EEC303 2022/23.

a) i) $X(z) = \sum_n x(n) z^{-n}$.

$y(k) = \{1, 0, 0, -0.1, 1, 0, \ldots 0\}$.

$Y(z) = 1 - 0.1 z^{-3} + z^{-4}$.

ii) $w = 100\pi$. $f = \frac{w}{2\pi} = 50\,Hz$. $f_s \geq 2f = 100\,Hz$.

$100\,Hz > 60\,Hz$ unsuitable.

iii). $\cancel{\frac{F(z)}{z} = \frac{z+1}{z(z+0.2)(}}$ $\frac{F(z)}{z} = \frac{z+1}{z(z+0.1)(z+0.2)} = \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2}$

① $z=0$. $0+1 = A \cdot 0.1 \cdot 0.2 + 0 + 0 \Rightarrow A = 50$.

② $z=-0.1$ $0.9 = 0 + B(-0.1) \cdot 0.1 + 0 \Rightarrow B = \frac{0.9}{-0.01} = -90$

③ $z=-0.2$ $0.8 = 0 + 0 + C(-0.2)(-0.1) \Rightarrow C = \frac{0.8}{0.02} = 40$.

$\therefore \cancel{F(z)}$ $\therefore f(x) = 50\delta(x) - 90(-0.1)^x u(x) + 40(-0.2)^x u(x)$.

b) $f(k) = \begin{cases} 4 & k = 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$ $\because x(n-k) \overset{Z}{\longleftrightarrow} z^{-k} X(z)$

$\therefore f(k-2) \overset{Z}{\longleftrightarrow} z^{-2} F(z)$.

$Z\{f(k-2)\} \cancel{F(z)} = \sum_{k-2=0}^{\infty} f(k-2) z^{-(k-2)}$.

$= \sum_{k-2=0}^{\infty} 4 z^{-(k-2)}$.

$\because \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ $Z\{f(k)\} = z^2 Z\{f(k-2)\}$.

$\therefore \sum_{k-2=0}^{\infty} 4 z^{-(k-2)} = \frac{4}{1-z^{-1}}$.

$\therefore F(z) = z^2 \cdot Z\{f(k-2)\} = \frac{4z^2}{1-z^{-1}}$

c) $x(k) - a x(k-1) = u(k)$.

i) $Z\{x(k) - a x(k-1)\} = Z\{u(k)\}$

$X(z) - a \cdot z^{-1} X(z) = \frac{z}{z-1}$

$X(z)(1 - a z^{-1}) = \frac{z}{z-1}$

$X(z) = \frac{z}{z-1} \cdot \frac{1}{1-az^{-1}} = \frac{z}{(z-1)(1-az^{-1})} = \frac{z^2}{(z-1)(z-a)}$

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$X(z) = \frac{z^2}{(z-1)(z-a)}$ $\frac{X(z)}{z} = \frac{z^2}{z(z-1)(z-a)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-a}$

① $z=0$. $z^2 = 0 = A \cdot (-1)(-a) \Rightarrow A = 0$.

② $z=1$ $z^2 = 1 = 0 + B \cdot 1 \cdot (1-a) \Rightarrow B = \frac{1}{1-a}$

③ $z=a$ $z^2 = a^2 = 0 + 0 + C \cdot a \cdot (a-1) \Rightarrow C = \frac{a^2}{a(a-1)} = \frac{a}{a-1}$

$\therefore X(z) = \frac{1}{1-a} \frac{z}{z-1} + \frac{a}{a-1} \frac{z}{z-a}$.

$x(k) = \frac{1}{1-a}(1)^k u(k) + \frac{a}{a-1}(a)^k u(k) = \frac{1}{1-a} u(k) + \frac{a}{a-1} \cdot a^k u(k)$

iii) final value theorem: $x_{final} = \lim_{\cancel{z} \to 1} (z-1) X(z)$.

$x(k)_{final} = \lim_{z\to 1} (z-1) X(z) = \lim_{z \to 1} (z-1) \cdot \frac{z^2}{(z-1)(z-a)} = \frac{1}{1-a}$

2, a) characteristic equation: $z^2 - z + 0.25 + k = 0$.

$b^2 - 4ac = 1 - 4(0.25+k)$ $z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

① $k=0$. $z^2 - z + 0.25 = 0$. $z = \frac{1}{2} = 0.5$.

② $k=0.25$ $z^2 - z + 0.5 = 0$ $z_1 = \frac{1}{2} + \frac{1}{2}i$ $z_2 = \frac{1}{2} - \frac{1}{2}i$

③ $k=0.5$ $z^2 - z + 0.75 = 0$. $z_1 = \frac{1}{2} + \frac{\sqrt{2}}{2}i$ $z_2 = \frac{1}{2} - \frac{\sqrt{2}}{2}i$. $|z| = \frac{\sqrt{3}}{2} = 0.866 < 1$

④ $k=0.75$. $z^2 - z + 1 = 0$. $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ $z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. $|z| = 1$.

