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Question continued.

b) A Liverpool based radio channel provides radio services in most Northern cities in the UK. The channel is planning to expand its service to four cities in the Midlands by designing a new broadcasting network. To provide a good quality service, the channel needs a new transmission tower, which will transmit radio frequency to the pre-existing smaller towers in those cities. The new tower to be constructed can cover areas within a radius of K km. Thus, the new tower must be located within K km of each of those existing towers. The problem is to determine the tower location that minimises the total distance from the new tower to each of the existing towers.

new K km

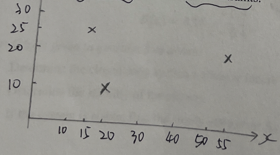
The location of each city can be calculated using two dimensional coordinates (x, y) from a given reference point as follows:

Table Q4

City	x	y
1	10	45
2	15	25
3	20	10
4	55	20

(10, 45)
(15, 25)
(20, 10)
(55, 20)

Formulate the problem stated above as a formal optimisation problem by identifying the variables, the objective function and the constraints. **7**



Total
25

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1. a) i) Find the z-transform of the following discrete-time signal:

$$y(k) = \{1,\ 0,\ 0,\ -0.1,\ 1,\ 0,\ 0,\ \ldots\ 0\}.$$ **2**

ii) In order to design a digital control system, an output signal having frequency bandwidth of 200 Hz needs to be sampled. Can the sampling frequency be chosen as 250 Hz? Explain your answer. **2**

iii) A digital control system features the following z-transfer function,

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$ **7**

Using the partial fraction expansion method, find its inverse z-transform.

b) Find the z-transform of the following causal sequence,

$$f(k) = \begin{cases} 4 & k = 2, 3, \ldots \\ 0 & otherwise \end{cases}$$ **5**

c) i) A discrete-time system is described by the following difference equation,

$$x(k) - ax(k-1) = u(k)$$ **4**

where: $-1 < a < 1$, $x(k) = 0$ for $k < 0$, and $u(k)$ is a unit sampled step signal.

Determine the z-transform, $X(z)$.

ii) Determine the corresponding sequence, $x(k)$. **3**

iii) Using the final value theorem, find the final value of $x(k)$. **2**

Total

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2. a) A DC motor digital control system has been designed using a fast-sampling digital controller for position control. The armature winding inductance can be considered negligible. The controller design uses just proportional control, and its performance is described by the following z-transfer function, where $\theta(z)$ is the motor output position and $V(z)$ is the desired motor position: **6**

$$\theta(z) = \frac{K(z - 0.5)}{z^2 - z + 0.25 + K} V(z)$$

characteristic

$\boxed{\text{Sketch}}$ the discrete-time root locus for K values equal to 0, 0.25, 0.5, 0.75 and 1. Clearly show the unit circle. State between which values of K the system becomes unstable.

b) The transfer function of a control plant is given by, **8**

$$P(s) = \frac{1}{2s+1} e^{-2s}$$

In order to directly design a digital controller, $C(z)$, in the z-domain, the plant, $P(s)$, needs to be discretised as, $P(z)$. $\boxed{\text{Find}}$ the zero-order hold (ZOH) equivalent $P(z)$ of $P(s)$, with a sampling period of $T = 1$ second.

Question continues overleaf.

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Question continued.

c) The state and output equations for an electrical circuit with an input voltage source, u, and an output current, y, are given as, 4

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{R_1C} & 0\\ 0 & -\frac{R_2}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{R_1C}\\ {}^{1}/_{L}\end{bmatrix}u$$

$$y = \begin{bmatrix}-{}^{1}/_{R_1} & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \left({}^{1}/_{R_1}\right)u$$

The capacitor voltage and inductor current are the state variables.

V_C　　I_L.

Determine the relationship between C, L, R_1 and R_2 that will cause the system to be deemed as uncontrollable.

d) An optimisation problem is defined by the following cost function: 7

Minimise $f(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)$

s. t.

and

$$g_1(x,y,z) = x - y = 0$$

$$g_2(x,y,z) = x + y + z - 1 = 0$$

Using the Lagrange multiplier method, find the solution for X (where $X = [x,y,z]^T$) for this optimisation problem.

