Motion Planning by Search in Derivative Space and Convex Optimization with Enlarged Solution Space (Extended Material)

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I. Proofs of Propositions and Theorem

A. Proof of Proposition 2

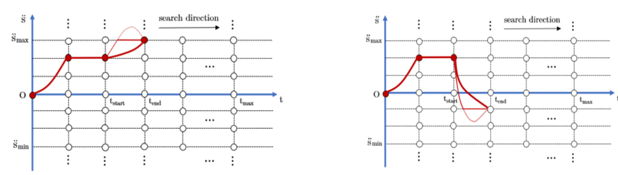
The fitting points between two sample points on $\ddot{S}$-T are generated by

$$\begin{aligned} \min_{\ddot{s}[1],\ldots,\ddot{s}[n_s]} \; & w_2 \sum_{k=1}^{n_s} (\ddot{s}[k])^2 + w_3 \sum_{k=0}^{n_s} (\ddot{s}[k+1] - \ddot{s}[k])^2 \\ \text{s.t.} \quad & \ddot{s}[0] = \ddot{s}_{\text{start}}, \;\; \ddot{s}[n_s+1] = \ddot{s}_{\text{end}}, \end{aligned} \tag{1}$$

where w_2 and w_3 are tunable parameters. The solution of problem (1) is also guaranteed to satisfy the constraints of $\ddot{s} \in [\ddot{s}_{\text{min}}, \ddot{s}_{\text{max}}]$, as shown in the following proposition.

Proposition 1. *Suppose that $\ddot{s}_{start}, \ddot{s}_{end} \in [\ddot{s}_{min}, \ddot{s}_{max}]$. Then, the solution of problem (1) satisfies the constraints of $\ddot{s}$, i.e., $\ddot{s}[1], \ldots, \ddot{s}[n_s] \in [\ddot{s}_{min}, \ddot{s}_{max}]$.*

Proof. We complete the proof by contradiction. Suppose that there exsits $\ddot{s}[k] > \ddot{s}_{\text{max}}$. Replacing $\ddot{s}[k]$ with $\ddot{s}_{\text{max}}$ will achieve a smaller objective function, which renders a contradiction. Then, it follows that $\ddot{s}[k] \le \ddot{s}_{\text{max}}$. Similarly, we have $\ddot{s}[k] \ge \ddot{s}_{\text{min}}$ (Fig.1(b)) and $\ddot{s}[1], \ldots, \ddot{s}[n_s] \in [\ddot{s}_{\text{min}}, \ddot{s}_{\text{max}}]$. □



(a) (b)

Fig. 1: Illustrations of the range of fitting points.

B. Proof of Theorem 1

Lemma 1. *Let $M \in \mathbb{R}^{(n+1)\times(n+1)}$ denote the transition matrix from the Bernstein basis $\{b_n^0(t), b_n^1(t), \ldots, b_n^n(t)\}$ to the monomial basis $\{1, t, \ldots, t^n\}$. We have $M_{i,0} = 1, \; 0 \le M_{i,j} \le 1, \; i = 0, 1, \ldots, n, \; j = 0, 1, \ldots, n$.*

Proof. It follows that

$$\begin{aligned} t^i = t^i(t+1-t)^{n-i} &= \sum_{j=0}^{n-i} C_{n-i}^j t^{n-j}(1-t)^j \\ &= \sum_{j=0}^{n-i} \frac{C_{n-i}^j}{C_n^{n-j}} C_n^{n-j} t^{n-j}(1-t)^j. \end{aligned}$$

Hence, the elements of matrix M satisfy

$$M_{n-j,i} = \begin{cases} \frac{C_{n-i}^j}{C_n^j}, & i+j \le n \\ 0, & i+j > n. \end{cases} \tag{2}$$

We have $M_{i,0} = 1, \; 0 \le M_{i,j} \le 1, \; i, \; j = 0, 1, \ldots, n$. □

Theorem 1. *For a trajectory, if it has control points in each time interval satisfying $c_i^k \in \Omega^k$, where $\Omega^k = \{c^k | \underline{p_0^k} + h_k \underline{p_1^k} M_{i,1} \leq c_i^k \leq \overline{p_0^k} + h_k \overline{p_1^k} M_{i,1}, i = 0,1,\ldots,n, k = 0,1,\ldots,m\}$, $s(t)$ is guaranteed to be safe. The upper bounds and lower bounds form a trapezoidal corridor $\mathcal{S}^{tra}$.*

