STONE DUALITY FOR COMPLETENESS OF BOOLEAN ALGEBRA

Nov 12th ,2019 Jialu Bao

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Based on works by Simon Docherty

HIGH-LEVEL

(KRIPKE) FRAME

- It's from Kripke semantics, which are first conceived for modal logic and then adapted to other logics.
- In our context, a frame is a triple $\mathscr{K} = (X, \geq)$ *where*

  X is a set of worlds

  $\geq$ *is a preorder(reflexive and transitive) on X*

- A frame and a valuation determining whether a world satisfies a formula —> a model

(KRIPKE) FRAME

- It's from Kripke semantics, which are first conceived for modal logic and then adapted to other logics.
- More generally, a frame is a triple $\mathscr{K} = (X, \geq, \circ)$ *where*

  X *is a set of worlds*

  $\geq$ *is a preorder(reflexive and transitive) on* X

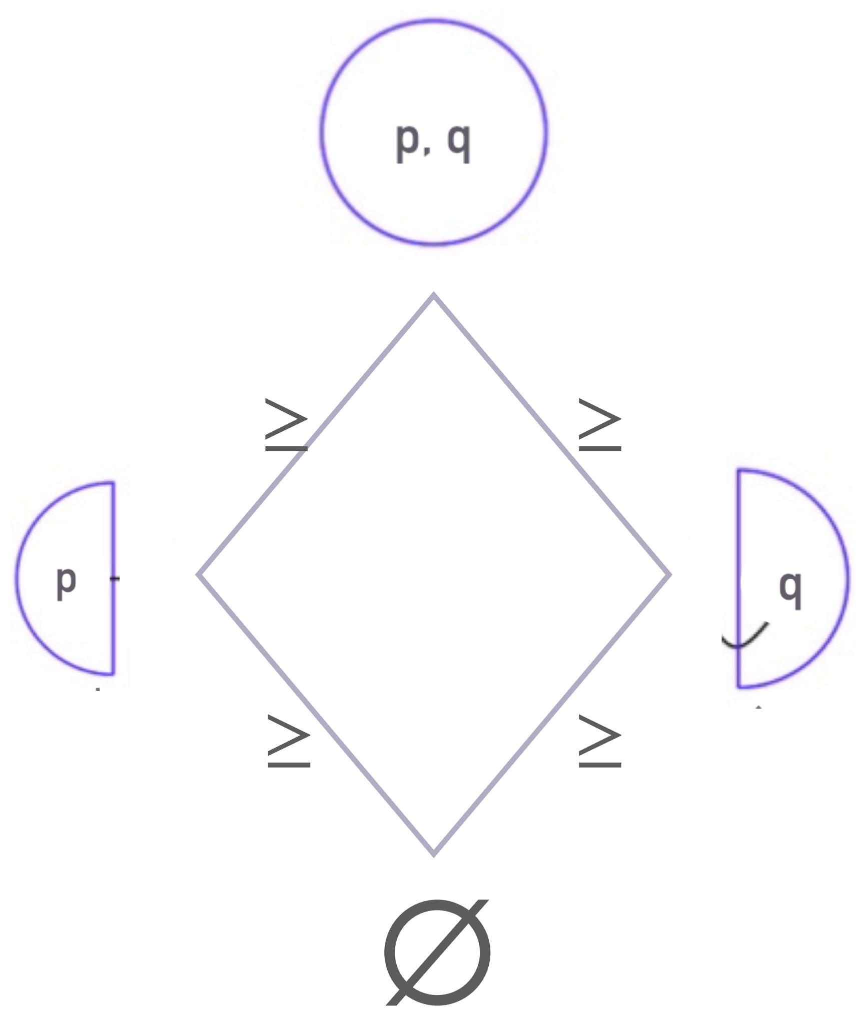
  For bunch implication $\circ : X \times X \to 2^X$ *is a binary operation*

  For modal logic, $\circ$ *is a binary relation* $X \times X$

- A frame and a valuation determining whether a world satisfies a formula —> a model

EXAMPLE. (INFORMAL)

- We can define semantics in a frame
  - eg. $p \wedge q$



BOOLEAN ALGEBRA

- Algebra: a set with some operators, satisfying some equations
- A Boolean Algebra is an algebra $\mathbb{A} = (A, \wedge, \vee, \rightarrow, \top, \bot)$ such that $(A, \wedge, \vee, \top, \bot)$ is a bounded distributive lattice and —> is a binary operation satisfying for all a, b, c

  $a \vee (a \rightarrow \bot)$

  $a \wedge b \leq c \textit{ iff } a \leq b \rightarrow c$

BOOLEAN ALGEBRA

- A Boolean Algebra is an algebra $\mathbb{A} = (A, \wedge, \vee, \rightarrow, \top, \bot)$ on some partially ordered set A such that

  $\wedge, \vee$ *are commutative and associative binary operators*

  $\wedge$ *Absorption* $: a \vee (a \wedge b) = a$

  $\vee$ *Absorbtion* $: a \wedge (a \vee b) = a$

  Distributive $: a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

  $\top \neq \bot$

  $\top$ *is the identity of* $\wedge : \top \wedge a = a$

  $\bot$ *is the identity of* $\vee : \bot \vee a = a$

  $a \vee (a \rightarrow \bot)$

  $a \wedge b \leq c \text{ iff } a \leq b \rightarrow c$

Distributive Lattice

Bounded Distributive Lattice

Exclude the middle

EXAMPLES:

- Two-Element Boolean Algebra

  $\mathbb{A} = (\{0,1\}, \wedge, \vee, \rightarrow, 1, 0)$ *where* $\vee, \wedge$ *are defined by*

