Upwind Scheme

BVP

$$(*1)\quad \begin{cases} Lu = -u'' + bu' + cu = f, & x \in \mathcal{O} = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Asm

① $c \geq 0$, ② $\exists!$ smooth soln

Goal

① Implement ~~CFD~~ UFD on $(*1)$

② Convergence rate

Recall . CFD of (*1) is

$$\begin{cases} u_0^h = 0 \\ -r_i u_{i-1}^h + s_i u_i^h - t_i u_{i+1}^h = f_i \qquad i = 1, 2, \dots N-1 \\ u_N^h = 0 \end{cases}$$

where

$$r_i = \frac{1}{h^2} + \frac{b_i}{2h}$$

$$s_i = \frac{2}{h^2} + c_i$$

$$t_i = \frac{1}{h^2} - \frac{b_i}{2h}$$

Convergence rate

$$\| u^h - R^h u \|_\infty \le K h^2$$

UFD : Basic idea

$$u'' \leftarrow \delta_h \delta_{-h} u$$

$\cancel{u' \leftarrow \bar{\delta}_h u}$ ~~for cFD~~

$$u' \leftarrow \begin{cases} \delta_h u, & \text{if } b(x) < 0 \\ \delta_{-h} u, & \text{if } b(x) \geq 0 \end{cases}$$

$$b u' = (b^+ - b^-) u'$$

$$\approx b^+ \delta_{-h} u - b\, \delta_h u$$

$$-u'' + b u' + c u = f$$

$$\wr\wr$$

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + b^+ \cdot \frac{u_i - u_{i-1}}{h} - b^- \frac{u_{i+1} - u_i}{h} + c u_i = f_i$$

$$-\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+b^+\cdot\frac{u_i-u_{i-1}}{h}-b^-\frac{u_{i+1}-u_i}{h}+cu_i=f_i$$

$$-u_{i-1}\left(\frac{1}{h^2}+\frac{b^+}{h}\right)$$

$$\frac{|b|}{h}$$

$$+u_i\left(\frac{2}{h^2}+\frac{b^+}{h}+\frac{b^-}{h}+c\right)$$

$$-u_{i+1}\left(\frac{1}{h^2}+\frac{b^-}{h}\right)=f_i$$

Thm UFD soln u^h solves

$$\begin{cases} u_0^h = 0 \\ -r_i u_{i-1}^h + s_i u_i^h - t_i u_{i+1}^h = f_i \quad i = 1, 2, \dots N-1 \\ u_N^h = 0 \end{cases}$$

where

	UFD	CFD
$r_i =$	$\frac{1}{h^2} + \frac{b^+}{h}$	$\frac{1}{h^2} + \frac{b_i}{2h}$
$S_i =$	$\frac{2}{h^2} + \frac{\lvert b \rvert}{h} + c$	$\frac{2}{h^2} + c_i$
$t_i =$	$\frac{1}{h^2} + \frac{b^-}{h}$	$\frac{1}{h^2} - \frac{b_i}{2h}$

Convergence rate

$$\| u^h - R^h u \|_\infty \le K h^2$$

Rk

① UFD, $r_i, s_i, t_i \geq 0$.

CFD, $r_i, s_i, t_i \geq 0$ only if h is small

② So UFD has better stability.

③ But convergence rates are the same

④ The proof are the same.