第 11 周习题
常微分方程 B

April 26, 2022

1. Express $f(t)$ in terms of the unit step function $u_c(t)$:

$$f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & 2 \le t < 5, \\ 7-t, & 5 \le t < 7, \\ 0, & t \ge 7. \end{cases}$$

2. Find the Laplace transform for each of the following functions.

(1) $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \ge 2. \end{cases}$

(2) $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

(3) $f(t) = \displaystyle\int_0^t (t-\tau)^2 \cos(2\tau)\, d\tau$

Hint. Note that the last function is a convolution integral!

3. Let f be a periodic function with period T on $0 \le t < \infty$, *i.e.* $f(t+T) = f(t)$ for all $t \ge 0$, where $T > 0$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}.$$

Use this result to find the Laplace transform of $f(t) = |\sin t|$.

4. Find the inverse Laplace transform of the given functions:

(1) $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

(2) $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2}$

(3) $H(s) = \dfrac{1}{(s+1)^2(s^2+4)}$

Hint. You may want to use the convolution theorem for $H(s)$: if $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $g(t) = \mathcal{L}^{-1}\{g(s)\}$, then $\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$.

5. Use the Laplace transform to solve the given initial value problem:

(1) $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t-2\pi); \quad y(0) = 0, \quad y'(0) = 0$

(2) $y'' + 3y' + 2y = \begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases} \quad ; \quad y(0) = 0, \quad y'(0) = 0;$

(3) $y'' + 2y' + 2y = \delta(t-\pi); \quad y(0) = 1, \quad y'(0) = 0$

(4) $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t); \quad y(0) = 0, \quad y'(0) = 1/2$

(5) $4y'' + 4y' + 17y = g(t); \quad y(0) = 0, \quad y'(0) = 0$

Hint. For the final problem, the solution can be expressed in terms of a convolution integral.