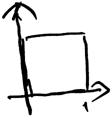
Multidimensiontal PDE with Value Iteration

WPI

Problem: nd BVP

- Domain

$$O = \{x \in \mathbb{R}^d : 0 < x_i < 1, i = 1, 2, \ldots d\}.$$



- Equation on O:

$$(\frac{1}{2}\Delta - \lambda)v(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial v(x)}{\partial x_i} + \ell(x) = 0.$$

$$\Delta v(x) = \sum_{i=1}^{d} \partial_{ii} v(x)$$

- Dirichlet data on ∂O:

$$v(x) = g(x).$$

A class of benchmarks

Set. $d=2$ $\lambda=1$, $b=0$
$$\ell = -2 + (x_1 - \tfrac{1}{2})^2 + (x_2 - \tfrac{1}{2})^2$$
$$= +x_1^2 + x_2^2 - x_1 - x_2 - \tfrac{3}{2}$$

Consider a class of PDE with coefficients satisfying,

$$d - \lambda \|x - \frac{1}{2}\mathbf{1}\|_2^2 + b(x) \cdot (2x - \mathbf{1}) + \ell(x) = 0,$$

where $\mathbf{1}$ is an $\mathbb{R}^d$-vector with each element being 1. The exact solution is

$$\vec{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$v(x) = \|x - \frac{1}{2}\mathbf{1}\|_2^2 = \sum_{i=1}^{d}(x_i - \frac{1}{2})^2.$$

- Verify the above exact solution.

$$\partial_i v(x) = \frac{\partial}{\partial x_i}\left(x_i - \frac{1}{2}\right)^2$$
$$= 2x_i - 1$$

$$\partial_{ii} v(x) = 2$$
$$\frac{1}{2}\Delta v(x) = \frac{1}{2}\sum_{i=1}^{d} 2 = d$$

A benchmark with 2-d

- Equation on $O = (0,1)^2 := (0,1) \times (0,1) =$

$$\frac{1}{2}\Delta v(x) - v(x) + x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2} = 0.$$

- Dirichlet data on ∂O:

$$v(x) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2.$$

Exact solution is

$$v(x) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2.$$

Finite difference operators

- FFD

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{he_i} v(x) := \frac{v(x + he_i) - v(x)}{h}.$$

- BFD

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{-he_i} v(x) := \frac{v(x - he_i) - v(x)}{-h}.$$

- CFD

$$\frac{\partial}{\partial x_i} v(x) \approx \bar{\delta}_{he_i} v(x) = \frac{1}{2}(\delta_{-he_i} + \delta_{he_i}) v(x). = \frac{v(x+he_i) - v(x-he_i)}{2h}$$

- SCFD

$$\frac{\partial^2}{\partial x_i \partial x_j} v(x) \approx \frac{1}{2}(\underline{\delta_{he_i}\delta_{-he_j} v(x)} + \underline{\delta_{he_j}\delta_{-he_i} v(x)})$$

$$\shortparallel$$

$$\frac{\partial^2}{\partial x_j \partial x_i} v(x)$$

Explicit forms of FD operators

- CFD

$$\frac{\partial}{\partial x_i} v(x) \approx \bar{\delta}_{he_i} v(x) = \frac{v(x+he_i) - v(x-he_i)}{2h}. = O(h^2)$$

- SCFD with $i = j$:

$$\frac{\partial^2}{\partial x_i \partial x_j} v(x) \approx \delta_{he_i} \delta_{-he_i} v(x) = \frac{v(x+he_i) - 2v(x) + v(x-he_i)}{h^2} = O(h^2)$$

- What is the convergence rate of the above FD operators?
- For 2-d Benchmark, derive CFD with $h = 1/4$, i.e . write the linear system of the form $L^h u^h = f^h$.

$$\frac{1}{2} \Delta v(x) = \frac{1}{2} \sum_{i=1}^{d} \partial_{ii} v(x) \qquad \frac{\partial^2}{\partial x_i \partial x_i}$$

Sketch of CFD on PDE

- In multidimensional case, L^h is not simply tridiagonal and it's hard for programing.
- One can rewrite $L^h v^h = f^h$ by

$$v^h = (I - L^h)v^h + f^h := F^h v^h.$$

$$v^h = Iv^h - L^h v^h + f^h$$
$$= (I - L^h)v^h + f^h$$
$$= F^h v^h$$

- Implement value iteration:

$$v^{(n+1)} = F^h v^{(n)}.$$

Then, we expect $v^{(n)} \to v^h$.

