1. Study Group

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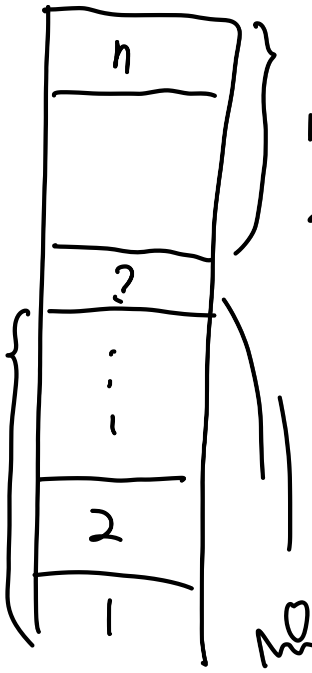
2. (a) $M(x,m) = M(x-1, m-1) + M(x-1, m) + 1$




After we drop an egg on floor $f$, it will break or not. If it breaks, then we should use $x-1$ drops and $m-1$ eggs to decide $l$ below floor $f$, else we should use $x-1$ drops and $m$ eggs to decide $l$ above floor $f$

To ensure the two can both be successful, $f-1$ should be at most $M(x-1, m-1)$ while $n-f$ should be at most $M(x-1, m)$.

So $n = (n-f) + (f-1) + 1 \leq M(x-1, m-1) + M(x-1, m) + 1$

to guarantee we can solve the problem.

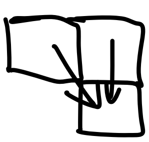
So $M(x, m) = M(x-1, m-1) + M(x-1, m) + 1$

(b) Our algorithm is also based on dynamic programming.

Using the subproblem just described, base cases are:

$M(x, 0) = 0$ for all $x$ and $M(0, m) = 0$ for all $m$.

We compute $M(1, 1), M(1,2), \cdots, M(1, m)$



then $M(2,1), M(2,2), \cdots, M(2,m)$

$\cdots$

until $M(x,1), \cdots, M(x,m)$. Given $(1 \le x \le m)$

So finally we can get $M(x,m)$.

Runtime : There are $O(xm)$ subproblems

Each subproblem takes $O(1)$ to solve.

So ~~total~~ time complexity is $O(xm)$

(c) We only need to find the first $M(x,m) \ge n$ for $1 \le x \le m$. So we should do as (b) does to compute all $M(x,m)$ from $x=1$ until we get some $M(x',m) \ge n$ and we just output $x'$.

(d) The runtime of Alg in (c) is obviously $\Theta(x'm)$
On the other hand we know $x' \le n$.

So runtime of Alg in (c) is $O(nm)$.

The runtime of alg last week is $O(n^2 m)$.

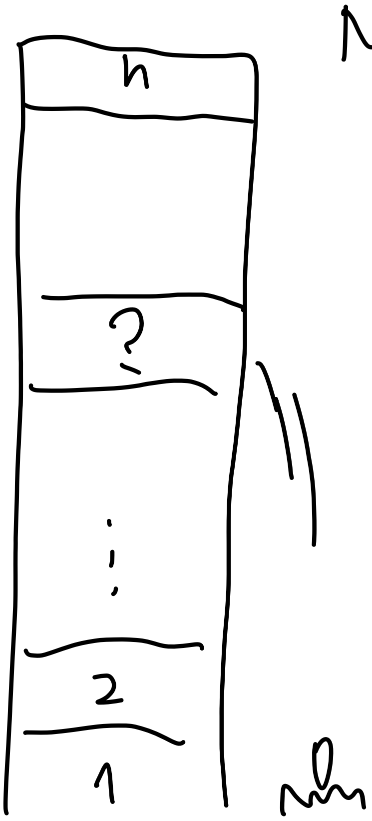
So the alg (c) gives is faster than that of last week.

(e) Since we only need $M(x-1, m-1)$ and $M(x-1, m)$ to compute $M(x, m)$, we only need to store 2 lines of matrix M at once. The space complexity is $O(m)$.

