Motion Planning by Search in Derivative Space and Convex Optimization with Enlarged Solution Space (Extended Material)

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I. PROOFS OF PROPOSITIONS AND THEOREM

A. Proof of Proposition 2

The fitting points between two sample points on $\ddot{S}$-T are generated by

$$\min_{\ddot{s}[1],\ldots,\ddot{s}[n_s]} w_2 \sum_{k=1}^{n_s} \left(\ddot{s}[k]\right)^2 + w_3 \sum_{k=0}^{n_s} \left(\ddot{s}[k+1] - \ddot{s}[k]\right)^2 \tag{1}$$
$$\text{s.t.} \quad \ddot{s}[0] = \ddot{s}_{\text{start}}, \ \ddot{s}[n_s+1] = \ddot{s}_{\text{end}},$$

where w_2 and w_3 are tunable parameters. The solution of problem (1) is also guaranteed to satisfy the constraints of $\ddot{s} \in [\ddot{s}_{\min}, \ddot{s}_{\max}]$, as shown in the following proposition.

Proposition 1. *Suppose that $\ddot{s}_{start}, \ddot{s}_{end} \in [\ddot{s}_{min}, \ddot{s}_{max}]$. Then, the solution of problem* (1) *satisfies the constraints of $\ddot{s}$, i.e., $\ddot{s}[1], \ldots, \ddot{s}[n_s] \in [\ddot{s}_{min}, \ddot{s}_{max}]$.*

Proof. We complete the proof by contradiction. Suppose that there exsits $\ddot{s}[k] > \ddot{s}_{\max}$. Replacing $\ddot{s}[k]$ with $\ddot{s}_{\max}$ will achieve a smaller objective function, which renders a contradiction. Then, it follows that $\ddot{s}[k] \leq \ddot{s}_{\max}$. Similarly, we have $\ddot{s}[k] \geq \ddot{s}_{\min}$ (Fig.1(b)) and $\ddot{s}[1], \ldots, \ddot{s}[n_s] \in [\ddot{s}_{\min}, \ddot{s}_{\max}]$. $\square$
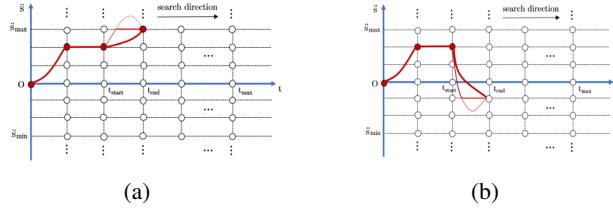

Fig. 1: Illustrations of the range of fitting points.

B. Proof of Theorem 1

Lemma 1. *Let $M \in \mathbb{R}^{(n+1)\times(n+1)}$ denote the transition matrix from the Bernstein basis $\{b_n^0(t), b_n^1(t), \ldots, b_n^n(t)\}$ to the monomial basis $\{1, t, \ldots, t^n\}$. We have $M_{i,0} = 1$, $0 \leq M_{i,j} \leq 1$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, n$.*

Proof. It follows that

$$t^i = t^i(t+1-t)^{n-i} = \sum_{j=0}^{n-i} C_{n-i}^j t^{n-j}(1-t)^j$$
$$= \sum_{j=0}^{n-i} \frac{C_{n-i}^j}{C_n^{n-j}} C_n^{n-j} t^{n-j}(1-t)^j.$$

Hence, the elements of matrix M satisfy

$$M_{n-j,i} = \begin{cases} \frac{C_{n-i}^j}{C_n^j}, & i+j \leq n \\ 0, & i+j > n. \end{cases} \tag{2}$$

We have $M_{i,0} = 1$, $0 \leq M_{i,j} \leq 1$, $i, j = 0, 1, \ldots, n$. $\square$

Theorem 1. *For a trajectory, if it has control points in each time interval satisfying $c_i^k \in \Omega^k$, where $\Omega^k = \{c^k | \underline{p_0^k} + h_k \underline{p_1^k} M_{i,1} \le c_i^k \le \overline{p_0^k} + h_k \overline{p_1^k} M_{i,1}, i = 0, 1, \ldots, n, k = 0, 1, \ldots, m\}$, $s(t)$ is guaranteed to be safe. The upper bounds and lower bounds form a trapezoidal corridor $\mathcal{S}^{tra}$.*

