Homework 2, 2019 Fall

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Exercise 1:

(i) Let $f : [0,1] \to \mathbb{R}, f(x) = \sqrt{x}$. Show that f is uniformly continuous but not Lipschitz continuous.

(ii) Let $g : (0,1) \to \mathbb{R}, g(x) = \frac{1}{x}$. Show that g is not uniformly continuous.

Solution:

(i) Let $\epsilon > 0$ be given, choose $\delta = \epsilon^2$, for any $x, y \in [0,1]$ with $|x-y| < \delta$,

$$|f(x) - f(y)|^2 = |\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}| \cdot |\sqrt{x} + \sqrt{y}| = |x-y| < \epsilon^2$$

as $|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x} + \sqrt{y}|$ when $x, y \in [0,1]$. Thus $|f(x) - f(y)| < \epsilon$, which implies that f is uniformly continuous.

We argue by contradiction to show that f is not Lipschitz continuous. Suppose there exists a constant $c > 0$ such that $|f(x) - f(y)| \leq c|x-y|, \forall x, y \in [0,1]$. Let $y = 0$ and $x = \frac{1}{m^2}$ with $m \geq 1$, then

$$|f(x) - f(y)| = \Big|\frac{1}{m} - 0\Big| \leq c\Big|\frac{1}{m^2} - 0\Big|,$$

thus $c \geq m$ for all $m \geq 1$, which is a contradiction.

(ii) Choose $\epsilon = 1$, for any $1 > \delta > 0$, we set $x = \delta$ and $y = \delta/2$, then $|x-y| = \delta/2 < \delta$ and

$$|g(x) - g(y)| = \Big|\frac{1}{\delta} - \frac{2}{\delta}\Big| = \frac{1}{\delta} > \epsilon = 1.$$

Thus g is not uniformly continuous.

Or assume that $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in (0,1)$ with $|x-y| < \delta$, we have $|g(x) - g(y)| < \epsilon$. Choose $\epsilon = 1/2, x_m = \frac{1}{m+1}$ and $y = \frac{1}{m}$, then $|g(x_m) - g(y_m)| = 1 > \epsilon$, but for m large enough, $|x_m - y_m| < \delta$ since $\lim_{m\to\infty} |x_m - y_m| = 0$. Contradiction.

Exercise 2:

Find a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous, bounded, but not uniformly continuous.

Solution:

Set $f(x) = \sin(x^2)$, then f is continuous and bounded on $\mathbb{R}$. Next we show that f is not uniformly continuous. Let $\epsilon = 1/2$, for any $\delta > 0$, choose

$$x_n = \sqrt{2n\pi + \frac{\pi}{2}}, y_n = \sqrt{2n\pi}.$$

Then

$$x_n - y_n = \sqrt{2n\pi + \frac{\pi}{2}} - \sqrt{2n\pi} = \frac{\frac{\pi}{2}}{\sqrt{2n\pi + \frac{\pi}{2}} + \sqrt{2n\pi}} \to 0$$

as $n \to \infty$. Thus for n large enough, $|x_n - y_n| < \delta$, but

$$|f(x_n) - f(y_n)| = |\sin(2n\pi + \frac{\pi}{2}) - \sin(2n\pi)| = 1 > \epsilon.$$

Thus f is not uniformly continuous.

Exercise 3:

Let X be a non-empty set and $B(X)$ be the space of bounded functions from X to $\mathbb{R}$ with the metric

$$\rho(f,g) = \sup_{x\in X} |f(x) - g(x)|.$$

Show that $B(X)$ is complete.

Solution:

Let $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $B(X)$. Let $\epsilon > 0$ be given, there exists a $N \in \mathbb{N}$ such that

$$\sup_{x\in X} |f_n(x) - f_m(x)| < \epsilon, \quad \forall n, m \geq N.$$

Thus for $y \in X$, $|f_n(y) - f_m(y)| < \epsilon$ when $n, m \geq N$. This shows that $\{f_m(y)\}_{m\geq 1}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $f(y) \in \mathbb{R}$ such that $f_m(y) \to f(y)$ as $m \to \infty$. This define a function $f : X \to \mathbb{R}$.