⑤ $k = \cancel{0.1}\,1$ $z^2 - z + 1.25 = 0$. $z_1 = \frac{1}{2} + i$ $z_2 = \frac{1}{2} - i$. $|z| = \frac{\sqrt{5}}{2} > 1$

between 0.75 and 1.

b) $\cancel{ZOH = \frac{1-e^{-2s}}{s}}$ $ZOH(s) = \frac{1-e^{-sT}}{s}$ $T = 1$ second $ZOH(s) = \frac{1-e^{-s}}{s}$.

$P(z) = \mathcal{Z}\left\{\frac{1-e^{-s}}{s} \cdot P(s)\right\} = (1-z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\} = (1-z^{-1}) \cdot \mathcal{Z}\left\{\frac{e^{-2s}}{s(2s+1)}\right\}$.

$\mathcal{Z}\left\{\frac{e^{-2s}}{s(2s+1)}\right\} = \mathcal{Z}\left\{\frac{e^{-2s}}{2s^2+s}\right\} = \frac{1}{2}\mathcal{Z}\left\{\frac{e^{-2s}}{s^2+\frac{1}{2}s}\right\} = \frac{1}{2} \cdot z^{-2} \mathcal{Z}\left\{\frac{1}{\cancel{s^2+\frac{1}{2}s}}\right\}$ $s(s+\frac{1}{2})$

$= z^{-2} \mathcal{Z}\left\{\frac{\frac{1}{2}}{s(s+\frac{1}{2})}\right\} = z^{-2} \cdot \frac{z(1-e^{-\frac{1}{2}})}{(z-1)(z-e^{-\frac{1}{2}})}$

$\therefore P(z) = (1-z^{-1}) \cdot z^{-2} \cdot \frac{z(1-e^{-\frac{1}{2}})}{(z-1)(z-e^{-\frac{1}{2}})} = \frac{z-1}{z} \cdot z^{-2} \cdot \frac{z(1-e^{-\frac{1}{2}})}{(z-1)(z-e^{-\frac{1}{2}})}$

$= z^{-2} \frac{1-e^{-\frac{1}{2}}}{z-e^{-\frac{1}{2}}}$

c) $A = \begin{bmatrix} -\frac{1}{R_1C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix}$ $B = \begin{bmatrix} \frac{1}{R_1C} \\ \frac{1}{L} \end{bmatrix}$

uncontrollable: $|Q_M| = |[B \quad AB]| = 0$.

$AB = \begin{bmatrix} -\frac{1}{R_1^2C^2} \\ -\frac{R_2}{L^2} \end{bmatrix}$ $|Q_M| = |[B \quad AB]| = \left|\begin{bmatrix} \frac{1}{R_1C} & -\frac{1}{R_1^2C^2} \\ \frac{1}{L} & -\frac{R_2}{L^2} \end{bmatrix}\right| = \frac{-R_2}{R_1CL^2} + \frac{1}{R_1^2C^2L} = 0$

$R_1^2R_2C^2L = R_1CL^2$

$\therefore R_1R_2 = \frac{L}{C}$

②

$f(x,y,z)=\frac{1}{2}(x^2+y^2+z^2)$

$\mathcal{L}f(x,y,z,\lambda_1,\lambda_2)=\frac{1}{2}(x^2+y^2+z^2)+\lambda_1(x-y)+\lambda_2(x+y+z-1)$

$\therefore \frac{\partial \mathcal{L}}{\partial x}=x+\lambda_1+\lambda_2=0.$ ①　　①②$\Rightarrow \begin{cases} x+y+2\lambda_2=0. \\ x-y+2\lambda_1=0. \end{cases}$

$\frac{\partial \mathcal{L}}{\partial y}=y-\lambda_1+\lambda_2=0.$ ②　　④$\Rightarrow \begin{cases} 2x+2\lambda_2=0 \\ 2\lambda_1=0 \end{cases} \Rightarrow \begin{matrix} x=-\lambda_2 \\ \lambda_1=0. \end{matrix}$

$\frac{\partial \mathcal{L}}{\partial z}=z+\lambda_2=0.$ ③　　$\because$ ③ $\Rightarrow z=-\lambda_2$ $\therefore x=z=y.$

$\frac{\partial \mathcal{L}}{\partial \lambda_1}=x-y=0.$ ④　　in ⑤ $\Rightarrow 3x-1=0$ $\therefore x=y=z=\frac{1}{3}$

