ELEC 303.

$i_c = C\frac{dV_c}{dt}$.　$V_L = L\frac{di_L}{dt}$

单词: ① Constraint(s) 约束. ② gradient. ③ orthogonal

isopleth
optimisation
iteratively
Hesse matrix

1. Steepest Descent

$X^{k+1} = X^k - \lambda \nabla f(X^k)$

$\frac{df(X^{k+1})}{d(\lambda^k)} = 0$. $P^k = \nabla f(X^k)$

$\lambda^* = \frac{(P^*)^T P^k}{(P^*)^T A P^k}$

2. Newton Method.

$x^{k+1} = x^k - \frac{f'(x^k)}{f''(x^k)}$

$\boxed{\begin{aligned}\nabla f(X) &= C + QX \\ \nabla^2 f(x) &= Q\end{aligned}}$

$\frac{1}{2}X^TQX + C^TX$

$X^{k+1} = X^k - \underbrace{[\nabla^2 f(X^k)]^{-1}}\nabla f(X^k)$

steady-state error

$e_{ss} = \lim_{z\to 1}(z-1)E(z)$

$E(z) = R(z) - Y(z)$
$= R(z) - T(z)R(z)$

3. Conjugate Gradient

$P^0 = -\nabla f(X^0)$, $X^1 = X^0 + \lambda P^0$.

$P^1 = -\nabla f(X^1) + \beta^0 P^0$

$\beta^k = \frac{\|\nabla f(X^{k+1})\|^2}{\|\nabla f(X^k)\|^2}$

$X^2 = X^1 + \lambda^k P^1$

$\lambda^{k*} = \min f(X^{k+1}) \Rightarrow \frac{df(X^{k+1})}{d\lambda} = 0$.

4. Lagrange Multiplier.

① 1 Equality: $\mathcal{L}(X,\lambda) = f(X) + \lambda g(x)$

② m Equality:

$\mathcal{L}(X,\lambda) = f(X) + \lambda^T G(X) = f(X) + \sum_{i=1}^{m} \lambda_i g_i$

③ m Inequality: Introduce m slack variables, v^2

$g_i(X) + v_i^2 = 0$, $\mathcal{L}(x_1, x_2, \dots, x_n, v_1, \dots, v_m, \lambda_1, \dots, \lambda_m)$
$= f(X) + \sum_{i=1}^{m} \lambda_i (g_i(X) + v_i^2)$

5. $X(z) = \sum_n x(n)\cdot z^{-n}$.　$z = re^{j\omega} = e^{sT}$

$\boxed{6}$ $a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a} = \frac{1}{1-az^{-1}}$

$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$; $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$, $a \neq 1$

$\sum_{k=0}^{n} a^k = \frac{1}{1-a}$, $|a|<1$; $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$(a^x)' = a^x \ln a$, $(e^x)' = e^x$, $(\log_a x)' = \frac{1}{x\ln a}$

$(\ln x)' = \frac{1}{x}$　$\cos(ak) = \frac{1}{2}e^{jak} + \frac{1}{2}e^{-jak}$

7. $x(n-k) \leftrightarrow z^{-k}X(z)$　$x(n+k) \leftrightarrow z^k X(z)$

$a^n x(n) \leftrightarrow X(a^{-1}z)$

$n x(n) \leftrightarrow -z\frac{dX(z)}{dz}$; $n^m x(n) \leftrightarrow (-z\frac{d}{dz})^m X(z)$

$x(n+1) \leftrightarrow zX(z) - zX(0)$

$f(k+n) \leftrightarrow z^n F(z) - z^n f(0) - z^{n-1}f(1) - \dots - zf(n-1)$.

8. $x(0) = \lim_{z\to\infty} X(z)$; $f(\infty) = \lim_{k\to\infty} f(k) = \lim_{z\to 1}(z-1)F(z)$
$= \lim_{z\to 1}(\frac{z-1}{z})F(z) = \lim_{z\to 1}(1-z^{-1})F(z)$

9. Inverse z Transform

① Long Division. 分母 $\overline{)\text{分子}}$　$2 \quad -2\cdot z^{-1} \quad 4\cdot z^{-2} \dots$

② PFE: i) Real Roots $b^2 - 4ac < 0$.　$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

ii) Complex Roots. $\frac{A}{z} + \frac{B}{z-0.1} + \frac{C}{z+0.5-j0.5} + \frac{C^*}{z+0.5+j0.5}$

