Kou Jump Diffusion Model

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Erdos Institute Fall 2025: Quantitive Finance

Motivations and Contributions

- Equity daily log returns show leptokurtic feature and skewness not captured by Black-Scholes models
- Volatility smile phenomenon is observed in option markets.
- Jump risk matters for hedging. Delta hedging leaves jump risk.
- Need a tractable model incorporating above feature.

We: (i) introduce Kou jump diffusion model; (ii) simulate stock paths; (iii) price European call options; (iv) test three hedging strategies; (v) calibrate to SPY data and backtest, compare skewness and excess kurtosis

Model: Kou's double exponential jump diffusion (2002)

Stock Price:

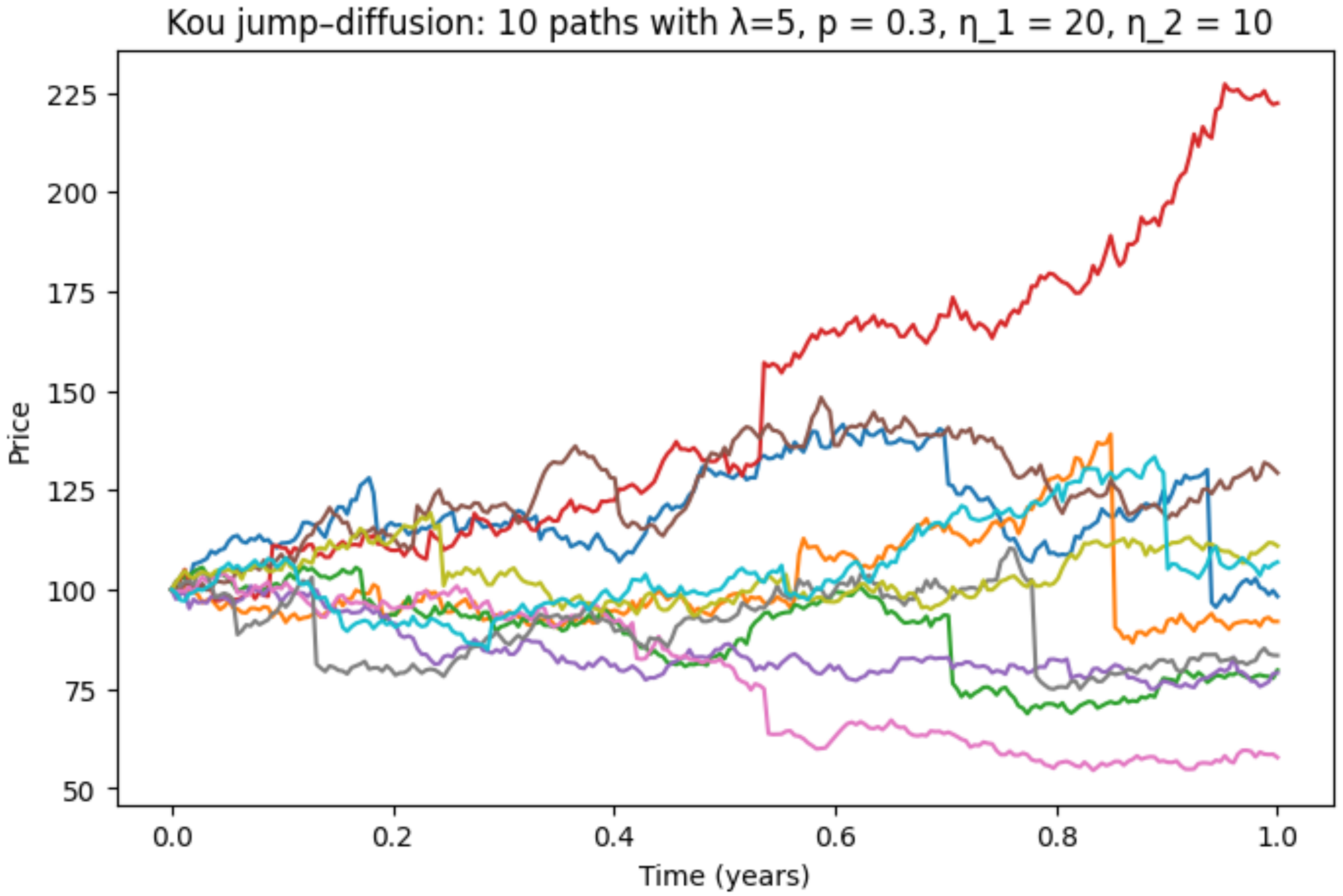
$$S_t = S_0 \exp\{(r - \frac{1}{2}\sigma^2 - \lambda\zeta)t + \sigma W(t)\} \prod_{i=1}^{N(t)} V_i$$

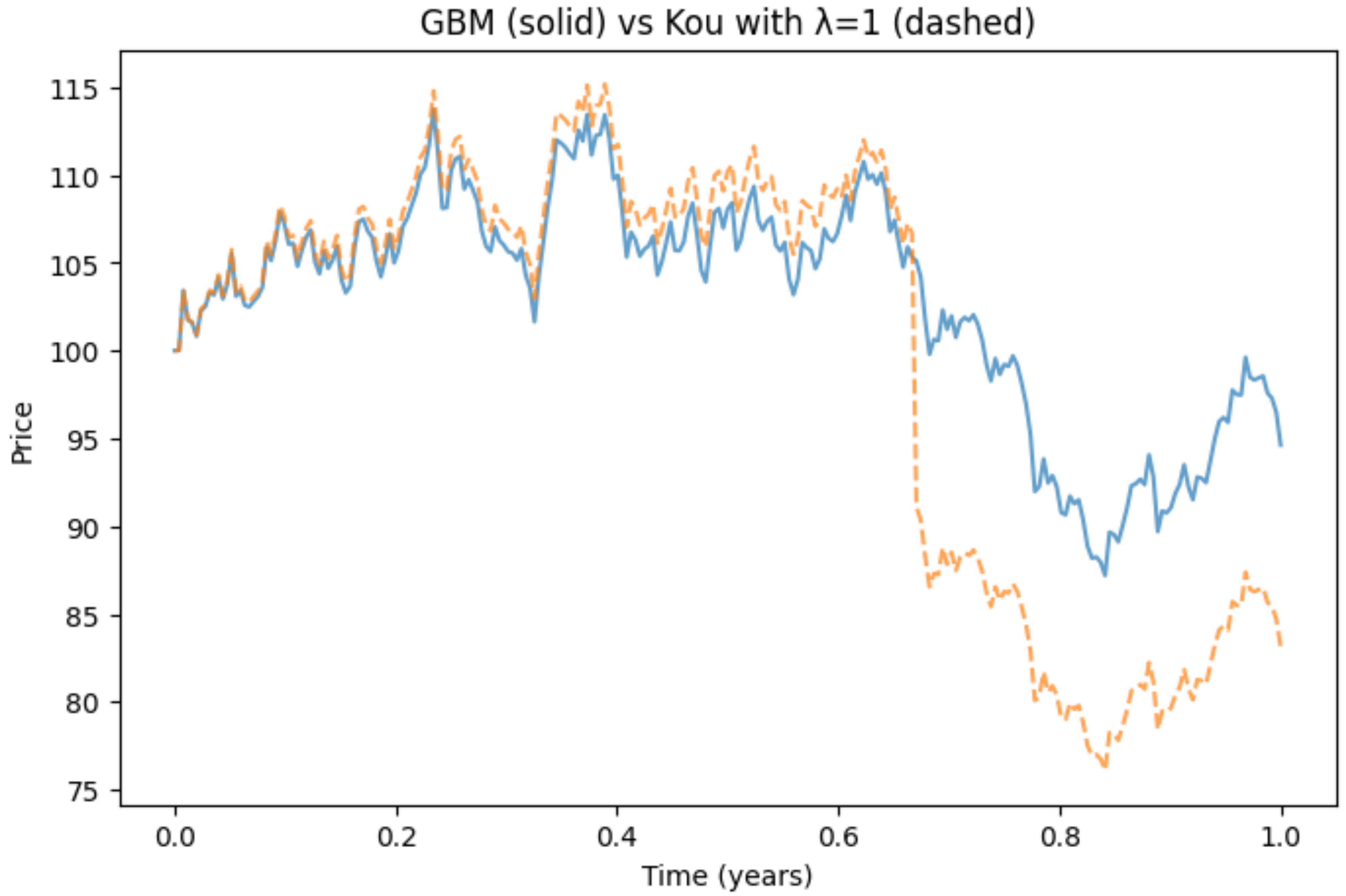
- Brownian motion: $W(t) \sim Normal(0,1)$
- Jump counting process: $N(t) \sim Poisson(\lambda t)$
- Jump amplitude: $Y_i = \log V_i$ has asymmetric double exponential density:

