Please write carefully and clearly in complete sentences. Your explanations are your only representative when your work is being graded.

1. Solve the following problems:

   (a) $y^{(4)} + 2y'' + y = 8\sin t - 16\cos t$;

   (b) $y''' - 3y'' + 3y' - y = 6e^t$;

   (c) $y' + y^2 \sin x = 0$.

   Solutions.

   (a) The characteristic equation is $r^4 + 2r^2 + 1 = 0$. The roots are $i, i, -i, -i$, and the general solution of the homogeneous equation is

   $$y_c(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

   Our initial assumption for a particular solution is $Y(t) = A\sin t + B\cos t$, but we must multiply this choice by t^2 to make this choice by t^2 to make it different from all solutions of the homogeneous equation. Thus our assumption is $Y(t) = At^2 \sin t + Bt^2 \cos t$. Plugging this into the original equation, we obtain $A = -1,\ B = 2$. Hence, the general solution of the nonhomogeneous equation is

   $$y(t) = y_c(t) + Y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t - t^2 \sin t + 2t^2 \cos t,$$

   where $c_1,\ c_2,\ c_3,\ c_4$ are arbitrary constants.

   (b) The characteristic equation is $r^3 - 3r^2 + 3r - 1 = (r-1)^3$. So, the general solution of the homogeneous equation is

   $$y_c(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t,$$

   where $c_1,\ c_2,\ c_3$ are arbitrary constants. Since e^t, te^t, t^2e^t are all solution-s of the homogeneous equation, we need to find a particular solution of the form $Y(t) = At^3e^t$ of the nonhomogeneous equation, where A is an undetermined coefficient. Plugging this into the original equation, we obtain $A = 1$. In conclusion, the general solution of the nonhomogeneous equation is

$$y(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + t^3 e^t,$$

where c_1, c_2, c_3 are arbitrary constants.

(c) If $y \neq 0$, then

$$-\frac{1}{y^2} dy = \sin x dx \Rightarrow d(y^{-1}) = -d(\cos x) \Rightarrow y^{-1} + \cos x = c,$$

where c is an arbitrary constant. Also, $y = 0$ is a solution.

2. Use the method of undetermined coefficients to find a particular solution of

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

Solution. To use the method of undetermined coefficients, we write $\mathbf{g}(t)$ in the form

$$\mathbf{g}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t.$$

Note that -1 is an eigenvalue of the coefficient matrix, and therefore we must include both $\mathbf{a}te^{-t}$ and $\mathbf{b}e^{-t}$ in the form of a solution. Therefore the solution of the system can be assumed to be of the form

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are vectors to be determined. Plugging the above $\mathbf{x}(t)$ into the original system, we obtain

$$\begin{aligned} \mathbf{x}' &= -\mathbf{a}te^{-t} + (\mathbf{a} - \mathbf{b})e^{-t} + \mathbf{c} \\ &= \mathbf{A}\mathbf{a}te^{-t} + \left(\mathbf{A}\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) e^{-t} + \left(\mathbf{A}\mathbf{c} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) t + \mathbf{A}\mathbf{d}. \end{aligned}$$

Equating all the corresponding coefficients, we obtain

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{b} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{5}{3} \end{bmatrix},$$

where k is an arbitrary constant. We can simply take $k = 0$, and thus a particular solution of the previous system is given as follows:

$$\mathbf{v}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

3. Determine the type and stability of the critical point $(4, 3)$ of the locally linear system

$$\frac{dx}{dt} = 33 - 10x - 3y + x^2,$$
$$\frac{dy}{dt} = -18 + 6x + 2y - xy.$$

Solution. With $F(x,y) = 33 - 10x - 3y + x^2$, $G(x,y) = -18 + 6x + 2y - xy$ and $x_0 = 4, y_0 = 3$ we have

$$J(x,y) = \begin{bmatrix} -10 + 2x & -3 \\ 6 - y & 2 - x \end{bmatrix}, \ J(4,3) = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}.$$

The associated linear system

$$\frac{du}{dt} = -2u - 3v,$$
$$\frac{dv}{dt} = 3u - 2v$$

has characteristic equation $(\lambda + 2)^2 + 9 = 0$, with complex conjugate roots $\lambda = -2 \pm 3i$. Hence, $(0, 0)$ is an asymptotically stable spiral of the linear system. It follows that $(4, 3)$ is an asymptotically stable spiral point of the original locally linear system.

4. Construct a suitable Liapunov function to determine the stability of the origin for the system

$$\frac{dx}{dt} = x^3 - 2y^3,$$
$$\frac{dy}{dt} = xy^2 + x^2y + \frac{1}{2}y^3.$$

Solution. We can take $V(x,y) = x^2 + 2y^2$ and $\dot{V} = 2(x^2 + y^2)^2$. Both of them are positive definite, therefore the origin is unstable.

5. Consider the system

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \mathbf{x}$$

Find a fundamental matrix.

Solution. A fundamental matrix is given as follows:

$$\Psi(t) = \begin{bmatrix} -3e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}.$$

6. Transform $u'' - 2u' - 3u = 0$ into a system of first order equations.

Solution.

$$\begin{cases} x_1' = x_2 \\ x_2' = 2x_2 + 3x_1. \end{cases}$$

7. It two functions $f(t)$ and $g(t)$ are linearly independent on the interval $(-1, 1)$, then the Wronskian $W[f, g](t) \neq 0$ for all $(-1, 1)$. True or False?

Solution. False.

8. Write down the definition of the matrix exponential e^{At}.

Solution.

$$e^{\mathbf{At}} = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{A}^k t^t}{k!}.$$

9. All solutions of $y' + 2y = \sin t$ converges to a particular solution as $t \to \infty$. True or False.

Solution. True.

10. Can be a saddle point asymptotically stable in a particular direction? Yes or No?

Solution. Yes.

11. Suppose that $f(x, y)$ and $f_y(x, y)$ are continuous on the xy-plane. Is it possible that $y_1(x) = \sin x$ and $y_2(x) = \cos x$ are solutions of $y'(x) = f(x, y)$? Yes or no?

Solution. No.

12. Use Laplace transform to solve the following initial value problem

$$\begin{cases} y^{(4)} + 2y'' + y = 4te^t, \\ y(0) = y'(0) = y''(0) = y'''(0) = 0. \end{cases}$$

Solution.

$$Y(s) = \frac{4}{(s-1)^2(s^2+1)^2} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{Cs+D}{(s^2+1)^2} + \frac{Es+F}{s^2+1}.$$

Solve for $B = -2,\ C = 2,\ D = 0,\ E = 2,$ and $F = 1$. Therefore, the solution of the given initial value problem is

$$y(t) = (t-2)e^t + (t+1)\sin t + 2\cos t.$$