Question 4: Show that $\tilde{u}'(I_n \otimes J_T)\tilde{u} = \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} \tilde{u}_{it}\tilde{u}_{is} = \sum_{i=1}^{N}\sum_{i=1}^{T} \tilde{u}_{it}\left(\sum_{s=1}^{T}\hat{u}_{is}\right)$

Note that :

- $(I_n \otimes J_T)$ is a $NT * NT$ matrix : $\begin{pmatrix} J_T & 0 & 0 & \cdots & 0 \\ 0 & J_T & 0 & \cdots & 0 \\ 0 & 0 & J_T & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_T \end{pmatrix}$

- $\hat{u}$ is $NT \times 1$ vector

Hence $\tilde{u}'(I_n \otimes J_T)\tilde{u} = [\tilde{u}_{11} \ \tilde{u}_{12} \cdots \tilde{u}_{1T} \ \tilde{u}_{21} \cdots \tilde{u}_{2T} \cdots \tilde{u}_{NT}](I_n \otimes J_T)\tilde{u}$

* $\tilde{u}'$ by $(I_n \otimes J_T)$ $= \left[\sum_{s=1}^{T}\hat{u}_{1s} \ \sum_{s=1}^{T}\tilde{u}_{2s} \cdots \sum_{s=1}^{T}\tilde{u}_{Ts}\right] \cdot \tilde{u}$

* by $\tilde{u}$ $= \left[\sum_{s=1}^{T}\hat{u}_{1s} \ \sum_{s=1}^{T}\tilde{u}_{2s} \cdots \sum_{s=1}^{T}\hat{u}_{Ts}\right]\left[\hat{u}_{11} \ \hat{u}_{12} \cdots \tilde{u}_{NT}\right]'$

$= \hat{u}_{11}\sum_{s=1}^{T}\hat{u}_{1s} + \tilde{u}_{12}\sum_{s=1}^{T}\hat{u}_{2s} + \cdots + \tilde{u}_{NT}\cdot\sum_{s=1}^{T}\hat{u}_{Ts}$

* summation on N $= \sum_{i=1}^{N}\left(\hat{u}_{i1}\sum_{s=1}^{T}\hat{u}_{is} + \hat{u}_{i2}\sum_{s=1}^{T}\tilde{u}_{is} + \cdots + \tilde{u}_{iT}\sum_{s=1}^{T}\hat{u}_{is}\right)$

* summation on T $= \sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{u}_{it}\left(\sum_{s=1}^{T}\tilde{u}_{is}\right)$

* properties of summation $= \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{u}_{it}\hat{u}_{is}$

Question 5 : Answer question 2.1

Prove that $\tilde{\beta}$ given (2.7): $\tilde{\beta} = (X'QX)^{-1}X'Qy$ can be obtained from OLS on (2.5): $y = \alpha \iota_{NT} + X\beta + Z_\mu \mu + \nu = Z\delta + Z_\mu \mu + \nu$, using results on partitioned inverse. This can easily be obtained using the Frisch – Waugh – Lovell theorem of Davidson and MacKinnon.

Hint: This theorem states that the OLS estimate of β from (2.5) will be identical to the OLS estimate of β from (2.6): $Qy = QX\beta + Q\nu$. Also, the least squares residuals will be the same.

- Define $Q = I - P$

$Qy = \tilde{y}$, $Qx = \tilde{x}$, the transformed error component model $Qy = Qx\beta + Q\nu$ can be written as: $\tilde{Y}_i = \tilde{X}_i \beta + \tilde{\eta}_i$

The FE estimator is therefore an OLS estimator of $\tilde{Y} = \tilde{X}\beta + \tilde{\eta}$

$$\hat{\beta}_{FE} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}\tilde{Y} = (X'QX)^{-1}X'Qy = \tilde{\beta}$$

- LSDV, within estimator

$Y_i = X_i'\beta + \alpha_i 1_T + \eta_i$, stack $\rightarrow Y = X\beta + D\alpha + \eta = X\beta + (I_n \otimes 1_T)\alpha + \eta$

$\tilde{\beta} = (X'Q_D X)^{-1} X' Q_D y$ where $Q_D = I - P_D$, $P_D = D(D'D)^{-1}D'$, $D = I_n \otimes 1_T$

To obtain Q_D:
$$\begin{aligned} Q_D &= I - P_D \\ &= I - D(D'D)^{-1}D' \\ &= I - (I_n \otimes 1_T)\left[(I_n \otimes 1_T)'(I_n \otimes 1_T)\right]^{-1}(I_n \otimes 1_T)' \\ &= I - I_n \otimes P_T \\ &= I_n \otimes Q_T \end{aligned}$$

$$\begin{aligned} \beta_{LSDV} &= (X'Q_D X)^{-1} X' Q_D y = \left[X'(I_n \otimes Q_T)X\right]^{-1} X'(I_n \otimes Q_T) y \\ &= (X'QX)^{-1}X'Qy \end{aligned}$$