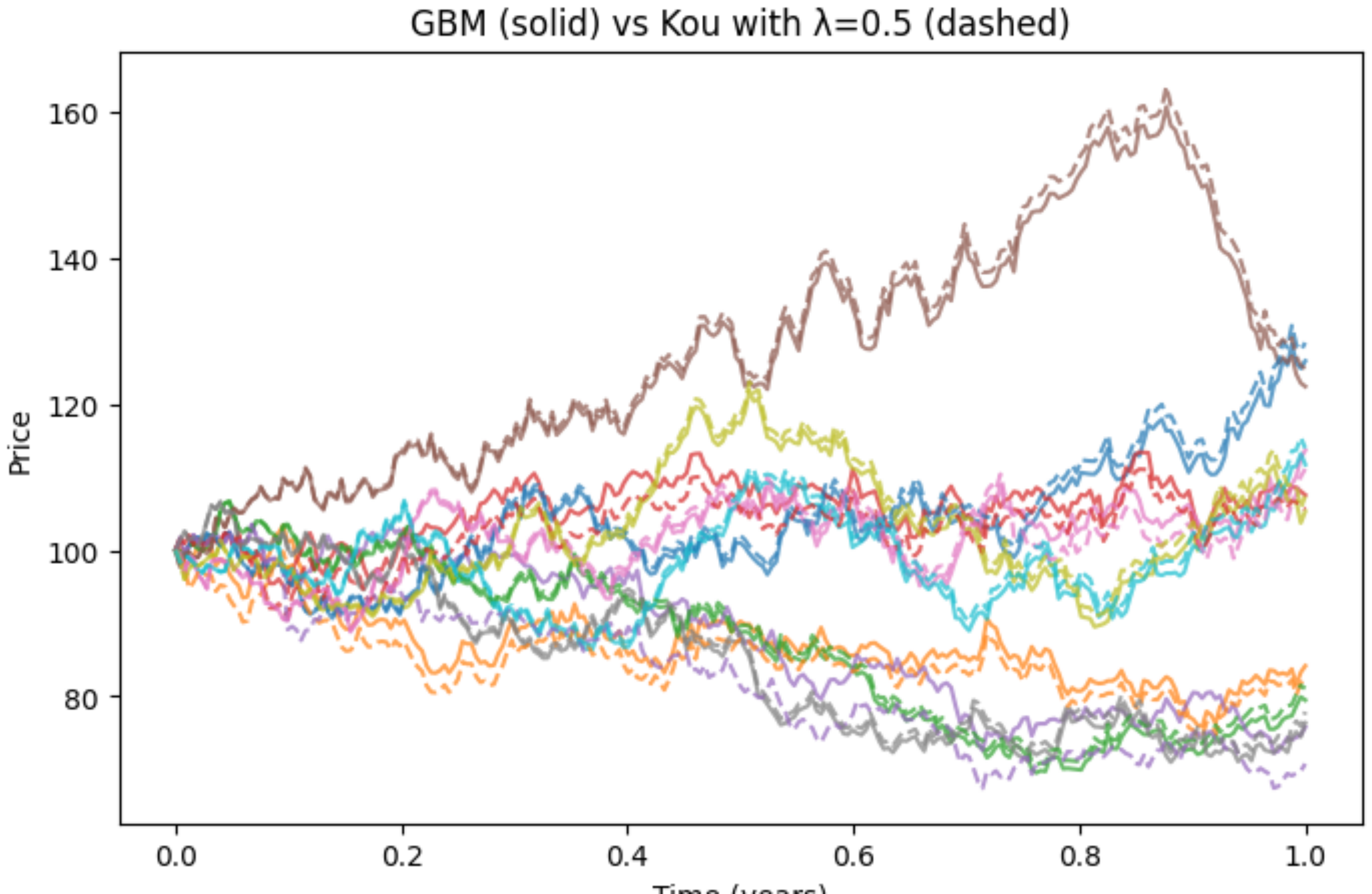
$$f_{Y_i}(y) = \underbrace{p \cdot \eta_1 e^{-\eta_1 y} \mathbf{1}_{\{y \geq 0\}}}_{\text{upward jump}} + \underbrace{(1-p) \cdot \eta_2 e^{\eta_2 y} \mathbf{1}_{\{y < 0\}}}_{\text{downward jump}}, \quad \eta_1 > 1, \eta_2 > 0$$

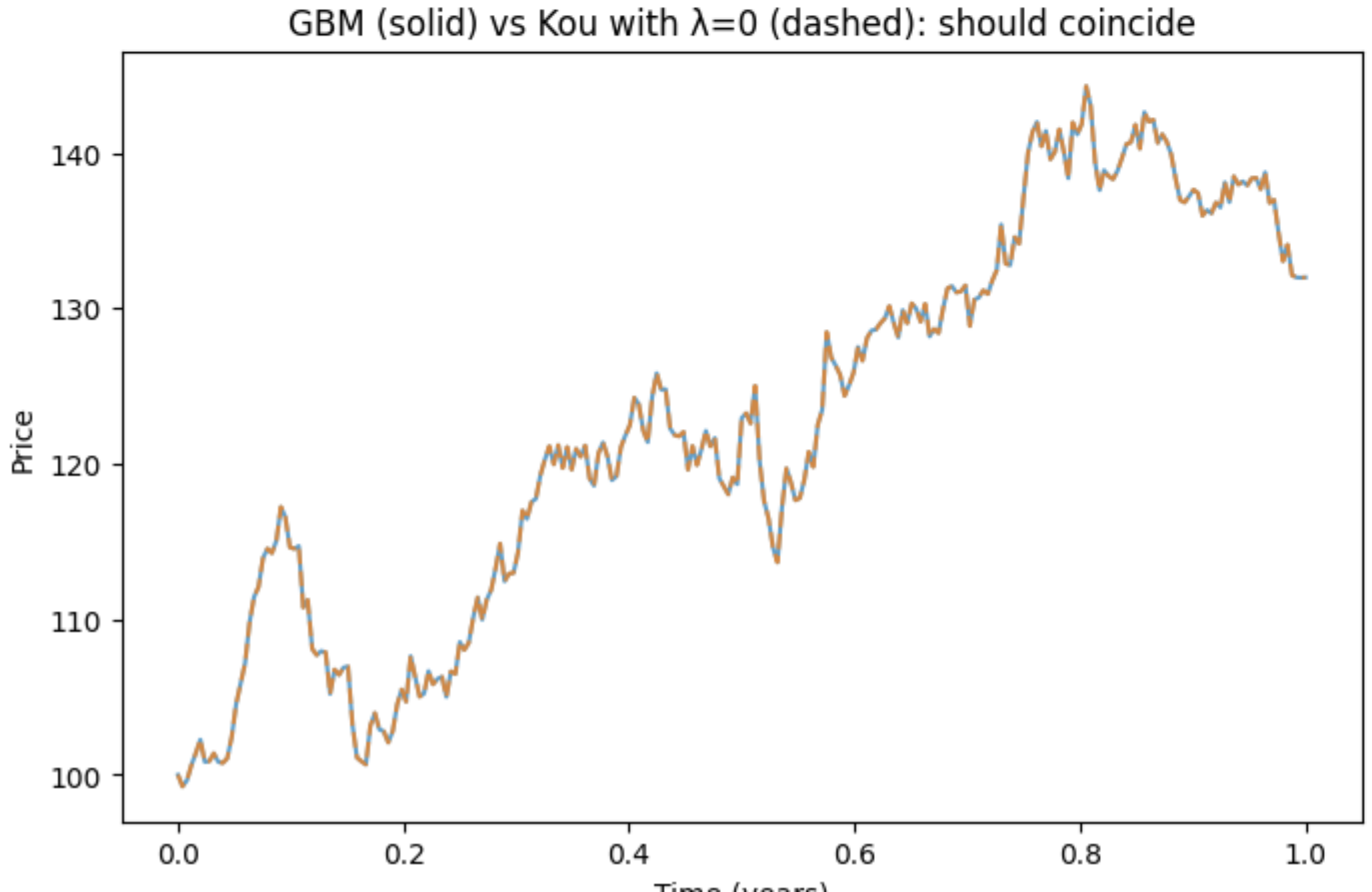
- $\zeta = \mathbb{E}[e^{Y_i} - 1]$

Stock path simulations









European call option pricing

$$C_T(k) = e^{-rT}\mathbb{E}[\max\{S_T - K, 0\}], \quad k = \ln K$$

- Closed form exits [Kou02], but very complicated
- Laplace transform approach [KPW05]:

