Homework 4, MA573

Monotonicity in volatility

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Exercise1:

Prove or disprove: Suppose f is convex and X is submartingale, prove that $g(t) = \mathbb{E}[f(X_t)]$ is increasing.

Solution:

This proposition is not right. We can give a counter example. Assume $f(x) = -x$, then $f(x)$ is a convex function since $\forall x_1, x_2 \in \mathbb{R}$, and $\forall t \in [0,1]$, we have $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$ holds. Then we can verify the proposotion that $g(t)$ is increasing. Since $\forall s \leq t$,

$$\mathbb{E}[f(X_t) - f(X_s)|\mathcal{F}_s] = \mathbb{E}[-X_t + X_s|\mathcal{F}_s] = X_s - \mathbb{E}[X_t|\mathcal{F}_s].$$

Since X_t is a submartigale with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$, then we have

$$\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s,$$

thus we know that $\forall s \leq t$, $\mathbb{E}[f(X_t) - f(X_s)|\mathcal{F}_s] \leq 0$. Then we take expectation on the both side and by the definition of conditional expectation, we have

$$\mathbb{E}[f(X_t) - f(X_s)] \leq 0.$$

Then we know that $\forall s \leq t$, $g(t) \leq g(s)$, such that $g(t)$ is decreasing.

Exercise2:

Let $t \mapsto e^{-rt}S_t$ be a martingale, then prove that

$$C(t) = \mathbb{E}[e^{-rt}(S_t - K)^+]$$

is increasing.

Solution:

Since $e^{-rt}S_t$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$, and the function $f(x) = x^+$ is a convex function, then by the Jensen's inequality in the conditional expectation, we have $\forall s \leq t$,

$$\mathbb{E}[e^{-rt}(S_t - K)^+|\mathcal{F}_s] \geq (\mathbb{E}[e^{-rt}(S_t - K)|\mathcal{F}_s])^+ = (e^{-rs}S_s - e^{-rt}K)^+,$$

And we have

$$(e^{-rs}S_s - e^{-rt}K)^+ = \big((e^{-rs}(S_s - K) + K(e^{-rs} - e^{-rt})\big)^+ \geq (e^{-rs}(S_s - K))^+$$

since $K(e^{-rs}-e^{-rt})\geq 0$, $\forall s\leq t$ and $K>0$. Thus we know that $\mathbb{E}[e^{-rt}(S_t-K)^+|\mathcal{F}_s]\geq (e^{-rs}(S_s-K))^+$, then we take expectation on the both side, we have that

$$\mathbb{E}[e^{-rt}(S_t-K)^+]\geq \mathbb{E}[(e^{-rs}(S_s-K))^+].$$

Such that $\forall s\leq t$, we have $C(s)\leq C(t)$, then $C(t)$ is increasing.

Exercise3:

Suppose $r=0$ and S is martingale, prove that $P(t)=\mathbb{E}[(S_t-K)^-]$ is increasing.

Solution:

Since S_t is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$, we have $\forall s\leq t$,

$$\mathbb{E}[(S_t-K)^- - (S_s-K)^-|\mathcal{F}_s]=\mathbb{E}[(S_t-K)^-|\mathcal{F}_s]-(S_s-K)^-,$$

which equivalent to

$$\mathbb{E}[(S_t-K)^- - (S_s-K)^-|\mathcal{F}_s]=(K-S_s)\mathbb{I}_{\{S_t<K\}}-(K-S_s)^+,$$

then we have

$$\mathbb{E}[(S_t-K)^- - (S_s-K)^-|\mathcal{F}_s]=\begin{cases} K-S_s-(K-S_s)^+, & S_t<K \\ -(K-S_s)^+, & S_t\geq K, \end{cases}$$

Then take expectation on the both side, we have

$$\mathbb{E}[(S_t-K)^- - (S_s-K)^-]=\begin{cases} \mathbb{E}[K-S_s-(K-S_s)^+], & S_t<K \\ \mathbb{E}[-(K-S_s)^+], & S_t\geq K, \end{cases}$$

since $K-S_s-(K-S_s)^+\leq 0$ and $-(K-S_s)^+\leq 0$, then we have $\forall s\leq t$,$\mathbb{E}[(S_t-K)^- - (S_s-K)^-]\leq 0$, thus $P(t)\leq P(s)$. So, we proved that $P(t)$ is decreasing.

solution 2:

Or we can just verify the relationship between $\mathbb{E}[(S_t-K)^-|\mathcal{F}_s]$ and $(S_s-K)^-$. If $S_t>K$,

$$\mathbb{E}[(S_t-K)^-|\mathcal{F}_s]=0\leq (S_s-K)^-.$$

If $S_t\leq K$,

$$\mathbb{E}[(S_t-K)^-|\mathcal{F}_s]=K-S_s\leq (S_s-K)^-.$$

Thus we have $\mathbb{E}[(S_t-K)^-|\mathcal{F}_s]\leq (S_s-K)^-$. Taking expectation on the both side, we have

$$\mathbb{E}[(S_t-K)^-]\leq \mathbb{E}[(S_s-K)^-].$$

So we know that $\forall s\leq t$, $P(t)\leq P(s)$, then $P(t)$ is decreasing.