Proof. On the S-T graph for $t \in [T_k, T_{k+1}]$, it holds that

$$\underline{p_0^k} + h_k \underline{p_1^k} \frac{t - T_k}{h_k} < \overline{p_0^k} + h_k \overline{p_1^k} \frac{t - T_k}{h_k}. \tag{3}$$

According to Lemma 1, $M_{i,1}$ satisfies $0 \le M_{i,1} \le 1$. Thus, we have $T_k \le T_k + h_k M_{i,1} \le T_{k+1}$ and let $t = T_k + h_k M_{i,1}$. Then we obtain

$$\underline{p_0^k} + h_k \underline{p_1^k} M_{i,1} < \overline{p_0^k} + h_k \overline{p_1^k} M_{i,1},$$

and $\exists\, c_i^k, s.t.\ \underline{p_0^k} + h_k \underline{p_1^k} M_{i,1} \le c_i^k \le \overline{p_0^k} + h_k \overline{p_1^k} M_{i,1}$. As for the safety of $s(t)$, $\forall t_0 \in [T_k, T_{k+1}]$,

$$s(t_0) \le \sum_{i=0}^{n} (\overline{p_0^k} + h_k \overline{p_1^k} M_{i,1}) b_n^i \left(\frac{t - T_k}{h_k} \right)$$

$$\le \overline{p_0^k} \sum_{i=0}^{n} b_n^i \left(\frac{t - T_k}{h_k} \right) + h_k \overline{p_1^k} \sum_{i=0}^{n} M_{i,1} b_n^i \left(\frac{t - T_k}{h_k} \right)$$

$$= \overline{p_0^k} + h_k \overline{p_1} \frac{t - T_k}{h_k}.$$

Similarly, we have $s(t_0) \ge \underline{p_0^k} + h_k \underline{p_1^k} \frac{t - T_k}{h_k}$. Therefore, we have $s(t_0) \in \mathcal{S}_k^{tra} = \mathcal{S}_k$ and $s(t) \in \mathcal{S}^{tra} = \mathcal{S}$, i.e., $s(t)$ is safe and the corridors are trapezoidal. $\square$

II. QP FORMULATION

This part illustrates how to formulate the Bézier polynomial optimization as a QP problem as

$$\mathbf{P}: \quad \min_{\mathbf{c}} \ \mathbf{c}^T \mathbf{Q_c} \mathbf{c} + \mathbf{q_c}^T \mathbf{c} + \text{const}$$

$$\text{s.t.} \ \mathbf{A}_{eq} \mathbf{c} = \mathbf{b}_{eq} \tag{4}$$

$$\mathbf{A}_{ie} \mathbf{c} \le \mathbf{b}_{ie}.$$

First, we express the Bézier curve as a polynomial

$$s_k(t) = h_k \sum_{i=0}^{n} c_i^k b_n^i \left(\frac{t - T_k}{h_k} \right)$$

$$= h_k \sum_{i=0}^{n} p_i^k \left(\frac{t - T_k}{h_k} \right)^i = h_k f_k \left(\frac{t - T_k}{h_k} \right), \tag{5}$$

where $f_k(t) = \sum_{i=0}^{n} p_i^k t^i$, $k = 0, 1, \ldots, m$ is a polynomial curve. Let $M \in \mathbb{R}^{(n+1) \times (n+1)}$ denote the transition matrix from the Bernstein basis $\{b_n^0(t), b_n^1(t), \ldots, b_n^n(t)\}$ to the monomial basis $\{1, t, t^2, \ldots, t^n\}$. Then, we have $\mathbf{c^k} = M \mathbf{p^k}$ with $\mathbf{c^k} = [c_0^k, \ldots, c_n^k]^T$ and $\mathbf{p^k} = [p_0^k, \ldots, p_n^k]^T$.

According to lemma 1, it holds that $|M| > 0$ and M is invertible. Hence, if the objective function

$$J = w_1 \sum_{k=1}^{m} (c_n^k - s_{\text{ref}}[\sum_{l=1}^{k} m_l])^2 + w_2 \int_0^T (\dot{s}(t) - \dot{s}_{\text{ref}})^2 \, dt + w_3 \int_0^T \ddot{s}(t)^2 dt$$

$$+ w_4 \int_0^T \dddot{s}(t)^2 dt + w_5 \left(c_n^m - s_{\text{ref}}[\sum_{l=1}^{m} m_l] \right)^2 \tag{6}$$

