Supplemental Materials with Computer Codes
for Math 252 Calculus III, Spring 2018
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The supplemental materials with a summary in Table 1 are provided to help students review the following topics:

(1) vector fields; (2) gradient and normal vector; (3) curl and circulation;

(4) divergence and flux; (5) line integrals; (6) double integrals;

(7) fundamental theorem of line integrals;

(8) conservative fields and independence of path;

(9) Green's theorem in both the tangential and normal forms;

(10) a comparison amongst Green's, Stokes' and Divergence theorems.

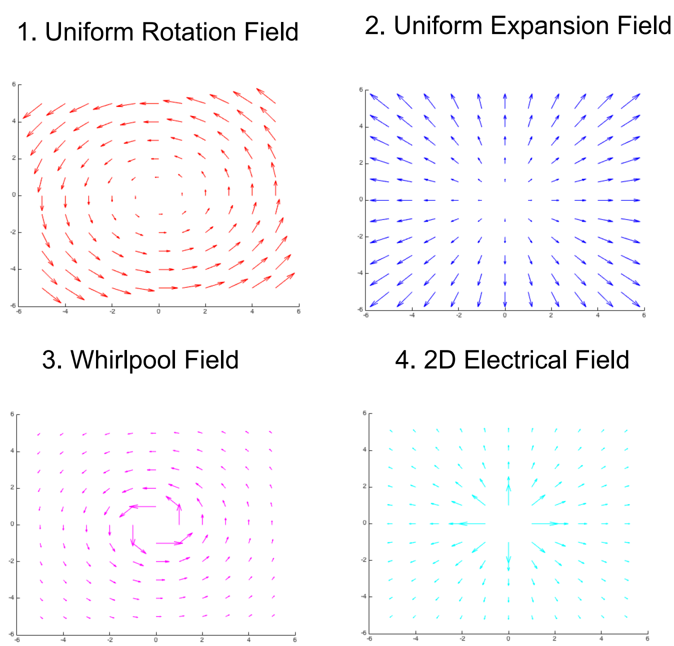

Figure 1: Four vector fields described in Eqs. A1-A4.

Let **C** be a circle, $x^2+y^2=a^2$, **C'** be a circle, $x^2+y^2=\epsilon^2$, **D** be the region $0 \le x^2+y^2 \le a^2$, and D_s be the region $0 < \epsilon^2 \le x^2+y^2 \le a^2$. Let $\vec{F}=(P,Q)$ be one of the vector fields in Figure 1 as follows:

(1) Uniform rotation field, $(-y,x)$; **(A1)**

(2) Uniform expansion field, (x,y); **(A2)**

(3) Whirlpool field, $(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$; and **(A3)**

(4) 2D electrical field, $(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. **(A4)**

Solve the following problems using Eqs. (A1-A4).

1: Let $\vec{F}$ be the vector field in Eq. A1.
(a) Find $\oint_C \vec{F}\cdot\vec{T}ds$;
(b) Find $\int\int_D \nabla\times\vec{F}\cdot\vec{k}dA$;
(c) Compare the results in (a) and (b).

(a) Using Eq. (A1), we have

$$\oint_C \vec{F}\cdot\vec{T}ds = \oint_C Pdx + Qdy = \oint_C xdy - ydx \tag{1.1}$$

Assume

$$x = acos(\theta) \quad and \quad y = asin(\theta).$$

Then, we have

$$dx = -asin(\theta)d\theta \quad and \quad dy = acos(\theta)d\theta.$$

Thus, the line integral above becomes

$$\oint_C xdy - ydx = \int_0^{2\pi} a^2cos^2\theta d\theta + a^2sin^2\theta d\theta = a^2\int_0^{2\pi} d\theta = 2\pi a^2. \tag{1.2}$$

A Matlab code is written as:

$$syms \;\; theta \;\; a \;\; r \;\; x \;\; y$$
$$x = a*cos(theta)$$
$$y = a*sin(theta)$$
$$dx = diff(x)$$
$$dy = diff(y)$$
$$result = int(x*dy - y*dx, theta, 0, 2*pi)$$
$$\boxed{the \;\; answer \;\; is \;\; 2\pi a^2}$$

The corresponding Python code is:

$$import \;\; sympy \;\; as \;\; sp$$
$$theta, a, r, x, y = sp.symbols(\text{“}theta \;\; a \;\; r \;\; x \;\; y\text{”})$$
$$x = a*sp.cos(theta)$$
$$y = a*sp.sin(theta)$$
$$dx = sp.diff(x, theta)$$
$$dy = sp.diff(y, theta)$$
$$f = x*dy - y*dx$$
$$result = sp.integrate(f, (theta, 0, 2*sp.pi))$$
$$print(result)$$

$$\boxed{the\ answer\ is\ 2\pi a^2}$$

(b)

$$\int\int_D \nabla\times\vec{F}\cdot\vec{k}dA = \int\int (\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$$

With Eq. A1, we have

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y) = 2.$$

Assuming

$$x = rcos(\theta) \quad and \quad y = rsin(\theta),$$

the double integral above becomes

$$\int\int (\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy = \int\int 2dxdy = 2\int_0^{2\pi}\int_0^a rdrd\theta$$

$$= 2\int_0^{2\pi}\frac{a^2}{2}d\theta = 2\pi a^2. \tag{1.3}$$

A Matlab code (with $curlFn$ indicating $\nabla\times\vec{F}\cdot\vec{k}$) is written as:

```
syms x y z
F = [-y, x, 0]
curlF = curl(F, [x, y, z])
curlFn = dot(curlF, [0, 0, 1])
result = int(int(r * curlFn, r, 0, a), theta, 0, 2 * pi)
```

$$\boxed{the\ answer\ is\ 2\pi a^2}$$

The corresponding Python code is:

```
from sympy.physics.vector import *
R = ReferenceFrame('R')
F = -R[1] * R.x + R[0] * R.y + 0 * R.z
curlF = curl(F, R)
curlFn = dot(curlF, R.x + R.y + R.z)
result = sp.integrate(sp.integrate(curlFn * r, (r, 0, a)), (theta, 0, 2 * sp.pi))
print(result)
```

$$\boxed{the\ answer\ is\ 2\pi a^2}$$

(c) Equations (1.1)-(1.3) lead to

$$\int\int_D \nabla \times \vec{F} \cdot \vec{k} dA = \oint_C \vec{F} \cdot \vec{T} ds. \tag{1.4}$$

The above states Green's theorem in the plane, which helps to transform line integrals into double integrals, or conversely, double integrals into line integrals. More specifically, **Green's theorem in the tangential form** in Eq. (1.4) states that the double integral of the vertical component of a curl vector, $\nabla \vec{\times} F \cdot \vec{k}$, over the region D enclosed by C is equal to the line integral of the tangential component of the vector, $\vec{F} \cdot \vec{T}$, along C (i.e., "circulation").

(Optional) Eq. (1.4) can help explain the physical meaning of a curl vector, $\nabla \times \vec{F}$. The mean value theorem for double integrals says that if D is simply connected, then there exists at least one point $M(x_0, y_0)$ in D such that we have

$$\int\int_D g(x,y) dx dy = g(x_0, y_0) A, \tag{1.5}$$

where $g = \nabla \times \vec{F} \cdot \vec{k}$ and A is the area of D. From Eqs. (1.4)-(1.5), we obtain

$$g(x_0, y_0) = \frac{1}{A} \int\int_D \nabla \times \vec{F} \cdot \vec{k} dA = \frac{1}{A} \oint_C \vec{F} \cdot \vec{T} ds. \tag{1.6}$$

We can select a point $N(x_1, y_1)$ in D and let D shrink down onto N so that the maximum distance $d(D)$ from the points of D to N goes to zero. Therefore, $M(x_0, y_0)$ must approach N. Hence, Eq. (1.6) becomes

$$\nabla \times \vec{F}(x_1, y_1) \cdot \vec{k} = \lim_{d(D) \to 0} \frac{1}{A} \oint_C \vec{F} \cdot \vec{T} ds, \tag{1.7}$$

which relates the the vertical component of a curl vector to the ratio of the circulation to the area.

2: Let $\vec{F}$ be the vector field in Eq. A2.
(a) Find $\oint_C \vec{F} \cdot \vec{n} ds$;
(b) Find $\int\int_D \nabla \cdot \vec{F} dA$;
(c) Compare the results in (a) and (b).

(a) Using Eq. (A2), we have

$$\oint_C \vec{F} \cdot \vec{n} ds = \oint_C P dy - Q dx = \oint_C x dy - y dx \tag{2.1}$$

Assume

$$x = a cos(\theta) \quad and \quad y = a sin(\theta).$$

Then, we have

$$dx = -a sin(\theta) d\theta \quad and \quad dy = a cos(\theta) d\theta.$$

The line integral above becomes

$$\oint_C x dy - y dx = \int_0^{2\pi} a^2 cos^2\theta d\theta + a^2 sin^2\theta d\theta = a^2 \int_0^{2\pi} d\theta = 2\pi a^2. \tag{2.2}$$

A Matlab code is written as:

$$syms \quad theta \quad a \quad r \quad x \quad y$$

$$x = a * cos(theta)$$

$$y = a * sin(theta)$$

$$dx = diff(x)$$

$$dy = diff(y)$$

$$result = int(x * dy - y * dx, theta, 0, 2 * pi)$$

$$\boxed{the \ answer \ is \ 2\pi a^2}$$

The corresponding Python code is:

$$import \quad sympy \quad as \quad sp$$

$$theta, a, x, y = sp.symbols(\text{“}theta \quad a \quad x \quad y\text{”})$$

$$x = a * sp.cos(theta)$$

$$y = a * sp.sin(theta)$$

$$dx = sp.diff(x, theta)$$

$$dy = sp.diff(y, theta)$$

$$f = x * dy - y * dx$$

$$result = sp.integrate(f, (theta, 0, 2 * sp.pi))$$

$$print(result)$$

$$\boxed{the \ answer \ is \ 2\pi a^2}$$

(b)

$$\int\int_D \nabla \cdot \vec{F} dA = \int\int (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) dxdy$$

With Eq. A2, we have

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2.$$

Assume

$$x = rcos(\theta) \quad and \quad y = rsin(\theta),$$

the above double integral becomes

$$\int\int (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) dxdy = \int\int 2dxdy = 2\int_0^{2\pi}\int_0^a rdrd\theta$$

$$= 2\int_0^{2\pi} \frac{a^2}{2} d\theta = 2\pi a^2. \tag{2.3}$$

A Matlab code (with div indicating $\nabla \cdot \vec{F}$) is written as:

$$syms \quad x \quad y \quad r \quad a$$
$$F = [x, y]$$
$$div = divergence(F)$$
$$result = div * int(int(r, r, 0, a), theta, 0, 2 * pi)$$

$$\boxed{the \ answer \ is \ 2\pi a^2}$$

The corresponding Python code is:

$$from \quad sympy.physics.vector \quad import \quad *$$
$$x, y = sp.symbols(“x \quad y”)$$
$$R = ReferenceFrame('R')$$
$$F = R[0] * R.x + R[1] * R.y$$
$$div = divergence(F, R)$$
$$result = div * sp.integrate(sp.integrate(r, (r, 0, a)), (theta, 0, 2 * sp.pi))$$
$$print(result)$$

$$\boxed{the \ answer \ is \ 2\pi a^2}$$

(c) Equations (2.1)-(2.3) lead to

$$\int\int_D \nabla \cdot \vec{F} dA = \oint_C \vec{F} \cdot \vec{n} ds. \tag{2.4}$$

The above states a second vector form of Green's theorem in the plane. Similarly, it helps to transform line integrals into double integrals, or conversely, double integrals into line integrals. In contrast to Eq. (1.4), **Green's theorem in the normal form**, in Eq. (2.4), says that the double integral of the divergence of a vector field, $\nabla \cdot \vec{F}$, over the region D enclosed by C is equal to the line integral of the normal component of the vector, $\vec{F} \cdot \vec{n}$, along C (i.e., "flux").
(Optional) Eq. (2.4) can help explain the physical meaning of the divergence of a vector field, $\nabla \cdot \vec{F}$. The mean value theorem for double integrals says that if D is simply connected, then there exists at least one point $M(x_0, y_0)$ in D such that we have

$$\int\int_D g(x, y) dx dy = g(x_0, y_0) A, \tag{2.5}$$

where $g = \nabla \cdot \vec{F}$ and A is the area of D. From Eqs. (2.4)-(2.5), we obtain

$$g(x_0, y_0) = \frac{1}{A} \int\int_D \nabla \cdot \vec{F} dA = \frac{1}{A} \oint_C \vec{F} \cdot \vec{n} ds. \tag{2.6}$$

We can select a point $N(x_1, y_1)$ in D and let D shrink down onto N so that the maximum distance $d(D)$ between the points from D to N goes to zero. Therefore, $M(x_0, y_0)$ must approach N. Hence, Eq. (2.6) becomes

$$\nabla \cdot \vec{F}(x_1, y_1) = \lim_{d(D)\to 0} \frac{1}{A} \oint_C \vec{F} \cdot \vec{n} ds, \tag{2.7}$$

which relates the divergence of a vector to the ratio of the flux to the area.

3: Let $\vec{F}$ be the vector field in Eq. A3.
(a) Find $\oint_C \vec{F} \cdot \vec{T} ds$;
(b) Find $\int\int_{D_s} \nabla \times \vec{F} \cdot \vec{k} dA$;
(c) Compare the results in (a) and (b).

(a) Using Eq. A3, we have

$$\oint_C \vec{F} \cdot \vec{T} ds = \oint_C P dx + Q dy = \oint_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy. \tag{3.1}$$

Assume

$$x = acos(\theta) \quad and \quad y = asin(\theta).$$

Then, we have

$$dx = -asin(\theta)d\theta \quad and \quad dy = acos(\theta)d\theta.$$

The line integral above becomes

$$\oint_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy = \int_0^{2\pi} \frac{a^2 sin^2\theta}{a^2} d\theta + \frac{a^2 cos^2\theta}{a^2} d\theta = \int_0^{2\pi} d\theta = 2\pi. \tag{3.2}$$

A Matlab code is written as:

```
syms theta a x y
x = a*cos(theta)
y = a*sin(theta)
dx = diff(x)
dy = diff(y)
result = int((-y*dx + x*dy)/(x^2 + y^2), theta, 0, 2*pi)
```

$$\boxed{the\ answer\ is\ 2\pi}$$

The corresponding Python code is:

```
import sympy as sp
theta, a, x, y = sp.symbols("theta a x y")
x = a*sp.cos(theta)
```

$$y = a * sp.sin(theta)$$

$$dx = sp.diff(x, theta)$$

$$dy = sp.diff(y, theta)$$

$$f = (-y * dx + x * dy)/(x ** 2 + y ** 2)$$

$$result = sp.integrate(f, (theta, 0, 2 * sp.pi))$$

$$print(result)$$

$$\boxed{the\ answer\ is\ 2\pi}$$

(b) Since $\vec{F}$ in Eq. A3 has a singularity at $(x, y) = (0, 0)$, we consider a region D_s that is bounded by C_1 and C_2, which are counterclockwise-oriented and clockwise-oriented circles, respectively, as shown in Figure 2. Therefore, $\vec{F}$ has continuous partial derivatives on an open region that contains D_s. Using Eq. A3, we have

$$\nabla \times \vec{F} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0. \tag{3.3}$$

Therefore, we have

$$\int\int_{D_s} \nabla \times \vec{F} \cdot \vec{k} dA = 0. \tag{3.4}$$

A Matlab code (with $crulFn$ indicating $\nabla \times \vec{F} \cdot \vec{k}$) is written as:

$$syms\ \ x\ \ y\ \ z$$

$$F = [-y/(x\hat{}2 + y\hat{}2), x/(x\hat{}2 + y\hat{}2), 0]$$

$$curlF = curl(F, [x, y, z])$$

$$curlFn = simplify(dot(curlF, [1, 1, 1]))$$

$$\boxed{\text{the curlFn value is 0, so answer is 0}}$$

The corresponding Python code is:

$$from\ \ sympy.physics.vector\ \ import\ \ *$$

$$x, y, z = sp.symbols(\text{“}x\ \ y\ \ z\text{”})$$

$$R = ReferenceFrame('R')$$

$$F = -R[1]/(R[0] ** 2 + R[1] ** 2) * R.x + R[0]/(R[0] ** 2 + R[1] ** 2) * R.y + 0 * R.z$$

$$curlF = curl(F, R)$$

$$curlFn = sp.simplify(dot(curlF, R.x + R.y + R.z))$$

$$print(curlFn)$$

$\boxed{\text{the curlFn value is 0, so answer is 0}}$

(c) From equations (3.2) and (3.4), we observe different answers. Why? It is because $\vec{F}$ has a singularity at $(x, y) = (0, 0)$. If we assume $C = C_1$ and $C' = -C_2$ in Figure 2, Green's Theorem can be extended to the region D_s with the positively orentied boundary $C \cup (-C')$, leading to

$$\int\int_{D_s} \nabla \times \vec{F} \cdot \vec{k} dA = \oint_C \vec{F} \cdot \vec{T} ds + \oint_{-C'} \vec{F} \cdot \vec{T} ds.$$

From the above equation and Eq. (3.4), we obtain

$$\oint_C \vec{F} \cdot \vec{T} ds = \oint_{C'} \vec{F} \cdot \vec{T} ds. \tag{3.5}$$

(Optional) Note that $\nabla \cdot \vec{F} = 0$ when $r \neq 0$ and $r = \sqrt{x^2 + y^2}$, as shown below.

$$\nabla \cdot \vec{F} = (\frac{\partial P}{\partial X} + \frac{\partial Q}{\partial y}) = \frac{\partial}{\partial x}(\frac{-y}{x^2+y^2}) + \frac{\partial}{\partial y}(\frac{x}{x^2+y^2}) = \frac{2xy}{x^2+y^2} + \frac{-2xy}{x^2+y^2} = 0. \tag{3.6}$$

(Optional) Can we can obtain a potential function f such that $\vec{F} = \nabla f$?

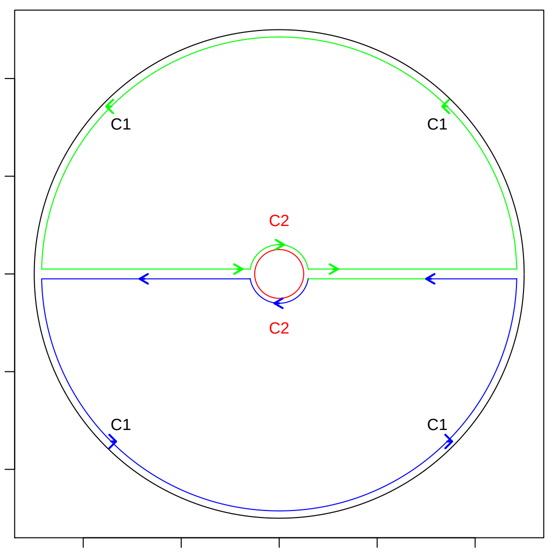

Figure 2: A diagram for the curves C and C', which are indicated by C_1 and $-C_2$, respectively.

4: Let $\vec{F}$ be the vector field in Eq. A4.
(a) Find $\oint_C \vec{F} \cdot \vec{n} ds$;
(b) Find $\int\int_{D_s} \nabla \cdot \vec{F} dA$;
(c) Compare the results in (a) and (b).

(a) Using Eq. (A4), we have

$$\oint_C \vec{F} \cdot \vec{n} ds = \oint_C P dy - Q dx = \int_C \frac{x}{x^2+y^2} dy - \frac{y}{x^2+y^2} dx. \tag{4.1}$$

Assume

$$x = acos(\theta) \quad and \quad y = asin(\theta).$$

We have

$$dx = -asin(\theta)d\theta \quad and \quad dy = acos(\theta)d\theta.$$

Thus, the line integral above becomes

$$\oint_C \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx = \int_0^{2\pi} \frac{a^2cos^2\theta}{a^2}d\theta + \frac{a^2sin^2\theta}{a^2}d\theta = \int_0^{2\pi} d\theta = 2\pi. \tag{4.2}$$

A Matlab code is written as:

$$syms \quad theta \quad a \quad x \quad y$$

$$x = a*cos(theta)$$

$$y = a*sin(theta)$$

$$dx = diff(x)$$

$$dy = diff(y)$$

$$result = int((x*dy - y*dx)/(x\hat{}2 + y\hat{}2), theta, 0, 2*pi)$$

$$\boxed{the \ answer \ is \ 2\pi}$$

The corresponding Python code is:

$$import \quad sympy \quad as \quad sp$$

$$theta, x, y = sp.symbols(``theta \quad x \quad y")$$

$$x = a*sp.cos(theta)$$

$$y = a*sp.sin(theta)$$

$$dx = sp.diff(x, theta)$$

$$dy = sp.diff(y, theta)$$

$$f = (x*dy - y*dx)/(x**2 + y**2)$$

$$result = sp.integrate(f, (theta, 0, 2*sp.pi))$$

$$print(result)$$

$$\boxed{the \ answer \ is \ 2\pi}$$

(b) Since $\vec{F}$ in Eq. A4 has a singularity at $(x, y) = (0,0)$, we consider a region D_s that is bounded by C_1 and C_2, which are counterclockwise-oriented and clockwise-oriented circles, respectively, as shown in Figure 2. Therefore, $\vec{F}$ has continuous partial derivatives on an open region that contains D_s. Using Eq. A4, we have

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}\Big(\frac{x}{x^2+y^2}\Big) + \frac{\partial}{\partial y}\Big(\frac{y}{x^2+y^2}\Big) = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0, \tag{4.3}$$

which leads to

$$\int\int_{D_s} \nabla \cdot \vec{F} dA = 0. \tag{4.4}$$

A Matlab code (with div indicating $\nabla \cdot \vec{F}$) is written as:

```
syms x y
F = [x/(x^2 + y^2), y/(x^2 + y^2)]
div = simplify(divergence(F))
```

$\boxed{\text{the div is 0, so answer is 0}}$

The corresponding Python code is:

```
import sympy as sp
from sympy.physics.vector import *
x, y = sp.symbols("x y")
R = ReferenceFrame('R')
F = R[0]/(R[0]**2 + R[1]**2)*R.x + R[1]/(R[0]**2 + R[1]**2)*R.y + 0*R.z
div = sp.simplify(divergence(F,R))
print(div)
```

$\boxed{\text{the div is 0, so answer is 0}}$

(c) From equations (4.2) and (4.4), we observe different answers, which are related to the singularity at $(x, y) = (0,0)$. If we assume $C = C_1$ and $C' = -C_2$ in Figure 2, Green's Theorem in the normal form can be extended to the region D_s with the positively oriented boundary $C \cup (-C')$, leading to

$$\int\int_{D_s} \nabla \cdot \vec{F} dA = \oint_C \vec{F} \cdot \vec{n} ds + \oint_{-C'} \vec{F} \cdot \vec{n} ds.$$

With the above equation and Eq. (4.4), we have

$$\oint_C \vec{F} \cdot \vec{n} ds = \oint_{C'} \vec{F} \cdot \vec{n} ds. \tag{4.5}$$

(Optional) Note that $\nabla \times \vec{F} = 0$ when $r \neq 0$ and $r = \sqrt{x^2+y^2}$, as shown below.

$$\nabla \times \vec{F} \cdot \vec{k} = \left(\frac{\partial Q}{\partial X} - \frac{\partial P}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{x^2+y^2} - \frac{-2xy}{x^2+y^2} = 0. \tag{4.6}$$

(Optional) Here we discuss how to obtain a potential function for the vector field in Eq. A4, $(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. By the definition of a potential function, $\vec{F} = \nabla f$, we have

$$f_x = \frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}, \tag{4.7a}$$

and

$$f_y = \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}. \tag{4.7b}$$

From Eq. (4.7a), we have $f = ln\sqrt{x^2+y^2} + g(y)$. We then calculate f_y, which is equal to $\frac{y}{x^2+y^2} + g_y$, and plug it into Eq. (4.7b) to obtain g_y=0 and, thus, $g = c$, here c is a constant. Without loss of generality, we can choose $c = 0$ and thus have the potential function as follows:

$$f = ln\sqrt{x^2+y^2}. \tag{4.8}$$

5: Let $\vec{F}$ be the vector in Eq. A2 and f be the potential function. Let C_0 be any curve from point A (x_1, y_1) to point B (x_2, y_2). Therefore, C_0 can be either $C_1 + C_2$ or C_3, as shown in Figure 3.
(a) Find $\int_{C_1+C_2} \vec{F} \cdot d\vec{r}$;
(b) Find $\int_{C_3} \vec{F} \cdot d\vec{r}$;
(c) Find a potential function such that $\vec{F} = \nabla f$;
(d) Calculate $\int_A^B \nabla f \cdot d\vec{r}$;
(e) Compare the results from (a)-(d).

(a)

$$\int_{C_1+C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} (xdx + ydy) + \int_{C_2} (xdx + ydy)$$
$$= \int_{x_1}^{x_2} xdx + \int_{y_1}^{y_2} ydy = \frac{1}{2}\left(x_2^2 - x_1^2 + y_2^2 - y_1^2\right) \tag{5.1}$$

A Matlab code is written as:

$$syms \;\; x \;\; y \;\; x1 \;\; y1 \;\; x2 \;\; y2 \;\; t$$
$$resulta = int(x, x1, x2) + int(y, y1, y2)$$

$$\boxed{the\ answer\ is:\ \left(x_2^2 - x_1^2 + y_2^2 - y_1^2\right)/2}$$

The corresponding Python code is:

$$import\ \ sympy\ \ as\ \ sp$$

$$x, y, x1, x2, y1, y2, t = sp.symbols(\text{“}x\ \ y\ \ x1\ \ x2\ \ y1\ \ y2\ \ t\text{”})$$

$$resulta = sp.integrate(x, (x, x1, x2)) + sp.integrate(y, (y, y1, y2))$$

$$print(resulta)$$

$$\boxed{the\ \ answer\ \ is:\ \left(x_2^2 - x_1^2 + y_2^2 - y_1^2\right)/2}$$

(b) Along the line segment C_3, we have $(x, y) = (x_1, y_1) + t(x_2 - x_1, y_2 - y_1)$, $t = 0 - 1$, and $(dx, dy) = (x_2 - x_1, y_2 - y_1)dt$. The line integral along C_3 is:

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_{C_3} (xdx + ydy) = \int_0^1 \left[\left(x_1 + t(x_2 - x_1)\right)(x_2 - x_1)dt + \left(y_1 + t(y_2 - y_1)\right)(y_2 - y_1)dt\right]$$

$$= x_1(x_2 - x_1)t\Big|_0^1 + (x_2 - x_1)^2 \frac{t^2}{2}\Big|_0^1 + y_1(y_2 - y_1)t\Big|_0^1 + (y_2 - y_1)^2 \frac{t^2}{2}\Big|_0^1$$

$$= x_1(x_2 - x_1) + \frac{1}{2}(x_2 - x_1)^2 + y_1(y_2 - y_1) + \frac{1}{2}(y_2 - y_1)^2$$

$$= \frac{1}{2}\left(x_2^2 - x_1^2 + y_2^2 - y_1^2\right). \tag{5.2}$$

A Matlab code is written as:

$$resultb = int(((x1 + t * (x2 - x1)) * (x2 - x1) + (y1 + t * (y2 - y1)) * (y2 - y1)), t, 0, 1)$$

$$\boxed{the\ \ answer\ \ is:\ \left(x_2^2 - x_1^2 + y_2^2 - y_1^2\right)/2}$$

The corresponding Python code is:

$$f = (x1 + t * (x2 - x1)) * (x2 - x1) + (y1 + t * (y2 - y1)) * (y2 - y1)$$

$$resultb = sp.integrate(f, (t, 0, 1))$$

$$print(resultb)$$

$$\boxed{the\ \ answer\ \ is:\ \left(x_2^2 - x_1^2 + y_2^2 - y_1^2\right)/2}$$

(c) For the vector field in Eq. A2, $\vec{F} = (x, y)$, and the definition of a potential function, $\vec{F} = \nabla f$, we have

$$f_x = \frac{\partial f}{\partial x} = x, \tag{5.3a}$$

and

$$f_y = \frac{\partial f}{\partial y} = y. \tag{5.3b}$$

From Eq. (5.3a), we have $f = x^2/2 + g(y)$, giving $f_y = g_y$. Plugging $f_y = g_y$ into Eq. (5.3b), we obtain $g = y^2/2 + c$, where c is a constant. Without loss of generality, we can choose $c = 0$ and thus have the potential function as follows.

$$f = \frac{1}{2}(x^2 + y^2). \tag{5.4}$$

A Matlab code is written as:

$$fx = x$$

$$fy = y$$

$$f = int(fx, x) + int(fy, y)$$

$$\boxed{the\ answer\ is : (x^2 + y^2)/2}$$

The corresponding Python code is:

$$fx = x$$

$$fy = y$$

$$f = sp.integrate(fx, x) + sp.integrate(fy, y)$$

$$\boxed{the\ answer\ is : (x^2 + y^2)/2}$$

(d) Therefore,

$$\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \nabla f \cdot dr = f(A) - f(B) = \frac{1}{2}(x_2^2 + y_2^2 - x_1^2 - y_1^2). \tag{5.5}$$

A Matlab code is written as:

$$resultd = subs(f, [x, y], [x2, y2]) - subs(f, [x, y], [x1, y1])$$

$$\boxed{the\ answer\ is : (x_2^2 + y_2^2 - x_1^2 - y_1^2)/2}$$

The corresponding Python code is:

$$resultd = f.subs([(x, x2), (y, y2)]) - f.subs([(x, x1), (y, y1)])$$

$$print(resultd)$$

$$\boxed{the\ answer\ is : (x_2^2 + y_2^2 - x_1^2 - y_1^2)/2}$$

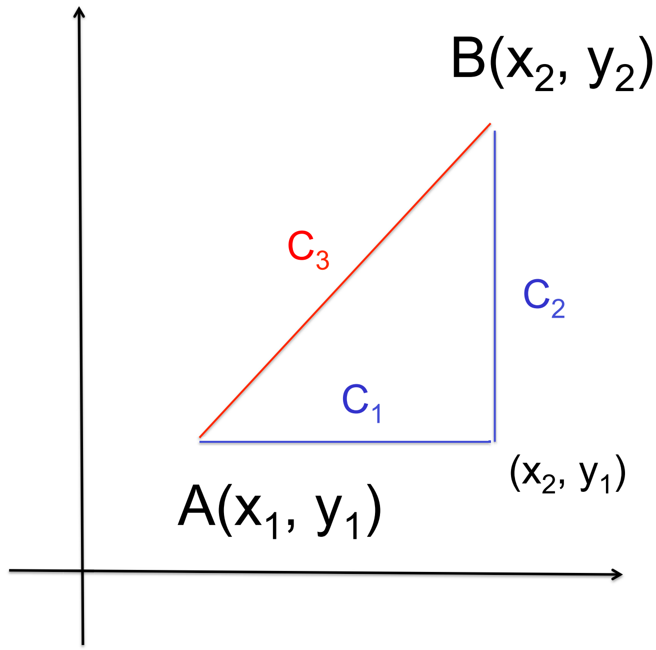

Figure 3: A diagram for the line segments C_1, C_2 and C_3.

(e) The integrals in (a) and (b) provide the same answer as that in (d) for the integral using the potential function. These results indicate the path independence of line integrals.