For $\epsilon > 0$ fixed and $n, m \geq N$, for $|f_n(y) - f_m(y)| < \epsilon$, let $m \to \infty$, then

$$|f_n(y) - f(y)| \leq \epsilon, \quad n \geq N.$$

In particular, $|f(y)| \leq \epsilon + |f_N(y)| \leq \epsilon + \sup_{x\in X} |f_N(x)|$, thus f is bounded. By the arbitrary of $y \in X$,

$$\sup_{y\in X} |f_n(y) - f(y)| \leq \epsilon, \quad n \geq N,$$

which yields that $f_n \to f$ in $B(X)$. Therefore $B(X)$ is complete.

Exercise 4:

Let (X, ρ) be a metric space and x_n be a sequence in X. Assume that there is a sequence a_n in $\mathbb{R}$ such that $\rho(x_n, x_{n+1}) \leq a_n$ and $\sum a_n$ converges. Show that x_n is a Cauchy sequence.

Solution:

Let $\epsilon > 0$ be given. Since $\sum a_n$ converges, there exists a $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} a_n < \epsilon$. For each $n \in \mathbb{N}$, $\rho(x_n, x_{n+1}) \leq a_n$, we have $a_n \geq 0$. Thus $\forall m \geq n \geq N$,

$$\begin{aligned}
\rho(x_n, x_m) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{m-1}, x_m) \\
&\leq a_n + a_{n+1} + \cdots + a_{m-1} \\
&\leq \sum_{k=N}^{\infty} a_k \\
&\leq \epsilon,
\end{aligned}$$

which shows that x_n is a Cauchy sequence.

<u>Exercise 5:</u>

Let (X, ρ) be a metric space and A a non-empty subset of X. Define

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Show that A is closed if and only if $A = \{x \in X : d(x, A) = 0\}$.

Solution:

Suppose A is closed. For any $x \in A$, $x \in X$, we have $d(x, A) = 0$, thus $A \subset \{x \in X : d(x, A) = 0\}$. If A is closed, for any $x \in \{x \in X : d(x, A) = 0\}$, let $\epsilon > 0$ be given, there exists $y \in A$ such that $d(x, y) < \epsilon$. Thus $x \in \bar{A} = A$, we have $\{x \in X : d(x, A) = 0\} \subset A$. Therefore we have $A = \{x \in X : d(x, A) = 0\}$. Or let $y \in A = \{x \in X : d(x, A) = 0\}$, we have $\inf\{d(y, a) : a \in A\} = 0$, there exists a sequence a_m in A such that $\lim_{m\to\infty} d(a_m, y) = 0$, thus $\lim_{m\to\infty} a_m = y$. Since A is closed, we have $y \in \bar{A} = A$. We conclude that $\{x \in X : d(x, A) = 0\} \subset A$.

Conversely suppose that $A = \{x \in X : d(x, A) = 0\}$. Let $y \in \bar{A}$, there is a sequence a_m in A such that $\lim_{m\to\infty} a_m = y$, so $\lim_{m\to\infty} d(a_m, y) = 0$, which implies that $\inf\{d(y, a) : a \in A\} = 0$. Thus $y \in \{x \in X : d(x, A) = 0\}$. Since $A = \{x \in X : d(x, A) = 0\}$, we have $y \in A$. Thus $A = \bar{A}$. A is closed.

<u>Exercise 6:</u>

Let (X, ρ) be a metric space and A a non-empty subset of X. Define

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Show that $f : X \to \mathbb{R}^+, f(x) = d(x, A)$ is Lipschitz continuous.

