Chapter 3: The Two-way Error Component Regression Model

(3.1) $u_{it} = \mu_i + \lambda_t + \nu_{it}, \quad i=1,\cdots,N;\ t=1,\cdots,T$

In vector form:

(3.2) $u = Z_\mu \mu + Z_\lambda \lambda + \nu$ where $Z_\mu = I_N \otimes \iota_T$, $Z_\lambda = \iota_N \otimes I_T$

Note that $Z_\lambda Z_\lambda' = J_N \otimes I_T$ and the projection on Z_λ is $Z_\lambda (Z_\lambda' Z_\lambda)^{-1} Z_\lambda' = \bar{J}_N \otimes I_T$

$(\bar{J}_N \otimes I_T) y = \bar{y}_{.t} = \sum_{i=1}^N y_{it}/N$

3.2 The fixed effects model

The Within transformation given by Wallace and Hussain (1969):

(3.3) $Q = E_N \otimes E_T = (I_N - \bar{J}_N) \otimes (I_T - \bar{J}_T)$

$$= I_N \otimes I_T - I_N \otimes \bar{J}_T - \bar{J}_N \otimes I_T + \bar{J}_N \otimes \bar{J}_T$$

This transformation "sweeps" the μ_i, λ_t effects.

$\tilde{y} = Qy \Rightarrow \tilde{y}_{it} = (y_{it} - \bar{y}_{i.} - \bar{y}_{.t} + \bar{y}_{..})$ where $\bar{y}_{..} = \sum_i \sum_t y_{it}/NT$

Within estimator: $\tilde{\beta} = (X'QX)^{-1} X'Qy$

(3.4) $\bar{y}_{.t} = \alpha + \beta \bar{x}_{.t} + \lambda_t + \bar{\nu}_{.t}$ where we have utilize $\sum_i \mu_i = 0$ to avoid dummy variable trap

(3.5) $(y_{it} - \bar{y}_{i.} - \bar{y}_{.t} + \bar{y}_{..}) = (x_{it} - \bar{x}_{i.} - \bar{x}_{.t} + \bar{x}_{..})\beta + (\nu_{it} - \bar{\nu}_{i.} - \bar{\nu}_{.t} + \bar{\nu}_{..})$

Getting $\tilde{\beta}$, we go forward to get $\tilde{\alpha} = \bar{y}_{..} - \tilde{\beta}\bar{x}_{..}$, then

(3.6) $\tilde{\mu}_i = (\bar{y}_{i.} - \bar{y}_{..}) - \tilde{\beta}(\bar{x}_{i.} - \bar{x}_{..})$

(3.7) $\tilde{\lambda}_t = (\bar{y}_{.t} - \bar{y}_{..}) - \tilde{\beta}(\bar{x}_{.t} - \bar{x}_{..})$

Not that within estimator cannot estimate the effect of tim-invariant and individual-invariant variables because Q wipes out these variables.

Testing for fixed effects

$H_0: \mu_1 = \cdots = \mu_{N-1} = 0$ and $\lambda_1 = \cdots = \lambda_{T-1} = 0$

(3.8) $F_1 = \dfrac{(RRSS - URSS)/(N+T-2)}{URSS/(N-1)(T-1)-K} \overset{H_0}{\sim} F_{(N+T-2),\,(N-1)(T-1)-K}$

① Test for the existence of individual effects allowing for time effects

$H_2: \mu_1 = \cdots = \mu_{N-1} = 0$ allowing $\lambda_t \neq 0$ for $t = 1, \cdots, T-1$

② Test for the existence of time effects allowing for individual effects

$H_3: \lambda_1 = \cdots = \lambda_{T-1} = 0$ allowing $\mu_i \neq 0$; $i = 1, \cdots, (N-1)$

$F_2 \overset{H_2}{\sim} F_{N-1,\,(N-1)(T-1)-K}$

$F_3 \overset{H_3}{\sim} F_{T-1,\,(N-1)(T-1)-K}$

3.3 The Random Effects Model

If $\mu_i \sim IID(0, \sigma_\mu^2)$, $\lambda_t \sim IID(0, \sigma_\lambda^2)$, $v_{it} \sim IID(0, \sigma_v^2)$ independent of each other
X_{it} is independent of μ_i, λ_t and v_{it} for all i, t → Random Effect Model

Variance-covariance matrix :

(3.10) $\Omega = E(uu') = Z_\mu E(\mu\mu') Z_\mu' + Z_\lambda E(\lambda\lambda') Z_\lambda' + \sigma_v^2 I_{NT}$

$$= \sigma_\mu^2 (I_N \otimes J_T) + \sigma_\lambda^2 (J_N \otimes I_T) + \sigma_v^2 (I_N \otimes I_T)$$

