exercise 1:
Find a function $f : (0,\infty) \times [0,1] \to \mathbb{R}$ such that $x \to f(x,t)$ is in $\mathcal{L}^1((0,\infty))$ for all t in $[0,1]$, $\frac{\partial f}{\partial t}(x,t)$ exists for all $x > 0$ and t in $[0,1]$, and $x \to \frac{\partial f}{\partial t}(x,t)$ is in $\mathcal{L}^1((0,\infty))$ for all t in $[0,1]$, and setting $F(t) = \int_0^\infty f(x,t)dx$, F is fails to be differentiable at some point in $[0,1]$.
Hint: $t^a e^{-tx}$, for some adequate value of a.

exercise 2:
Find (with proof)

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{(\sin x)^n}{x^2}\, dx$$

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exercise 3:
(a). Let f_n be a sequence of measurable functions on $[a,b]$ valued in $\mathbb{R}$ and converging uniformly to zero. Use the D.C.T. to show that $\displaystyle\int_a^b f_n$ converges to 0.
(b). Let f_n be a sequence of measurable functions in $\mathcal{L}^1(\mathbb{R})$ converging uniformly to zero. Does $\displaystyle\int_{\mathbb{R}} f_n$ converge to 0?

exercise 4:
(i). Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a metric space. Let $f : X \times Y \to \mathbb{R}$. Assume that for all y in Y, $x \to f(x,y)$ is in $L^1(X)$, and that for almost all x in X, $y \to f(x,y)$ is continuous on Y. Assume that there is an h in $L^1(X)$ such that $|f(x,y)| \leq h(x)$, for all y in Y, for almost all x in X. Set

$$F(y) = \int_X f(x,y)d\mu(x).$$

Show that F is continuous on Y.
(ii). Let g be in $L^1(\mathbb{R})$. Set

$$G(y) = \int_{-\infty}^{\infty} \cos(xy)g(x)dx,$$

for y in $\mathbb{R}$. Show that G is continuous.

exercise 5:
Let (X,d) be a metric space and x_n a Cauchy sequence in X. Show that there is a subsequence z_n of x_n such that $d(z_{n+1}, z_n) \leq 2^{-n}$.

exercise 6:
Let $(X, \mathcal{A}, \mu)$ be a measure space. We proved in class that $L^1(X)$ is complete. Show that

$L^2(X)$ is complete. **Hint:** Apply the triangle inequality in $L^2(X)$ to $G = \sum_{n=1}^{\infty} |g_n - g_{n+1}|$.

<u>exercise 7</u>:
Let V be an open subset of $\mathbb{R}$ and f be in $C(V) \cap L^\infty(V)$. Show that $\|f\|_\infty = \sup_V |f|$, where $\| \|_\infty$ is the norm on L^∞ defined in class.

<u>exercise 8</u>:
For t in $[0, \infty)$, define $F(t) = \int_0^\infty e^{-x^2 - \frac{t^2}{x^2}} dx$.
a. Show that F is continuous on $[0, \infty)$.
b. Show that F is differentiable on $(0, \infty)$ and form a differential equation for F.
c. Using that $F(0) = \frac{\sqrt{\pi}}{2}$, express $F(t)$ using known functions.

<u>exercise 9</u>:
Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) < +\infty$.
a. Let f be in $L^\infty(X)$. Show that for all $p \geq 1$, $|f|^p$ is in $L^1(X)$, and $\limsup_{p \to \infty} (\int |f|^p)^{1/p} \leq \|f\|_\infty$.
b. Let ϵ be positive. What can be said about the set $A = \{x \in X : |f(x)| \geq \|f\|_\infty - \epsilon\}$? Infer that $\lim_{p \to \infty} (\int |f|^p)^{1/p} = \|f\|_\infty$.