Solutions to Midterm Test:

Partial solutions of midterm test are provided here. If there exist any mistakes, please do not hesitate to send your feedback to the following email:11855017@mail.sustech.edu.cn

Problem 1. –a: False;

Solution: Just substitute $C\phi$ to the equation to verify:

$$C\phi' - 2C\phi = Ct^2e^t \neq t^2e^t,$$

if $C \neq 1$ □

Problem 1. –b: Ture;

Solution: The general solution is equal to the sum of homogeneous solution and particular solution. □

Problem 1. –c: False;

Solution: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ □

Problem 1. –d: False;

Solution: $x' = x^2$ with initial value $x(0) = 1$ The solution is $x = \frac{1}{1-t}$ with the interval $(-\infty, 1)$ □

Problem 1. –e: True;

Solution: By the definition of equilibrium point. □

Problem 1. –f: True;

Solution: $x = 0$ is an equilibrium point. Consider $f = -\frac{1}{5}sin(e^x - 1)$ in the neighborhood of 0. It is an asymptotic stable point since it is positive in the left and negative in the right. □

Problem 1. –g: True;

Solution: By the definition,

$$\frac{d\psi(t)}{dt} = \frac{d\phi(t+C)}{dt}$$

By the chain rule,

$$\frac{d\psi(t)}{dt} = \frac{d\phi(T)}{dt} = \frac{d\phi(T)}{dT}\frac{dT}{dt} = \frac{d\phi(T)}{dT} = f(\phi(T)) = f(\phi(t+C)) = f(\psi(t))$$

where $T = t + C$ and ϕ is a solution of ODE. □

Problem 1. –h: True;

Solution: For $c > 0$, the solution always has an exponential decay. (On the other side,since the physical meaning of c is the resistance, the system is always losing energy without input). $\square$

Problem 1. –i: False;

Solution: f,g should be two solutions of differential equation. Counter-example: $f(t) = t$, $g(t) = t^2$ and $W = t^2$ $\square$

Problem 1. –j: False;

Solution:

$$L(y') = sL(y) - y(0)$$

$\square$

Problem 2. –a:

Solution: This is a linear equation and we can use the integrating factor. Consider the integrating factor $\mu = e^{-x}$. Then,

$$(e^{-x}y)' = 2x$$

Integrating both sides,

$$e^{-x}y = x^2 + C$$

$$y = e^x x^2 + Ce^x$$

$\square$

Problem 2. –b:

Solution: By the initial condition, we have $C = 3$. Then, $y = e^x x^2 + 3e^x$ $\square$

Problem 3. –a:

Solution: $\frac{\partial \mu 5y}{\partial y} = \frac{\partial \mu 4x}{\partial x}$
i.e.

$$5(b+1)x^a y^b = 4(a+1)x^a y^b$$

We have the conditions : $4a - 5b = 1$ $\square$

Problem 3. –b:

Solution: Take $a = 0$, we have $b = \frac{-1}{5}$.
Then,

$$5y^{\frac{4}{5}}dx + 4xy^{-\frac{1}{5}}dy = 0$$

Finally, we have $F(x, y) = xy^{\frac{4}{5}} = C$. where C is an arbitary constant. $\square$

Problem 4. –a:

Solution:

$$f(y) = 6 + y - y^2 = (3 - y)(2 + y) = 0$$

The equilibrium points are $x = 3$ and $x = -2$ □

Problem 4. –b:

Solution: $x = -2$ is unstable and $x = 3$ is asymptotically stable. □

Problem 5. –a:

Solution: Solve the characteristic equation: $\lambda^2 + 1 = 0$

Find two homogeneous solutions, $x_1 = cost$ and $x_2 = sint$ □

Problem 5. –b:

Solution: Use the method of parameter to find the particular solution. Assume

$$x_p = v_1(t)x_1 + v_2(t)x_2$$

Then, we get the following two equations:

$$v_1'x_1 + v_2'x_2 = 0$$

$$v_1'x_1' + v_2'x_2' = g(t)$$

(Please see lecture 11,if you want to know the details.)

Thus,

$$v_1' = \frac{-x_2 g(t)}{W}$$

$$v_2' = \frac{x_1 g(t)}{W}$$

where $W = 1$ is the Wronskian of x_1, x_2

Finally, we have

$$v_1(t) = In|cost|$$

&

$$v_2(t) = t$$

Then,

$$x(t) = C_1 cost + C_2 sint + In|cost|cost + tsint$$

By the initial condition, we have $C_1 = 1, C_2 = 2$

and the general solution is :

$$x(t) = cost + 2sint + In|cost|cost + tsint$$

□