Consider ODE

$$-u'' + u = x,\ \forall x \in (0,1),\ u(0) = u(1) = 0.$$

Answer the following questions:

1. Prove that
$$u(x) = x - \frac{\exp(x-1) - \exp(-x-1)}{1 - \exp(-2)}$$
is the unique solution.
2. Using upwind finite difference scheme, find out the matrix L^h and vector $R^h f$, such that the numerical solution satisfies $L^h u^h = R^h f$.
3. Convert $L^h u^h = R^h f$ to Markovian Reward Process.
4. Write a pseudocode for value iteration.
5. Write a pseudocode for first visit Monte-Carlo method.
6. Prove the consistency
7. Prove the stability.
8. Find convergence rate.

Hint:

1. The equation to solve is $u \in C^2(0,1) \cap C[0,1]$ satisfying $Lu = \hat{f}$, where the differential operator is
$$Lu(x) = \begin{cases} -u''(x) + u(x), & 0 < x < 1 \\ u(x), & x = 0 \end{cases} \quad \forall u \in C^2$$
The source term is
$$\hat{f}(x) = \begin{cases} f(x), & 0 < x < 1 \\ 0, & x = 0, 1 \end{cases}$$
Let $h = 1/n$. UFD is the same as CFD since there is no first order term, and UFD is to solve for $u \in \mathbb{R}^{n+1}$ from $L^h u = R^h \hat{f}$, where
$$L^h u = \begin{bmatrix} u_0 \\ -ru_0 + su_1 - tu_2 \\ \vdots \\ -ru_{i-1} + su_i - tu_{i+1} \\ \vdots \\ -ru_{n-2} + su_{n-1} - tu_n \\ u_n \end{bmatrix} \quad \forall u \in \mathbb{R}^{n+1},$$
with
$$r = 1/h^2, s = 2/h^2 + 1, t = 1/h^2.$$
and
$$R^h f = [f(x_0), f(x_1), \dots, f(x_{n-1}), f(x_n)], \forall f \in C([0,1])$$

2. One needs to show the following for the consistency: there exists $\alpha > 0$ such that

$$\|L^h R^h v - R^h L v\|_\infty \le K h^\alpha, \forall v \in C^2(0,1) \cap C[0,1], \forall \text{ small } h.$$

If $i = 0$, then

$$(L^h R^h v)_i = (R^h v)_0 = v(x_0) = 0,$$

while

$$(R^h L v)_i = L v(x_0) = v(x_0) = 0.$$

If $i = n$, then

$$(L^h R^h v)_i = (R^h v)_n = v(x_n) = 0,$$

while

$$(R^h L v)_i = L v(x_n) = v(x_n) = 0.$$

If $1 \le i \le n-1$, then

$$(L^h R^h v)_i = -\delta_h \delta_{-h} v(x_i) + v(x_i)$$

while

$$(R^h L v)_i = -v''(x_i) + v(x_i).$$

Therefore, Taylor theorem implies that

$$\|L^h R^h v - R^h L v\|_\infty \le K h^2, \forall v \in C^2(0,1) \cap C[0,1], \forall \text{ small } h.$$

3. One needs to show the following for the stability: there exists K such that

$$\|v\|_\infty \le K \|L^h v\|_\infty, \forall v \in \mathbb{R}^{n+1}.$$

If $|v_0| = \|v\|_\infty$, then

$$\|L^h v\|_\infty \ge |(L^h v)_0| = |v_0| = \|v\|_\infty.$$

If $|v_n| = \|v\|_\infty$, then

$$\|L^h v\|_\infty \ge |(L^h v)_n| = |v_n| = \|v\|_\infty.$$

If $|v_i| = \|v\|_\infty$ for some $1 \le i \le n-1$, then

$$\|L^h v\|_\infty \ge |(L^h v)_i| = |-r v_{i-1} + s v_i - t v_{i+1}| \ge s|v_i| - r|v_{i-1}| - t|v_{i+1}| \ge (s - r - t)|v_i| \ge |v_i| = \|v\|_\infty.$$

4. Let u^h be the numerical solution.

$$\begin{aligned} \|u^h - R^h u\|_\infty &\le \|L^h(u^h - R^h u)\|_\infty \\ &= \|L^h u^h - L^h R^h u\|_\infty \\ &= \|R^h \hat{f} - L^h R^h u\|_\infty \\ &= \|R^h L u - L^h R^h u\|_\infty = O(h^2). \end{aligned}$$