Proof. On the S-T graph for $t \in [T_k, T_{k+1}]$, it holds that

$$\underline{p_0^k} + h_k \underline{p_1^k} \frac{t - T_k}{h_k} < \overline{p_0^k} + h_k \overline{p_1^k} \frac{t - T_k}{h_k}. \tag{3}$$

According to Lemma 1, $M_{i,1}$ satisfies $0 \leq M_{i,1} \leq 1$. Thus, we have $T_k \leq T_k + h_k M_{i,1} \leq T_{k+1}$ and let $t = T_k + h_k M_{i,1}$. Then we obtain

$$\underline{p_0^k} + h_k \underline{p_1^k} M_{i,1} < \overline{p_0^k} + h_k \overline{p_1^k} M_{i,1},$$

and $\exists\ c_i^k, s.t.\ \underline{p_0^k} + h_k \underline{p_1^k} M_{i,1} \leq c_i^k \leq \overline{p_0^k} + h_k \overline{p_1^k} M_{i,1}$. As for the safety of $s(t)$, $\forall t_0 \in [T_k, T_{k+1}]$,

$$\begin{aligned} s(t_0) &\leq \sum_{i=0}^{n} (\overline{p_0^k} + h_k \overline{p_1^k} M_{i,1}) b_n^i \left(\frac{t - T_k}{h_k}\right) \\ &\leq \overline{p_0^k} \sum_{i=0}^{n} b_n^i \left(\frac{t - T_k}{h_k}\right) + h_k \overline{p_1^k} \sum_{i=0}^{n} M_{i,1} b_n^i \left(\frac{t - T_k}{h_k}\right) \\ &= \overline{p_0^k} + h_k \overline{p_1} \frac{t - T_k}{h_k}. \end{aligned}$$

Similarly, we have $s(t_0) \geq \underline{p_0^k} + h_k \underline{p_1^k} \frac{t - T_k}{h_k}$. Therefore, we have $s(t_0) \in \mathcal{S}_k^{tra} = \mathcal{S}_k$ and $s(t) \in \mathcal{S}^{tra} = \mathcal{S}$, i.e., $s(t)$ is safe and the corridors are trapezoidal. □

II. QP Formulation

This part illustrates how to formulate the Bézier polynomial optimization as a QP problem as

$$\begin{aligned} \mathbf{P}: \quad & \min_{\mathbf{c}} \ \mathbf{c}^T \mathbf{Q_c} \mathbf{c} + \mathbf{q_c}^T \mathbf{c} + \text{const} \\ & \text{s.t.} \ \mathbf{A}_{eq} \mathbf{c} = \mathbf{b}_{eq} \\ & \quad\quad \mathbf{A}_{ie} \mathbf{c} \leq \mathbf{b}_{ie}. \end{aligned} \tag{4}$$

First, we express the Bézier curve as a polynomial

$$\begin{aligned} s_k(t) =& h_k \sum_{i=0}^{n} c_i^k b_n^i \left(\frac{t - T_k}{h_k}\right) \\ =& h_k \sum_{i=0}^{n} p_i^k \left(\frac{t - T_k}{h_k}\right)^i = h_k f_k \left(\frac{t - T_k}{h_k}\right), \end{aligned} \tag{5}$$

where $f_k(t) = \sum_{i=0}^{n} p_i^k t^i, \quad k = 0,1,\ldots,m$ is a polynomial curve. Let $M \in \mathbb{R}^{(n+1)\times(n+1)}$ denote the transition matrix from the Bernstein basis $\{b_n^0(t), b_n^1(t), \ldots, b_n^n(t)\}$ to the monomial basis $\{1, t, t^2, \ldots, t^n\}$. Then, we have $\mathbf{c^k} = M\mathbf{p^k}$ with $\mathbf{c^k} = [c_0^k, \ldots, c_n^k]^T$ and $\mathbf{p^k} = [p_0^k, \ldots, p_n^k]^T$.

According to lemma 1, it holds that $|M| > 0$ and M is invertible. Hence, if the objective function

$$\begin{aligned} J = w_1 \sum_{k=1}^{m} (c_n^k - s_{\text{ref}}[\sum_{l=1}^{k} m_l])^2 + w_2 \int_0^T \left(\dot{s}(t) - \dot{s}_{\text{ref}}\right)^2 \text{d}t + w_3 \int_0^T \ddot{s}(t)^2 \text{d}t \\ + w_4 \int_0^T \dddot{s}(t)^2 \text{d}t + w_5 \left(c_n^m - s_{\text{ref}}[\sum_{l=1}^{m} m_l]\right)^2 \end{aligned} \tag{6}$$

