MA573 - PROBLEM SETS

1. Finite Difference Operator

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth even function satisfying $f(0) = 0$. Our objective is to approximate the second order derivative $f''(0)$.

(1) Prove that $f'(0) = 0$.

(2) Ziyue proposes the following estimator for $f''(0)$: for a step size h

$$a_h = \frac{2f(h)}{h^2}.$$

Please justify that Ziyue's estimation has its convergence $O(h^2)$.

(3) Is there anyway to improve the above convergence to $O(h^4)$ in the form of

$$b_h = \frac{c_1 f(h) + c_2 f(2h)}{h^2}$$

for some constants c_1 and c_2?

(4) If the above function f is odd and other properties remain the same, how do you want to find the $f''(0)$ efficiently?

2. Estimation of π

Thomas proposed the following estimator for the value π.

$$\hat{\pi} = c \cdot I(X_1^2 + X_2^2 < 1/2)$$

where X_1 and X_2 are two independent uniform random variables on $(-1, 1)$.

(1) find the constant c so that $\hat{\pi}$ is unbiased estimator to π;

(2) Write pseudocode.

(3) What is its MSE of $\hat{\pi}$ with c given in your above answer?

(4) Can you propose a better algorithm?

3. Euler method for CEV

Consider pricing a call of (T, K) underlying CEV

$$dS_t = 0.03 S_t dt + 0.2 S_t^{1/2} dW_t, \ S_0 = 10.$$

(1) Euler method with step size Δ is based on iteration

$$X_{t+\Delta} = X_t + 0.03 X_t \Delta + 0.2\sqrt{X_t \Delta} \cdot Z_t, \ t = 0, \Delta, 2\Delta, \ldots.$$

What is the potential danger to implement this algorithm?

(2) Revise the above algorithm, and write its pseudocode.

4. Importance sampling on digital option

Asset price under Equivalent Martingale Measure (EMM) $\mathbb{Q}$ follows Asset price under $\mathbb{Q}$ follows

$$S_t = S_0 \exp\{\mu t + \sigma W_t\}.$$

Consider Digital put with its payoff

$$h(S_T) = I(S_T < S_0 e^{-b}).$$

We want to find the forward price:

$$v = \mathbb{E}^{\mathbb{Q}}[h(S_T)].$$

Parameters are given as

$$r = .03, \sigma = .2, \mu = r - \frac{1}{2}\sigma^2 = .01, T = 1, b = .39.$$

(1) Prove that the exact price is $v = \mathbb{E}[I(Z < -2)]$ with $Z \sim \mathcal{N}(0, 1)$.
(2) Write a pseudocode for OMC to find the price
(3) Importance Sampling, denoted by IS(α), is to use Monte Carlo to the following price formula:
$$v = \mathbb{E}[k(\bar{Z}) I(\bar{Z} < -2)], \ \bar{Z} \sim \mathcal{N}(\alpha, 1).$$
Find the function $k(z)$. Your answer may depend on α.
(4) Write a pseudocode for $IS(\alpha = 1)$.

5. Stochastic approximation

Let $D = \{X_i : i \in \mathbb{N}\}$ be a data set of iid sequence from a random generator of distribution $\mathcal{N}(b, \sigma^2)$ for some unknown parameters b and σ. Our goal is to estimate b using so called stochastic approximation (SA) with a given learning rate $\alpha \in (0, 1)$:

- initialize b_0
- iterate $b_{n+1} = b_n + \alpha(x_k - b_k)$.

We want to examine the convergence $b_n \to b$. For simplicity, let's fix $\alpha = 0.01$. We denote the error

$$\epsilon_n = b_n - b.$$

(1) Write pseudocode for SA.
(2) Prove that $\lim_n \mathbb{E}[\epsilon_n] = 0$.
(3) Prove $\lim_n \|\epsilon_n\|_2 \geq \frac{\alpha\sigma}{2-\alpha}$, if the limit exists.
(4) Can you prove or disprove that $b_n \to b$ in L^2?

6. UFD on BVP

Consider ODE

$$-\epsilon u'' + u = x, \ \forall x \in (0, 1), \ u(0) = u(1) = 0,$$

with $\epsilon = 1$. Answer the following questions:

(1) Prove that
$$u(x) = x - \frac{\exp(\frac{x-1}{\sqrt{\epsilon}}) - \exp(-\frac{x+1}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})}$$
is the unique solution.
(2) Using upwind finite difference scheme, find out the matrix L^h and vector $R^h f$, such that the numerical solution satisfies $L^h u^h = R^h f$.
(3) Convert $L^h u^h = R^h f$ to Markovian Reward Process.
(4) Write a pseudocode for value iteration.
(5) Write a pseudocode for first visit Monte-Carlo method.
(6) Prove the consistency and stability.