∧	0	1
0	0	0
1	0	1

∨	0	1
0	0	1
1	1	1

- Power set $\mathbb{A} = (2^X, \cap, \cup, \rightarrow, X, \varnothing)$

STONE REPRESENTATION THEOREM

- Two-Element Boolean Algebra

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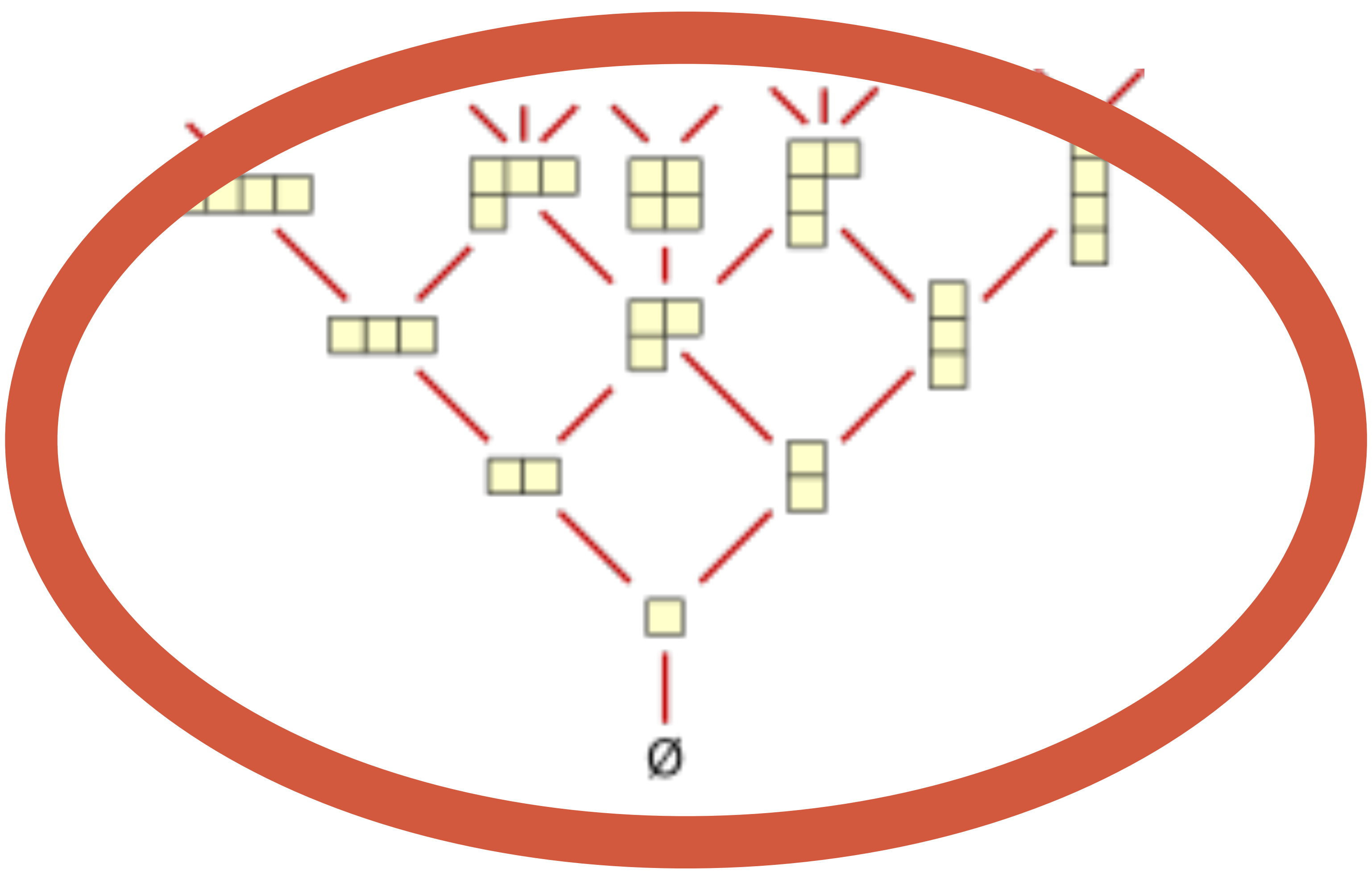
- Power set $\mathbb{A} = (2^X, \cap, \cup, \rightarrow, X, \varnothing)$
- Every Boolean algebra is isomorphic to a sub algebra of a power set algebra.