$\frac{\partial \mathcal{L}}{\partial \lambda_2}=x+y+z-1=0.$ ⑤　　$\therefore \lambda_2=-\frac{1}{3}.$

3. TF: $\frac{Y(z)}{R(z)}=T(z)=\cancel{C(z)\cdot P(z)}\ \frac{C(z)\cdot P(z)}{1+C(z)\cdot P(z)}.$　　$C(z)=k\cancel{\frac{z}{z-1}}$

$P(z)=\mathcal{Z}\{ZOH(s)\cdot P(s)\}.$

$P(z)=\mathcal{Z}\left\{\frac{1-e^{-sT}}{s}\cdot P(s)\right\}=(1-z^{-1})\mathcal{Z}\left\{\frac{P(s)}{s}\right\}=(1-z^{-1})\mathcal{Z}\left\{\frac{1}{s^2(s+1)}\right\}.\ \cancel{=(1-z^{-1})}$

$=(1-z^{-1})\left(\frac{z[e^{-1}z+1-e^{-1}-e^{-1}]}{(z-1)^2(z-e^{-1})}\right)=\frac{z-1}{z}\ \frac{z\cdot(e^{-1}z-2e^{-1}+1)}{(z-1)^2(z-e^{-1})}$

$\cancel{=\frac{1}{(z-1)(z-e^{-1})}=\frac{1}{z^2-(1+e^{-1})z+e^{-1}}}\ \frac{e^{-1}(z-2)+1}{(z-1)(z-e^{-1})}=\frac{e^{-1}(z-2)+1}{z^2-(1+e^{-1})z+e^{-1}}$

$\cancel{\therefore C(z)P(z)=\frac{kz}{(z-1)}\ \frac{1}{(z-1)(z-e^{-1})}=\frac{k(z)}{(z-1)^2(z-e^{-1})}}$

$\cancel{\therefore T(z)=\frac{kz}{(z-1)^2(z-e^{-1})}\cdot\frac{(z-1)^2(z-e^{-1})}{(z-1)^2(z-e^{-1})+kz}}=\frac{kz}{(z-1)^2(z-e^{-1})+kz}\ \cancel{=\frac{kz}{z^2-(1+e^{-1})z+e^{-1}+kz}}$

$\cancel{e^{-1}=0.3679\Rightarrow 1+e^{-1}=1.3679}\quad \cancel{=\frac{kz}{(z^2-2z+1)(z-e^{-1})+kz}}$

$\cancel{\therefore T(z)}\ \frac{k(e^{-1}(z-2)+1)}{z^2-(1+e^{-1})z+e^{-1}}=C(z)P(z)\quad \cancel{=\frac{kz}{z^3-2z^2+z-e^{-1}z^2-2e^{-1}z-e^{-1}+kz}}$

$\cancel{=\frac{k}{z^2-(2+e^{-1})z+1-2e^{-1}+k-e^{-1}z^{-1}}}$

$T(z)=\frac{C(z)P(z)}{1+C(z)P(z)}=\frac{ke^{-1}(z-2)}{z^2-(1+e^{-1})z+e^{-1}}\cdot\frac{z^2-(1+e^{-1})z+e^{-1}}{z^2-(1+e^{-1})z+e^{-1}+k(e^{-1}(z-2)+1)}$

$=\frac{ke^{-1}(z-2)}{z^2+z(ke^{-1}-1-e^{-1})\cancel{+e^{-1}k}+e^{-1}+(1-2e^{-1})k}$　　$\therefore$ Ch eq: $z^2+z(ke^{-1}-1-e^{-1})\cancel{+e^{-1}k}+e^{-1}+(1-2e^{-1})k=0.$

$e^{-1}=0.3679.\quad 1+e^{-1}=1.3679.\quad 1-2e^{-1}=0.26424$

$\therefore z^2+z(0.3679k-1.3679)+0.2642k+0.3679=0.$

b) Jury test: $P(z)=a_nz^n+a_{n-1}z^{n-1}+\dots+a_1z+a_0,\ a_n>0.$

① $|a_0|<a_n.$　$a_0=0.2642k+0.3679$　$a_n=0.3679k-1.3679.\Rightarrow |0.2642k+0.3679|<0.3679k-1.3679.$

② $P(z)|_{z=1}>0.$ $P(z)|_{z=1}=1+0.3679k-1.3679+0.2642k+0.3679=0.6321k>0\Rightarrow k>0.$