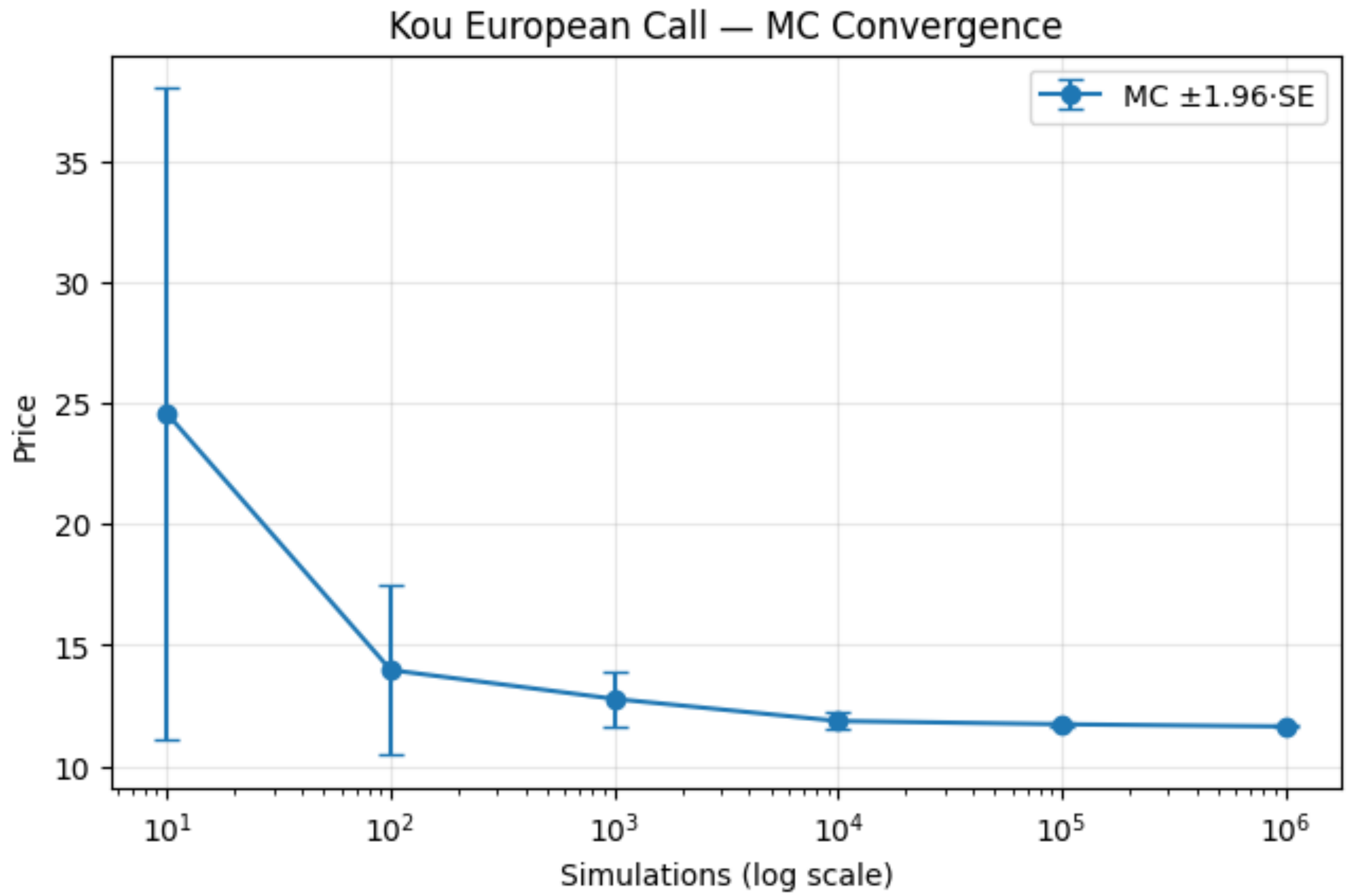
$$\widehat{C_T}(\xi) = \int_{-\infty}^{\infty} e^{-\xi k} C_T(k) \mathrm{d}k = e^{-rT} \frac{S_0^{\xi+1}}{\xi(\xi+1)} \exp\big(G(\xi+1)T\big),$$

where G is the cumulant generating function of $(\ln S_T - \ln S_0)/T$, given by

$$G(\theta) = \theta\left(r - \lambda\zeta - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\sigma^2\theta^2 + \lambda\left(p\frac{\eta_1}{\eta_1 - \theta} + (1-p)\frac{\eta_2}{\eta_2 + \theta} - 1\right), \quad \theta \in (-\eta_2, \eta_1)$$

Have fast and accurate algorithms for its inversion [Pet04, KPW05]

Comparison of option pricing under Kou (Laplace transform), Monte-Carlo, Black-Scholes.

