Homework 4, MA573

Monotonicity in volatility

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Exercise1:

Prove or disprove: Suppose f is convex and X is submartingale, prove that $g(t) = \mathbb{E}[f(X_t)]$ is increasing.

Solution:

This proposition is not right. We can give a counter example. Assume $f(x) = -x$, then $f(x)$ is a convex function since $\forall x_1, x_2 \in \mathbb{R}$, and $\forall t \in [0,1]$, we have $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$ holds. Then we can verify the proposotion that $g(t)$ is increasing. Since $\forall s \leq t$,

$$\mathbb{E}[f(X_t) - f(X_s)|\mathcal{F}_s] = \mathbb{E}[-X_t + X_s|\mathcal{F}_s] = X_s - \mathbb{E}[X_t|\mathcal{F}_s].$$

Since X_t is a submartigale with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$, then we have

$$\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s,$$

thus we know that $\forall s \leq t$, $\mathbb{E}[f(X_t) - f(X_s)|\mathcal{F}_s] \leq 0$. Then we take expectation on the both side and by the definition of conditional expectation, we have

$$\mathbb{E}[f(X_t) - f(X_s)] \leq 0.$$

Then we know that $\forall s \leq t$, $g(t) \leq g(s)$, such that $g(t)$ is decreasing.

If we know f is a increasing function, then by the Jensen's inequality, we have for all $s \leq t$,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] \geq f(\mathbb{E}[X_t|\mathcal{F}_s]),$$

and since X_t is a submartingale, we have $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$. As f is a increasing function, then

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] \geq f(\mathbb{E}[X_t|\mathcal{F}_s]) \geq f(X_s),$$

by taking expectation on the both side, we have

$$\mathbb{E}[f(X_t)] \geq \mathbb{E}[f(X_s)], \forall s \leq t.$$

Then we know $g(t)$ is increasing.

Exercise2:

Let $t \mapsto e^{-rt}S_t$ be a martingale, then prove that

$$C(t) = \mathbb{E}[e^{-rt}(S_t - K)^+]$$

is increasing.

Solution:

Since $e^{-rt}S_t$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$, and the function $f(x) = x^+$ is a convex function, then by the Jensen's inequality in the conditional expectation, we have $\forall s \le t$,

$$\mathbb{E}[e^{-rt}(S_t - K)^+|\mathcal{F}_s] \ge (\mathbb{E}[e^{-rt}(S_t - K)|\mathcal{F}_s])^+ = (e^{-rs}S_s - e^{-rt}K)^+,$$

And we have

$$(e^{-rs}S_s - e^{-rt}K)^+ = \big((e^{-rs}(S_s - K) + K(e^{-rs} - e^{-rt})\big)^+ \ge (e^{-rs}(S_s - K))^+$$

since $K(e^{-rs} - e^{-rt}) \ge 0$, $\forall s \le t$ and $K > 0$. Thus we know that $\mathbb{E}[e^{-rt}(S_t - K)^+|\mathcal{F}_s] \ge (e^{-rs}(S_s - K))^+$, then we take expectation on the both side, we have that

$$\mathbb{E}[e^{-rt}(S_t - K)^+] \ge \mathbb{E}[(e^{-rs}(S_s - K))^+].$$

Such that $\forall s \le t$, we have $C(s) \le C(t)$, then $C(t)$ is increasing.

Exercise3:

Suppose $r = 0$ and S is martingale, prove that $P(t) = \mathbb{E}[(S_t - K)^-]$ is increasing.

Solution:

Since S_t is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$, and the function $f(x) = x^-$ is a convex function, we have $\forall s \le t$,

$$\mathbb{E}[(S_t - K)^-|\mathcal{F}_s] \ge (\mathbb{E}[S_t - K|\mathcal{F}_s])^- = (S_s - K)^-,$$

then we know that $(S_t - K)^-$ is a submartingale. Taking expectation on the both side, we have for all $s \le t$,

$$\mathbb{E}[(S_t - K)^-] \ge \mathbb{E}[(S_s - K)^-].$$

So, $P(t)$ is increasing.