③ $z = -0.5 + j0.5 \Rightarrow C$ & C^*

iii) Repeated Roots. $A_i = \frac{1}{(i-1)!}\frac{d^{i-1}}{dz^{i-1}}(z-a)^r X(z)\Big|_{z=a}$

eg: $\frac{1}{z^3(z-0.5)} = \frac{A_{11}}{z^3} + \frac{A_{12}}{z^2} + \frac{A_{13}}{z} + \frac{A_a}{z-0.5}$

10. $ZOH(s) = \frac{1-e^{-sT}}{s}$　$G(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\underbrace{G(s)}_{\text{plant}}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right]$

11. $\underset{\uparrow\text{state variables}}{\dot{\vec{x}}} = A\vec{x} + B\vec{u}$. $\vec{y} = C\vec{x} + D\vec{u}$.

uncontrollable. $|Q_c| = |[B \ AB]| = 0$.

Poles: $|sI_n - A| = 0$.　zero: $\begin{vmatrix} sI_n - A & -B \\ C & D \end{vmatrix} = 0$.　$z = e^{sT}$　$T^2 \to 0$　$s = \frac{z-1}{T}$

characteristic

12. quadratic function. $f(X) = \frac{1}{2}X^TQX + B^TX + C$.

gradient vector: $\nabla f = [\vdots] = QX + B$

gradient direction: $S = \frac{\nabla f(X)}{\|\nabla f(X)\|}$　$S_k \perp S_{k+1}$

stationary point: $\nabla f = 0$. $X = -Q^{-1}B$.

① positive $a_{11} > 0$ $|A| > 0$ ② negative $a_{11} < 0$ $|A_{kk}| > 0$ k even, $|A_{kk}| < 0$ k odd.

③ positive semi: $a_{11} \geq 0$ $|A| = 0$, others ≥ 0.

④ negative semi: $a_{11} \leq 0$, $|A_{kk}| \geq 0$ even, $|A_{kk}| \leq 0$ odd, $|A_{kk}| = 0$. A singular

13. Nyquist–Shannon sampling theorem: $f_s \geq 2f_{max}$, alias(ing)

14. A cost function also known as an objective function, it guides the optimisation algorithm towards the optimal solution. By maximizing or minimizing the cost function, the algorithm iteratively improves the solution.

- An isopleth is a line on a diagram that represents a constant value of a particular variable. In the context of optimization, an isopleth typically connects points where the function takes the same value.
- necessary: $f'(x) = 0$. sufficient: $f''(x) > 0$. $x^* \to$ mini, < 0 max
-

	Rise Time	Overshoot	Settling time	steady state Error.
proportional	↓	↑	negligible	↓
integral	↓	↑	↑	↓ eliminate
derivative	negligible	↓	↓	No effect

- an integral action will eliminate the steady-state error for a constant input. It accumulates the error over time and adjusts the control effort to reduce the accumulated error to zero. Derivative control action gives an estimate of the error in the future.
- ① Accuracy and Precision ② Flexibility and Programmability ③ Integration and Scalability.

$X(z)=\sum_{n}^{\infty} x(n) z^{-n}$

$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \ a\neq 1$

$\sum_{k=0}^{n} a^k = \frac{1}{1-a}, \ |a|<1$

$a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}}$

$z=re^{j\omega}$

$\frac{z}{z-a}$

$x(n-k) \xleftrightarrow{Z} z^{-k}X(z)$

$a^n x(n) \longleftrightarrow X(a^{-1}z)$

$n\,x(n) \longleftrightarrow -z\frac{dX(z)}{dz}$

$x(n+1) \longleftrightarrow zX(z)-zx(0)$

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$\underset{u(n)}{a^n} \longleftrightarrow \frac{z}{z-a}$

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

Appendix:

Table A1. The Laplace and z-Transform Table

$F(s)$ is the Laplace transform of $f(t)$, and $F(z)$ is the z-transform of $f(kT)$. Note: $f(t)=0$ for $t<0$