FILTERS

- A filter on a distributive lattice A is a non-empty subset F of A satisfying

Upwards Closure: $a \in F$ *and* $b \geq a$ *implies* $b \in F$

Meet Closure: $a, b \in F$ *implies* $a \wedge b \in F$



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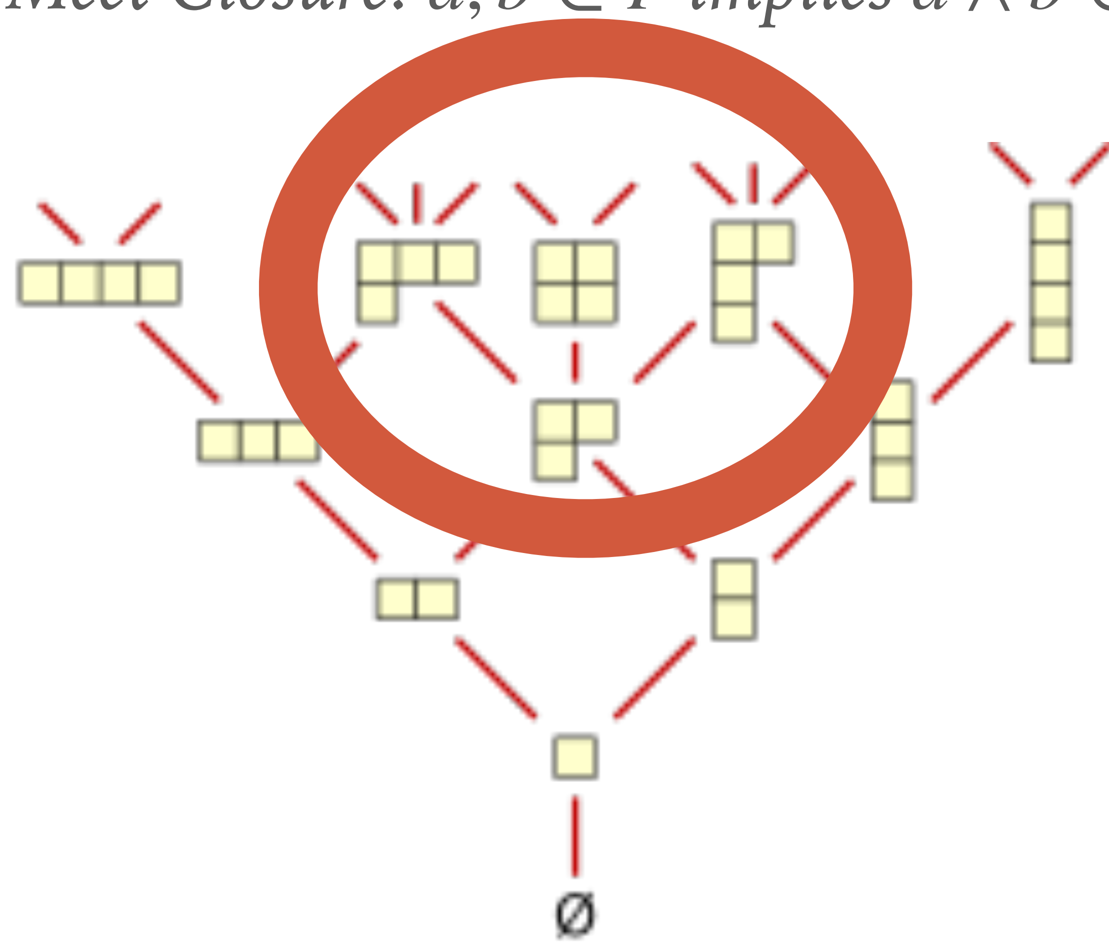

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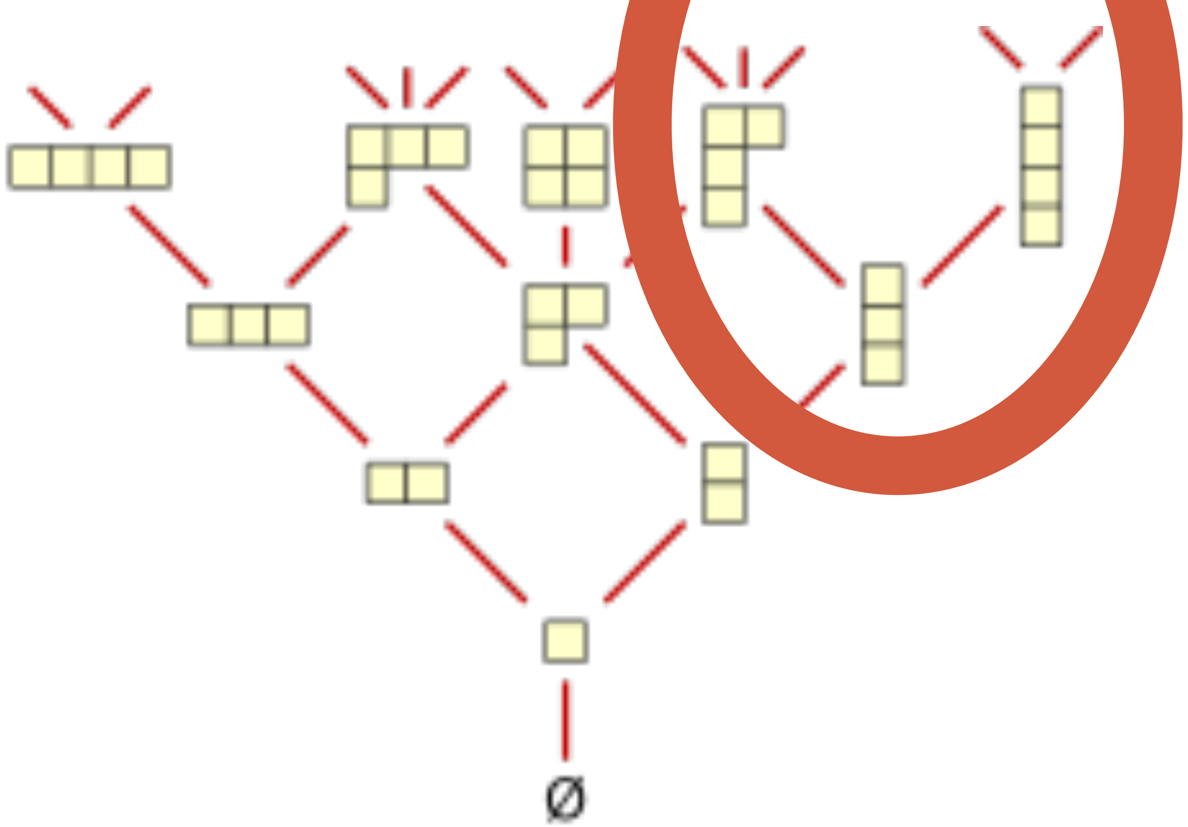

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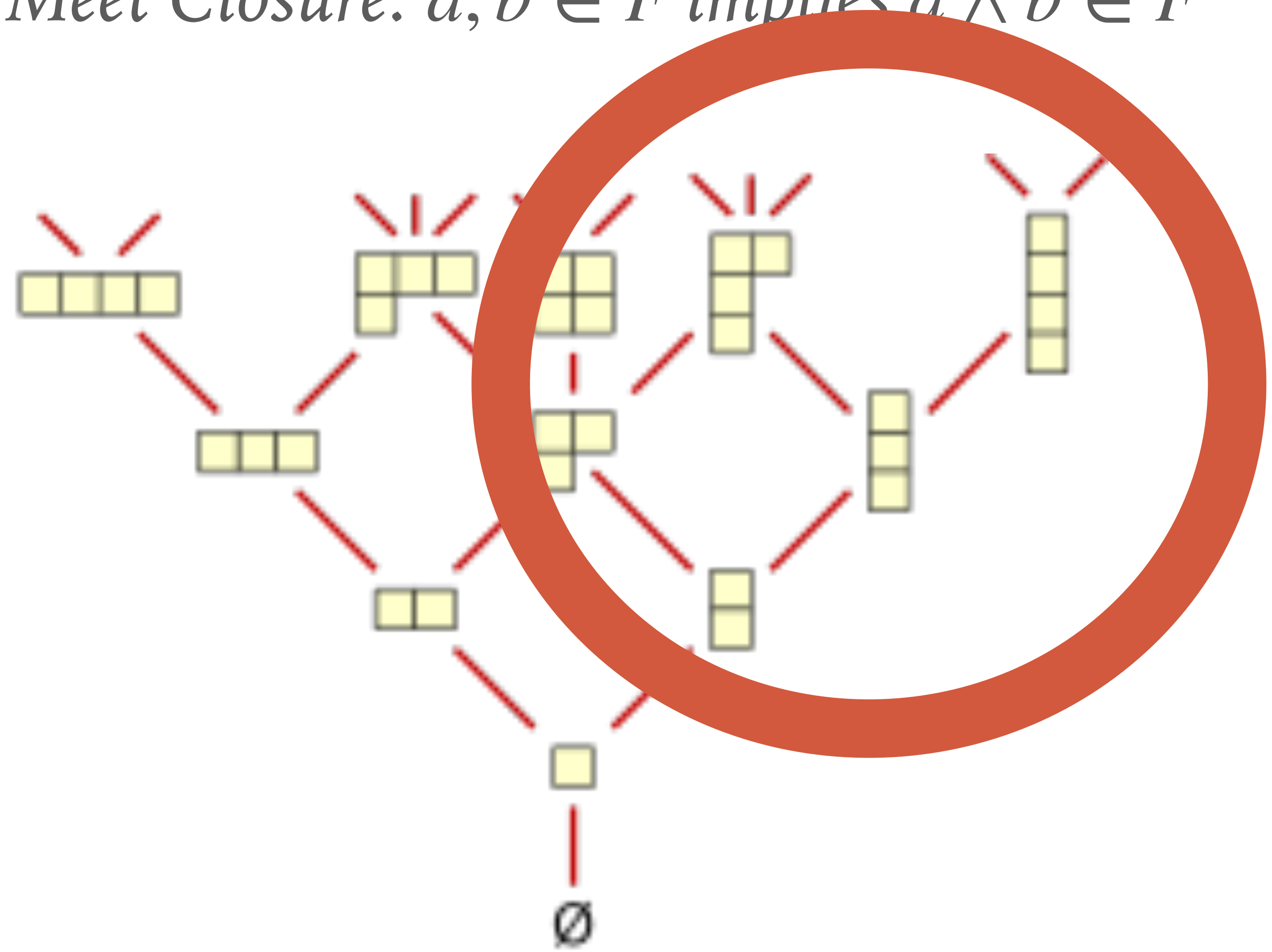

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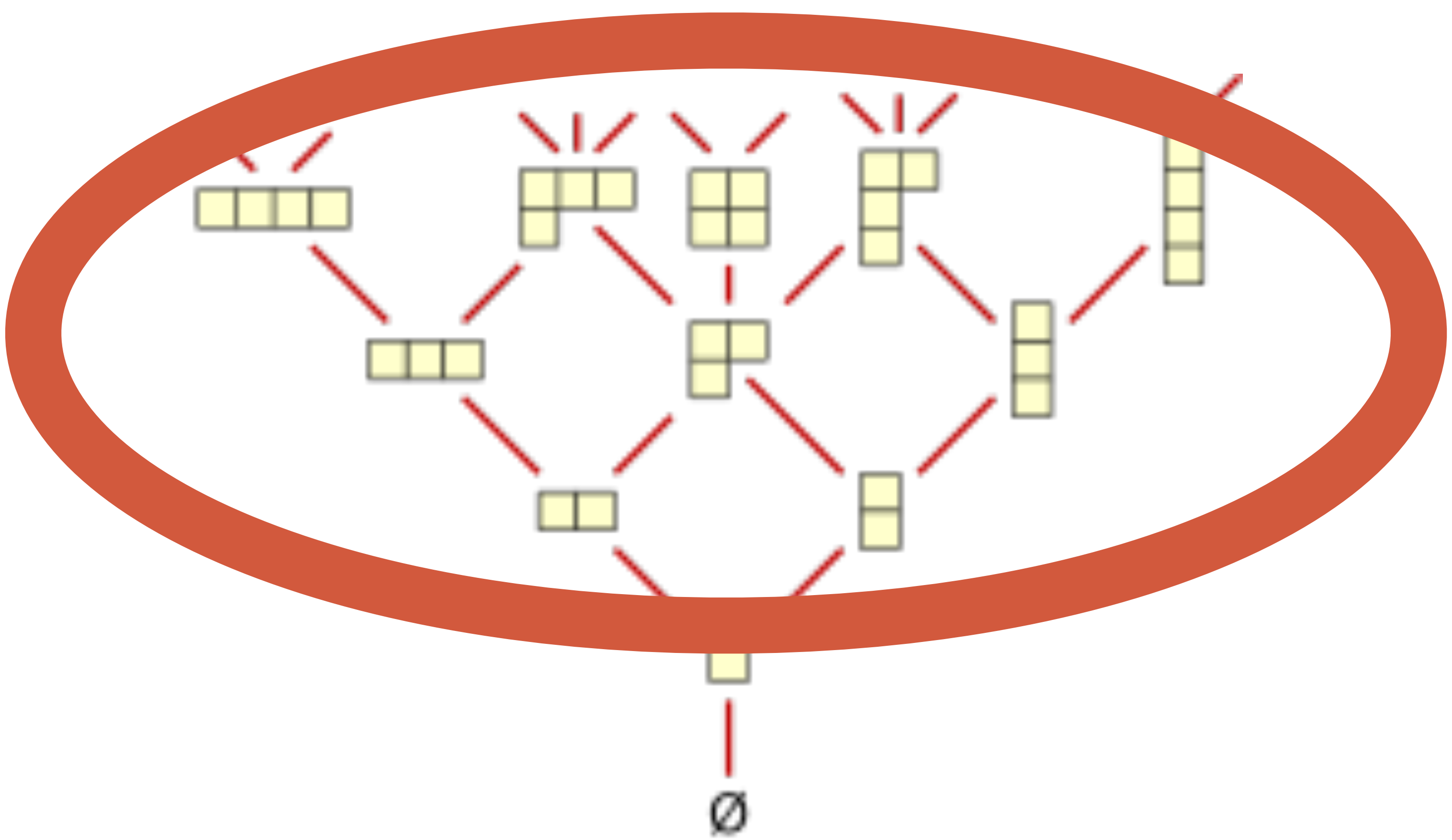

