Quiz 15

Date: 2022-05-27 Name: SID:

Use ***DIFFERENT*** methods to solve the given equations separately.

1. $x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}, x(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix};$
2. $x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix}, x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix};$

3. In this problem we use the method illustrated in Example 1. We have the transformation matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}.$$

Inverting $\mathbf{T}$ we find that

$$\mathbf{T}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}.$$

If we let $\mathbf{x} = \mathbf{Ty}$ and substitute into the differential equation, we obtain

$$\mathbf{y}' = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\mathbf{y} + \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} =$$

$$\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{y} + \frac{1}{5}\begin{pmatrix} e^{-2t} + 2e^{t} \\ 4e^{-2t} - 2e^{t} \end{pmatrix}.$$

This corresponds to the two scalar equations $y_1' + 3y_1 = (1/5)e^{-2t} + (2/5)e^{t}$ and $y_2' - 2y_2 = (4/5)e^{-2t} - (2/5)e^{t}$, which may be solved by the methods of Section 2.1. For the first equation the integrating factor is e^{3t} and we obtain $(e^{3t}y_1)' = (1/5)e^{t} + (2/5)e^{4t}$, so $e^{3t}y_1 = (1/5)e^{t} + (1/10)e^{4t} + c_1$. For the second equation the integrating factor is e^{-2t}, so $(e^{-2t}y_2)' = (4/5)e^{-4t} - (2/5)e^{-t}$, hence $e^{-2t}y_2 = -(1/5)e^{-4t} + (2/5)e^{-t} + c_2$. Thus

$$\mathbf{y} = \begin{pmatrix} 1/5 \\ -1/5 \end{pmatrix}e^{-2t} + \begin{pmatrix} 1/10 \\ 2/5 \end{pmatrix}e^{t} + \begin{pmatrix} c_1e^{-3t} \\ c_2e^{2t} \end{pmatrix}.$$

$c_1 = c_2 = 1$

Finally, multiplying by $\mathbf{T}$, we obtain

$$\mathbf{x} = \mathbf{Ty} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{-2t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}e^{t} + c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}.$$

The last two terms are the general solution of the corresponding homogeneous system, while the first two terms constitute a particular solution of the nonhomogeneous system.

6. The eigenvalues of the coefficient matrix are $r_1 = 1$ and $r_2 = -1$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}.$$

Use the method of undetermined coefficients. Since the right hand side is related to one of the fundamental solutions, set $\mathbf{v}=\mathbf{a}\,te^{t}+\mathbf{b}\,e^{t}$. Substitution into the ODE yields

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}(e^{t} + te^{t}) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}e^{t} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}te^{t} + \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}e^{t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t}.$$

In scalar form, we have

$$(a_1 + b_1)e^{t} + a_1te^{t} = (2a_1 - a_2)te^{t} + (2b_1 - b_2)e^{t} + e^{t}$$
$$(a_2 + b_2)e^{t} + a_2te^{t} = (3a_1 - 2a_2)te^{t} + (3b_1 - 2b_2)e^{t} - e^{t}.$$

Equating the coefficients in these two equations, we find that

$$a_1 = 2a_1 - a_2$$
$$a_1 + b_1 = 2b_1 - b_2 + 1$$
$$a_2 = 3a_1 - 2a_2$$
$$a_2 + b_2 = 3b_1 - 2b_2 - 1.$$

It follows that $a_1 = a_2$. Setting $a_1 = a_2 = a$, the equations reduce to

$$b_1 - b_2 = a - 1$$
$$3b_1 - 3b_2 = 1 + a.$$

Combining these equations, it is necessary that $a = 2$. As a result, $b_1 = b_2 + 1$. Choosing $b_2 = k$, some arbitrary constant, a particular solution is

$$\mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}te^{t} + \begin{pmatrix} k+1 \\ k \end{pmatrix}e^{t} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}te^{t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t}.$$

Since the second vector is a fundamental solution, the general solution can be written as

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t}.$$

$C_1 = 1 \quad C_2 = 0$

$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}te^{t}$