6: Let $\vec{F}$ represent the 3D vector field,

$$\vec{F} = \frac{(x, y, z)}{r^3}, \qquad r = \sqrt{x^2 + y^2 + z^2},$$

S_a be the surface $x^2 + y^2 + z^2 = a^2$, S_ϵ be the surface $x^2 + y^2 + z^2 = \epsilon^2$, and D_3 be the region $0 < \epsilon^2 \leq x^2 + y^2 + z^2 \leq a^2$.
(a) Find the net outward flux, i.e., $\iiint_{D_3} \nabla \cdot \vec{F} dV$;
(b) Find the outward flux across the sphere S_a, i.e., $\iint_{S_a} \vec{F} \cdot \vec{n} dS$;
(c) Compare the results in (a) and (b).

(a) Let (P, Q, R) represent the vector $\vec{F}$, where $\nabla \cdot \vec{F} = P_x + Q_y + R_z$. We first calculate P_x, Q_y and R_z as follows.

$$P_x = \frac{\partial}{\partial x} \frac{x}{r^3} = \frac{y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^{5/2}} = \frac{y^2 + z^2 - 2x^2}{r^5}.$$

Similarly, we can obtain

$$Q_y = \frac{x^2 + z^2 - 2y^2}{r^5} \quad \text{and} \quad R_z = \frac{x^2 + y^2 - 2z^2}{r^5}.$$

Therefore, we have

$$\nabla \cdot \vec{F} = P_x + Q_y + R_z = 0 \quad \text{when } r \neq 0, \tag{6.1}$$

leading to

$$\iiint_{D_3} \nabla \cdot \vec{F} dV = 0. \tag{6.2}$$

A Matlab code is written as:

$$syms \quad x \quad y \quad z \quad r$$

$$r = sqrt(x\hat{}2 + y\hat{}2 + z\hat{}2)$$

$$F = [x/r\hat{}3, y/r\hat{}3, z/r\hat{}3]$$

$$div = simplify(divergence(F))$$

or

$$Px = simplify(diff(x/r\hat{}3, x))$$

$$Qy = simplify(diff(y/r\hat{}3, y))$$

$$Rz = simplify(diff(z/r\hat{}3, z))$$

$$result = simplify(Px + Qy + Rz)$$

$$\boxed{\text{the div is 0, so answer is 0}}$$

The corresponding Python code is:

$$import \quad sympy \quad as \quad sp$$

$$from \quad sympy.physics.vector \quad import \quad *$$

$$R = ReferenceFrame('R')$$

$$r = sp.sqrt(R[0]**2 + R[1]**2 + R[2]**2)$$

$$F = R[0]/r**3*R.x + R[1]/r**3*R.y + R[2]/r**3*R.z$$

$$div = sp.simplify(divergence(F, R))$$

$$print(div)$$

or

$$r = sp.sqrt(x**2 + y**2 + z**2)$$

$$Px = sp.simplify(sp.diff(x/r**3, x))$$

$$Qy = sp.simplify(sp.diff(y/r**3, y))$$

$$Rz = sp.simplify(sp.diff(z/r**3, z))$$

$$result = sp.simplify(Px + Qy + Rz)$$

$$print(result)$$

$$\boxed{\text{the div is 0, so answer is 0}}$$

(b) By defining $g = x^2 + y^2 + z^2 = a^2$, a normal vector is determined as

$$\vec{n}_1 = \frac{\nabla g}{|\nabla g|} = \frac{(x, y, z)}{a}.$$

Thus, we have

$$\iint_{S_a} \vec{F} \cdot \vec{n}_1 dS = \iint_{S_a} \frac{(x, y, z)}{a^3} \cdot \frac{(x, y, z)}{a} dS = \iint_{S_a} \frac{dS}{a^2} = \frac{4\pi a^2}{a^2} = 4\pi. \tag{6.3}$$

A Matlab code is written as:

$$syms \quad x \quad y \quad z \quad real$$
$$a = sqrt(x\hat{\ }2 + y\hat{\ }2 + z\hat{\ }2))$$
$$n1 = [x, y, z]/a$$
$$F = [x, y, z]/a\hat{\ }3$$
$$Fn1 = simplify(dot(n1, F))$$
$$result = 4 * pi * a\hat{\ }2 * Fn1$$

$$\boxed{the \ answer \ is: \ 4\pi}$$

The corresponding Python code is:

$$a = sp.sqrt(R[0] ** 2 + R[1] ** 2 + R[2] ** 2)$$
$$n1 = R[0]/a * R.x + R[1]/a * R.y + R[2]/a * R.z$$
$$F = R[0]/a ** 3 * R.x + R[1]/a ** 3 * R.y + R[2]/a ** 3 * R.z$$
$$Fn1 = sp.simplify(dot(n1, F))$$
$$result = 4 * sp.pi * a ** 2 * Fn1$$
$$print(result)$$

$$\boxed{the \ answer \ is: \ 4\pi}$$

(c)

$$\iiint_{D_3} \nabla \cdot \vec{F} dV = 0 = \iint_{S_a} \vec{F} \cdot \vec{n}_1 dS + \iint_{S_\epsilon} \vec{F} \cdot (-\vec{n}_0) dS,$$

where $-\vec{n}_0$ is the normal vector of the surface S_ϵ, as shown in Figure 4. Thus, we have

$$\iint_{S_a} \vec{F} \cdot \vec{n}_1 dS = \iint_{S_\epsilon} \vec{F} \cdot \vec{n}_0 dS = 4\pi. \tag{6.4}$$

7: Here, for a comparison with problem 6, we re-formulate the problem 4 as follows. Let $\vec{F}$ represent the 2D vector field,

$$\vec{F} = \frac{(x, y)}{r^2}, \quad r = \sqrt{x^2 + y^2},$$

and D_s be the region $0 < \epsilon^2 \leq x^2 + y^2 \leq a^2$.
(a) Find the net outward flux, i.e., $\iint_{D_s} \nabla \cdot \vec{F} dS$;
(b) Find the outward flux across the **circle** $x^2 + y^2 = a^2$, i.e., $\oint_C \vec{F} \cdot \vec{n} ds$;
(c) Compare the results in (a) and (b).