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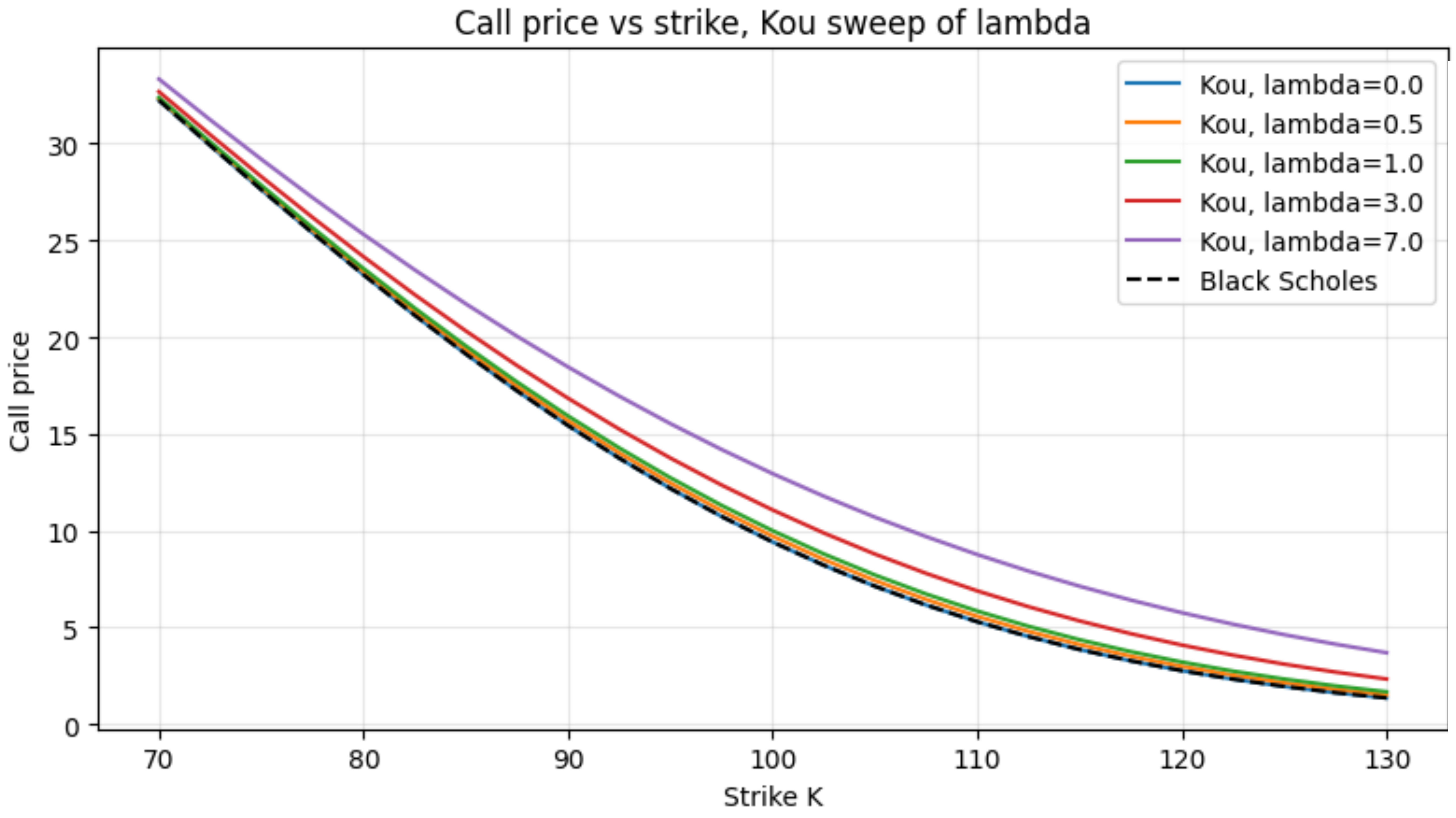
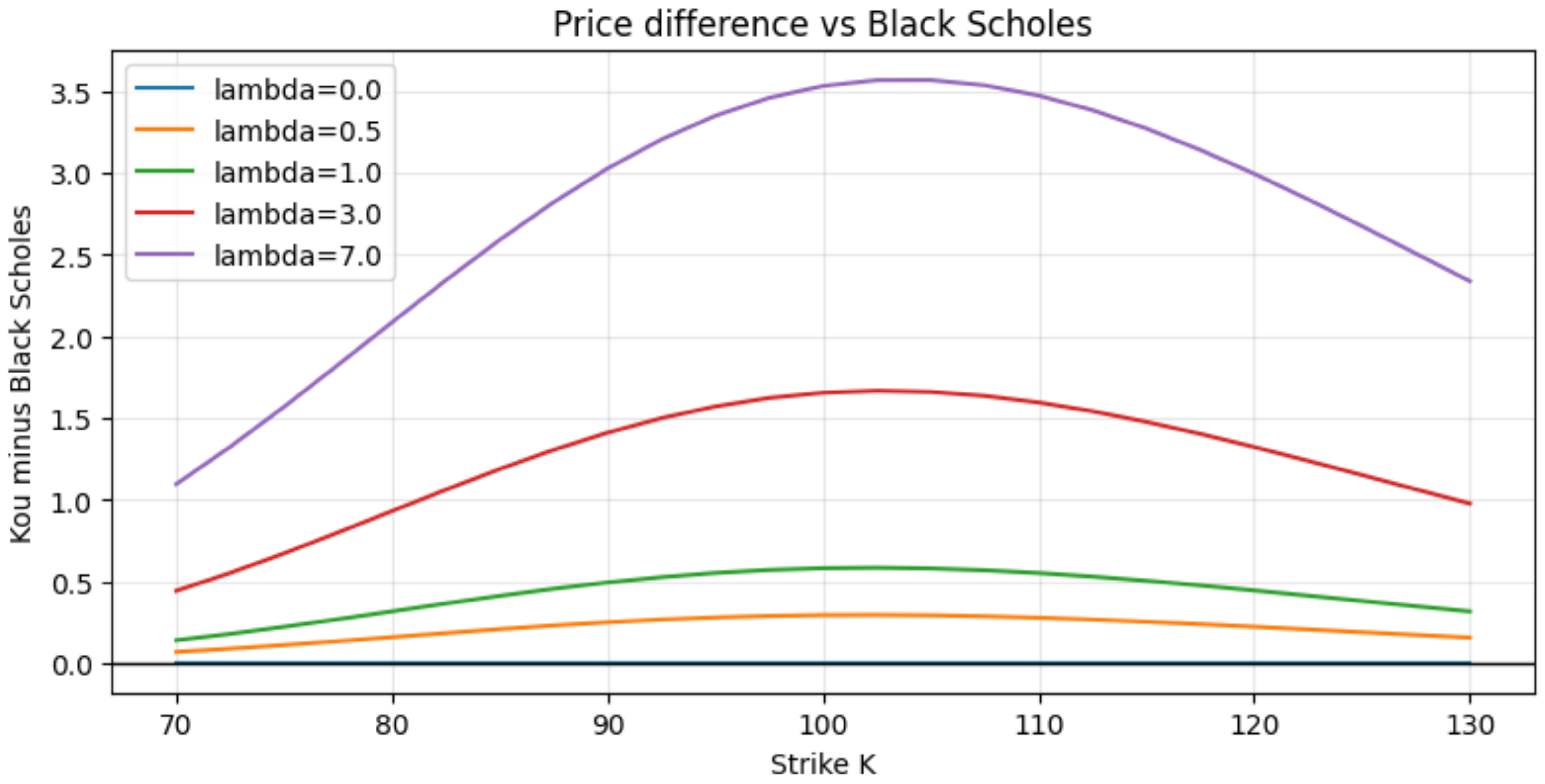
Monte-Carlo price (Kou_path): Number of similations=10  ->  Call estimated=24.561926  (Standard Error=6.849805)
Monte-Carlo price (Kou_path): Number of similations=100  ->  Call estimated=13.969964  (Standard Error=1.790611)
Monte-Carlo price (Kou_path): Number of similations=1,000  ->  Call estimated=12.779124  (Standard Error=0.580427)
Monte-Carlo price (Kou_path): Number of similations=10,000  ->  Call estimated=11.865646  (Standard Error=0.178497)
Monte-Carlo price (Kou_path): Number of similations=100,000  ->  Call estimated=11.732102  (Standard Error=0.055937)
Monte-Carlo price (Kou_path): Number of similations=1,000,000  ->  Call estimated=11.640647  (Standard Error=0.017580)
Price under Kou-Laplace transform: 11.625124
Price under Black-Scholes: 9.413403
```

Comparison Continued: call price vs strike plots under Kou and Black-Scholes with same parameters S_0, T, r, σ.

Choose $p = 0.6$, $\eta_1 = 25$, $\eta_2 = 12.5$: tend to have more upward jumps and upward jump size is larger than the downward jump size





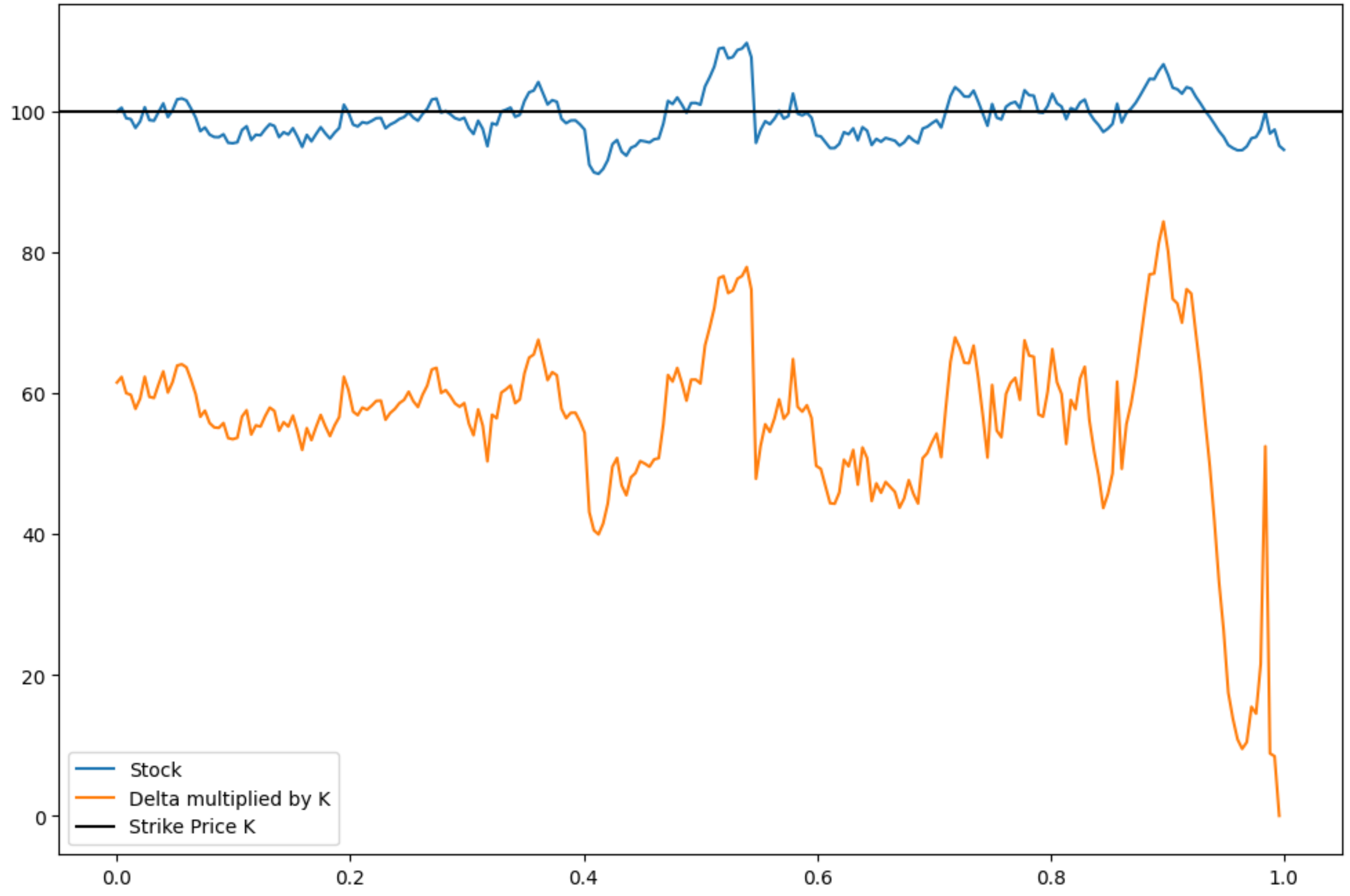
Delta of the call option

$$\Delta(C_T(k)) = \frac{\partial}{\partial S_0} C_T(k) = L_\xi^{-1}\left(e^{-rT}\frac{S_0^\xi}{\xi}\exp(G(\xi+1)T)\right)$$