Number	$F(s)$	$f(kT)$ $f(k)$	$F(z)$
1	1	$\delta(t)$, unit impulse	1
2	e^{-k_0Ts} time delay	$\delta(t-k_0T)$	z^{-k_0} time delay $z^{-k}X(z)$ $z^{-k}\cdot 1$
3	$\frac{1}{s}$	$1(kT)$ $1^n(kT)$ $u(n)$, $x(n)$	$\frac{z}{z-1}$ $\frac{1}{1-z^{-1}}$
4	$\frac{1}{s^2}$ $n\,x(n)$	kT $kT\cdot\delta(t)$	$\frac{Tz}{(z-1)^2}$ $-z\frac{dX(z)}{dz}$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(kT)^2$	$\frac{T^2}{2}\left[\frac{z(z+1)}{(z-1)^3}\right]$
6	$\frac{1}{s+a}$ $a^n x(n)$	e^{-akT} $(e^{-a})^{kT}$	$\frac{z}{z-e^{-aT}}$ $X(a^{-1}z)$
7	$\frac{1}{(s+a)^2}$	kTe^{-akT} $(e^{-a})^{kT}\cdot kT$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
8	$\frac{a}{s(s+a)}\cdot\frac{1}{a}$	$1-e^{-akT}\cdot\frac{1}{a}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}\cdot\frac{1}{a}$ $\frac{1}{s(s+a)} \xrightarrow{S} \frac{1}{a}(1-e^{-akT})$
9	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(akT-1+e^{-akT})$	$\frac{z[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$ $\xrightarrow{Z} \frac{z(1-e^{-aT})}{a(z-1)(z-e^{-aT})}$
10	$\frac{b-a}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
11	$\frac{s}{(s+a)^2}$	$(1-akT)e^{-akT}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$
12	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bkT}-ae^{-akT}$	$\frac{z[z(b-a)-(be^{-aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
13	-	a^{kT} $a^{kT}\delta(kT)$	$\frac{z}{z-a}$

Newton-Raphson Equation

$x^{k+1}=x^k-[\nabla^2 f(x^k)]^{-1}\nabla f(x^k)$

$\cancel{x^{k+1}=x^k-\frac{f'(x)}{f''(x)}}$

$$X^{k+1} = X^k - A_k^{-1}\nabla f(X^k)$$

where

$$A_k^{-1} = [\nabla^2 f(X^k)]^{-1}$$

$x^{k+1}=x^k-\frac{f'(x^k)}{f''(x^k)}$

$(a^x)' = a^x \ln a \quad (e^x)' = e^x.$

$(\log_a x)' = \frac{1}{x \ln a} \quad (\ln x)' = \frac{1}{x}$

① $\delta[n]$ $1.$

② $u[n]$ $\frac{z}{z-1}$

③ b^n $\frac{z}{z-b}$

④ $b^{n-1}u[n-1]$ $\frac{1}{z-b}$

⑤ e^{-at} $\frac{z}{z-e^{-aT}}$

⑥ n $\frac{z}{(z-1)^2}$

$x(0) = \lim_{z \to \infty} X(z)$

$f(\infty) = \lim_{z \to 1} (z-1)F(z)$

$\frac{Y(z)}{U(z)} = \frac{G(z)}{1-G(z)H(z)}$

$P(z) = 1 - G(z)H(z)$ characteristic Eq.

$|sI - A| = 0.$

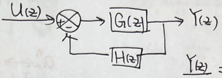
$f(X) = \frac{1}{2}X^TQX + C^TX.$

$x^* = Q^{-1}C$ (for quadratic functions)

open loop poles: $P(s) = |sI_n - A| = 0.$

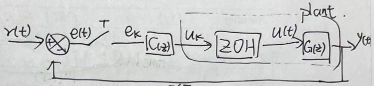
tf: $G(s) = C(sI_n - A)^{-1}B + D.$

Closed-loop:



$\frac{Y(z)}{U(z)} = \frac{G(z)}{1+G(z)\cdot H(z)}$

characteristic eq. $1+G(z)H(z) = P(z) = 0.$

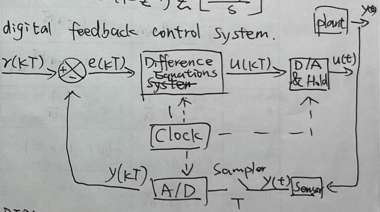


$ZOH = \frac{1-e^{-sT}}{s}.$

$G(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s} \cdot G(s)\right]$

$= (1-z^{-1})\,\mathcal{Z}\left[\frac{G(s)}{s}\right]$

digital feedback control system.



PID: