Convergence rate on CFD

Benchmark PDE

$$(*2)\quad \begin{cases} -u'' + u' + 2u = 2x^2 - 3 \\ u(0) = u(1) = 0 \end{cases}$$

Exact soln

$$u(x) = x^2 - x$$

(*2) Can be written as

$$\mathcal{L} u = f \quad \text{on } [0,1]$$

where

$$\mathcal{L} u = \begin{cases} -u'' + u' + 2u, & 0 < x < 1 \\ u, & x = 0, 1 \end{cases}$$

$$f(x) = \begin{cases} 2x^2 - 3 & 0 < x < 1 \\ 0, & x = 0, 1 \end{cases}$$

CFD on ✱2

✱ step size $h = \frac{1}{N}$

✱

$$L^h u^h = R^h f \iff \begin{cases} u_0^h = 0 \\ -r u_{i-1}^h + s u_i^h - t u_{i+1}^h = f_i \quad i = 1, 2, \cdots N-1 \\ u_N^h = 0 \end{cases}$$

where
$$\left.\begin{aligned} r &= \frac{1}{h^2} + \frac{1}{2h} \\ s &= \frac{2}{h^2} + 2 \\ t &= \frac{1}{h^2} - \frac{1}{2h} \end{aligned}\right\} \Rightarrow$$

① $r, s, t > 0$ for small h

② $s - r - t = 2$

✱ How fast $u^h \to u$ as $h \to 0$?

Precise def of convergence rate

- $h = \frac{1}{N}$
- $\forall g \in C([0,1])$, $\quad R^h g = (g(x_0), g(x_1) \cdots g(x_N))$

$$= (g_0, g_1 \cdots g_N)$$

- CFD

$$L^h u^h = R_h f$$

where

$$L^h = \begin{pmatrix} 1 & 0, & 0 & \cdots & & 0 \\ -r & s, & -t & & & \vdots \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -r & s & -t \\ \cdots & & & 0 & 0 & 1 \end{pmatrix} \qquad R_h f = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix}$$

RK stability + consistency $\Rightarrow$ convergence

Consistency

Def L^h is consistent with order $\alpha > 0$ to L

if

$$\| L^h R^h v - R^h L v \|_\infty \leq K h^\alpha , \quad \forall v \in C^2([0,1])$$

Rk $\|v\|_\infty = \max_i |v_i| \quad \forall v \in \mathbb{R}^N$.

(*2) is consistent

$$\mathcal{L} u = \begin{cases} -u'' + u' + 2u, & 0 < x < 1 \\ u, & x = 0, 1 \end{cases}$$

Lem L^h of (*2) is consistent of $\alpha = 2$,

$$f(x) = \begin{cases} 2x^2 - 3 & 0 < x < 1 \\ 0, & x = 0, 1 \end{cases}$$

pf.

$$L^h R^h v = \begin{pmatrix} 1, 0 \cdots 0 \\ L^h \\ \\ \end{pmatrix} \cdot \begin{pmatrix} V_0 \\ V_1 \\ \vdots \\ V_N \end{pmatrix}$$

① $(L^h R^h v)_0 = V_0$

$(R^h L V)_0 = LV(\underbrace{x_0}_{=0}) = V(0)$

$|(L^h R^h V)_0 - (R^h L V)_0| = 0$

② similarly

$|(L^h R^h V)_N - (R^h L V)_N| = 0$

③ for $1 \le i \le N-1$

$$\Rightarrow \begin{cases} (L^h R^h v)_i = -\delta_h \delta_{-h} v_i + \bar{\delta}_h v_i + 2 v_i \\ (R^h L v)_i = Lv(x_i) = -v''(x_i) + v'(x_i) + 2v(x_i) \end{cases}$$

$$\left| (L^h R^h v)_i - (R^h L v)_i \right| = O(h^2)$$

$\square$

Stability

Def L^h is stable if

$$\|v\|_\infty \le K \|L^h v\|_\infty , \quad \forall v \in \mathbb{R}^{N+1}$$

Rk. In general, stability is hard to check.

One needs M-matrix theory.

(*2) is stable

$$L^h = \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ -r & s & -t & & & \vdots \\ & \ddots & \ddots & \ddots & & \vdots \\ & & & -r & s & -t \\ & \cdots & & 0 & 0 & 1 \end{pmatrix}$$

① $r, s, t > 0$ for small h

② $s - r - t = 2$

Lem L^h is stable for small h.

pf.

① If $|v_0| = \|v\|_\infty$, then

$$\|L^h v\|_\infty \geq |(L^h v)_0| = |v_0| = \|v\|_\infty$$

② If $|v_N| = \|v\|_\infty$, then similarly

$$\|L^h v\|_\infty \geq \|v\|_\infty$$

③ If $v_i = \|v\|_\infty$ for some $1 \leq i \leq N-1$

$$(L^h v)_i = -r v_{i-1} + s v_i - t v_{i+1}$$

$$= \underset{\geq 0}{r}\underbrace{(v_i - v_{i-1})}_{\geq 0} + \underset{\geq 0}{t}\underbrace{(v_i - v_{i+1})}_{\geq 0} + \overbrace{(s - r - t)}^{2} v_i$$

$$\|L^h v\|_\infty \geq |(L^h v)_i| \geq 2|v_i| = \|v\|_\infty$$

④ If $-v_i = \|v\|_\infty$ for $1 \le i \le N-1$,

$$(L^h v)_i = -r v_{i-1} + s v_i - t v_{i+1}$$

$$\underbrace{= -r \underbrace{(v_{i-1} - v_i)}_{+} - t \underbrace{(v_{i+1} - v_i)}_{+}}_{-} + 2v_i \le 2v_i \le 0$$

$$\|L^h v\|_\infty \ge |(L^h v)_i| \geq 2|v_i| = 2\|v\|_\infty$$

Sum up ① — ④,

$$\|v\|_\infty \le \|L^h v\|_\infty \qquad \square$$

stability + consistency $\Rightarrow$ convergence

$|(L^h R^h v)_i - (R^h L v)_i| = O(h^2)$

$\|v\|_\infty \le \|L^h v\|_\infty$

$L^h u^h = R^h f$

$Lu = f$

Thm If h is small enough, then

$$\|u^h - R^h u\|_\infty \le K h^2$$

Pf By stability

$$\begin{aligned}
\|u^h - R^h u\|_\infty &\le \|L^h(u^h - R^h u)\|_\infty \\
&= \|L^h u^h - L^h R^h u\|_\infty \\
&= \|R^h f - L^h R^h u\|_\infty \\
&= \|R^h L u - L^h R^h u\|_\infty \\
&= O(h^2). \quad \text{by consistency.}
\end{aligned}$$

$\boxed{S}$

Ex (HW)

Ref. An ε-uniform approximation of singularly perturbed BVP (github) — Ex 5.2

Consider
$$\begin{cases} -\varepsilon u'' + u = x \\ u(0) = u(1) = 0 \end{cases} \qquad \varepsilon = 10^{-10}$$

① check exact soln
$$u(x) = x - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}}$$

② Prove CFD is consistent.

③ Compute CFD soln u^h with $h = \frac{1}{5}$.