CS 170 Homework 9

Due **11/1/2023, at 10:00 pm (grace period until 11:59pm)**

1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write "none".

2 Canonical Form LP

Recall that any linear program can be reduced to a more constrained *canonical form* where all variables are non–negative, the constraints are given by $\leq$ inequalities, and the objective is the maximization of a cost function.

More formally, our variables are x_i. Our objective is $\max c^\top x = \max \sum_i c_i x_i$ for some constants c_i. The jth constraint is $\sum_i a_{ij} x_i \leq b_j$ for some constants a_{ij}, b_j. Finally, we also have the constraints $x_i \geq 0$.

An example canonical form LP:

$$\text{maximize } 5x_1 + 3x_2$$

$$\text{subject to } \begin{cases} x_1 + x_2 - x_3 \leq 1 \\ -(x_1 + x_2 - x_3) \leq -1 \\ -x_1 + 2x_2 + x_4 \leq 0 \\ -(-x_1 + 2x_2 + x_4) \leq 5 \\ x_1, x_2, x_3, x_4 \geq 2 \end{cases}$$

For each of the subparts below, describe how we should modify it to so that it satisfies canonical form. If it is impossible to do so, justify your reasoning. Note that the subparts are independent of one another.

(a) Min Objective: $\min \sum_i c_i x_i$

(b) Lower Bound on Variable: $x_1 \geq b_1$

(c) Bounded Variable: $b_1 \leq x_1 \leq b_2$

(d) Equality Constraint: $x_2 = b_2$

(e) More Equality Constraint: $x_1 + x_2 + x_3 = b_3$

(f) Absolute Value Constraint: $|x_1 + x_2| \leq b_2$

(g) Another Absolute Value Constraint: $|x_1 + x_2| \geq b_2$

(h) Min Max Objective: $\min \max(x_1, x_2)$

Hint: use a dummy variable!

(i) Unbounded Variable: $x_4 \in \mathbb{R}$

Hint: how can you represent any real number as an operation on two positive numbers?

3 Baker

You are a baker who sells batches of brownies and cookies (unfortunately no brookies... for now). Each brownie batch takes 4 kilograms of chocolate and 2 eggs to make; each cookie batch takes 1 kilogram of chocolate and 3 eggs to make. You have 80 kilograms of chocolate and 90 eggs. You make a profit of 60 dollars per brownie batch you sell and 30 dollars per cookie batch you sell, and want to figure out how many batches of brownies and cookies to produce to maximize your profits.

(a) Formulate this problem as a linear programming problem; in other words, write a linear program (in canonical form) whose solution gives you the answer to this problem. Draw the feasible region, and find the solution using Simplex.

(b) Suppose instead that the profit per brownie batch is C dollars and the profit per cookie batch remains at 30 dollars. For each vertex you listed in the previous part, give the range of C values for which that vertex is the optimal solution.

4 Meal Replacement

Jonny is planning an "Introduction to CS Theory" overnight summer camp for penguins in Antarctica. Penguins can't solve problems well when they're hungry, so Jonny wants to secure an emergency source of food in case polar bears sneak in and eat everything in the igloo. Unfortunately, he is on a tight budget, and in order to accommodate as many penguins as possible, he needs to minimize the cost of food while still meeting the penguins' minimum dietary needs.

Every penguin needs to consume at least 600 calories of protein per day, 800 calories of carbs per day, and 500 calories of fat per day. Jonny has three options for food he's considering buying: salmon, bread, and squid. The composition of each food is provided in the following table:

Food Type	**Price per Pound**	**Protein Calories per Pound**	**Carb Calories per Pound**	**Fat Calories per Pound**
Salmon	6	400	0	150
Bread	1	50	300	25
Squid	8	300	100	200

Our goal is to find a combination of these options that meets the penguins' daily dietary needs while being as cheap as possible.

(a) Formulate this problem as a linear program.

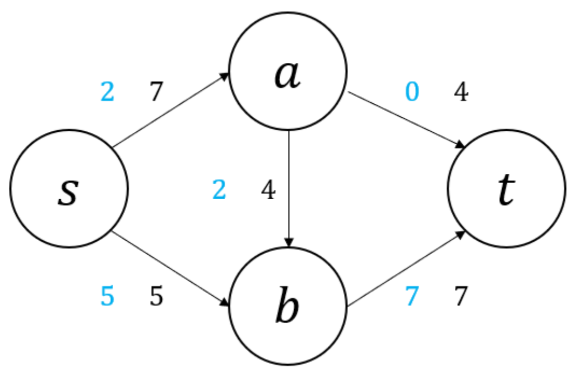
(b) Take the dual of your LP from part (a).

(c) Suppose now there is a pharmacist trying to assign a price to three pills, with the hopes of getting us to buy these pills instead of food. Each pill provides exactly one of protein, carbs, and fat.