can be written as

$$J = \sum_{k=0}^{m} \left[(\mathbf{p^k})^T Q^k \mathbf{p^k} + \mathbf{q^k} \mathbf{p^k}\right] + \text{const} \geq 0, \tag{7}$$

where Q^k is positive definite and known, then we have

$$
\begin{aligned}
J = & \begin{bmatrix} \mathbf{c^0} \\ \vdots \\ \mathbf{c^m} \end{bmatrix}^T \begin{bmatrix} (M^{-1})^T Q^0 M^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (M^{-1})^T Q^m M^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{c^0} \\ \vdots \\ \mathbf{c^m} \end{bmatrix} \\
& + \begin{bmatrix} \mathbf{q^0} \\ \vdots \\ \mathbf{q^m} \end{bmatrix}^T \begin{bmatrix} M^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & M^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{c^0} \\ \vdots \\ \mathbf{c^m} \end{bmatrix} + \text{const} \\
= & \mathbf{c}^T Q_c \mathbf{c} + \mathbf{q_c}^T \mathbf{c} + \text{const} \geq 0.
\end{aligned} \tag{8}
$$

Note that Q_c is also a positive-definite matrix. Since the constraints are all linear with $\mathbf{c}$, the original problem is a QP problem. Next we will illustrate that Eq. (7) holds and how to calculate Q_k and $\mathbf{q^k}$. We first calculate preliminary terms to achieve the cost function J. To begin with, it holds that

$$
\begin{aligned}
& \int_{T_k}^{T_{k+1}} \left(\frac{\mathrm{d}^l s(\tau)}{\mathrm{d}\tau^l} \right)^2 \mathrm{d}\tau = \int_0^{h_k} \left(\frac{\mathrm{d}^l s(\tau + T_k)}{\mathrm{d}\tau^l} \right)^2 \mathrm{d}\tau \\
= & \int_0^{h_k} \left(\frac{\mathrm{d}^l s(\tau + T_k)}{\mathrm{d}t^l} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^l \right)^2 \mathrm{d}\tau = \frac{1}{h_k^{2l-3}} \int_0^1 \left(\frac{\mathrm{d}^l f_k(t)}{\mathrm{d}t^l} \right)^2 \mathrm{d}t.
\end{aligned} \tag{9}
$$

As for $\int_0^1 \left(\frac{\mathrm{d}^l f_k(t)}{\mathrm{d}t^l} \right)^2 \mathrm{d}t$, it follows that

$$
\begin{aligned}
& \int_0^1 \left(\frac{\mathrm{d}^l f_k(t)}{\mathrm{d}t^l} \right)^2 \mathrm{d}t = \int_0^1 \sum_{i \geq l, j \geq l} p_i^k p_j^k t^{i+j-2l} \mathrm{d}t \\
= & \sum_{i \geq l, j \geq l} \frac{i(i-1)\cdots(i-l)j(j-1)\cdots(j-l)}{i+j+1-2l} p_i^k p_j^k,
\end{aligned} \tag{10}
$$

which is a quadratic form. We represent the i-th term in Eq. (6) as $w_i J_i, i = 1, 2, \ldots, 5$. Then, we have

$$
J_1 = \sum_{k=1}^{m} (c_n^k - s_{\text{ref}}[\sum_{l=1}^{k} m_l])^2 = \sum_{k=1}^{m} \left[(c_n^k)^2 - 2 s_{\text{ref}}[\sum_{l=1}^{k} m_l] c_n^k \right] + \text{const}, \tag{11}
$$

and

$$
J_2 = \sum_{k=0}^{m} \int_{T_k}^{T_{k+1}} \dot{s_k}(t)^2 \mathrm{d}t - 2\dot{s}_{\text{ref}} \int_0^T \dot{s}(t) \mathrm{d}t + \text{const} = \sum_{k=0}^{m} h_k \int_0^1 \dot{f}_k(t)^2 \mathrm{d}t - 2\dot{s}_{\text{ref}} c_n^m + \text{const}, \tag{12}
$$

$$
J_3 = \sum_{k=0}^{m} \frac{1}{h_k} \int_0^1 \ddot{f}_k(t)^2 \mathrm{d}t, \;\; J_4 = \sum_{k=0}^{m} \frac{1}{h_k^3} \int_0^1 \dddot{f}_k(t)^2 \mathrm{d}t, \;\; J_5 = (c_n^m)^2 - 2 s_{\text{ref}}[\sum_{l=1}^{m} m_l] c_n^m + \text{const}. \tag{13}
$$

Then we come to the Eq. (7) by replacing integral terms and using $J = \sum_{i=1}^{5} w_i J_i$.