③ $(-1)^nP(z)|_{z=-1}>0.$ $(-1)^2P(z)|_{z=-1}>0.\Rightarrow P(z)|_{z=-1}>0.$

④ $|b_0| > |b_{n-1}|$

$|c_0| > |c_{n-2}|$

$\vdots$

$|q_0| > |q_2|$

$b_0 = a_0 \times a_0 - a_n \times a_n$

$b_1 = a_0 \times a_1 - a_n \times a_{n-1}$

$c_0 = b_0 \times b_0 - b_{n-1} \times b_{n-1}$

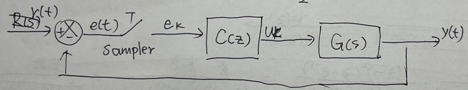
Row	z^0	z^1	z^2
1	$0.2642k+0.3679$	$0.3679k-1.3679$	$1.$
2.	$1.$	$0.3679k-1.3679.$	$0.2642k+0.3679.$
3.	$(0.2642k+0.3679)^2-1$	$(0.2642k+0.3679)\times(0.3679k-1.3679)-(0.3679k-1.3679)\times(0.2642k+0.3679)=0.$	
4.	0	$(0.2642k+0.3679)^2-1$	
5.	$\left[(0.2642k+0.3679)^2-1\right]^2$		

$|b_0| > |b_{n-1}| = 0. \Rightarrow (0.2642k+0.3679)^2 - 1 > 0.$

c) i) $G(s) = \dfrac{k}{s(s+1)}$ TF: $T(s) = \dfrac{k}{s(s+1)} \cdot \dfrac{s(s+1)}{s(s+1)+k} = \dfrac{k}{s(s+1)+k}$

$\therefore$ poles: $s(s+1)+k=0.$ $s^2+s+k=0.$ $b^2-4ac = 1-4k$

$\therefore s = \dfrac{-1 \pm \sqrt{1-4k}}{2}$

ii)


The continuous-time system is converted into a discrete-time system. If the sampling is too low, ~~alisa~~ aliasing can occur, leading to instability or degraded performance. The process of sampling and quantization can introduce noise and errors.

Introducing a sample and hold process can offer benifits such as easiler implementation of digital controllers.

d) steepest descent: $X^{k+1} = X^k - \lambda \nabla f(X^k)$

$\nabla f(x_1,x_2) = \begin{bmatrix} 1+4x_1+2x_2 \\ -1+2x_1+2x_2 \end{bmatrix}$ $X_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \nabla f(0,0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$(\lambda^k)^* = \dfrac{(P^k)^T \cdot P^k}{(P^k)^T A \cdot P^k}$.

$A = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} = \nabla^2 f(X^k)$

$\therefore X' = X^0 - \lambda \nabla f(X^0)$

$(\lambda^0)^* = \dfrac{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \dfrac{1+1}{\begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \dfrac{2}{2} = 1$

$\therefore X' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$X^2 = X' - \lambda \nabla f(X') = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4.a) i) $\nabla f(x) = \begin{pmatrix} x_1^2 - 4 \\ x_2^2 - 4 \end{pmatrix}$ let $\nabla f(x) = 0 \Rightarrow x_1 = \pm 2 \quad x_2 = \pm 2$

stationary point (2, 2) (2, -2) (-2, 2) (-2, -2).

ii). $f(x) = \frac{1}{2} X^T Q X + B^T X + C$.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad Q = \begin{bmatrix} 12 & 6 & -4 \\ 6 & -4 & 0 \\ -4 & 0 & 2 \end{bmatrix} \quad B^T = 0 \quad C = 0.$

$\therefore f(X) = \frac{1}{2} [x_1\ x_2\ x_3] \begin{bmatrix} 12 & 6 & -4 \\ 6 & -4 & 0 \\ -4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

iii) $A = \begin{bmatrix} 10 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 10 \end{bmatrix} \quad a_{11} = 10 > 0.$

$|A| = 10(20-16) - 4(40-32) - 8(-16+16)$
$= 40 - 32 - 0 = 8 > 0.$ positive definiteness.

iv) minimum. necessary. $f'(x) = 0$. sufficient $f''(x) > 0$.

b) i) $X^{k+1} = X^k - A_k^{-1} \nabla f(X^k) \quad A_k^{-1} = [\nabla^2 f(X^k)]^{-1}$

$\cancel{X^{k+1} = X^k - \frac{\nabla f(x)}{f''(x)}}$

$x^{k+1} = x^k - \frac{f'(x^k)}{f''(x^k)}$

$J'(u) = 2\alpha u. \quad J''(u) = 2\alpha.$

$\alpha = 1 \quad J'(u) = 2u \quad J''(u) = 2.$

$u_1 = u_0 - \frac{J'(u_0)}{J''(u_0)} = u_0 - \frac{2u_0}{2} = 0.$

ii) The Newton method requires the calculation of the Hessian ~~matri~~ martix.
For high-dimensional systems, this can be computationly expensive.
It requires second-order derivative information. (the Hessian).