Interpret the dual LP variables, objective, and constraints as an optimization problem from the pharmacist's perspective.

5 Practice With Residual Graphs

(a) Consider the following network and flow on this network. An edge is labelled with its flow value (in blue) and capacity (in black). e.g. for the edge (s, a), we are currently pushing 2 units of flow on it, and it has capacity 7.



Draw the residual graph for this flow.

(b) We are given a network $G = (V, E)$ whose edges have integer capacities c_e, and a maximum flow f from node s to node t. Explicitly, f is given to us in the representation of integer flows along every edge e, $\{f_e\}_{e \in E}$.

However, we find out that one of the capacity values of G was wrong: for edge (u, v), we used c_{uv} whereas it really should have been $c_{uv} - 1$. This is unfortunate because the flow f uses that particular edge at full capacity: $f_{uv} = c_{uv}$. To fix this, we could run Ford Fulkerson from scratch, but there's a faster way.

Describe an algorithm to fix the max-flow for this network in $O(|V| + |E|)$ time. **Give a three-part solution.**

6 (Extra Credit) Huffman and LP

Consider the following Huffman code for characters a, b, c, d: $a = 0, b = 10, c = 110, d = 111$.

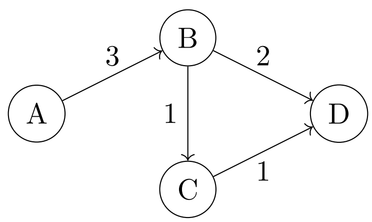
Let f_a, f_b, f_c, f_d denote the fraction of characters in a file (only containing these characters) that are a, b, c, d respectively. Write a linear program with variables f_a, f_b, f_c, f_d to solve the following problem: What values of f_a, f_b, f_c, f_d (that can generate this Huffman code) result in the Huffman code using the most bits per character?

7 (Extra Credit) Flow Decomposition

Let $G = (V, E)$ be a directed graph, and let f be any $s - t$ flow on this graph. Assume that there is no cycle in the graph where $f_e > 0$ for all edges in the cycle. Design an algorithm to decompose f into the sum of at most $|E|$ path flows. That is, your algorithm should find a set of $s - t$ paths $p_1 \ldots p_k$ and corresponding flow values $F_1 \ldots F_k$ such that:

- The number of paths k is at most $|E|$
- f is the sum of k path flows, where the ith flow sends F_i units of flow on path p_i. That is, for each edge e, if P_e is the set of paths in $p_1 \ldots p_k$ that contain e, then $\sum_{p_i \in P_e} F_i = f_e$.

For example, in the below graph (where each edge e is labelled with f_e), one can decompose the flow into $p_1 = ((A, B), (B, C), (C, D)), p_2 = ((A, B), (B, D))$ where $F_1 = 1$, $F_2 = 2$.



Provide the algorithm description and a brief explanation of why your algorithm finds at most $|E|$ paths.

Hint: How does Ford-Fulkerson work?

8 Coding Questions: Max Flow/Min Cut

For this week's coding questions, we'll walk through the **Edmonds Karp** algorithm for max flow and see how to compute the **Minimum Cut** given a solution to the max flow problem. There are two ways that you can access the notebook and complete the problems:

1. **On Local Machine**: `git clone` (or if you already cloned it, `git pull`) from the coding homework repo,

   https://github.com/Berkeley-CS170/cs170-fa23-coding

   and navigate to the `hw09` folder. Refer to the `README.md` for local setup instructions.

2. **On Datahub**: Click here and navigate to the `hw09` folder if you prefer to complete this question on Berkeley DataHub.

Notes:

- *Submission Instructions:* Please download your completed submission `.zip` file and submit it to the Gradescope assignment titled "Homework 9 Coding Portion".
- *OH/HWP Instructions:* Designated coding course staff will provide conceptual and debugging help during office hours and homework parties.
- *Edstem Instructions:* Conceptual questions are always welcome on the public thread. If you need debugging help first try asking on the public threads. To ensure others can help you, make sure to:
  1. Describe the steps you've taken to debug the issue prior to posting on Ed.
  2. Describe the specific error you're running into.
  3. Include a few small test cases, alongside both the output you expected to receive and your function's actual output.

  If staff tells you to make a private Ed post, make sure to include *all of the above items* plus your full function implementation. If you don't provide them, we will ask you to provide them.
- *Academic Honesty Guideline:* We realize that code for some of the algorithms we ask you to implement may be readily available online, but we strongly encourage you to not directly copy code from these sources. Instead, try to refer to the resources mentioned in the notebook and come up with code yourself. That being said, we **do acknowledge** that there may not be many different ways to code up particular algorithms and that your solution may be similar to other solutions available online.