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- A filter is proper if it additionally satisfies $\bot \notin F$
- A proper filter is prime if it further satisfies

  $a \vee b \in F$ *implies* $a \in F$ *or* $b \in F$

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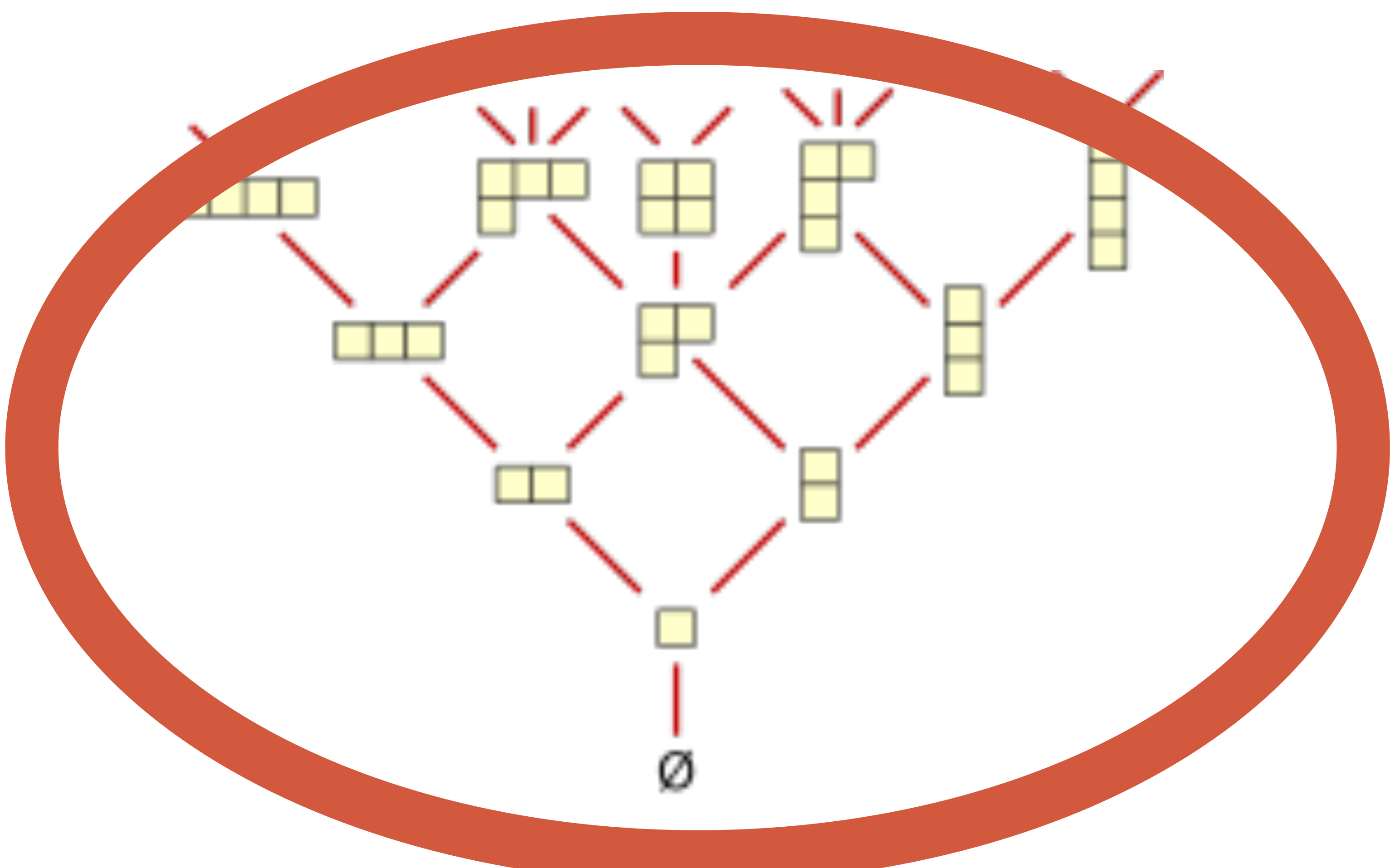

