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1. a) i) Find the z-transform of the following discrete-time signal: **2**

$$y(k) = \{1,\ 0,\ 0,\ -0.1,\ 1,\ 0,\ 0,\ \ldots\ 0\}.$$

ii) A reference signal is given by $r(t) = \cos(100\pi t)$. Is it suitable to sample this signal at a frequency of 60 Hz? Explain your answer. **2**

iii) A digital control system features the following z-transfer function, **7**

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

Using the partial fraction expansion method, find its inverse z-transform.

b) Find the z-transform of the following causal sequence, **5**

$$f(k) = \begin{cases} 4 & k = 2, 3, \ldots \\ 0 & otherwise \end{cases}$$

c) A discrete-time system is described by the following difference equation,

$$x(k) - ax(k-1) = u(k)$$

where: $-1 < a < 1$, $x(k) = 0$ for $k < 0$, and $u(k)$ is a unit sampled step signal.

i) Determine the z-transform, $X(z)$. **4**

ii) Determine the corresponding sequence, $x(k)$. **3**

iii) Using the final value theorem, find the final value of $x(k)$. **2**

Total

25

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2. a) A DC motor digital control system has been designed using a $\boxed{\text{fast-sampling}}$ digital controller for position control. The armature winding inductance can be considered negligible. The controller design uses just proportional control, and its performance is described by the following z-transfer function, where $\theta(z)$ is the motor output position and $V(z)$ is the desired motor position: **6**

$$\theta(z) = \frac{K\,(z-0.5)}{z^2 - z + 0.25 + K}\,V(z)$$

z-plane.

Sketch the discrete-time root locus for K values equal to 0, 0.25, 0.5, 0.75 and 1. Clearly show the unit circle. State between which values of K the system becomes unstable.

b) The transfer function of a control plant is given by, **8**

$$P(s) = \frac{1}{2s+1}e^{-2s}$$

In order to directly design a digital controller, $C(z)$, in the z-domain, the plant, $P(s)$, needs to be discretised as, $P(z)$. Find the zero-order hold (ZOH) equivalent $P(z)$ of $P(s)$, with a sampling period of $T = 1$ second.

Question continues overleaf.

Question continued.

c) The state and output equations for an electrical circuit with an input voltage source, u, and an output current, y, are given as, **4**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-\frac{1}{R_1C} & 0\\ 0 & -\frac{R_2}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\frac{1}{R_1C}\\ 1/L\end{bmatrix}u$$

$$y=\begin{bmatrix}-1/R_1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\left(1/R_1\right)u$$

The capacitor voltage and inductor current are the state variables.

Determine the relationship between C, L, R_1 and R_2 that will cause the system to be deemed as $\boxed{\text{uncontrollable.}}$

d) An optimisation problem is defined by the following cost function: **7**

Minimise $f(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)$

s. t.

$$g_1(x,y,z) = x - y = 0$$

and

$$g_2(x,y,z) = x + y + z = 1$$

Using the Lagrange multiplier method, find the solution for X (where $X = [x,y,z]^T$) for this optimisation problem.

Total

25

3. An analogue model used for controller design is shown in Figure Q3a.

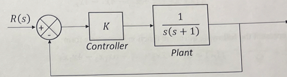

Figure Q3a

The corresponding digital implementation is shown in Figure Q3b where the sampling period is $T = 1$ second. The plant transfer function is $\frac{1}{s(s+1)}$ and the feedback controller is just a simple gain $K > 0$.

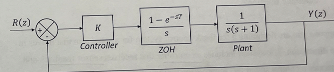

Figure Q3b

a) Show that the $\boxed{\text{characteristic equation}}$ of the system in Figure Q3b is given by **6**

$$z^2 + z(0.3679K - 1.3679) + 0.2642K + 0.3679 = 0.$$

b) Determine the allowable range for K ($K > 0$) if the system in Figure Q3b is to be stable (use Jury test). **6**

c) i) Determine the poles of the closed-loop system shown in Figure Q3a. **4**

ii) Hence discuss the possible effects on stability of introducing a sample and hold process into an analogue system. **1**

d) Minimise $f(x_1,x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ by completing one iteration of the steepest descent method, starting from point $X_1 = [0\,,0]^T$. **8**