The disturbances are homoskedastic with $var(u_{it}) = \sigma_\mu^2 + \sigma_\lambda^2 + \sigma_v^2$ for all i and t

(3.11) $cov(u_{it}, u_{js}) = \sigma_\mu^2 \quad i=j,\ t \neq s$
$\qquad = \sigma_\lambda^2 \quad i \neq j,\ t = s$ and zero otherwise

This means that the correlation coefficient

$$(3.12)\quad \text{correl} = \sigma_\mu^2/(\sigma_\mu^2+\sigma_\lambda^2+\sigma_\nu^2) \qquad i=j,\ t\neq s$$
$$= \sigma_\lambda^2/(\sigma_\mu^2+\sigma_\lambda^2+\sigma_\nu^2) \qquad i\neq j,\ t=s$$
$$= 1 \qquad i=j,\ t=s$$
$$= 0 \qquad i\neq j,\ t\neq s$$

In order to get Ω^{-1},

$$(3.13)\quad \Omega = \sum_{i=1}^{4} \lambda_i Q_i \quad \text{where } \lambda_1 = \sigma_\nu^2 \qquad Q_1 = E_N \otimes E_T$$
$$\lambda_2 = T\sigma_\mu^2 + \sigma_\nu^2 \qquad Q_2 = E_N \otimes \bar{J}_T$$
$$\lambda_3 = N\sigma_\lambda^2 + \sigma_\nu^2 \qquad Q_3 = \bar{J}_N \otimes E_T$$
$$\lambda_4 = T\sigma_\mu^2 + N\sigma_\lambda^2 + \sigma_\nu^2 \qquad Q_4 = \bar{J}_N \otimes \bar{J}_T$$

$$(3.14)\quad \Omega^r = \sum_{i=1}^{4} \lambda_i^r Q_i$$

$$(3.15)\quad \sigma_\nu \Omega^{-1/2} = \sum_{i=1}^{4} (\sigma_\nu/\lambda_i^{1/2}) Q_i$$

And the typical element of $y^* = y_{it} - \theta_1 \bar{y}_{i.} - \theta_2 \bar{y}_{.t} + \theta_3 \bar{y}_{..}$

where $\theta_1 = 1-(\sigma_\nu/\lambda_2^{1/2})$, $\theta_2 = 1-(\sigma_\nu/\lambda_3^{1/2})$ and $\theta_3 = \theta_1+\theta_2+(\sigma_\nu/\lambda_4^{1/2})$

BQU estimator of the variance components arise naturally from $Q_i u \sim (0, \lambda_i Q_i)$

$\Rightarrow$ (3.17) $\hat{\lambda}_i = u' Q_i u / tr(Q_i)$

These "ANOVA" estimators are minimum variance unbiased (MVU) under normality of the disturbances.

Amemiya (1971):

$$(3.18)\quad \begin{pmatrix} \sqrt{NT}\,(\hat{\sigma}_\nu^2 - \sigma_\nu^2) \\ \sqrt{N}\,(\hat{\sigma}_\mu^2 - \sigma_\mu^2) \\ \sqrt{T}\,(\hat{\sigma}_\lambda^2 - \sigma_\lambda^2) \end{pmatrix} \sim N\left(0,\ \begin{pmatrix} 2\sigma_\nu^4 & 0 & 0 \\ 0 & 2\sigma_\mu^4 & 0 \\ 0 & 0 & 2\sigma_\lambda^4 \end{pmatrix}\right)$$

And then again, Swamy and Arora (1972) suggest running 3 Least squares regressions and estimating the variance components from the corresponding mean square errors of these regressions.

① One-way model SA (1972)

$$\begin{pmatrix} Qy \\ Py \end{pmatrix} = \begin{pmatrix} QZ \\ PZ \end{pmatrix} \delta + \begin{pmatrix} Qu \\ Pu \end{pmatrix}$$

③ Two-way model SA (1972)

$$\begin{pmatrix} Q_1 y \\ Q_2 y \\ Q_3 y \end{pmatrix} = \begin{pmatrix} Q_1 X \\ Q_2 X \\ Q_3 X \end{pmatrix} \beta + \begin{pmatrix} Q_1 u \\ Q_2 u \\ Q_3 u \end{pmatrix} \quad \cdots (3.22)$$

where $Q_1 = E_N \otimes E_T$

$Q_2 = E_N \otimes \bar{J}_T \rightarrow$ individual average across time.

