DISCRETIZATION OF ELLIPTIC LINEAR PDE AND NEURAL NETWORK

1. Problem setup

1.1. **HJB.** We want to solve a d-dimensions linear PDE given below:

- Domain
$$O = \{x \in \mathbb{R}^d : 0 < x_i < 1, i = 1, 2, \dots d\}.$$
- Equation on O:
$$(\frac{1}{2}\Delta - \lambda)v(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial v(x)}{\partial x_i} + \ell(x) = 0.$$
- Dirichlet data on ∂O:
$$v(x) = g(x).$$

1.2. **Examples.**

1.2.1. *Multidimensional PDE with quadratic function as its solution.* Consider a class of PDE with coefficients satisfying,
$$d - \lambda \|x - \frac{1}{2}\mathbf{1}\|_2^2 + b(x) \cdot (2x - \mathbf{1}) + \ell(x) = 0,$$
where $\mathbf{1}$ is an $\mathbb{R}^d$-vector with each element being 1. The exact solution is
$$v(x) = \|x - \frac{1}{2}\mathbf{1}\|_2^2 = \sum_{i=1}^{d} (x_i - \frac{1}{2})^2.$$

2. Discretization

2.1. **FDM.** We introduce some notions of finite difference operators. Commonly used first order finite difference operators are FFD, BFD, and CFD. Forward Finite Difference (FFD) is
$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{he_i} v(x) := \frac{v(x + he_i) - v(x)}{h}.$$
Backward Finite Difference (BFD) is
$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{-he_i} v(x) := \frac{v(x - he_i) - v(x)}{-h}.$$
Central Finite Difference (CFD) is
$$\frac{\partial}{\partial x_i} v(x) \approx \bar{\delta}_{he_i} v(x) := \frac{1}{2}(\delta_{-he_i} + \delta_{he_i}) v(x).$$

It can be verified that the CFD has the following explicit form:

$$\bar{\delta}_{he_i} v(x) = \frac{v(x+he_i) - v(x-he_i)}{2h}.$$

Second order finite difference operators are the followings:

$$\frac{\partial^2}{\partial x_i^2} v(x) \approx \delta_{-he_i}\delta_{he_i} v(x) = \frac{v(x+he_i) - 2v(x) + v(x-he_i)}{h^2}.$$

Although the next operator will not be used below, we will write it for its completeness. If $i \neq j$, we use

$$\tfrac{\partial^2}{\partial x_i \partial x_j} v(x) \quad \approx \tfrac{1}{2}(\delta_{he_i}\delta_{-he_j} v(x) + \delta_{he_j}\delta_{-he_i} v(x))$$

2.2. **CFD on PDE.** Approximations for PDE are

$$\frac{\partial v(x)}{\partial x_i} \leftarrow \delta_{\pm he_i} v(x)$$

and

$$\frac{\partial^2 v(x)}{\partial x_i^2} \leftarrow \delta_{-he_i}\delta_{he_i} v(x).$$

For simplicity, if we set

$$\gamma = \frac{d}{d+h^2\lambda},\ p^h(x \pm he_i|x) = \frac{1}{2d}(1 \pm hb_i(x)),\ \ell^h(x) = \frac{h^2\ell(x)}{d},$$

then it yields DPP

$$v(x) = \gamma\Big\{\ell^h(x) + \sum_{i=1}^{d} p^h(x+he_i|x)v(x+he_i) + p^h(x-he_i|x)v(x-he_i)\Big\}.$$

2.3. **UFD on PDE.** Upwind finite difference(UFD) is the following:

$$\frac{\partial v(x)}{\partial x_i} \leftarrow \delta_{he_i} v(x) \cdot I(b_i(x) \geq 0) + \delta_{-he_i} v(x) \cdot I(b_i(x) < 0)$$

and

$$\frac{\partial^2 v(x)}{\partial x_i^2} \leftarrow \delta_{-he_i}\delta_{he_i} v(x).$$

Then, with

$$c = d + h\sum_i |b_i(x)|,\ \gamma = \frac{c}{c+h^2\lambda},\ \ell^h = \frac{\ell(x)h^2}{c},\ p^h(x \pm he_i|x) = \frac{1+2hb_i^{\pm}(x)}{2c},$$

then it yields DPP

$$v(x) = \gamma\Big\{\ell^h(x) + \sum_{i=1}^{d} p^h(x+he_i|x)v(x+he_i) + p^h(x-he_i|x)v(x-he_i)\Big\}.$$