FDM on PDE

PDE

$$(*1)\quad \begin{cases} Lu = -u'' + bu' + cu = f \\ u(0) = u(1) = 0 \end{cases}$$

Asm

① $c \geq 0$, ② $\exists!$ smooth soln

Goal

① Implement CFD on $(*1)$

② Convergence rate

Rk ① L is linear differential operator

b/c $L ku = k L u \quad \forall k \in \mathbb{R}$

② 1-d b/c domain $= [0, 1]$

③ Elliptic eqn with boundary value.

Bench mark ODE

$$(*2)\quad \begin{cases} -u'' + u' + 2u = 2x^2 - 3 \\ u(0) = u(1) = 0 \end{cases}$$

Exact soln

$$u(x) = x^2 - x$$

ex Determine coefficients of (*2)

soln $b = 1, \quad c = 2, \quad f(x) = 2x^2 - 3$

$$Lu = -u'' + u' + 2u$$

Finite Difference operator

- step size $h > 0$
- FFD

$$O(h) + u'(x) = \delta_h u(x) \triangleq \frac{u(x+h) - u(x)}{h}$$

- BFD

$$O(h) + u'(x) = \delta_{-h} u(x) = \frac{u(x) - u(x-h)}{h}$$

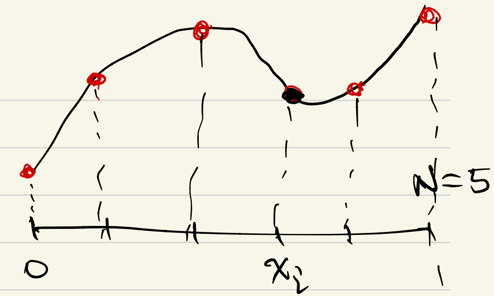
- CFD

$$O(h^2) + u'(x) = \bar{\delta}_h u(x) \triangleq \frac{1}{2}(\delta_h + \delta_{-h}) u(x) = \frac{u(x+h) - u(x-h)}{2h}$$

- SCFD

$$O(h^2) + u''(x) = \delta_h \delta_{-h} u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

Other notations



⊛ step size $h = \frac{1}{N}$

Grid $G^h = \{ x_i = ih \mid i = 0, 1, \cdots N \}$

⊛ $\forall g \in C(0,1)$, projection on G^h

$$R^h g = \left(f(x_0), f(x_1) \cdots , f(x_N) \right)$$

$$\triangleq \left(f_0, f_1 \cdots f_N \right) \in \mathbb{R}^{N+1}$$

⊛ $$\bar{\delta}_h g_i \triangleq \bar{\delta}_h g(x_i) = \frac{g_{i+1} - g_{i-1}}{2h} \approx g'(x_i)$$

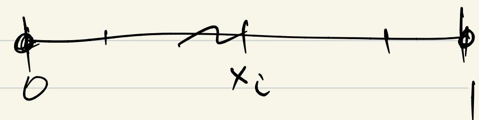
$$\delta_h \delta_{-h} g_i \triangleq \delta_h \delta_{-h} g(x_i) = \frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} \approx g''(x_i)$$

General framework

Idea:
replace diff operator by its FD est.

$$F(x, u, u', u'') = 0 \qquad (*3)$$

FDM with $h = \frac{1}{N}$



$$\begin{cases} u_0^h = 0 \\ F(x_i, u_i^h, \bar{\delta}_h u_i^h, \delta_h \delta_{-h} u_i^h) = 0, \quad i = 1, 2, \dots, N-1 \\ u_N^h = 0 \end{cases}$$

solve for $N+1$ unknowns $(u_0^h, u_1^h, \dots, u_N^h)$.
from $N+1$ eqns.

Ex

prove that CFD of (*1) is

$$\begin{cases} u_0^h = 0 \\ -r_i u_{i-1}^h + s_i u_i^h - t_i u_{i+1}^h = f_i \quad i = 1, 2, \dots N-1 \\ u_N^h = 0 \end{cases}$$

where

$$r_i = \frac{1}{h^2} + \frac{b_i}{2h}$$

$$s_i = \frac{2}{h^2} + c_i$$

$$t_i = \frac{1}{h^2} - \frac{b_i}{2h}$$

Proof. $u_i = u^h_i$ for simplicity

$$(*1)\ \begin{cases} Lu = -u'' + bu' + cu = f \\ u(0) = u(1) = 0 \end{cases}$$

$$-\overline{\delta_h}\,\overline{\delta_{-h}}\, u_i + b_i \overline{\delta_h}\, u_i + c_i u_i = f_i$$

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + b_i \frac{u_{i+1} - u_{i-1}}{2h} + c_i u_i = f_i$$

$$u_{i-1} \cdot \left(-\frac{1}{h^2} - \frac{b_i}{2h}\right)$$

$$+ u_i \cdot \left(\frac{2}{h^2} + c_i\right)$$

$$+ u_{i+1} \cdot \left(-\frac{1}{h^2} + \frac{b_i}{2h}\right)$$

$$= f_i$$

$\boxed{\text{S}}$

Implementation by python.

- scipy.linalg to compute $Ax=b$
- $L^h u = R^h f$.

$$R^h f = \begin{vmatrix} 0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ 0 \end{vmatrix}, \qquad L^h = \begin{vmatrix} 1, & 0, & 0, & \cdots & & & 0 \\ -r_1 & s_1 & -t_1, & 0 & \cdots & & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & & & \vdots \\ & & & & -r_{N-1} & s_{N-1} & -t_{N-1} \\ 0 & & \cdots & & 0 & 0 & 1 \end{vmatrix}$$

- L^h is $(N+1)$ by $(N+1)$ Tridiagonal matrix.

Ex

consider (*2) with $h = \frac{1}{4}$

(*2) $\begin{cases} -u'' + u' + 2u = 2x^2 - 3 \\ u(0) = u(1) = 0 \end{cases}$

(*) Find L^h, $R^h f$

$r_i = \frac{1}{h^2} + \frac{b_i}{2h} = 16 + 2 = 18$

$s_i = \frac{2}{h^2} + c_i = 32 + 2 = 34$

$t_i = \frac{1}{h^2} - \frac{b_i}{2h} = 16 - 2 = 14$

Soln $G^h = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$

$$R_h f = \begin{bmatrix} 0 \\ 2\left(\frac{1}{4}\right)^2 - 3 \\ 2\left(\frac{2}{4}\right)^2 - 3 \\ 2\left(\frac{3}{4}\right)^2 - 3 \\ 0 \end{bmatrix}$$

$$L^h = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -18 & 34 & -14 & 0 & 0 \\ 0 & -18 & 34 & -14 & 0 \\ 0 & 0 & -18 & 34 & -14 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Ex (HW1)

consider CFD on (*2)

① compute CFD soln u^h for $h=2^{-2}$

② plot a figure with curves

$$\begin{cases} x \to u^h(x) \\ x \to u(x) \end{cases}$$

③ for $h = 2^{-2} \cdots\cdots 2^{-6}$

compute $\varepsilon^h = \sup_{0 \le i \le N} |u(x_i) - u_i^h|$

plot log-log chart, what's the convergence rate?