is a filter but is not proper

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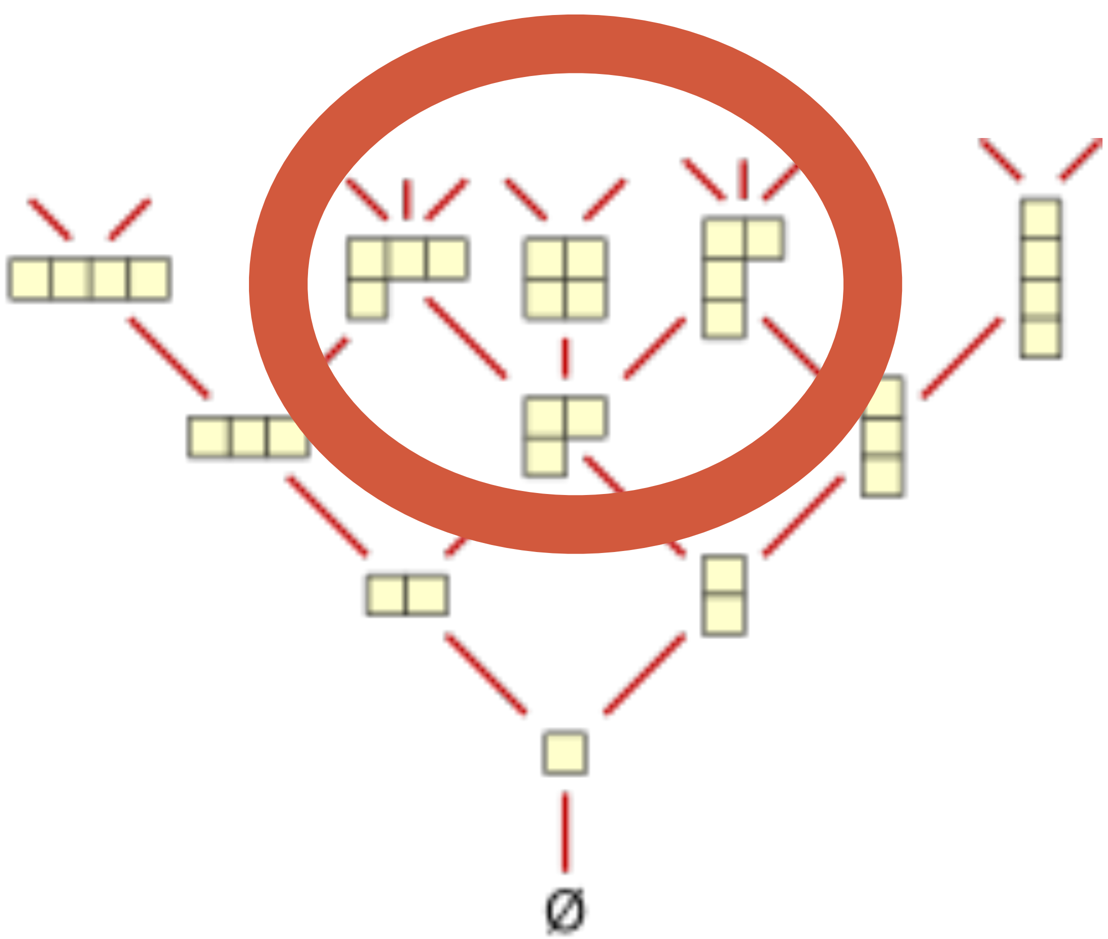
  $a \vee b \in F$ implies $a \in F$ or $b \in F$