where L_ξ^{-1} denotes the Laplace inversion with respect to ξ

- Sample of computing delta and plot how delta varies over time:

	T	S0	K	KouDelta	BSDelta	DeltaDifference
0	0.25	90.0	90.0	0.553073	0.539828	0.013246
1	0.25	90.0	100.0	0.188402	0.170142	0.018260
2	0.25	90.0	110.0	0.035344	0.028279	0.007065
3	0.25	100.0	90.0	0.867111	0.875669	-0.008558
4	0.25	100.0	100.0	0.553073	0.539828	0.013246
5	0.25	100.0	110.0	0.215863	0.196801	0.019061
6	0.25	110.0	90.0	0.970465	0.982429	-0.011964
7	0.25	110.0	100.0	0.846817	0.853853	-0.007036
8	0.25	110.0	110.0	0.553073	0.539828	0.013246
9	1.00	90.0	90.0	0.590586	0.579260	0.011326
10	1.00	90.0	100.0	0.396201	0.371909	0.024293
11	1.00	90.0	110.0	0.237530	0.210885	0.026644
12	1.00	100.0	90.0	0.761450	0.766327	-0.004876
13	1.00	100.0	100.0	0.590586	0.579260	0.011326
14	1.00	100.0	110.0	0.414532	0.391062	0.023469
15	1.00	110.0	90.0	0.871534	0.885580	-0.014046
16	1.00	110.0	100.0	0.747161	0.750655	-0.003494
17	1.00	110.0	110.0	0.590586	0.579260	0.011326



Hedging Strategies in Incomplete Markets

Hedge risk of selling a call option by trading in assets $X_t = (X_t^1, \cdots, X_t^m)$ with hedge ratio $\phi_t = (\phi_t^1, \cdots, \phi_t^m)$, terminal payoff Y, initial capital c

- Delta Hedging: single asset $X_t = S_t$ and $\phi_t = \Delta(C_t(k))$
- Local Risk-Minimizing (LRM): single asset $X_t = S_t$ and take ϕ_t that 'locally' minimizes the expected quadratic hedging error.