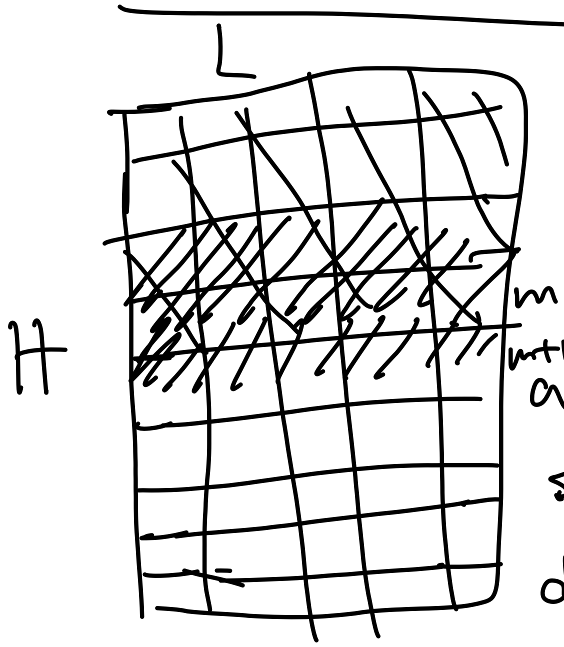
(f) No need to redefine M, just compute $M(x, 2m)$ in (c) alg to find the first $x'$ s.t. $M(x, 2m) > n$. Output $x'$.

$$M(x, i) = \underset{\substack{\downarrow \\ \text{No change!}}}{M(x-1, i)} + M(x-1, i-1) + 1.$$

$(0 \le i \le 2m)$



3. Algorithm Description: (State Compression)




① Subproblem: L-bit number to represent which lines have knights. We can solve the problem about $x \times L$ chessboard and then solve $(x+1) \times L$ chessboard problem. $dp[x][l_1][l_2]$ is the answer of problem about first $x$ rows with the $(m-1)$-th row is $l_1$ and $m$-th row is $l_2$.

② Recurrence:

$$dp[x][l_1][l_2] = \sum_{\substack{l' \text{ with} \\ \text{no confliction} \\ \text{with } l_1 \text{ and } l_2}} dp[x-1][l'][l_1] \bmod 1337 \quad (x \geq 3)$$

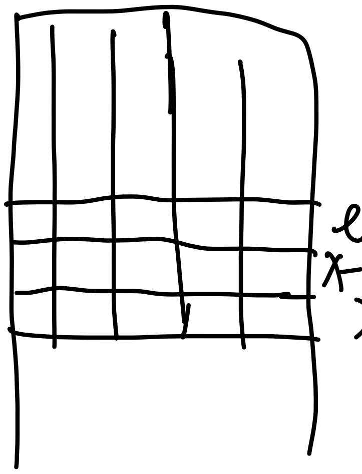
Base Cases: $$dp[2][l_1][l_2] = \begin{cases} 1, & \text{for no-conflicted } (l_1, l_2) \\ 0, & \text{otherwise} \end{cases}$$

③ Ordering: Just compute dp in increasing order of $x$. For a given $x$, compute all $dp[x][l_1][l_2]$ for all no-conflicted $(l_1, l_2)$.

Proof of Correctness: Situation of 2-row: the solution is just $l_1$ and $l_2$, so $dp[2][l_1][l_2]$

$$= \begin{cases} 1, & l_1 \text{ has no confliction with } l_2 \\ 0, & \text{otherwise} \end{cases}$$

Situation of $X$-row $(X>2)$: A solution for




$X$-row with $l_1$ and $l_2$ can be got by a solution for $(X-1)$-row with $l'$ and $l_1$ which have no confliction with $l_2$.

So $dp[X][l_1][l_2] = \sum_{\substack{l',\ (l', l_1, l_2) \\ \text{has no confliction}}} dp[X-1][l'][l_1] \mod 1337$

(It's a bijective map between 2 solution)

Runtime Analysis: $H$ rows

Each subproblem we need to compute all $(l_1, l_2)$ pairs, which is $O(2^{2L})$. Computing all $(X-1)$-row takes $O(2^L)$. Checking confliction takes $O(L)$

So each row we need to take $O(2^{3L} \cdot L)$ to compute.

Total Runtime: $O(2^{3L} \cdot L \cdot H)$.

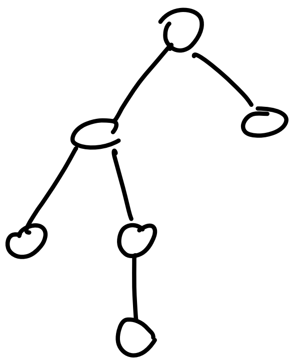
Space Complexity: Only $H \cdot 2^{2L}$ subproblems.

