Importance sampling

Example

Our goal is to compute,

$$\alpha = \int_0^1 h(x)dx$$

where

$$h(x) = 100 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x).$$

Of course, the exact value shall be

$$\alpha = 1.99.$$

- (Q.) Pretended not to know the exact value, use OMC?

OMC

procedure MCINTEGRAL(N) $\triangleright$ N is total number of samples
    $s \leftarrow 0$ $\triangleright$ s is the sum of samples
    for $i = 1 \dots N$ **do**
        generate $Y \sim U(0, 1)$
        $s \leftarrow s + h(Y)$
    return $\frac{s}{N}$ $\triangleright$ return the average

MSE of OMC

- $X_i \sim U(0,1)$ iid.
- OMC is

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} h(X_i)$$

- *Prove* $MSE(\hat{\alpha}_n) = \frac{97.0299}{n}$.[1]

[1] Goal. Use Importance Sampling to reduce MSE.

Importance Sampling (IS) - 1

- We want

$$\alpha = \mathbb{E}[h(X)|X \sim p] = \int_0^1 h(x)p(x)dx.$$

- IS considers, with a pdf p_1

$$\alpha = \int_0^1 h(x)\frac{p(x)}{p_1(x)}p_1(x)dx = \mathbb{E}\left[h(X)\frac{p(X)}{p_1(X)}\middle|X \sim p_1(x)\right]$$

[2]

[2] How to choose a smart p_1?

Importance Sampling (IS) - 2

Since we observe that the interval $(0, 1/100)$ is much more *important* than $(1/100, 1)$, our choice of p_1 is the following:

$$p_1(x) = \frac{1}{C}(\alpha \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x)),$$

where $\alpha = 2$, denoted by $\mathrm{IS}(\alpha = 2)$.

- prove that $C = 1.01$.

Importance Sampling (IS) - 3

procedure ISINTEGRAL(N) ▷ N is total number of samples
  $s \leftarrow 0$ ▷ s is the sum of samples
  for $i = 1 \ldots N$ **do**
    generate $Y \sim p_1$ ▷ use ITM
    $s \leftarrow s + h(Y) \cdot \frac{p(Y)}{p_1(Y)}$
  return $\frac{s}{N}$ ▷ return the average

MSE of IS($\alpha = 2$)

- Prove $MSE = 47.53/n$.
- Can you find better α?

Example on Digital Put (DP)

- Asset price under $\mathbb{Q}$:

$$S_t = S_0 \exp\{\mu t + \sigma W_t\}.$$

- Payoff

$$h(S_T) = I(S_T < S_0 e^{-b}).$$

- Want forward price:

$$v = \mathbb{E}^{\mathbb{Q}}[h(S_T)].$$

- Parameters:

$$r = .03, \sigma = .2, \mu = r - \frac{1}{2}\sigma^2 = .01,\ T = 1,\ b = .39.$$

IS(α) on DP

- $\phi_\alpha \sim \mathcal{N}(-\alpha, 1)$.
- $$\begin{aligned} v &= \int I(x < -2)\phi_0(x)dx \\ &= \int I(x < -2)\exp\{\tfrac{1}{2}\alpha^2 + \alpha x\}\phi_\alpha(x)dx \\ &= \mathbb{E}[I(X < -2)\exp\{\tfrac{1}{2}\alpha^2 + \alpha x\}],\ X \sim \phi_\alpha \end{aligned}$$

Hw on DP

- Prove that the exact price is 0.02275.
- Use OMC find the price
- Use IS(α) find the price.
- Can you show your approach is optimal?
- Prove or demonstrate IS is more efficient to OMC.