4.a) $f(x) = -12x_1^2 - 6x_1x_2 - 5x_2^2$.

$f(x) = \frac{1}{2} X^T Q X + B^T X + C$.

i) Gradient vector $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -24x_1 - 6x_2 \\ -6x_1 - 10x_2 \end{bmatrix}$.

ii) stationary point. $\nabla f = 0$. $\Rightarrow \begin{bmatrix} -24x_1 - 6x_2 \\ -6x_1 - 10x_2 \end{bmatrix} = 0$.

$\therefore x_1 = 0,\ x_2 = 0$.

iii) $Q = \begin{vmatrix} -24 & -6 \\ -6 & -10 \end{vmatrix}$ $B^T = 0$ $C = 0$.

$q_{11} = -24 < 0$. $|Q| = 240 - 36 > 0$. k even.

$\therefore$ f(x) is negative define. and Q is a negative define matrix.

iv) Necessary condition: $\nabla f = 0$.

sufficient condition: $\nabla^2 f \Rightarrow \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 x_2} \\ \frac{\partial f}{\partial x_2 x_1} & \frac{\partial f}{\partial x_2^2} \end{bmatrix}$

$\nabla^2 f = \begin{bmatrix} -24 & -6 \\ -6 & -10 \end{bmatrix} < 0$. $x^* \rightarrow$ maximum.

b) i) $X^{k+1} = X^k - A_k^{-1} \nabla f(X^k)$ $A_k^{-1} = [\nabla^2 f(X^k)]^{-1}$

~~$f(X) = \frac{1}{2} X^T Q X + B^T X + C$ $\nabla f(X) = \frac{1}{2} X^T Q + B^T$ $\nabla^2 f(X^k) =$~~

$f(X) = \frac{1}{2} X^T Q X + C^T X$ $\nabla f(X) = QX + C$ $\nabla^2 f(X) = Q$.

$\nabla f(X) = 0$. $X^1 = -Q^{-1}C$. $X^1 = X^0 - [\nabla^2 f(X^0)]^{-1} \nabla f(X^0)$

$= X^0 - Q^{-1}(QX^0 + C)$

$= X^0 - Q^{-1} \cdot Q X^0 - Q^{-1}C = -Q^{-1}C$.

$X^2 = X^1 - [\nabla^2 f(X^1)]^{-1} \nabla f(X^1) = -Q^{-1}C$.

$X^* = -Q^{-1}C$.

ii) $\max f(x) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$. $Q = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix}$ $C^T = [4 \quad 6]$

$f(x) = \frac{1}{2} X^T Q X + C^T X$.

$X^* = -Q^{-1}C$. $Q^{-1} = \frac{\mathrm{adj}(Q)}{\det(Q)}$ $\det Q = 16 - 4 = 12$. $\mathrm{adj}(Q) = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$

$\therefore Q^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$ $X^* = -\begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = -\begin{bmatrix} -\frac{4}{3} + 1 \\ \frac{4}{6} - 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \end{bmatrix}$ $x_1 = \frac{1}{3}$ $x_2 = \frac{4}{3}$

$$\max f(x) = 4\times\frac{7}{3} + 6\times\frac{4}{3} - 2\times\left(\frac{7}{3}\right)^2 - 2\times\frac{7}{3}\times\frac{4}{3} - 2\times\left(\frac{4}{3}\right)^2.$$

$$X^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{4}{3} \end{bmatrix}$$

iii) the gradient direction refers to the direction in which the objective function incresses most rapidly. the gradient of a function $f(x)$ at point X is $\nabla f(x)$. This vector points in the direction of the steepest ascent. For miniization problems, the negative gradient direction $-\nabla f(x)$ is used, as it points in the direction of the steepest descent.