can be written as

$$J = \sum_{k=0}^{m} \left[(\mathbf{p^k})^T Q^k \mathbf{p^k} + \mathbf{q^k} \mathbf{p^k} \right] + \text{const} \ge 0, \tag{7}$$

where Q^k is positive definite and known, then we have

$$J = \begin{bmatrix} \mathbf{c^0} \\ \vdots \\ \mathbf{c^m} \end{bmatrix}^T \begin{bmatrix} (M^{-1})^T Q^0 M^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (M^{-1})^T Q^m M^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{c^0} \\ \vdots \\ \mathbf{c^m} \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{q^0} \\ \vdots \\ \mathbf{q^m} \end{bmatrix}^T \begin{bmatrix} M^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & M^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{c^0} \\ \vdots \\ \mathbf{c^m} \end{bmatrix} + \mathrm{const} \tag{8}$$

$$= \mathbf{c}^T Q_c \mathbf{c} + \mathbf{q_c}^T \mathbf{c} + \mathrm{const} \geq 0.$$

Note Q_c is also a positive-definite matrix. Since the constraints are all linear with $\mathbf{c}$, the original problem is a QP problem. Next we will illustrate that equation (7) holds and how to calculate Q_k and $\mathbf{q^k}$. We first calculate preliminary terms to achieve the cost function J. To begin with, it holds that

$$\int_{T_k}^{T_{k+1}} \left(\frac{\mathrm{d}^l s(\tau)}{\mathrm{d}\tau^l} \right)^2 \mathrm{d}\tau = \int_0^{h_k} \left(\frac{\mathrm{d}^l s(\tau + T_k)}{\mathrm{d}\tau^l} \right)^2 \mathrm{d}\tau$$

$$= \int_0^{h_k} \left(\frac{\mathrm{d}^l s(\tau + T_k)}{\mathrm{d}t^l} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^l \right)^2 \mathrm{d}\tau = \frac{1}{h_k^{2l-3}} \int_0^1 \left(\frac{\mathrm{d}^l f_k(t)}{\mathrm{d}t^l} \right)^2 \mathrm{d}t. \tag{9}$$

As for $\int_0^1 \left(\frac{\mathrm{d}^l f_k(t)}{\mathrm{d}t^l} \right)^2 \mathrm{d}t$, it follows that

$$\int_0^1 \left(\frac{\mathrm{d}^l f_k(t)}{\mathrm{d}t^l} \right)^2 \mathrm{d}t = \int_0^1 \sum_{i \geq l, j \geq l} p_i^k p_j^k t^{i+j-2l} \mathrm{d}t$$

$$= \sum_{i \geq l, j \geq l} \frac{i(i-1)\cdots(i-l)j(j-1)\cdots(j-l)}{i+j+1-2l} p_i^k p_j^k, \tag{10}$$

which is a quadratic form. We represent the i-th term in Eq. (6) as $w_i J_i$, $i = 1, 2, \ldots, 5$. Then, we have

$$J_1 = \sum_{k=1}^m (c_n^k - s_{\mathrm{ref}}[\sum_{l=1}^k m_l])^2 = \sum_{k=1}^m \left[(c_n^k)^2 - 2s_{\mathrm{ref}}[\sum_{l=1}^k m_l] c_n^k \right] + \mathrm{const}, \tag{11}$$

and

$$J_2 = \sum_{k=0}^m \int_{T_k}^{T_{k+1}} \dot{s}_k(t)^2 \mathrm{d}t - 2\dot{s}_{\mathrm{ref}} \int_0^T \dot{s}(t) \mathrm{d}t + \mathrm{const} = \sum_{k=0}^m h_k \int_0^1 \dot{f}_k(t)^2 \mathrm{d}t - 2\dot{s}_{\mathrm{ref}} c_n^m + \mathrm{const}, \tag{12}$$

$$J_3 = \sum_{k=0}^m \frac{1}{h_k} \int_0^1 \ddot{f}_k(t)^2 \mathrm{d}t, \ J_4 = \sum_{k=0}^m \frac{1}{h_k^3} \int_0^1 \dddot{f}_k(t)^2 \mathrm{d}t, \ J_5 = (c_n^m)^2 - 2s_{\mathrm{ref}}[\sum_{l=1}^m m_l] c_n^m + \mathrm{const}. \tag{13}$$

Then we come to the Eq. (7) by replacing integral terms and using $J = \sum_{i=1}^5 w_i J_i$.