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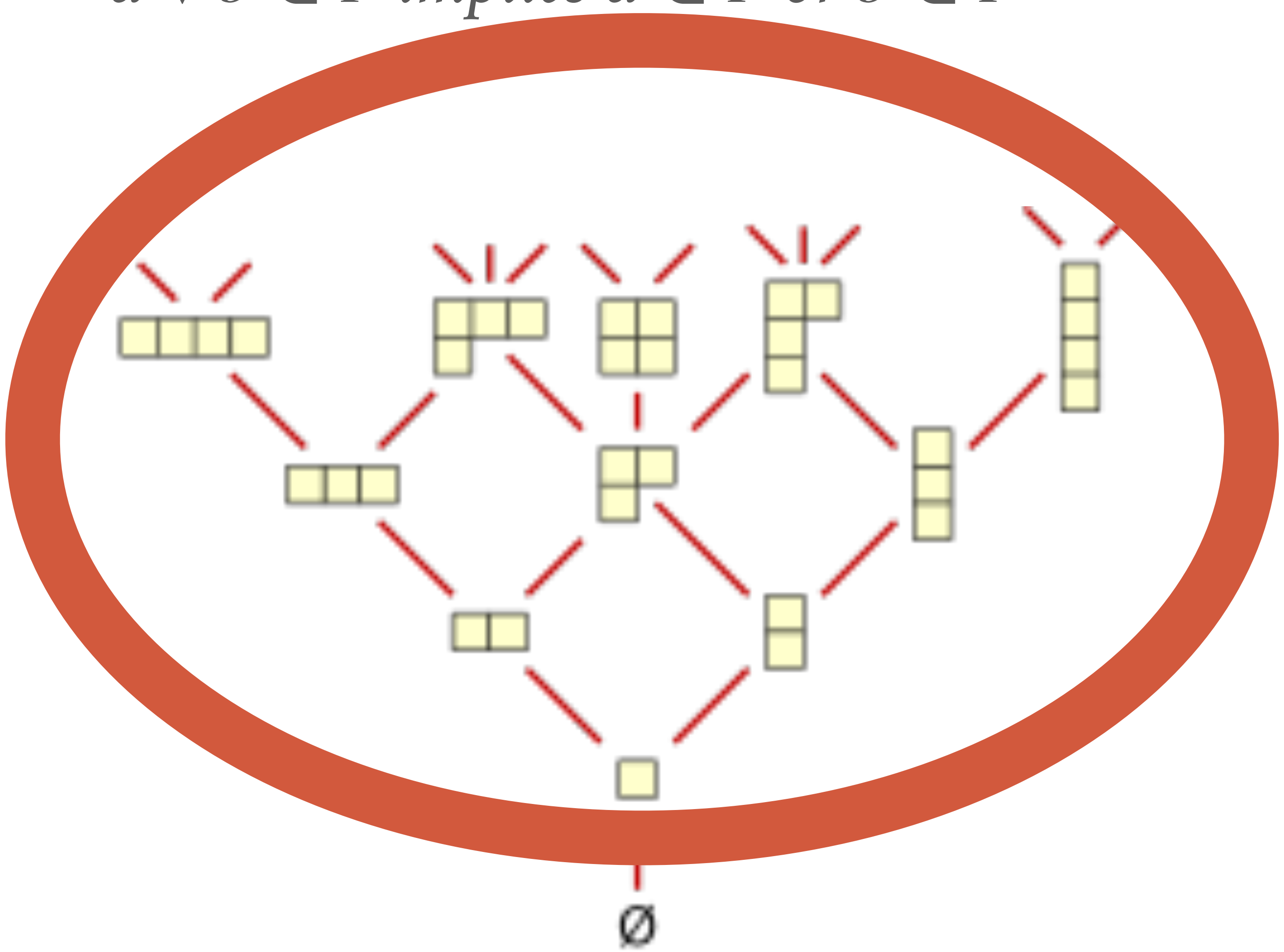

prime filter

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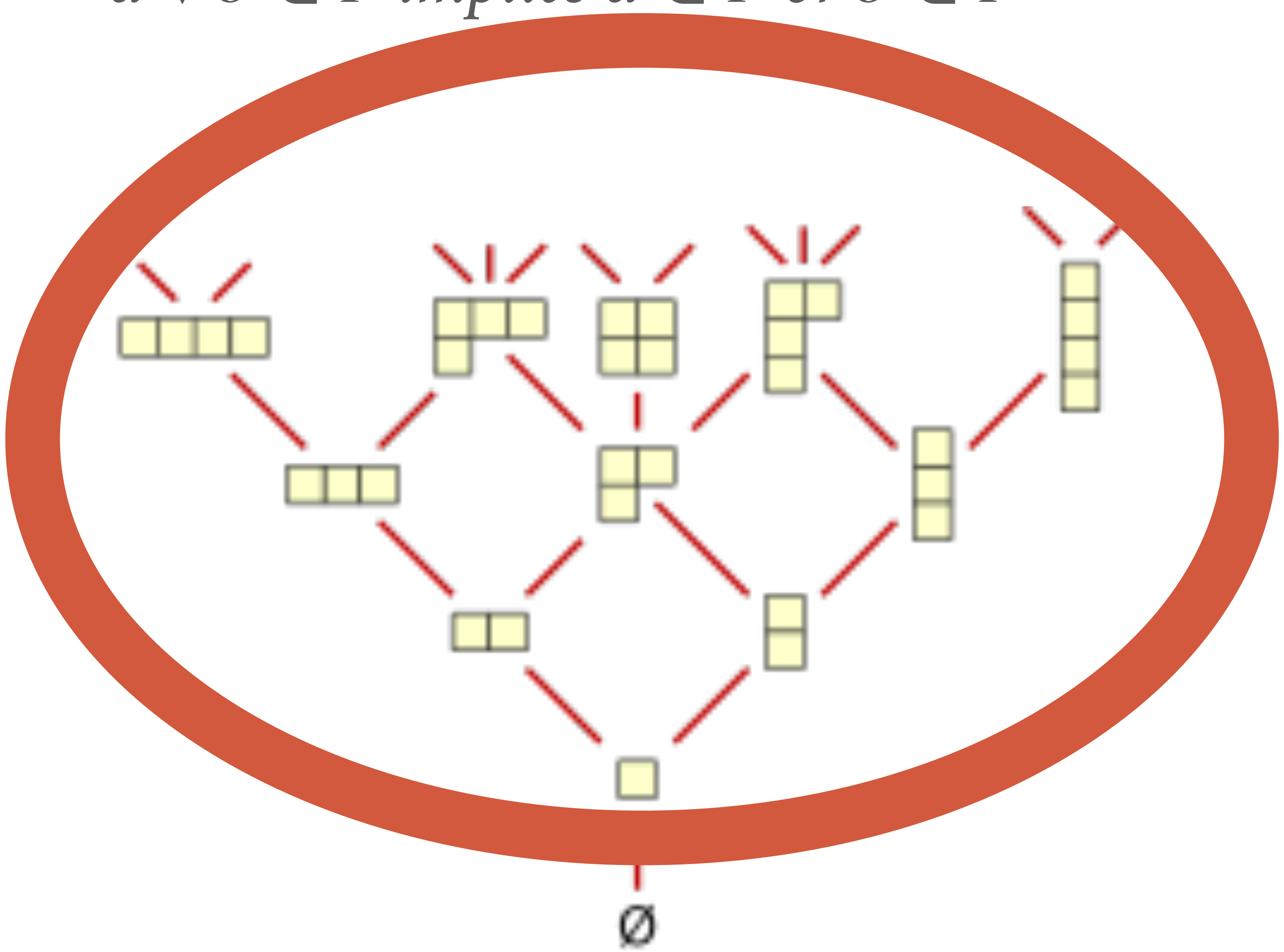

prime filter

FILTERS

- A filter can be seen as a propositional theory, closed under a simple form of logical consequence.
  - If $\perp$ is in the theory then that theory is 'improper' as it is inconsistent and thus contains every proposition.
  - Given a boolean algebra A, we can define a frame,