$$\mathbb{E}[\big(c - Y + \sum_{i=1}^{m} \int_0^T \phi_t^i dX_t^i\big)^2]$$

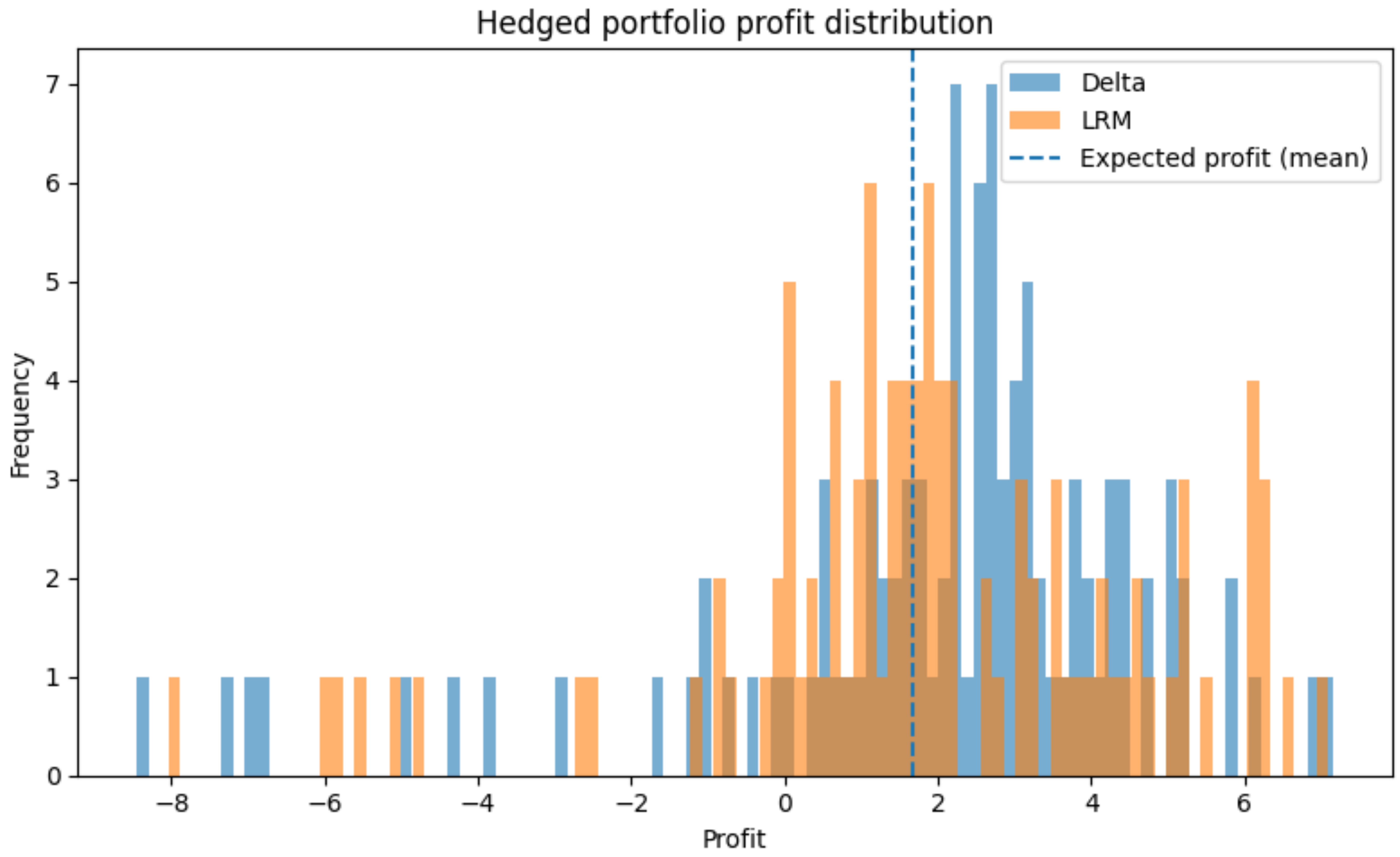
— Closed formula for ϕ_t is obtained in [GT96] (orthogonal projection)

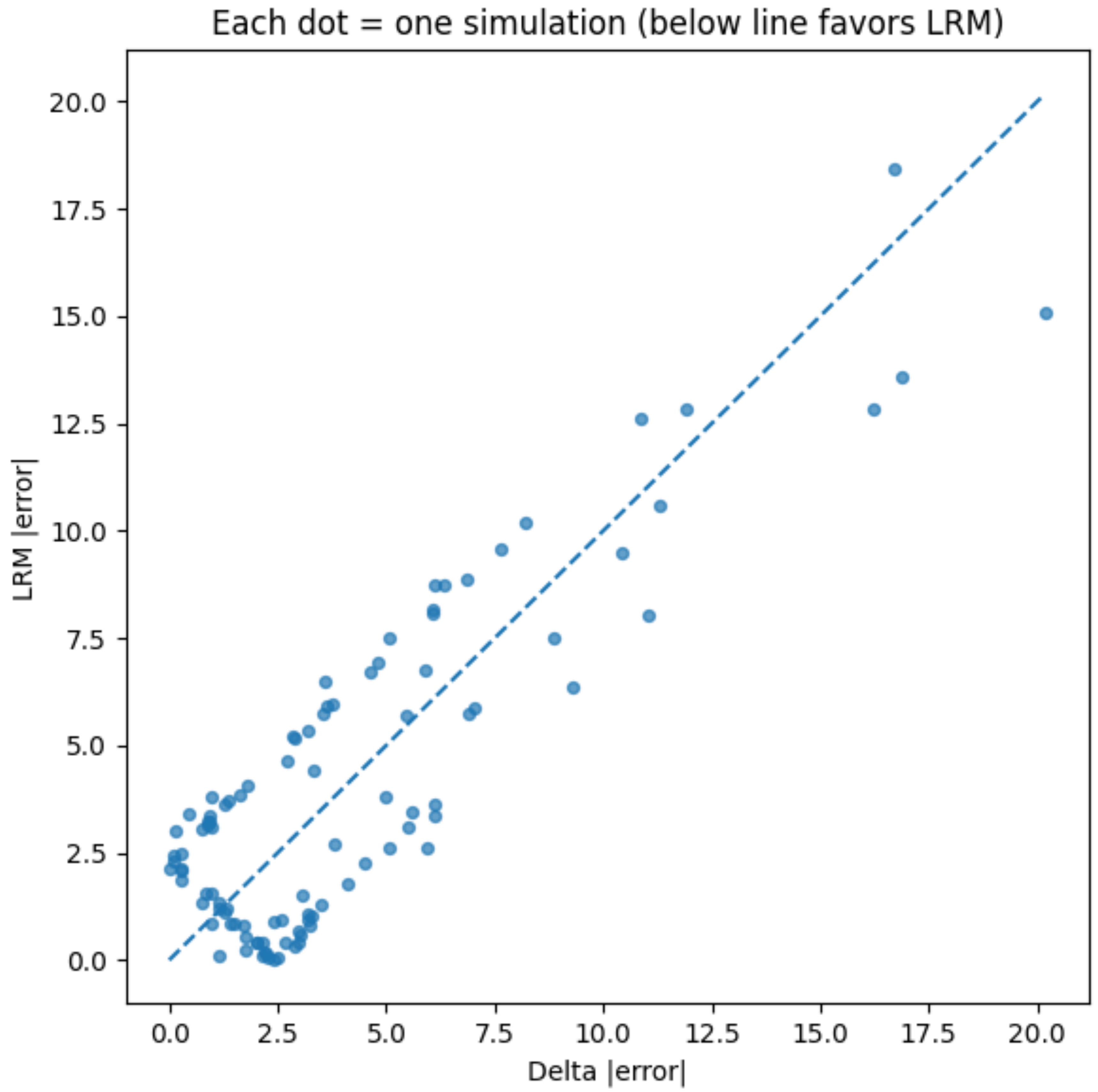
- Hedge with Options: two assets $X_t^1 = S_t$, $X_t^2 = D_T(k)$ another call option, take ϕ_t that minimizes the expected quadratic hedging error.

— Closed formula for ϕ_t is obtained in [CTV07]

Hedging Experiment

- Portfolio profit distribution with 252 delta and LRM hedges across 100 simulations





Interpretation: delta hedging only incorporates the diffusive part while LRM hedges both diffusion and jumps partly.

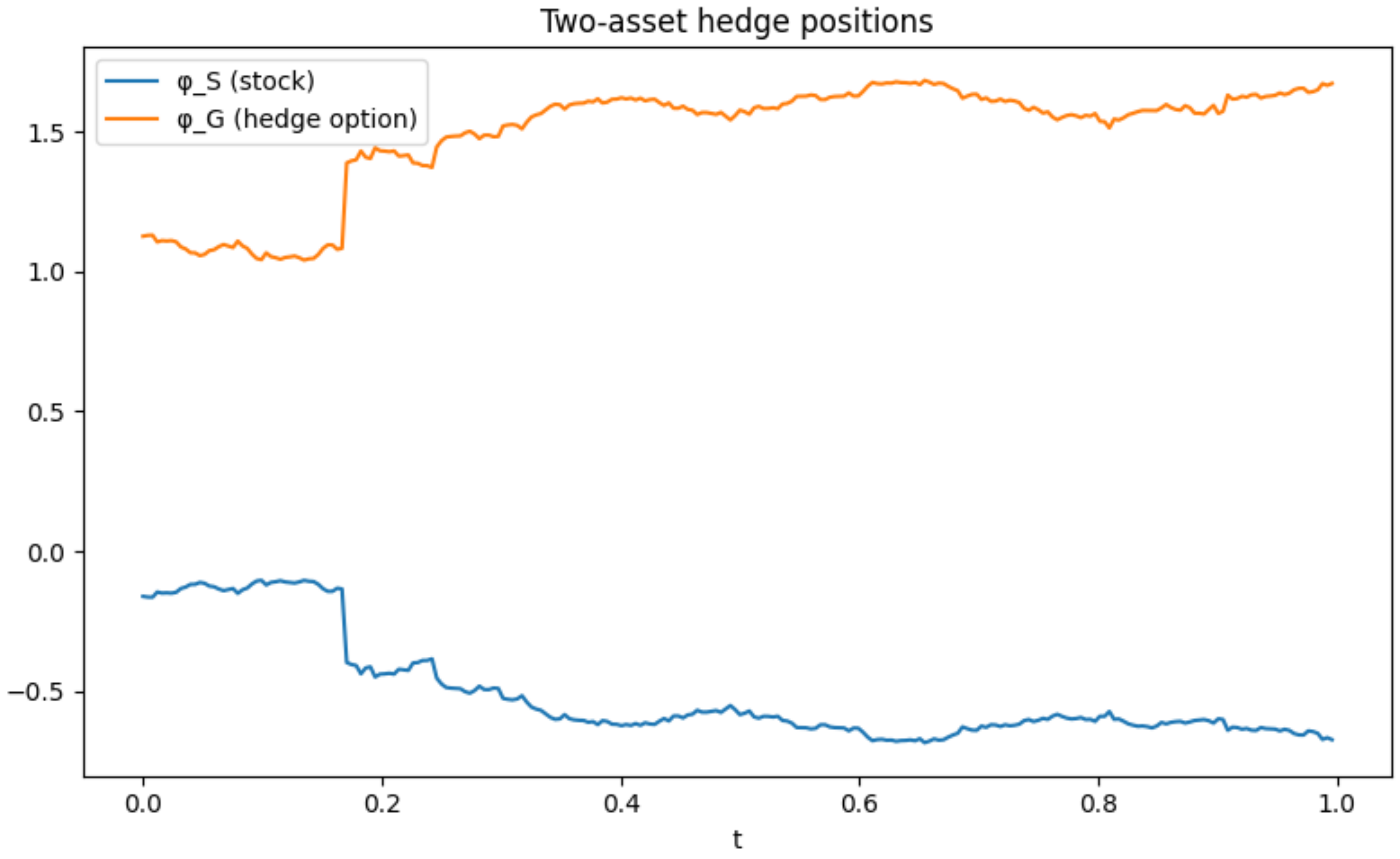
	mean	std
Delta	1.973439	2.886406e+00
LRM	1.790167	2.842087e+00
No hedge	7.318068	2.488567e+01
Expected Profit	1.669261	4.463264e-16

Hedging Experiment

- Simulate a stock path using parameters

$$S_0 = 100,\ T = 1,\ \sigma = 0.2,\ \lambda = 3,\ p = 0.8,\ \eta_1 = 15,\ \eta_2 = 20$$

- Optimal quadratic hedging with stock and another European call option with $K = 95,\ T = 1$. Plot its hedge ratios $\phi_t = (\phi_t^1, \phi_t^2)$:



- Hedging performance comparison:

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Delta profit : -5.3856
LRM profit   : -3.4543
2-asset profit: 1.6835
Expected profit (pricing diff): 1.6142
```