I want to solve, for $a, b \in \mathbb{R}$.

$$ax = b$$

① $x = b/a = a^{-1} \cdot b$

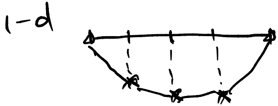
or

② $x = x - ax + b = (1-a)x + b \stackrel{\Delta}{=} g(x)$

$$\boxed{\begin{array}{l} \text{① Init } x_0 \\ \text{② Repeat} \\ \qquad x_{n+1} = g(x_n) \end{array}} \longrightarrow \text{VI.}$$

$x_n \to x$

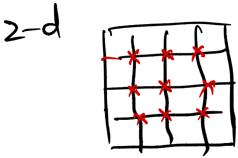
CFD solution



For simplicity, if we set

$$\gamma = \frac{d}{d + h^2\lambda}, \;\; p^h(x \pm he_i|x) = \frac{1}{2d}(1 \pm hb_i(x)), \;\; \ell^h(x) = \frac{h^2\ell(x)}{d},$$

then CFD solution $v^h : G^h \mapsto \mathbb{R}$ is



- For $x \notin \partial O^h$,

$$v(x) = g(x).$$

- For $x \in O^h = O \cap G^h$

$$v(x) = \gamma\Big\{\ell^h(x) + \sum_{i=1}^{d} p^h(x+he_i|x)v(x+he_i) + p^h(x-he_i|x)v(x-he_i)\Big\}.$$

  which we write as

$$v(x) = F^h v(x).$$

$$\left(\frac{1}{2}\Delta - \lambda\right)v(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial v(x)}{\partial x_i} + \ell(x) = 0.$$

$$\partial_i v(x) = \frac{v(x+he_i) - v(x-he_i)}{2h}$$

$v(x+he_i) \to v_i^+$, $v(x-he_i) \to v_i^-$

$$\partial_{ii} v(x) = \frac{v(x+he_i) - 2v(x) + v(x-he_i)}{h^2}$$

$v(x) \to v$

$$\frac{1}{2}\sum_{i=1}^{d} \frac{v_i^+ - 2v + v_i^-}{h^2} - \lambda v + \sum_{i=1}^{d} b_i \frac{v_i^+ - v_i^-}{2h} + \ell = 0$$

$$v\left(\frac{d}{h^2} + \lambda\right) = \sum_{i=1}^{d} v_i^+\left(\frac{1}{2h^2} + \frac{b_i}{2h}\right) + \sum_{i=1}^{d} v_i^-\left(\frac{1}{2h^2} - \frac{b_i}{2h}\right) + \ell$$

$$v(d + \lambda h^2) = \sum_{i=1}^{d} v_i^+\left(\frac{1}{2} + \frac{hb_i}{2}\right) + \sum_i v_i^-\left(\frac{1}{2} - \frac{b_i h}{2}\right) + \ell h^2$$

$$v\,\frac{d+\lambda h^2}{d} = \frac{1}{d}\sum_{i=1}^{d} v_i^+\left(\frac{1}{2} + \frac{hb_i}{2}\right) + \sum_i v_i^-\left(\frac{1}{2} - \frac{b_i h}{2}\right) + \frac{\ell h^2}{d}$$

Value iteration - Jacobi

Algo.

1: **procedure** VI($\hat{\epsilon}, \hat{n}$) ▷ $\hat{n}$: max iteration, $\hat{\epsilon}$: tolerance
2: Initial guess: $\{v(x) : x \in O^h\}$.
3: $flag \leftarrow 1, n \leftarrow 0$ ▷ n is number of iteration
4: **while** flag **do**
5: $\epsilon \leftarrow 0;\ n \leftarrow n+1$
6: **for** $x \in O^h$ **do**
7: $u(x) \leftarrow v(x)$
8: $v(x) \leftarrow F^h u(x)$.
9: $\epsilon \leftarrow \max\{\epsilon, |u(x) - v(x)|\}$.
10: **if** $\epsilon < \hat{\epsilon}$ **then**
11: flag $= 0$
12: **return** $\{v(x) : x \in O^h\}$.

for $x \in O^h$ do
$u(x) \leftarrow v(x)$
for $x \in O^h$ do
$v(x) \leftarrow F^h u(x)$
$\varepsilon \leftarrow \max(\varepsilon, |u(x) - v(x)|)$
If $\varepsilon < \hat{\varepsilon}$ then
flag $= 0$

- If line 8 is replaced by $v(x) \leftarrow F^h v(x)$, then it becomes Gauss-Seidel iteration.

Ex on Benchmark with 2d

- Write explicit form of $v^h = F^h v^h$ for CFD.
- Prove that, for any $v : O^h \mapsto \mathbb{R}$, iteration has contraction, i.e. there exists $\hat{\gamma} \in (0, 1)$ such that

$$\|F^h v\|_\infty \leq \hat{\gamma} \|v\|_\infty.$$

- Prove that, value iteration converges, i.e. $v^{(n)} \to v^h$ as $n \to \infty$.
- Use value iteration (Jacobi) to compute $\|v^h - R^h v\|_\infty$ for $h = 1/8$.
- Use Gauss-Seidel value iteration to compute $\|v^h - R^h v\|_\infty$ for $h = 1/8$.
- (optional) What is the highest dimension can you handle with value iteration?
- Write explicit form of $v^h = F^h v^h$ for UFD.