(a) Based on what has been discussed in problem 4, we have

$$\nabla \cdot \vec{F} = 0 \text{ when } r \neq 0, \tag{7.1}$$

and

$$\iint_{D_s} \nabla \cdot \vec{F} dS = 0. \tag{7.2}$$

A Matlab code is written as:

```
syms x y r
r = sqrt(x^2 + y^2)
F = [x/r^2, y/r^2]
div = simplify(divergence(F))
```

or

```
Px = simplify(diff(x/r^2, x))
Qy = simplify(diff(y/r^2, y))
result = simplify(Px + Qy)
```

$$\boxed{\text{the div is 0, so answer is 0}}$$

The corresponding Python code is:

```
import sympy as sp
from sympy.physics.vector import *
R = ReferenceFrame('R')
r = sp.sqrt(R[0]**2 + R[1]**2)
F = R[0]/r**2*R.x + R[1]/r**2*R.y
div = sp.simplify(divergence(F,R))
print(div)
```

or

```
r = sp.sqrt(x**2 + y**2)
```

$$Px = sp.simplify(sp.diff(x/r**2, x))$$
$$Qy = sp.simplify(sp.diff(y/r**2, y))$$
$$result = sp.simplify(Px + Qy)$$
$$print(result)$$

$$\boxed{\text{the div is 0, so answer is 0}}$$

(b) By defining $g = x^2 + y^2 = a^2$, a normal vector is determined as

$$\vec{n}_1 = \frac{\nabla g}{|\nabla g|} = \frac{(x, y)}{a}.$$

Thus, we have

$$\oint_C \vec{F} \cdot \vec{n}_1 ds = \oint_C \frac{(x, y)}{a^2} \cdot \frac{(x, y)}{a} ds = \oint_C \frac{x^2 + y^2}{a^3} ds = \frac{1}{a} \oint_C ds = \frac{2\pi a}{a} = 2\pi. \tag{7.3}$$

A Matlab code is written as:

$$a = sqrt(x\hat{}2 + y\hat{}2)$$
$$n1 = [x, y]/a$$
$$F = [x, y]/a\hat{}2$$
$$Fn1 = simplify(dot(n1, F))$$
$$result = 2 * pi * a * Fn1$$

$$\boxed{the\ answer\ is : 2\pi}$$

The corresponding Python code is:

$$a = sp.sqrt(R[0]**2 + R[1]**2)$$
$$n1 = R[0]/a * R.x + R[1]/a * R.y$$
$$F = R[0]/a**2 * R.x + R[1]/a**2 * R.y$$
$$Fn1 = sp.simplify(dot(n1, F))$$
$$result = 2 * sp.pi * a * Fn1$$

$$\boxed{the\ answer\ is : 2\pi}$$

(c)

$$\iint_{D_s} \nabla \cdot \vec{F} dS = 0 = \oint_C \vec{F} \cdot \vec{n}_1 ds + \oint_{C'} \vec{F} \cdot (-\vec{n}_0) ds,$$

here $-\vec{n}_0$ is the normal vector of the circle C': $x^2 + y^2 = \epsilon^2$. Therefore, we have

$$\oint_C \vec{F} \cdot \vec{n}_1 ds = \oint_{C'} \vec{F} \cdot \vec{n}_0 ds = 2\pi. \tag{7.4}$$

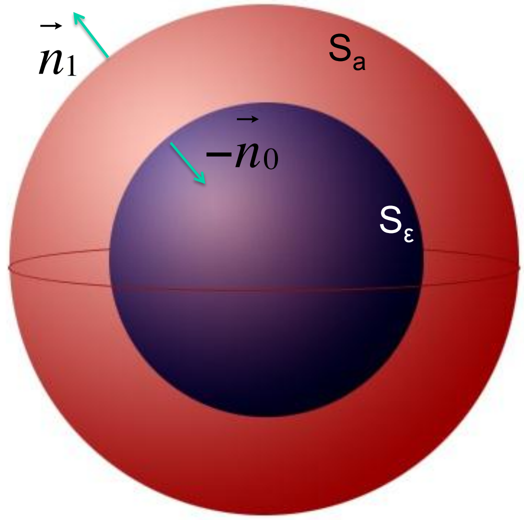

Figure 4: A diagram for the surfaces S_a and S_ϵ and their normal vectors, $\vec{n}_1$ and $\vec{n}_0$, respectively.

Table 1: A Summary

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) **i**, **j**, and **k** are unit vectors in the directions of increasing x, y, and z. **P, Q, and R** are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.

2. If the integral is independent of the path from A to B, its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem (Tangential Form) $\quad \iint_R \nabla \times \vec{F} \cdot \vec{k}\, dxdy = \oint \vec{F} \cdot d\vec{r}$

Stokes' Theorem $\quad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$

Green's Theorem (Normal Form) $\quad \iint_R \nabla \cdot \vec{F} dxdy = \oint \vec{F} \cdot \vec{n}\, ds$

Divergence Theorem $\quad \iiint_E \operatorname{div} \mathbf{F}\, dV = \iint \vec{F} \cdot \vec{n}\, dS$