Finite difference method on ODE

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Question:

Consider ODE

$$-\epsilon u^{''} + u = x, \, \forall x \in (0,1) \, u(0) = u(1) = 0, \tag{1}$$

with $\epsilon = 10^{-10}$. This example is taken from Example 5.2 of

- Qingshuo Song, George Yin, Zhimin Zhang, An epsilon-uniform finite element method for singularly perturbed boundary value problems.

Instead of FEM, we are going to discuss CFD solution of (1). Answer the following questions:

(i) Prove that

$$u(x) = x - \frac{\exp(\frac{x-1}{\sqrt{\epsilon}}) - \exp(-\frac{x+1}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})}$$

is the unique solution.

(ii) Using CFD on (1), find out the matrix L^h and vector $R^h f$, such that the numerical solution satisfies $L^h u^h = R^h f$.

(iii) Prove the consistency and stability of L^h.

(iv) Compute CFD solution u^h with $h = \frac{1}{5}$. Compare with the FEM solution of the paper, which one is better?

Solution:

(i) Firstly we verify that the $u(x)$ given above is the solution of (1). Since

$$u^{'}(x) = 1 - \frac{\exp(\frac{x-1}{\sqrt{\epsilon}})\frac{1}{\sqrt{\epsilon}} - \exp(-\frac{x+1}{\sqrt{\epsilon}})(-\frac{1}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})}$$

and then

$$u^{''}(x) = -\frac{\exp(\frac{x-1}{\sqrt{\epsilon}})\frac{1}{\epsilon} - \exp(-\frac{x+1}{\sqrt{\epsilon}})\frac{1}{\epsilon}}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})},$$

we have

$$-\epsilon u^{''} = \frac{\exp(\frac{x-1}{\sqrt{\epsilon}}) - \exp(-\frac{x+1}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})}.$$

Thus we know that $-\epsilon u^{''} + u = x$. And when $x = 0$,

$$u(0) = 0 - \frac{\exp(-\frac{1}{\sqrt{\epsilon}}) - \exp(-\frac{1}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})} = 0$$

and when $x = 1$,

$$u(1) = 1 - \frac{e^0 - \exp(-\frac{2}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})} = 0.$$

So, we know that $u(x)$ is a solution of ODE: $-\epsilon u'' + u = x$, $\forall x \in (0,1)\, u(0) = u(1) = 0$. By the existence and uniqueness theorem of the solution of non-homogeneous differential equation with second order constant coefficient, we know that $u(x)$ is the unique solution of the ODE (1).

We can also solve the ODE (1) directly. Firstly we consider the problem

$$-\epsilon u'' + u = 0, \tag{2}$$

the solution of (2) is

$$u(x) = c_1 e^{\frac{x}{\sqrt{\epsilon}}} + c_2 e^{-\frac{x}{\sqrt{\epsilon}}},$$

where c_1 and c_2 are constants. Since $u(x) = x$ is a special solution of $-\epsilon u'' + u = x$, we know that the solution of (1) can be

$$u(x) = x + c_1 e^{\frac{x}{\sqrt{\epsilon}}} + c_2 e^{-\frac{x}{\sqrt{\epsilon}}}.$$

With the condition $u(0) = u(1) = 0$, we have

$$\begin{cases} c_1 + c_2 = 0 \\ 1 + c_1 e^{\frac{1}{\sqrt{\epsilon}}} + c_2 e^{-\frac{1}{\sqrt{\epsilon}}} = 0 \end{cases}$$

thus we can get

$$c_1 = -\frac{\exp(-\frac{1}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})}, \; c_2 = \frac{\exp(-\frac{1}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})}.$$

Then we have

$$u(x) = x - \frac{\exp(\frac{x-1}{\sqrt{\epsilon}}) - \exp(-\frac{x+1}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})}$$

is the solution of (1). By the existence and uniqueness of the solution of non-homogeneous differential equation with second order constant coefficient, we know that $u(x)$ is the unique solution of the ODE (1).

(ii) For $h = \frac{1}{N}$, we denote $f(x) = x$, by the CFD method we have

$$-\epsilon \delta_h \delta_{-h} u_i^h + u_i^h = f_i,$$

which is equivalent to

$$-\epsilon \frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{h^2} + u_i^h = f_i.$$

Then we have

$$-\frac{\epsilon}{h^2} u_{i+1}^h + \Big(1 + \frac{2\epsilon}{h^2}\Big) u_i^h - \frac{\epsilon}{h^2} u_{i-1}^h = f_i.$$

We denote $\frac{\epsilon}{h^2}=r$ and $1+\frac{2\epsilon}{h^2}=s$, then for $1\le i\le N-1$ we have

$$\begin{cases} u_0^h=0\\ -ru_{i+1}^h+su_i^h-ru_{i-1}^h=f_i\\ u_N^h=0\end{cases}$$

Thus we have $L^hu^h=R^hf$, where $R^hf=(0,\frac{1}{N},\dots,\frac{N-1}{N},0)^{\mathrm{T}}$ and

$$L^h=\begin{pmatrix} 1 & 0 & 0 & & & & \\ -r & s & -r & & & & \\ & -r & s & -r & & & \\ & & & & \ddots & & & \\ & & & & -r & s & -r \\ & & & & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