Total

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4. a) i) Find the stationary points of the following function:

$$f(X) = \frac{1}{3}(x_1^3 + x_2^3) - 4x_1 - 4x_2$$ **6**

ii) Represent the following quadratic function in a matrix form:

$$f(X) = 6x_1^2 - 2x_2^2 + x_3^2 + 6x_1x_2 - 4x_1x_3$$ **4**

iii) Determine the definiteness of the following quadratic function using Sylvester criterion: **4**

$$f(X) = 5x_1^2 + x_2^2 + 5x_3^2 + 4x_1x_2 - 8x_1x_3 - 4x_2x_3$$

iv) What are the necessary and sufficient conditions to find the minimum of a quadratic function? **3**

b) Suppose you are designing a feedback control system for a robot arm that moves in one dimension. The system includes a controller that receives sensor feedback and generates a control signal (u) to move the arm to a desired position (x_d). The control signal is determined using a cost function (J) that minimises both the position error and the control effort. The cost function is given by

e

$$J(u) = (x_d - x)^2 + \alpha u^2$$

where x_d is the desired position of the arm, x is the current position of the arm, u is the control signal, and α is a weighting factor that balances the trade-off between position error and control effort. Assume $\alpha = 0.1$.

i) Using the Newton method and an initial guess for the control signal of $u_0 = 2$, apply one iteration of the Newton method to find the next approximation for the control signal (u_1). **6**

ii) What are some limitations of the Newton method in solving optimisation problems for control systems? **2**

Total
25

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1. a) i) $X(z) = \sum_n x(n) z^{-n}$.

$y(k) = \{1, 0, 0, -0.1, 1, 0, \dots 0\}$.

$Y(z) = 1 - 0.1 z^{-3} + z^{-4}$.

ii) $\omega = 100\pi$. $f = \frac{\omega}{2\pi} = 50\,Hz$. $f_s \geq 2f = 100\,Hz$.

$100\,Hz > 60\,Hz$ unsuitable.

iii). $\cancel{\frac{F(z)}{z} = \frac{z+1}{z(z+0.2)(}}$ $\frac{F(z)}{z} = \frac{z+1}{z(z+0.1)(z+0.2)} = \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2}$

① $z = 0$. $0+1 = A \cdot 0.1 \cdot 0.2 + 0 + 0 \Rightarrow A = 50$.

② $z = -0.1$ $0.9 = 0 + B(-0.1) \cdot 0.1 + 0 \Rightarrow B = \frac{0.9}{-0.01} = -90$

③ $z = -0.2$ $0.8 = 0 + 0 + C(-0.2)(-0.1) \Rightarrow C = \frac{0.8}{0.02} = 40$.

$\therefore \cancel{F(z)}$ $\therefore f(x) = 50\delta(x) - 90(-0.1)^x u(x) + 40(-0.2)^x u(x)$.

b) $f(k) = \begin{cases} 4 & k = 2, 3, \dots \\ 0 & \text{otherwise.} \end{cases}$ $\because x(n-k) \overset{Z}{\longleftrightarrow} z^{-k} X(z)$

$\therefore f(k-2) \overset{Z}{\longleftrightarrow} z^{-2} F(z)$.

$Z\{f(k-2)\} \cancel{F(z)} = \sum_{k-2=0}^{\infty} f(k-2) z^{-(k-2)}$.

$= \sum_{k-2=0}^{\infty} 4 z^{-(k-2)}$.

$Z\{f(k)\} = z^2 Z\{f(k-2)\}$.

$\because \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

$\therefore \sum_{k-2=0}^{\infty} 4 z^{-(k-2)} = \frac{4}{1 - z^{-1}}$.