Calibration

- Import two-year data of SPY and plot its one-year stock path

Price	Close	High	Low	Open	Volume
Ticker	SPY	SPY	SPY	SPY	SPY
Date					
2023-11-08	426.234192	427.053026	423.914147	426.526622	61746000
2023-11-09	422.910126	427.423485	422.481208	427.384485	83174400
2023-11-10	429.509521	429.821467	422.900334	424.996192	89462200
2023-11-13	429.100098	430.211362	427.374701	428.164292	52236100
2023-11-14	437.425018	438.721498	434.851514	435.075731	97176900
...	...	...	...	...	...
2025-11-03	683.340027	685.799988	679.940002	685.669983	57315000
2025-11-04	675.239990	679.960022	674.580017	676.109985	78427000
2025-11-05	677.580017	680.859985	674.169983	674.979980	74402400
2025-11-06	670.309998	677.380005	668.719971	676.469971	85035300
2025-11-07	670.969971	671.070007	661.205017	667.909973	91404192

502 rows × 5 columns

Calibration

- Use gradient descent to find parameters of Kou's pricing model to best match SPY market price.

– Initial parameters: $\sigma = 0.2$, $\lambda = 3$, $p = 0.8$, $\eta_1 = 15$, $\eta_2 = 20$

Hedging result:

```
Delta-hedged: call payout discounted - total stock profit:  48.2260
LRM-hedged: call payout discounted - total stock profit:  47.9417
Kou call price     :  53.8259
```

– Calibrated parameters: $\sigma = 0.05$, $\lambda = 5.56$, $p = 0.01$, $\eta_1 = 6.38$, $\eta_2 = 19.53$

Backtest hedging result:

```
Delta-hedged: call payout discounted - total stock profit:  49.9480
LRM-hedged: call payout discounted - total stock profit:  54.2886
Kou call price     :  57.2624
```

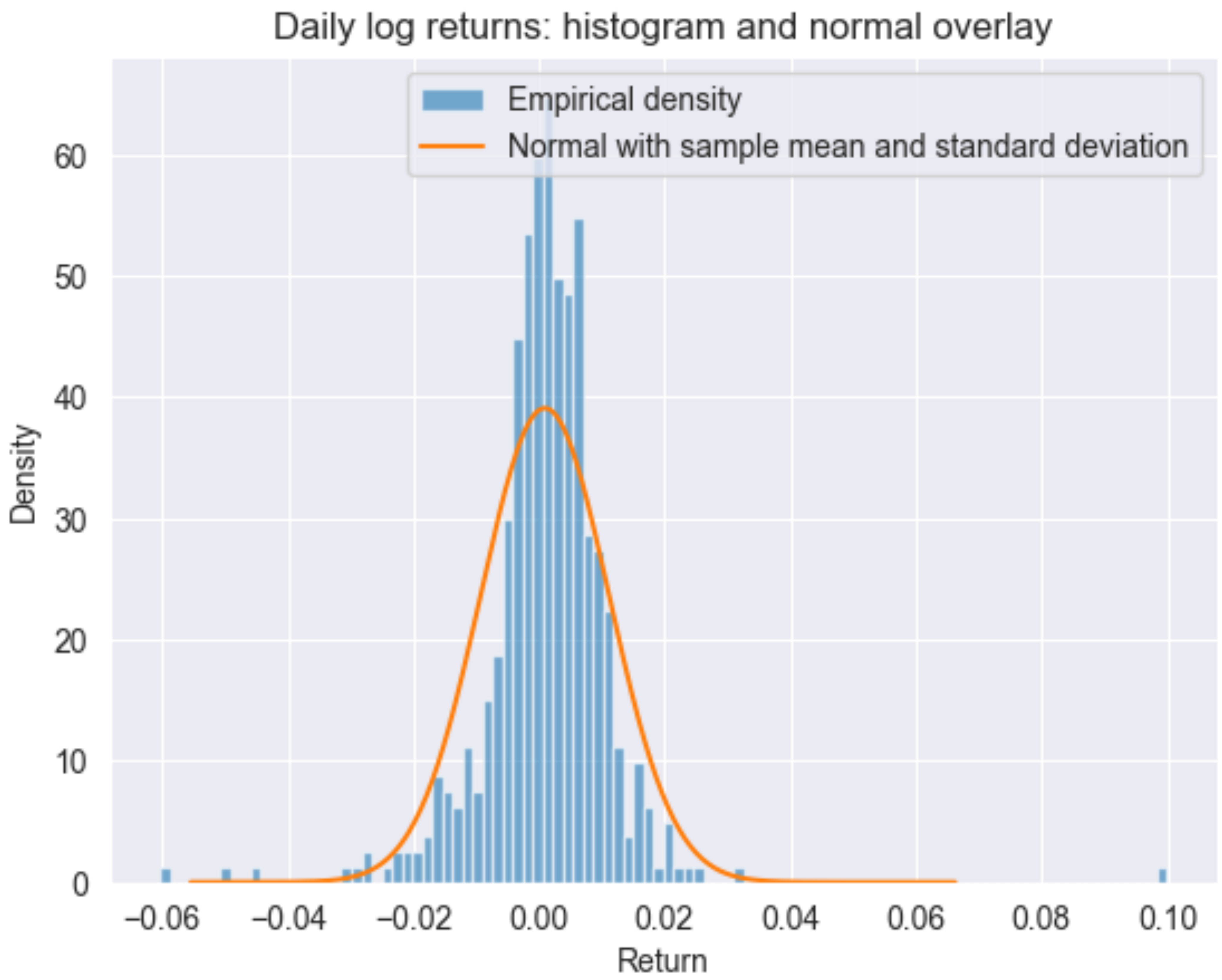
Calibration

- Leptokurtosis test for daily log returns

$$\text{Kurtosis: } K = \mathbb{E}[\frac{(X-\mu)^4}{\sigma^4}] \qquad \text{Skewness: } S = \mathbb{E}[\frac{(S-\mu)^3}{\sigma^3}]$$

Leptokurtic if $K > 3$: higher peak and two heavier tails than those of the normal distribution, e.g. double exponential distribution

SPY:

