrac{1}{3}+\dfrac{2}{s^2+2^2}\cdot(-\dfrac{1}{6})-\dfrac{e^{-2\pi s}}{s^2+1}\cdot\dfrac{1}{3}$

$+\dfrac{2}{s^2+2^2}\cdot e^{-2\pi s}\cdot\dfrac{1}{6}$

$\mathcal{L}\{u(t-c)\cdot f(t-c)\}=e^{-cs}\cdot F(s)$.

$\therefore y=\frac{1}{3}\cdot\sin t+(-\frac{1}{6})\cdot\sin 2t+(-\frac{1}{3})\cdot u(t-2\pi)\cdot\sin(t-2\pi)$

$+\frac{1}{6}\cdot u(t-2\pi)\cdot\sin(2t-4\pi)$

$=\frac{1}{3}\sin t-\frac{1}{6}\sin 2t-\frac{1}{3}u(t-2\pi)\cdot\sin t$

$+\frac{1}{6}u(t-2\pi)\cdot\sin 2t$

$=\sin t\cdot\frac{1}{3}[1-u(t-2\pi)]+\sin 2t\cdot\frac{-1}{6}[1-u(t-2\pi)]$

$=[1-u(t-2\pi)]\cdot\frac{1}{6}(2\sin t-\sin 2t)$

(2) $\mathcal{L}\{y\}(s^2+3s+2)=\dfrac{1}{s}-\dfrac{e^{-10s}}{s}$

$\mathcal{L}\{y\}=\dfrac{1}{s^2+3s+2}\cdot\dfrac{1}{s}\cdot(1-e^{-10s})$

$=\dfrac{1}{(s+1)(s+2)}\cdot\dfrac{1}{s}\cdot(1-e^{-10s})$

$=(\dfrac{A}{s+1}+\dfrac{B}{s+2}+\dfrac{C}{s})(1-e^{-10s})$

$=\dfrac{1}{s+1}(-1)+\dfrac{1}{s+2}(+\dfrac{1}{2})+\dfrac{1}{s}\cdot\dfrac{1}{2}$

$+\dfrac{1}{s+1}e^{-10s}+\dfrac{1}{s+2}e^{-10s}(-\dfrac{1}{2})-\dfrac{1}{s}\cdot e^{-10s}\cdot\dfrac{1}{2}$

$\mathcal{L}\{u(t-c)f(t-c)\}=e^{-cs}\cdot F(s)$

$\therefore y=(-1)\cdot e^{-t}+\frac{1}{2}e^{-2t}+\frac{1}{2}+$

$u(t-10)\cdot e^{-(t-10)}+(-\frac{1}{2})\cdot u(t-10)\cdot e^{-2(t-10)}-\frac{1}{2}u(t-10)$

$=\frac{1}{2}(1-e^{-t}+e^{-2t})-\frac{1}{2}u(t-10)(1-2e^{10-t}+e^{20-2t})$

(3) $\mathcal{L}\{y''\}+2\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\pi)\}$

$$s^2\mathcal{L}\{y\}-sy(0)-\cancel{y'(0)}+2\left[s\mathcal{L}\{y\}-y(0)\right]+2\mathcal{L}\{y\}=e^{-\pi s}$$

$$s^2\mathcal{L}\{y\}-s+2\left[s\mathcal{L}\{y\}-1\right]+2\mathcal{L}\{y\}=e^{-\pi s}$$

$$\mathcal{L}\{y\}(s^2+2s+2)-s-2=e^{-\pi s}$$

$$\mathcal{L}\{y\}=(e^{-\pi s}+s+2)\cdot\frac{1}{s^2+2s+2}$$

$$=(e^{-\pi s}+s+2)\cdot\frac{1}{(s+1)^2+1}$$

$$=e^{-\pi s}\cdot\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}+\frac{1}{(s+1)^2+1}$$

