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<u>**Exercise 1:**</u>

Let $f_n$ be a sequence of continuous functions from $[0,1]$ to $\mathbb{R}$ which is uniformly convergent. Let $x_n$ be in $[0,1]$ such that $f_n(x_n) \geq f_n(x)$, for all $x$ in $[0,1]$.

(i) Is the sequence $x_n$ convergent?

(ii) Show that the sequence $f_n(x_n)$ is convergent.

**Solution:**

(i) No, the sequence $x_n$ may not convergent. Let $f_n(x) = 0$ for all $x \in [0,1]$. And for each $k \in \mathbb{N}$ we let the sequence $x_n$ is

$$x_n = \begin{cases} 0, & n = 2k \\ 1, & n = 2k-1. \end{cases}$$

Then we know that $x_n \in [0,1]$ and $f_n(x_n) = 0 = f_n(x)$ for all $x \in [0,1]$, but the sequence $x_n$ is divergent.

(ii) Suppose $f_n$ is uniformly converges to $f$ on $[0,1]$. Since $f_n$ is continuous, $f$ is also a continuous function on $[0,1]$. For all $y \in [0,1]$, there exist a $x$, such that $f(y) \leq f(x)$. And since $f_n$ is uniformly converges to $f$ on $[0,1]$, let $\epsilon > 0$ be given, there exists a $N_1 \in \mathbb{N}$ such that when $n > N_1$, for all $y \in [0,1]$, we have

$$|f_n(y) - f(y)| < \epsilon,$$

which is equivalent to $f(y) - \epsilon < f_n(y) < f(y) + \epsilon$. We use the $x_n$ to substitute the $y$, then we have $f_n(x_n) \leq f(x_n) + \epsilon \leq f(x) + \epsilon$.

On the other hand, for the above $x$, we have $f_n(x_n) \geq f_n(x)$. As $f_n$ is uniformly converges to $f$ on $[0,1]$, for the above $\epsilon > 0$, there exists a $N_2 \in \mathbb{N}$, when $n > N_2$, for the above $x$, we have $f_n(x) > f(x) - \epsilon$. And then we have $f_n(x_n) > f(x) - \epsilon$. Thus for the above $\epsilon$ and $x$, there exists a $N = \max\{N_1, N_2\} \in \mathbb{N}$ such that

$$f(x) - \epsilon < f_n(x_n) < f(x) + \epsilon, \quad \forall n \geq N.$$

Therefore, we know that the sequence $f_n(x_n)$ is converges to $f(x)$ for some $x \in [0,1]$.

**Exercise 2:**

Let $\mathbb{I}$ be the set of all irrational number ($\mathbb{I} \subset \mathbb{R}$).

(i) Using that $\mathbb{Q} = \mathbb{R} \setminus \mathbb{I}$ (the set of all rationals) is countable, show that given $\epsilon > 0$, there is a closed subset $F \subset \mathbb{I}$ such that $|\mathbb{I} \setminus F| < \epsilon$.

(ii) Is $F$ compact? Please explain why or why not.

**Solution:**

(i) We enumerate the rational number and denote it as $\mathbb{Q} = \mathbb{R} \setminus \mathbb{I} = \{a_n\}_{n=1}^{\infty}$. It is a countable set. Let $\epsilon > 0$ be given, and for each $a_n \in \mathbb{Q}$, we can find an open set as follows

$$a_n \in \left(a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}\right),$$

then we know that $\bigcup_{n=1}^{\infty} \left(a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}\right)$ is an open cover of $\mathbb{Q}$, and

$$\left|\bigcup_{n=1}^{\infty} \left(a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}\right)\right| \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

We denote $S = \bigcup_{n=1}^{\infty} \left(a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}\right)$, then $\mathbb{R} \setminus S \subset \mathbb{R} \setminus \mathbb{Q} = \mathbb{I}$. We set $F = \mathbb{R} \setminus S$, as $S$ is an open subset of $\mathbb{R}$, then $F$ is closed in $\mathbb{R}$. And we have

$$|\mathbb{I} \setminus F| = |\mathbb{I}| - |\mathbb{R} \setminus S| = |\mathbb{I}| - |\mathbb{R}| + |S| < \epsilon.$$

(ii) No, $F$is not a compact set. Suppose $F$ is compact, then $F$ is closed and bounded, thus $F$ has finite measure. Since we have $(\mathbb{I} \setminus F) \cup F$, then there exists a $M > 0$ such that

$$|\mathbb{I}| = |(\mathbb{I} \setminus F) \cup F| \leq |\mathbb{I} \setminus F| + |F| < \epsilon + M,$$

which is contradictory with $|\mathbb{I}| = \infty$. Thus $F$ is not compact.

**Exercise 3:**

Find with proof:

$$\lim_{n\to\infty}\int_0^1 \frac{1+nx^3}{(1+x^2)^n}\,dx$$

**Solution:**

For $x\in(0,1)$, we denote $f_n(x)=\frac{1+nx^3}{(1+x^2)^n}$. Firstly, for $x\in(0,1)$, since $(1+x^2)^n\geq 1+nx^2$, then we have

$$f_n(x)\leq\frac{1+nx^3}{1+nx^2}\leq 1\in L^1((0,1)).$$

And for $x\in(0,1)$, since $(1+x^2)^n\geq\frac{1}{2}n(n-1)x^4$, we have

$$f_n(x)=\frac{1+nx^3}{(1+x^2)^n}\leq\frac{2+2nx^3}{n(n-1)x^4}=\frac{\frac{2}{x^4}}{n(n-1)}+\frac{\frac{1}{x}}{n-1},$$

so for any fixed $x\in(0,1)$, we have $\lim_{n\to\infty}f_n(x)=0$, thus we know that $f_n(x)$ converges pointwise to 0 almost everywhere. By the dominate convergence theorem, we have

$$\lim_{n\to\infty}\int_0^1\frac{1+nx^3}{(1+x^2)^n}\,dx=\int_0^1\lim_{n\to\infty}\frac{1+nx^3}{(1+x^2)^n}\,dx=0.$$

**<u>Exercise 4:</u>**

Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) = 1$. Let $f$ be in $L^1(X)$ such that $f \geq 0$ almost everywhere.