$\therefore F(z) = z^2 \cdot Z\{f(k-2)\} = \frac{4z^2}{1 - z^{-1}}$.

c) $x(k) - a x(k-1) = u(k)$.

i) $Z\{x(k) - a x(k-1)\} = Z\{u(k)\}$

$X(z) - a \cdot z^{-1} X(z) = \frac{z}{z-1}$

$X(z)(1 - a z^{-1}) = \frac{z}{z-1}$

$X(z) = \frac{z}{z-1} \cdot \frac{1}{1 - a z^{-1}} = \frac{z}{(z-1)(1 - a z^{-1})} = \frac{z^2}{(z-1)(z-a)}$

1

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1. a) i) Find the z-transform of the following discrete-time signal: **3**

$$y(k) = \{1,\ 0,\ -0.5,\ 1,\ \dots\}$$

ii) In order to design a digital control system, an output signal having frequency bandwidth of 100 Hz needs to be sampled. Can the sampling frequency be chosen as 150 Hz? Explain your answer. **2**

iii) May a system have more than two state space representations? Explain your answer. **2**

iv) Find the inverse z-transform of the following z-function, $F(z)$, using the power series (long-division) method: **5**

$$F(z) = \frac{z + 0.2}{(z - 0.5)(z - 0.6)}$$

You are required to find the values of $f(kT)$ at $k = 0, 1, 2$.

b) i) What is the purpose of a cost function in optimisation problems? **2**

ii) What is an isopleth? **1**

c) An optimisation problem is defined by the following objective function: **10**

Minimise $f(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$

Subject to

$$g_1(x, y, z) = x - y = 0$$

and

$$g_2(x, y, z) = x + y + z - 1 = 0$$

Use the Lagrange multiplier method to find the solution for X (where $X = [x, y, z]^T$) in this optimisation problem.

Total
25

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2. a) When designing a controller, why is an integral action often included when the input signal to the system is a step signal? **2**

b) Write the transfer function of a proportional-integral-derivative controller in the z-domain. **5**

c) A digital controller design for a feedback system is shown in Figure Q2 where the sampling period is $T = 0.1$ seconds. The plant transfer function is

$$P(s) = \frac{s + 10}{s}$$

and the feedback controller, K, is a simple proportional gain ($K > 0$).

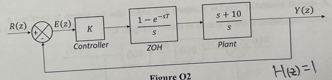

Figure Q2

i) Determine the discretised plant $P(z)$, i.e., the ZOH equivalent of $P(s)$. **8**

ii) Give the transfer function of the closed-loop system, i.e., from $R(z)$ to $Y(z)$. **5**

iii) Determine the range of K such that the closed-loop system is stable. **5**

Total
25

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3. a) A digitised antenna control system can be described by the following z-transfer function:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{3(z-0.4)}{z^3 + 1.6z^2 + 0.73z + 0.09} = \frac{3(z-0.4)}{(z+0.2)(z+0.5)(z+0.9)}$$

i) Find the difference equation relating $y(k)$ and $u(k)$. **4**

ii) If the system has a step input as follows: **2**

$$u(kT) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

and the initial state of the system is $y(0) = 0$.

Find the values of $y(1)$ and $y(2)$.

b) A digital controller $D(z)$ is designed and added into the system to construct a servo-controlled antenna system as shown in Figure Q3.

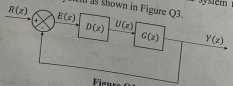

Figure Q3

The digital controller has a z-transfer function,

$$D(z) = 0.15\frac{z+0.9}{z-0.4}$$

Note: G(z) is given in question 3 a) above.

i) Determine the closed-loop system z-transfer function. **4**

ii) Determine the stability of the system. **4**

iii) If the system is stable, find the steady-state error for a unit step input. **8**

c) Give three advantages of using digital computers in the control system loop. **3**

Total 25

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4. a) i) What are the necessary and sufficient conditions to find the minimum or maximum of a quadratic function? **4**

ii) Given the quadratic function: **5**

$$f(X) = C^T X + \tfrac{1}{2} X^T Q X\,,$$

show that the optimum solution can be found in a single iteration by using the Newton method and that the optimum solution is given by $X^* = -Q^{-1}C$.

iii) In the Steepest Descent method, the updating equation is given as follows: **9**

$$X^{k+1} = X^k + \lambda \nabla f(X^k),$$

where λ is the step length.

Show that $\left(\nabla f(X^{k+1})\right)^T \nabla f(X^k) = 0$, i.e. the gradient vectors at the $(k+1)th$ and kth steps are orthogonal.

Question continues overleaf.