So space complexity is $O(H \cdot 2^{2L})$.

By rolling array optimization, it can be reduced to $O(2^{2L})$. (Same technique as T2 (e))

4. (a) <u>Algorithm Description</u>: ① Subproblem: We can firstly solve the problem with and without the root node of current subtree. Then use it to solve a bigger subtree problem. dp[v][k][0/1]: The max weight of ~~st~~ subtree with root v and at most k nodes in independent set, and 0: root isn't in set, 1: root is in set.

② Recurrence: $dp[v][k][1] = A[v] + \max_{\sum k_i = k-1} \left\{ \sum_{0 \le i \le \#\text{ of child of } v} dp[child_i \text{ of } v][k_i][0] \right\}$



$$dp[v][k][0] = \max_{\sum k_i = k} \left\{ \sum_{0 \le i \le \#\text{ of child of } v} dp[child_i \text{ of } v][k_i][0/1] \right\}$$

Base Cases: $dp[leaf][0][0/1] = 0$

$dp[leaf][k][0] = 0 \quad (k \ge 1)$

$dp[leaf][k][1] = \max\{A[leaf], 0\} \quad (k \ge 1)$

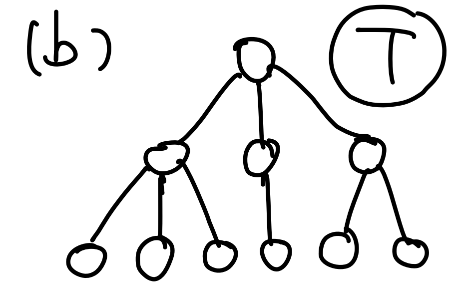
③ Ordering: Compute according to the post value in increasing order. For given v, compute all dp values of k.

<u>Runtime Analysis</u>: There are 2nk subproblems.

Each subproblem we need $O(k)$ to compute.
($O(k)$ ways to assign $k_i$)

Total time is $O(nk^2)$.

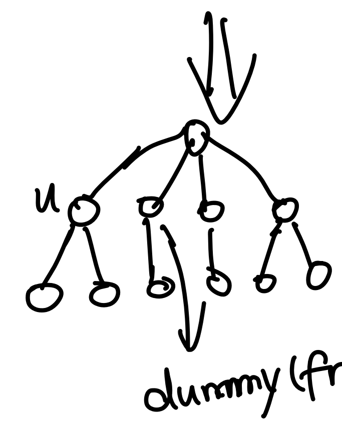
(b)



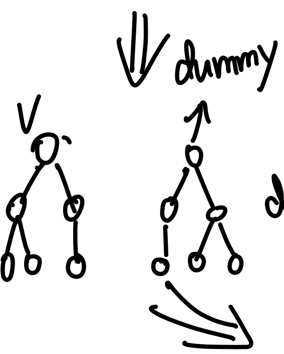

We do such modification on $T$:

While there's node $N$ having more than 2 children:

add dummy node as a sibling of $N$ to 'rob' at most 2 children of $N$. then make them same parent.



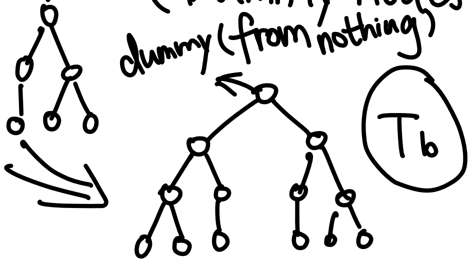

Obviously num of dummy nodes won't exceed num of all origin nodes, which is $O(n)$.




(Dummy nodes added at depth $i \leq$ # of nodes at depth $i+1$)

↓

a loose bound!




(c) <u>Algorithm Description</u>: Record which dummy node each node makes. Whenever we choose the origin node, we choose its dummy node too. Following this rule we just run alg in (a) on $T_b$. But dummy nodes don't consume # of nodes we can choose. Also, we never choose root of $T_b$.

<u>Runtime Analysis</u>: $O(2n)$ nodes so ~~total~~ runtime is $O(2nk^2) = O(nk^2)$