Solution:

Let x, y be in X and a in A, then

$$d(x, a) \leq d(x, y) + d(y, a).$$

Thus $f(x) \leq d(x, y) + d(y, a)$ for all a in A. Taking the inf of the right hand side for all $a \in A$, we have

$$f(x) \leq d(x, y) + f(y),$$

thus $f(x) - f(y) \leq d(x, y)$. As x and y are arbitrary in X, we also have $f(y) - f(x) \leq d(y, x) = d(x, y)$. Thus $|f(x) - f(y)| \leq d(x, y)$, for all x and y in X. Therefore f is Lipschitz continuous.

<u>Exercise 7:</u>

Show that a subset E of a metric space X is open if and only if there is a continuous real-valued function f on X for which $E = \{x \in X : f(x) > 0\}$.

Solution:

Firstly, if there is a continuous real-valued function f on X for which $E = \{x \in X : f(x) > 0\}$, we have $E = f^{-1}((0, \infty))$, which is the inverse image of f on $(0, \infty)$. Since $(0, \infty)$ is the open subset of $\mathbb{R}$ and f is continuous, $E = f^{-1}((0, \infty))$ is also open in X.

Conversely, if E is a open subset of a metric space X, we define

$$f(x) = d(x, E^c) = \min\{d(x, y) : y \in E^c\}.$$

Since E is open, E^c is closed in X. For any fixed $x \in E$, we have

$$f(x) = d(x, E^c) = \min\{d(x, y) : y \in E^c\} > 0.$$

Thus $E \subset \{x \in X : f(x) > 0\}$. For any $x \in \{x \in X : f(x) > 0\} = \{x \in X : d(x, E^c) > 0\}$, we have $\min\{d(x, y) : y \in E^c\} > 0$, thus $x \notin E^c$, $x \in E$. Hence we also have $\{x \in X : f(x) > 0\} \subset E$. Therefore $E = \{x \in X : f(x) > 0\}$.

Exercise 8:

Let L be a linear function between the two normed spaces (V_1, N_1) and (V_2, N_2). Show that the following conditions are equivalent:

(i) L is continuous at 0.

(ii) L is Lipschitz continuous.

(iii) $\exists C > 0, \forall x \in V_1, N_2(Lx) \leq CN_1(x)$.

Solution:

Firstly we show that (i) $\Rightarrow$ (iii). Since L is continuous at 0, let $\epsilon = 1$, there exists $\delta > 0$ such that for any $x \in V_1$ with $N_1(x) < \delta$, we have $N_2(Lx) < 1$. Let $y \neq 0$ be in V_1, set $z = \frac{\delta}{2}\frac{y}{N_1(y)}$, thus

$$N_1(z) = N_1\Big(\frac{\delta}{2}\frac{y}{N_1(y)}\Big) = \frac{\delta}{2} < \delta,$$

then we have $N_2(Lz) < 1$, which means

$$N_2(Lz) = N_2\Big(\frac{\delta}{2}\frac{Ly}{N_1(y)}\Big) = \frac{\delta}{2}\frac{N_2(Ly)}{N_1(y)} < 1.$$

Thus $N_2(Ly) < \frac{2}{\delta}N_1(y)$. If $y = 0$, we have $N_2(Ly) = 0 = N_1(y)$. Therefore if L is continuous at 0, $\exists C > 0, \forall x \in V_1, N_2(Lx) \leq CN_1(x)$.

Next we show that (iii) $\Rightarrow$ (ii). Let $x, y \in V_1$, by (iii), there exists a constant c such that

$$N_2(L(x-y)) \leq cN_1(x-y).$$

Since L be a linear function between (V_1, N_1) and (V_2, N_2), we have $L(x-y) = Lx - Ly$, thus

$$N_2(Lx - Ly)) \leq cN_1(x-y),$$

which implies that L is Lipschitz continuous.

Finally we show that (ii) $\Rightarrow$ (i). Since L is Lipschitz continuous, then L is continuous on V_1, we have L is continuous at 0.

Exercise 9:

Let A and B be two subsets of $\mathbb{R}^d$.