    $Pr(A) := (\textit{the set of prime filters of } A, \supseteq)$

    - we call it prime filter frame

STONE REPRESENTATION THEOREM

- Every Boolean algebra is isomorphic to a sub algebra of a power set algebra.
- Given a frame semantic $(X, \geqslant, \circ)$
  - define $Com(\mathscr{K}) = (\mathscr{P}_{\geq}(X), \cap, \cup, \Rightarrow_{\mathscr{K}}, X, \emptyset)$ *where*

    $\mathscr{P}_{\geq}(X) = \{A \subseteq X \mid if\ a \in A\ and\ b \geq a\ then\ b \in A\}$

    $A \Rightarrow_{B} \mathscr{K} = \{x \mid x' \geq x\ and\ x' \in A\ implies\ x' \in B\}$

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- A boolean algebra A is isomorphic to $Com(Pr(A))$
  - The isomorphism h can also be written as

    $h(a) = \{F \in Pr(A) \mid a \in F\}$

STONE REPRESENTATION THEOREM

- Every Boolean algebra is isomorphic to a sub algebra of a power set algebra.
- Given a frame semantic $(X, \succcurlyeq, \circ)$
  - define $Com(\mathscr{K}) = (\mathscr{P}_{\geq}(X), \cap, \cup, \Rightarrow_{\mathscr{K}}, X, \varnothing)$ *where*

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PROOF SKETCH

- A boolean algebra A is isomorphic to $Com(Pr(A))$
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    $h(a) = \{F \in Pr(A) \mid a \in F\}$

  - h is injective

    $h(a \to b) = h(a) \Rightarrow_{Pr(A)} h(b)$

- Pr(A) and topology on it forms a stone space
- With some topology, it's possible to show that the dual equivalence categories between Boolean algebra and stone space => Equivalence of Algebraic and Frame Semantics.

TOPOLOGICAL SPACE

- *A topologocial space is a pair* $\mathscr{K} = (X, \mathscr{O})$ *where* X *is a set and* $\mathscr{O} \subseteq 2^X$, *satisfying*

  $-\emptyset, X \in \mathscr{O}$;

  $-$*Given any* $O_i \in \mathscr{O}$ *indexed by* I, $\cup_{i \in I} O_i \in \mathscr{O}$;

  $-O_0, \ldots, O_n \in \mathscr{O}$ *implies* $\cap_{i=0}^{n} O_i \in \mathscr{O}$

PROOF SKETCH

- A boolean algebra A is isomorphic to $Com(Pr(A))$
  - The isomorphism h can also be written as

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EQUIVALENCE OF ALGEBRAIC AND FRAME SEMANTICS

- Equivalence of Algebraic and Frame Semantics.
- Given algebra A and frame K. Let [-] be an algebraic interpretation on A

valuation on $\mathscr{X}$. Define $\mathscr{V}_{[\![-]\!]} : \mathrm{Prop} \to \mathscr{P}(Pr(A))$ by $\mathscr{V}_{[\![-]\!]}(\mathrm{p}) = \theta_{\mathbb{A}}([\![\mathrm{p}]\!])$ and $[\![-]\!]_{\mathscr{V}}$ as the algebraic interpretation on $Com^{\mathscr{L}}(\mathscr{X})$ generated by $\mathscr{V}$. For all $\mathscr{L}$ formulae φ the following hold.

1. $x \vDash_{\mathscr{V}} \varphi$ *iff* $x \in [\![\varphi]\!]_{\mathscr{V}}$;
2. $[\![\varphi]\!] \in F$ *iff* $F \vDash_{\mathscr{V}_{[\![-]\!]}} \varphi$.

ALGEBRAIC COMPLETENESS

Theorem 7.2 (Algebraic Completeness (cf. [189])). *For all bunched logics* $\mathscr{L}$ *and* $\mathscr{L}$ *formulae* φ *and* ψ*, if* $[\![\varphi]\!] \leq [\![\psi]\!]$ *holds for all algebraic interpretations then* $\varphi \vdash \psi$ *is provable in the Hilbert system for* $\mathscr{L}$*.*

Proof. We reason contrapositively: suppose $\varphi \vdash \psi$ is not provable in the Hilbert system for $\mathscr{L}$. Consider the Lindenbaum-Tarski algebra over the propositional variables occurring in φ and ψ with the interpretation given by sending each formula to its equivalence class. $[\varphi]_\equiv \leq [\psi]_\equiv$ iff $[\varphi \to \psi]_\equiv = [\top]_\equiv$ iff $\top \vdash \varphi \to \psi$ provable iff $\varphi \vdash \psi$ (using the definition of Heyting implication and the deduction theorem for bunched logics), and so $[\varphi]_\equiv \not\leq [\psi]_\equiv$. □

THE TECHNIQUE

- Given a set L of inference rules and semantic frame F
  - Find a algebra A that is algebraically sound and complete of L
  - Define morphisms on F to mirror the operators in A
  - Show Com(F) is a algebra A
  - Show Pr(A) is a semantic frame F
  - => Equivalence of Algebraic and Frame Semantics.
  - A is algebraically sound and complete w.r.t L implies that F is sound and complete w.r.t. L too

GENERALIZATION

- Boolean algebra — Stone space

THANKS