1 Abstract

You will learn

- taking definite integral by ordinary Monte Carlo (OMC)
- exact sampling with python provided random number generators

2 Problem

Example 1 *Our goal is to compute, using OMC by exact sampling*

$$\alpha = \int_0^1 h(x)dx$$

where

$$h(x) = 100 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x).$$

The exact value shall be

$$\alpha = 1.99.$$

3 Analysis

3.1 OMC by exact sampling

The objective is to

- estimate

$$\alpha = \mathbb{E}[X], \quad X \sim p(x)$$

one can use random number generator by computer (if possible)

$$\{iid \ X_i \sim p(x) : i = 1, 2, \dots, n, \}.$$

Then, one can compute the approximation of α by

$$\hat{\alpha}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

We say $\hat{\alpha}_n$ as OMC by *exact sampling*, since the sample X_i produced by random generator has the same distribution as true distribution X, i.e.

$$X_i \sim X, \ \forall i.$$

Proposition 1 *Prove the properties of the OMC by exact sampling below:*

- X_1 *itself can be treated as an unbiased MC, because*

$$\mathbb{E}[X_1] = \alpha.$$

However, MSE is big, ie.

$$MSE(X_1) = Var(X) = \int x^2 p(x)dx.$$

- $\hat{\alpha}_n$ *is consistent almost surely due to LLN, i.e.*

$$\hat{\alpha}_n \to \alpha, \text{ almost surely as } n \to \infty.$$

Moreover, $\hat{\alpha}_n$ *is unbiased too, and*

$$MSE(\hat{\alpha}_n) = Var(\hat{\alpha}_n) = \frac{1}{n}Var(X) \to 0.$$

3.2 Evaluation of integral

Back to our Example 1, we write
$$\alpha = \mathbb{E}[X] = \mathbb{E}[h(Y)],$$

where $X = h(Y)$ and $Y \sim U(0,1)$. In other words, although X-sampling is not directly available in python, one can use $U(0,1)$ random generator (see *numpy.random.uniform*) to produce Y_i, then compute $h(Y_i)$ for the sample X_i.

Algorithm 1 Integral by MC - Example 1

1: **procedure** MCINTEGRAL(N)   ▷ N is total number of samples
2:     $s \leftarrow 0$ ▷ s is the sum of samples
3:     **for** $i = 1...N$ **do**
4:         generate two numbers Y from $U(0,1)$ ▷ use *numpy.random.uniform*
5:         $s \leftarrow s + h(Y)$
6:     **return** $\frac{s}{N}$ ▷ return the average

Example 2   • *Implement Algo 1 for estimator* $\hat{\alpha}_N$*;*

- *Estimate* $MSE(\hat{\alpha}_N)$ *for* $N = 2^5 \dots, 2^{10}$ *and plot log-log chart.*

Proposition 2 *Suppose*

- $\hat{Y}$ *is exact sampling of* Y*, ie,* $\hat{Y} \sim Y$

- h *is almost surely continuous w.r.t* Y*, i.e.*

$$\mathbb{P}Y^{-1}(D_h) = \mathbb{P}(Y \in D_h) = 0,$$

where D_h *is the set of discontinuous points of the function* h*.*

Then, $h(\hat{Y})$ *is unbiased estimator of* $\mathbb{E}h(Y)$*, i.e.* $\mathbb{E}h(\hat{Y}) = \mathbb{E}h(Y)$*.*

Example 3 *Prove that* $h(\hat{Y})$ *is unbiased estimator of* α *in Example 1.*