$$\mathcal{L}\{u(t-c)\cdot f(t-c)\}=e^{-cs}\cdot F(s).$$

$$\therefore y=u(t-\pi)\cdot e^{-\tau}\cdot\sin\tau+e^{-t}\cdot\cos t+e^{-t}\cdot\sin t$$

$$=u(t-\pi)\cdot e^{-t+\pi}\cdot\sin(t-\pi)+e^{-t}(\cos t+\sin t)$$

$$=-u(t-\pi)\cdot e^{-t+\pi}\cdot\sin t+e^{-t}(\cos t+\sin t)$$

(4) $\mathcal{L}\{y''\}+3\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-5)\}+\mathcal{L}\{u_{10}(t)\}$

$$s^2\mathcal{L}\{y\}-\cancel{sy(0)}-y'(0)+3\left[s\mathcal{L}\{y\}-\cancel{y(0)}\right]+2\mathcal{L}\{y\}$$

$$=\mathcal{L}\{y\}(s^2+3s+2)-\frac{1}{2}=e^{-5s}+\frac{e^{-10s}}{s}$$

$$\mathcal{L}\{y\}=\left[\left(e^{-5s}+\frac{1}{s}\cdot e^{-10s}\right)+\frac{1}{2}\right]\cdot\frac{1}{s^2+3s+2}$$

$$=\left(e^{-5s}+\frac{1}{s}\cdot e^{-10s}+\frac{1}{2}\right)\cdot\frac{1}{(s+1)(s+2)}$$

$$=\left(e^{-5s}+\frac{1}{2}\right)\left(\frac{A}{(s+1)}+\frac{B}{s+2}\right)+e^{-10s}\left(\frac{C}{s}+\frac{D}{s+1}+\frac{E}{s+2}\right)$$

$$=\left(e^{-5s}+\frac{1}{2}\right)\left(\frac{1}{s+1}+\frac{-1}{s+2}\right)+e^{-10s}\left(\frac{\frac{1}{2}}{s}+\frac{-1}{s+1}+\frac{\frac{1}{2}}{s+2}\right)$$

$$=e^{-5s}\cdot\frac{1}{s+1}-e^{-5s}\cdot\frac{1}{s+2}+\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{2}\cdot\frac{1}{s+2}$$

$$+\frac{1}{2}\cdot e^{-10s}\cdot\frac{1}{s}-e^{-10s}\cdot\frac{1}{s+1}+\frac{1}{2}\cdot e^{-10s}\cdot\frac{1}{s+2}$$

$$\mathcal{L}\{u(t-c)\cdot f(t-c)\}=e^{-cs}\cdot F(s).$$

$$\therefore y=u(t-5)\cdot e^{-\tau}-u(t-5)\cdot e^{-2\tau}+\frac{1}{2}\cdot e^{-t}-\frac{1}{2}\cdot e^{-2t}$$

$$+\frac{1}{2}\cdot u(t-10)\quad-u(t-10)\cdot e^{-\tau}+\frac{1}{2}u(t-10)\cdot e^{-2\tau}$$

$$=u(t-5)\cdot(e^{5-t}-e^{10-2t})+\frac{1}{2}e^{-t}-\frac{1}{2}e^{-2t}$$

$$+u(t-10)\left(\frac{1}{2}-e^{10-t}+\frac{1}{2}e^{20-2t}\right)$$

(5) $\mathcal{L}\{y\}=G(s)\cdot\frac{1}{4s^2+4s+17}$

$$=\frac{1}{8}G(s)\cdot\frac{2}{(s+\frac{1}{2})^2+2^2}$$

$$=\frac{1}{8}G(s)\cdot F(s)$$

$$\mathcal{L}^{-1}\{F(s)\}=e^{-\frac{1}{2}t}\cdot\sin 2t.$$

$$\therefore y=\frac{1}{8}\int_0^t e^{-\frac{1}{2}u}\cdot\sin 2u\cdot g(t-u)\,dt$$

$$=\frac{1}{8}(g*f)(t).$$