(i) If A is closed and B is compact, show that $A+B$ is closed.

(ii) If A is closed and B is closed, is $A+B$ is closed?

Solution:

(i) Let $\{a_m + b_m\}_m$ be a sequence in $A+B$, which converges to ℓ in $\mathbb{R}^d$. As B is compact, there exists a subsequence $\{b_{m_k}\}_k$ which converges to some $b \in B$. Thus a_{m_k} converges to $\ell - b$. Since A is closed, we have $\ell - b \in A$. Then $\ell = (\ell - b) + b \in A + B$. Hence $A+B$ is closed.

(ii) No, the counter example is as follows: we set

$$A = \{(x,y) \in \mathbb{R}^2 : xy = 1\} \cap \{x \geq 0\},$$

$$B = \{(x,y) \in \mathbb{R}^2 : xy = -1\} \cap \{x \geq 0\}.$$

The sequence $a_m = (\frac{1}{m}, m)$ is in A, and the sequence $b_m = (\frac{1}{m}, -m)$ is in B, where $m \in \mathbb{N}$. Then $a_m + b_m = (\frac{2}{m}, 0)$, which converges to $(0,0)$. But $(0,0) \notin A+B$ since for $(x,y) \in A$, we need $x > 0$ and for $(x', y') \in B$, we need $x' > 0$.

<u>Exercise 10:</u>

Let X be the space $[0,1)$ equipped with its usual metric. Find a cover of X by open sets which does not have a finite subcover.

Solution:

Let $V_m = [0, 1 - \frac{1}{m})$, V_m is open in X and $\bigcup_{m=2}^{\infty} V_m$ is an open cover of X. For all $x \in [0,1)$, there exists a p such that

$$x < 1 - \frac{1}{p}$$

for p large enough. If a finite subcover exists, as $V_m \subset X$ for all m, then there exists some $k \in \mathbb{N}$ such that

$$X = \bigcup_{m=2}^{k} V_m.$$

But we have $1 - \frac{1}{k+1} \in X$ and $1 - \frac{1}{k+1} \notin \bigcup_{m=2}^{k} V_m$.

Exercise 11:

Let $S = \{x \in \ell_2 : \|x\| = 1\}$.

(i) Show that S is closed and bounded.

(ii) Find with proof $\epsilon > 0$ such that S cannot be covered by finitely many balls with radius ϵ.

Solution:

(i) By the definition of S, we know that S is bounded by 1. $\forall x, y \in S$, since $\|x\| \leq \|x - y\| + \|y\|$ and $\|y\| \leq \|x - y\| + \|x\|$, we have $|\|x\| - \|y\|| \leq \|x - y\|$, thus the norm is continuous from ℓ^2 to $\mathbb{R}$. Since the image set $\{1\}$ is closed, then we know the inverse image of $\{1\}$ is also closed, which is actually S. So, S is bounded and closed.

(ii) Next, we verify that $\exists \epsilon > 0$, S cannot be covered by finitely many balls with radius ϵ. We define e_i as follows:

$$e_{i,m} = \begin{cases} 1, & m = i \\ 0, & m \neq i \end{cases},$$

thus $e_i \in \ell^2$. Clearly, $\forall i, j$, if $i \neq j$, we have $\|e_i - e_j\| = \sqrt{2}$. Suppose S can be covered by the finite balls with radius $\frac{\sqrt{2}}{2}$. Since the sequence $\{e_i\}_{i=1}^{\infty}$ is infinity many, at least one of such ball contains at least two elements e_j and e_k with $j \neq k$. Let x be the center of this ball, then we have $\|e_j - e_k\| \leq \|e_j - x\| + \|e_k - x\| < \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$. It contradicts with the fact that $\forall k, j$, if $k \neq j$, $\|e_i - e_j\| = \sqrt{2}$. Hence $\exists \epsilon > 0$, S cannot be covered by finitely many balls with radius ϵ. Then we know that S is not compact.