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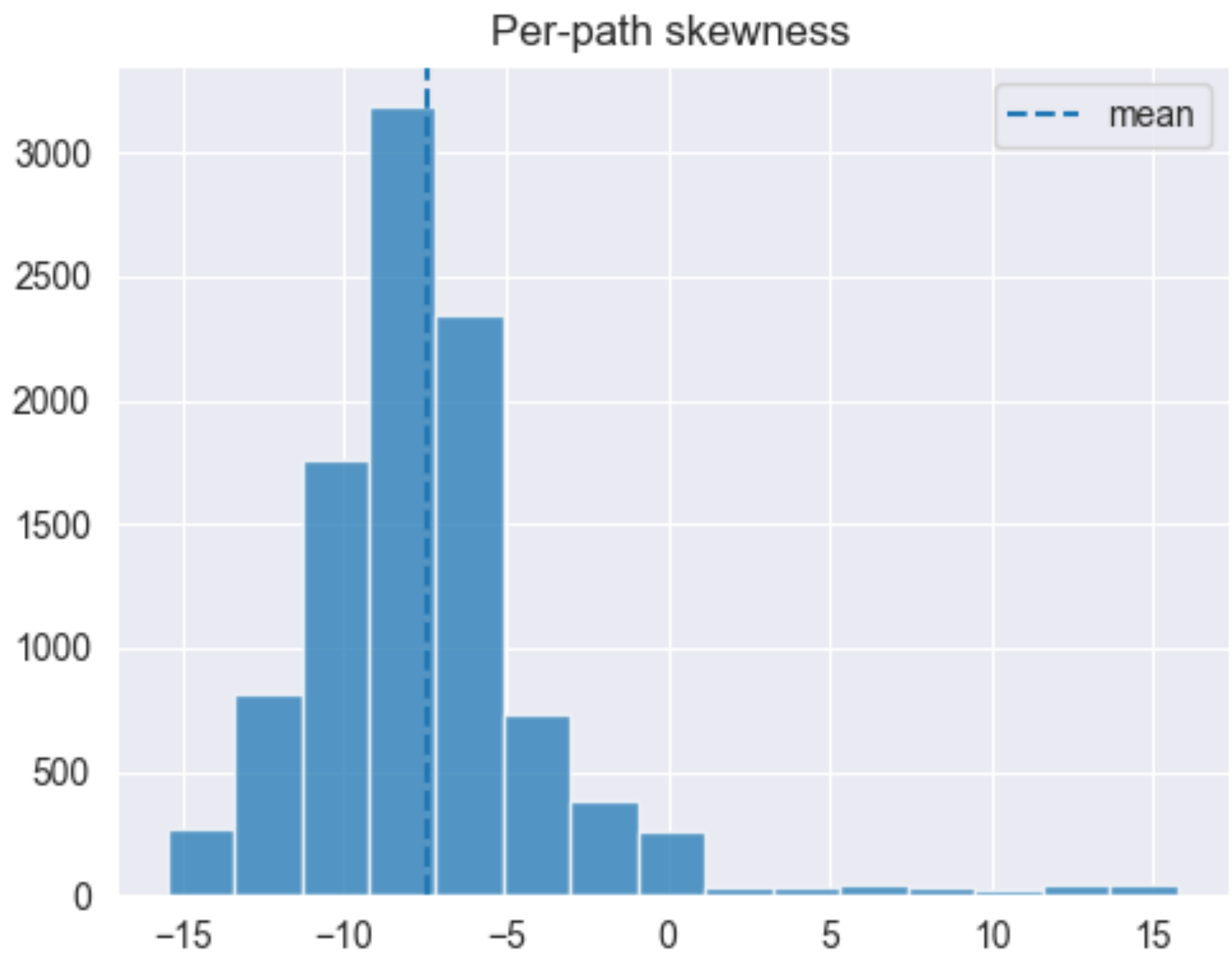
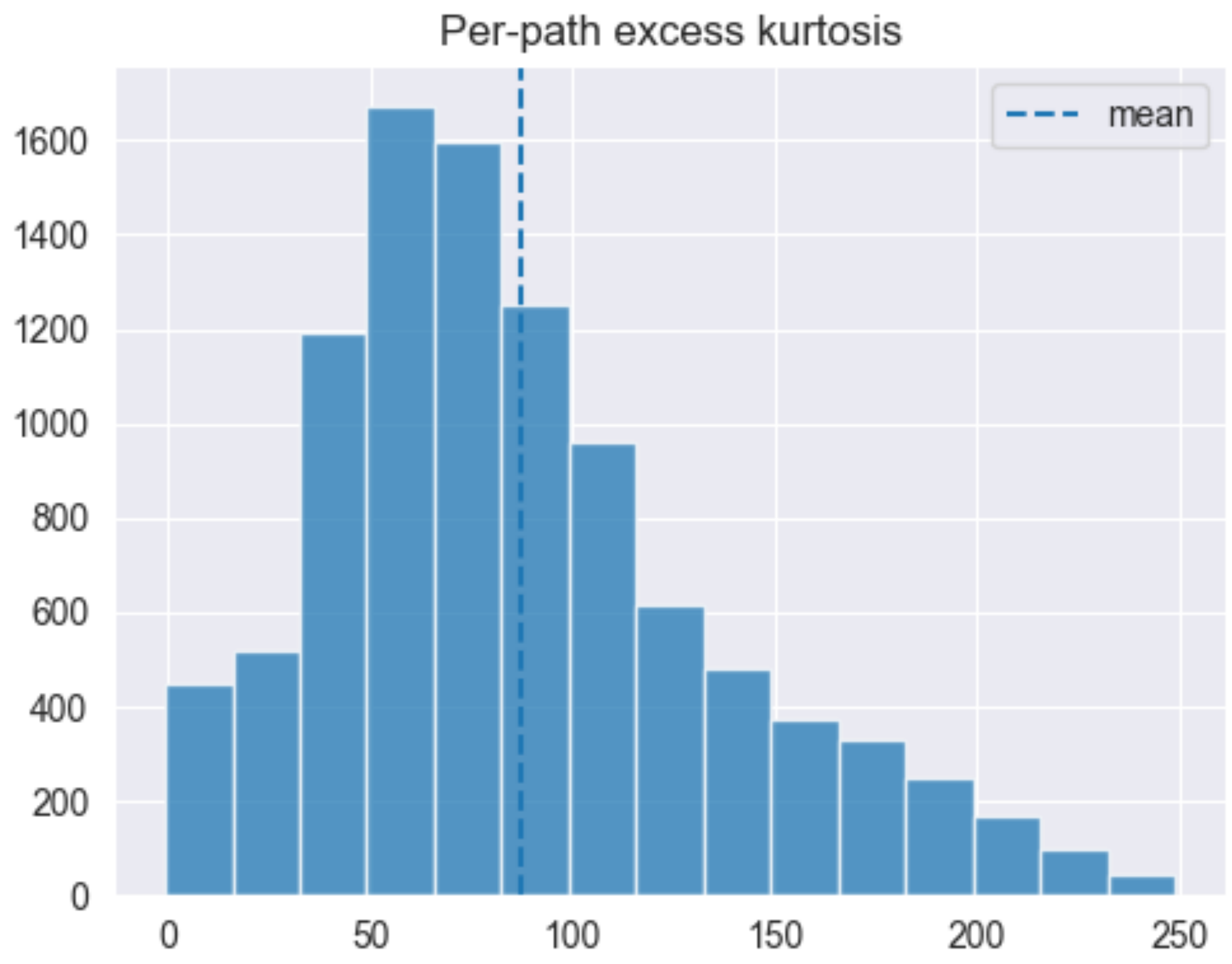
Sample mean: 0.000906
Sample standard deviation: 0.010207
Excess kurtosis (positive means leptokurtic): 21.622
Skewness: 0.773
Jarque–Bera test statistic: 9604.90, p-value: 0
```

Calibration

- Leptokurtosis test for daily log returns of 10000 simulated stock paths under Kou's model with calibrated parameters
  - Distribution of skewness and excess kurtosis $K - 3$:



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Paths: 10000, Steps per path: 252
Average skewness across paths:        -7.4193  (SE 0.0391)
Average excess kurtosis across paths:  87.0543  (SE 0.4913)
```

It captures the leptokurtic feature but

- the kurtosis is too large
- skewed in an opposite way

References

- [CTV07] Cont, R., Tankov, P., & Voltchkova, E. (2007). Hedging with options in models with jumps. In *Stochastic analysis and applications: The Abel Symposium 2005* (pp. 197-217). Berlin, Heidelberg: Springer Berlin Heidelberg.

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Thank You!