$Q_3 = \bar{J}_N \otimes E_T \rightarrow$ time average across individual

(3.19) $\hat{\lambda}_1 = \hat{\sigma}_\nu^2 = [y'Q_1 y - y'Q_1 X(X'Q_1 X)^{-1} X'Q_1 y] / [(N-1)(T-1)-k]$

(3.20) $\hat{\lambda}_2 = [y'Q_2 y - y'Q_2 X(X'Q_2 X)^{-1} X'Q_2 y] / [(N-1)-k]$

(3.21) $\hat{\lambda}_3 = [y'Q_3 y - y'Q_3 X(X'Q_3 X)^{-1} X'Q_3 y] / [(T-1)-k]$

$\Rightarrow \hat{\sigma}_\mu^2 = (\hat{\lambda}_2 - \hat{\sigma}_\nu^2)/T$

$\Rightarrow \hat{\sigma}_\lambda^2 = (\hat{\lambda}_3 - \hat{\sigma}_\nu^2)/N$

GLS on (3.22) yields the same estimator of β as GLS on (2.3)

(3.23) $\hat{\beta}_{GLS} = [X'Q_1 X/\sigma_\nu^2 + X'Q_2 X/\lambda_2 + X'Q_3 X/\lambda_3]^{-1} [X'Q_1 y/\sigma_\nu^2 + X'Q_2 y/\lambda_2 + X'Q_3 y/\lambda_3]$

$= [W_{XX} + \phi_2^2 B_{XX} + \phi_3^2 C_{XX}]^{-1} [W_{Xy} + \phi_2^2 B_{Xy} + \phi_3^2 C_{Xy}]$

(3.24) $\hat{\beta}_{GLS} = W_1 \tilde{\beta}_W + W_2 \hat{\beta}_B + W_3 \hat{\beta}_C$ where $W_1 = [W_{XX} + \phi_2^2 B_{XX} + \phi_3^2 C_{XX}]^{-1} W_{XX}$

$W_2 = [W_{XX} + \phi_2^2 B_{XX} + \phi_3^2 C_{XX}]^{-1} (\phi_2^2 B_{XX})$

$W_3 = [W_{XX} + \phi_2^2 B_{XX} + \phi_3^2 C_{XX}]^{-1} (\phi_3^2 C_{XX})$

3.4 Maximum Likelihood Estimation

(3.29) $\log L = \text{constant} - \frac{1}{2}\log|\Omega| - \frac{1}{2}(y - Z\gamma)'\Omega^{-1}(y - Z\gamma)$

(3.31) Breusch (1987): $L(\alpha, \beta, \sigma_\nu^2, \phi_2^2, \phi_3^2) = \text{constant} - (NT/2)\log\sigma_\nu^2 + (N/2)\log\phi_2^2$
$+ (T/2)\log\phi_3^2 - (1/2)\log[\phi_2^2 + \phi_3^2 - \phi_2^2\phi_3^2]$
$- (1/2\sigma_\nu^2) u'\Sigma^{-1} u$

(3.32) $L_c(\beta, \phi_2^2, \phi_3^2) = \text{constant} - (NT/2)\log[d'(Q_1 + \phi_2^2 Q_2 + \phi_3^2 Q_3)/d]$
$+ (N/2)\log\phi_2^2 + (T/2)\log\phi_3^2 - (1/2)\log[\phi_2^2 + \phi_3^2 - \phi_2^2\phi_3^2]$

(3.33) $\Rightarrow \hat{\beta} = [X'(Q_1 + \phi_2^2 Q_2 + \phi_3^2 Q_3)X]^{-1} X'(Q_1 + \phi_2^2 Q_2 + \phi_3^2 Q_3)y$

(3.37) $\Rightarrow \hat{\beta} = [X'(\bar{Q}_1 + \phi_2^2 Q_2)X]^{-1} X'(\bar{Q}_1 + \phi_2^2 Q_2)y$

where $\bar{Q}_1 = Q_1 + \phi_3^2 Q_3$

3.5 Prediction

Prediction for i^{th} individual, S periods ahead

(3.38) $u_{i,T+S} = \mu_i + \lambda_{T+S} + \nu_{i,T+S}$

(3.39) $E(u_{i,T+S}\, u_{jt}) = \sigma_\mu^2$ for $i = j$
$= 0$ for $i \neq j$

...

(3.44) $\hat{y}_{i,T+S} = Z'_{i,T+S}\,\hat{\delta}_{GLS} + \left(\frac{T\sigma_\mu^2}{T\sigma_\mu^2 + \sigma_\nu^2}\right)\hat{\bar{u}}_{i\cdot,GLS}$