Total
25

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Question continued.

i) Represent the above equations using state-space form as continuous time state and output equations, where the output is chosen such that $y(t) = i(t)$ and the system input is $u(t) = v(t)$. 7

ii) Given that: 4

$$\dot{x}(t) = A\,x(t) + Bu(t)$$

and

$$y(t) = C\,x(t)$$

show that the open-loop zeros are given by the solutions for s in:

$$\begin{vmatrix} sI - A & -B\\ C & 0\end{vmatrix} = 0$$

iii) Determine the open-loop zeros for the RLC circuit in question i) above. 4

b) Assume that a digital controller is sampling sufficiently fast such that 5

$$s \rightarrow \frac{z-1}{T}$$

where T is the controller sampling period. Use your result from Q3 a) i) above to determine the approximate discrete-time state difference and output equations for the RLC circuit.

~~v(t)=?~~ $y(t) = i(t)$.

c) i) What is the purpose of a cost function in optimisation problems? 1

ii) What is an isopleth? 1

iii) What are the necessary and sufficient conditions to find the minimum and the maximum of a quadratic function? 2

iv) Show, mathematically, the equivalency that exists between the maximisation and minimisation of some objective function, $f(X)$. 1

Total
25

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3. a) RLC components connected in series are supplied by a variable DC voltage. The voltage drop across the inductor, $v_L(t)$, is given by:

$$L\frac{di(t)}{dt} = v_L(t)$$

where $i(t)$ is the loop current.

The voltage drop across the resistor is given by Ohm's law, $v_R(t) = Ri(t)$, and the voltage drop across the capacitor, $v_c(t)$, can be related to the loop current, $i(t)$, and capacitance, C, by,

$$C\frac{dv_c(t)}{dt} = i(t)$$

For a series circuit $v_c(t) + v_L(t) + v_R(t) = v(t)$, where $v(t)$ is the applied voltage as shown in Figure Q3.

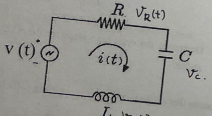

Figure Q3: RLC circuit connected in series

Question continues overleaf.

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4. a) Given a quadratic function:

$$f(X) = -12x_1^2 - 6x_1x_2 - 5x_2^2$$

where $X = [x_1 \quad x_2]^T$

i) Find the gradient vector. **4**

ii) Find the stationary point. **4**

iii) Determine the definiteness of the function. **4**

iv) What is the sufficient condition to determine whether the above function has an extreme point? Should the above function have a maximum or minimum value? **2**

b) i) Show that, in general, Newton's method when applied to a convex quadratic function will converge in exactly one step. **5**

ii) Apply Newton's method to the maximisation of: **4**

$$f(X) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

where $X = [x_1 \quad x_2]^T$.

iii) Describe what is meant by gradient direction in gradient methods of optimisation and what are their limitations in practice? **2**

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Appendix:

Table A1. The Laplace and z-Transform Table

$F(s)$ is the Laplace transform of $f(t)$, and $F(z)$ is the z-transform of $f(kT)$. Note: $f(t) = 0$ for $t < 0$.

Number	$F(s)$	$f(kT)$	$F(z)$
1	1	$\delta(t)$, unit impulse	1
2	$e^{-k_0 Ts}$	$\delta(t-k_0T)$	z^{-k_0} $x(n-k)$ $z^{-k}\cdot X(z)$ time delay
3	$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$ $n\cdot X(n) \Leftrightarrow -z\frac{d}{dz}F(z)$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(kT)^2$	$\frac{T^2}{2}\left[\frac{z(z+1)}{(z-1)^3}\right]$
6	$\frac{1}{s+a}$	e^{-akT} $(e^{-a})^{kT}$	$\frac{z}{z-e^{-aT}}$ $a^n x(n)$ $X(a^{-1}z)$
7	$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
8	$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
9	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(akT-1+e^{-akT})$	$\frac{z[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$
10	$\frac{b-a}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
11	$\frac{s}{(s+a)^2}$	$(1-akT)e^{-akT}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$
12	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bkT}-ae^{-akT}$	$\frac{z[z(b-a)-(be^{aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
13	-	a^k	$\frac{z}{z-a}$

Newton-Raphson Equation

$$X^{k+1} = X^k - A_k^{-1}\nabla f(X^k)$$

where

$$A_k^{-1} = [\nabla^2 f(X^k)]^{-1}$$