1 Abstract

Our goal is to learn

- Euler approximation for the solution of 1-d SDE
- Strong and weak convergence rate

2 Problem

Consider 1-d SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, X(0) = x_0$$

We shall find, for some small step size δ

$$X^\delta(t) \approx X(t), \forall t \geq 0$$

in some sense. For convenience, we assume μ and σ are infinitely smooth and bounded in all its values and derivatives.

3 Euler scheme

The above SDE can be written as the following integral form:

$$X_t = x_0 + \int_0^t \mu(X_s)ds + \sigma(X_s)dW_s$$

If we denote

$$X_{t,s} = X_s - X_t,$$

then

$$X_{t,t+\delta} = \int_t^{t+\delta} \mu(X_s)ds + \sigma(X_s)dW_s.$$

Ito formula says

$$\mu(X_s) = \mu(X_t) + \int_t^s (\mu'(X_r) + \frac{1}{2}\mu''(X_r))dr + \mu'(X_r)\sigma(X_r)dW_r$$

and

$$\sigma(X_s) = \sigma(X_t) + \int_t^s (\sigma'(X_r) + \frac{1}{2}\sigma''(X_r))dr + \sigma'(X_r)\sigma(X_r)dW_r.$$

If $|s - t| < \delta$ is small enough, we expect $\int_t^s ...$ is again small, and write

$$\mu(X_s) \approx \mu(X_t)$$

and

$$\sigma(X_s) \approx \sigma(X_t).$$

Thus, $X_{t,t+\delta}$ can be rewritten as

$$X_{t,t+\delta} \approx \mu(X_t)\delta + \sigma(X_t)W_{t,t+\delta}.$$

With the fact that $W_{i\delta,(i+1)\delta} \sim \sqrt{\delta}Z_i$ are iid normal random variables, Euler method repeats the following recursive formula:

$$X_{i\delta,(i+1)\delta}^{\delta} = \mu(X_{i\delta}^{\delta})\delta + \sigma(X_{i\delta}^{\delta})\sqrt{\delta}Z_i, \forall i = 0,\ldots,N-1.$$

We can write the following pseudocode.

Algorithm 1 Euler for SDE 1d to simulate $\hat{X} \approx \langle X, \Pi_{T,N} \rangle$

1: **inputs:** drfit $\mu(\cdot)$ and volatility $\sigma(\cdot)$
2: **procedure** EULER1D(x,T,N) ▷ x: initial state, T: terminal time,
3:   ▷ N: number of meshes
4:     $\delta \leftarrow T/N$; $X_0^{\delta} \leftarrow x$;   ▷ Init
5:     **for** $i = 0...N-1$ **do**
6:         $t_{i+1} \leftarrow t_i + \delta$
7:         $Z \leftarrow \mathcal{N}(0,1)$
8:         $X_{i+1}^{\delta} \leftarrow X_i^{\delta} + \mu(X_i^{\delta})\delta + \sigma(X_i^{\delta})\sqrt{\delta}Z$
9:     **return** $\{(t_i, X_i^{\delta}) : i = 0,1,\ldots,N\}$.   ▷ output is a discrete path

---

4 Strong convergence rate

Algo 1 gives a sequence of numbers:

$$(X_0^{\delta}, X_1^{\delta}, \ldots, X_N^{\delta}) := X^{\delta}.$$

To compare with continuous true path $(X_t : t \in [0,T])$, we first do the piecewise linear interpolation of X^{δ}, that is

$$L_t^{\delta} = \frac{(i+1)\delta - t}{\delta}X_i^{\delta} + \frac{t - i\delta}{\delta}X_{i+1}^{\delta}, \quad \text{if } i\delta \le t < (i+1)\delta.$$

Theorem 1 *RMSE of Euler approximation under uniform norm has convergence order $1/2$, i.e.*

$$\mathbb{E}\left[\sup_{0 \le t \le T}|X_t - L_t^{\delta}|\right] \le K\delta^{1/2}.$$

PROOF: see Theorem 2.7.3 of [2]. $\square$

ex. Show that

$$\mathbb{E}[|X_T - L_T^{\delta}|] \le K\delta^{1/2}.$$

4.1 A remark on constant interpolation of Euler solution

If we denote the piecewise constant interpolation by

$$C_t^{\delta} = X_i^{\delta}, \text{ if } i\delta \le t < (i+1)\delta,$$

then the above strong convergence fails. Let's use the following example to illustrate this issue.

Let $X = W$ be the Brownian motion itself. Euler yields

$$C_t^{\delta} = W_{[t/\delta]\delta}.$$

Therefore,

$$RMSE = \mathbb{E}\Big[\sup_{0 \le t \le T} |X_t - C_t^\delta|\Big] = \mathbb{E}\Big[\sup_{0 \le t \le T} |W_t - W_{[t/\delta]\delta}|\Big] = \mathbb{E}\Big[\sup_{i=0,...,N-1} Y_i\Big],$$

where

$$Y_i = \sup_{i\delta \le t < (i+1)\delta} |W_t - W_{i\delta}|.$$

Note $Y_i \ge |W_{i\delta,(i+1)\delta}| := \sqrt{\delta}|Z_i|$, then

$$RMSE \ge \sqrt{\delta}\mathbb{E}\Big[\sup_{i=0,...,N-1} |Z_i|\Big] > O(\delta^{1/2}).$$

5 Weak convergence rate

Given Y^δ and X, we define

$$e^g(\delta) = \Big|\mathbb{E}[g(X_T)] - \mathbb{E}[g(Y_T^\delta)]\Big|.$$

Then, we say Y_T^δ converges to X_T weakly if

$$\lim_{\delta \to 0} e^g(\delta) = 0, \ \forall g \in C_b.$$

We say weak convergence rae is γ, if

$$\exists K_g > 0, \ s.t. \ e^g(\delta) \le K_g \delta^\gamma$$

for any $g \in C_b$.

Theorem 2 C^δ *covnverges to X with $\gamma = 1$.*

PROOF: see section 9.7 of [1] □

5.1 Excercise

Example 1 *Consider CEV*

$$dS_t = 0.03 S_t dt + \sigma S_t^\gamma dW_t, \ S_0 = 100.$$

- *Write a pseudo code and python code for vanilla option price*

- *For $\sigma = 0.2$ and $\gamma = 1$, compute call price with $T = 1$ and $K = 97$. Does it recover option price given by BSM formula?*

- *(optional) can you demonstrate the weak/strong convergence rate*

- *Download option data from here and calibrate volatility and elasticity constant from 5 month call options.*

References

[1] P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992. 3

[2] Xuerong Mao. *Stochastic Differential Equations and Applications*. Horwood Pub Ltd, 2007. 2