(iii) Firstly, we show that L^h is consistent of $\alpha=2$. For any $v\in C^2([0,1])$, since $L^hR^hv=L^h(v_0,v_1,\dots,v_N)^{\mathrm{T}}$, we have $(L^hR^hv)_0=v_0=0$ and $(R^hLv)_0=Lv(x_0)=0$, then we can get $|(L^hR^hv)_0-(R^hLv)_0|=0$. Similarly, we have $|(L^hR^hv)_N-(R^hLv)_N|=0$. For any $1\le i\le N-1$, we have

$$(L^hR^hv)_i=-\epsilon\delta_h\delta_{-h}v_i+v_i$$

and

$$(R^hLv)_i=Lv(x_i)=-\epsilon v^{''}(x_i)+v(x_i),$$

then we know that

$$|(L^hR^hv)_i-(R^hLv)_i|=O(h^2).$$

Thus we know that L^h is consistent of $\alpha=2$.

Next we show that L^h in (4) is stable. For any $V\in\mathbb{R}^{N+1}$, if $|V_0|=\|V\|_\infty$, then

$$\|L^hV\|_\infty\ge|(L^hV)_0|=|V_0|=\|V\|_\infty.$$

Similarly, when $|V_N|=\|V\|_\infty$, then we have

$$\|L^hV\|_\infty\ge|(L^hV)_N|=|V_N|=\|V\|_\infty.$$

If $V_i=\|V\|_\infty$ for some $1\le i\le N-1$, we have

$$\begin{aligned}(L^hV)_i &= -rV_{i+1}+sV_i-rV_{i-1}\\ &= -r(V_{i+1}-V_i)-r(V_{i-1}-V_i)+(s-2r)V_i.\end{aligned}$$

Since $r>0$, $s>0$ and $s-2r=1$, we have $(L^hV)_i\ge V_i$, then we know that

$$\|L^hV\|_\infty\ge|(L^hV)_i|\ge|V_i|=\|V\|_\infty.$$

When $-V_i = \|V\|_\infty$ for some $1 \le i \le N-1$, we have

$$\begin{aligned}(L^hV)_i &= -rV_{i+1} + sV_i - rV_{i-1} \\ &= -r(V_{i+1} - V_i) - r(V_{i-1} - V_i) + (s-2r)V_i \\ &\le (s-2r)V_i = V_i < 0,\end{aligned}$$

thus we can also get that

$$\|L^hV\|_\infty \ge |(L^hV)_i| \ge |V_i| = \|V\|_\infty.$$

Then for any $V \in \mathbb{R}^d$, we have $\|V\|_\infty \le \|L^hV\|_\infty$. When h is small enough, we have

$$\begin{aligned}\|u^h - R^hu\|_\infty &\le \|L^h(u^h - R^hu)\|_\infty \\ &= \|L^hu^h - L^hR^hu)\|_\infty \\ &= \|R^hf - L^hR^hu)\|_\infty \\ &= \|R^hLu - L^hR^hu)\|_\infty \\ &= O(h^2).\end{aligned}$$

(iv) When $h = \frac{1}{5}$ and $\epsilon = 10^{-10}$, we can get the exact value of $s = 1 + 50 \times 10^{-10}$ and $r = 25 \times 10^{-10}$. And we have $R^hf = (0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 0)$ and

$$L^h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -r & s & -r & 0 & 0 & 0 \\ 0 & -r & s & -r & 0 & 0 \\ 0 & 0 & -r & s & -r & 0 \\ 0 & 0 & 0 & -r & s & -r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

Then by the $L^hu^h = R^hf$, we can get the numerical solution of (1).

```
# Get the exact value of s and r

e = 10**(-10)
s = 1 + 50 * e
r = 25 * e

print("the value of e is", e)
print("the value of s is", s)
print("the value of r is", r)

import numpy as np
from scipy.linalg import solve

a = np.array([[1, 0, 0 ,0, 0, 0], [-r, s, -r ,0, 0, 0],\
             [0, -r, s, -r, 0, 0], [0, 0, -r, s, -r, 0],\
             [0, 0, 0, -r, s, -r], [0, 0, 0 ,0, 0, 1]])
b = np.array([0, 1/5, 2/5, 3/5, 4/5, 0])
numerical_solution = solve(a, b)
print(numerical_solution)

# Get the exact value of the ODE

def exact_value(x, e):
  u = x - (np.exp((x - 1)/np.sqrt(e)) - np.exp(- (x+1)/np.sqrt(e))) / (1 -
     np.exp(- 2 / np.sqrt(e)))
  return u

exact_value_list = []
e = 10**(-10)
exact_value_list = [exact_value(i/5, e) for i in range(6)]
print(exact_value_list)

# Plot the exact value with the numerical solution of the ODE

import matplotlib.pyplot as plt
from pylab import plt
plt.style.use('seaborn')
%matplotlib inline
```

```
x = np.array([0, 1/5, 2/5, 3/5, 4/5, 1])
plt.plot(x, numerical_solution, 'o', label='Numerical solution of the ODE')
plt.plot(x, exact_value_list, label = 'Exact value of the ODE')

plt.ylabel('u(x)')
plt.xlabel('x')
plt.legend();
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