(i) show that

$$\lim_{p \to 0^+} \int f^p = \mu(\{x \in X : f(x) > 0\})$$

(ii) If $\mu(\{x \in X : f(x) > 0\}) < 1$, find

$$\lim_{p \to 0^+} \left( \int f^p \right)^{\frac{1}{p}}.$$

**Solution:**

(i) Since

$$\begin{aligned} \int_X f^p \, d\mu &= \int_{\{x \in X : f > 0\}} f^p \, d\mu + \int_{\{x \in X : f = 0\}} f^p \, d\mu \\ &= \int_{\{x \in X : f > 0\}} f^p \, d\mu, \end{aligned}$$

as $f$ be in $L^1(X)$ and $f \geq 0$ almost everywhere, by the Fatou's lemma,

$$\mu(\{x \in X : f(x) > 0\}) = \int \mathbb{I}_{\{x \in X : f > 0\}}(x) \, d\mu \leq \liminf_{p \to 0^+} \int_{\{x \in X : f > 0\}} f^p \, d\mu.$$

On the other hand, for each $n \in \mathbb{N}$, we know that

$$\begin{aligned} \int_{\{x \in X : f > 0\}} f^p \, d\mu &= \int_{\{x \in X : 0 < f < n\}} f^p \, d\mu + \int_{\{x \in X : f \geq n\}} f^p \, d\mu \\ &\leq \int_{\{x \in X : f \geq n\}} f^p \, d\mu + n^p \mu(\{x \in X : f(x) > 0\}). \end{aligned}$$

For $0 < p < 1$, when $x \in \{x \in X : f(x) > n\}$, we have $f^p < f$. Thus each $n \in \mathbb{N}$, we have

$$\begin{aligned} \limsup_{p \to 0^+} \int_{\{x \in X : f > 0\}} f^p \, d\mu &\leq \mu(\{x \in X : f(x) > 0\}) + \limsup_{p \to 0^+} \int_{\{x \in X : f \geq n\}} f^p \, d\mu \\ &\leq \mu(\{x \in X : f(x) > 0\}) + \int_{\{x \in X : f \geq n\}} f \, d\mu \\ &\leq \mu(\{x \in X : f(x) > 0\}) + \int_X f \, \mathbb{I}_{\{x \in X : f \geq n\}}(x) \, d\mu. \end{aligned}$$

Since $f \cdot \mathbb{I}_{\{x \in X : f \geq n\}}(x) \leq f \in L^1(X)$ and $\lim_{n \to \infty} f \, \mathbb{I}_{\{x \in X : f \geq n\}}(x) = 0$, by the dominate convergence theorem, we have

$$\limsup_{p \to 0^+} \int_{\{x \in X : f > 0\}} f^p \, d\mu \leq \mu(\{x \in X : f(x) > 0\}),$$

thus we know that

$$\lim_{p \to 0^+} \int f^p = \mu(\{x \in X : f(x) > 0\}).$$

(ii) Denote $S = \{x \in X : f > 0\}$, then

$$\begin{aligned} \int_X f^p \, d\mu &= \int_{\{x \in X : f > 0\}} f^p \, d\mu + \int_{\{x \in X : f = 0\}} f^p \, d\mu \\ &= \int_S f^p \, d\mu. \end{aligned}$$

And denote that $F(p) = \log(\int_S f^p \, d\mu)$, then we have

$$\lim_{p \to 0^+} \left( \int f^p \right)^{\frac{1}{p}} = \lim_{p \to 0^+} e^{\frac{F(p)}{p}}.$$

As $F(0) = \log(\mu(S))$,

$$\begin{aligned} \lim_{p \to 0^+} \left( \int f^p \right)^{\frac{1}{p}} &= \lim_{p \to 0^+} \exp\left\{ \frac{F(p) - \log(\mu(S)) + \log(\mu(S))}{p} \right\} \\ &= \lim_{p \to 0^+} (\mu(S))^{\frac{1}{p}} \exp\left\{ \frac{F(p) - \log(\mu(S))}{p - 0} \right\}. \end{aligned}$$

Since $F(p) = \log(\int_S f^p \, d\mu)$, we have

$$F'(p) = \frac{\int_S f^p \cdot \log f \, d\mu}{\int_S f^p \, d\mu},$$

thus $F'(0) = \frac{\int_S \log f \, d\mu}{\mu(S)}$. Then we know that

$$\begin{aligned} \lim_{p \to 0^+} \left( \int f^p \right)^{\frac{1}{p}} &= \lim_{p \to 0^+} (\mu(S))^{\frac{1}{p}} \exp\left\{ \lim_{p \to 0^+} \frac{F(p) - F(0)}{p - 0} \right\} \\ &= \lim_{p \to 0^+} (\mu(S))^{\frac{1}{p}} e^{F'(0)} \\ &= 0 \end{aligned}$$

as $\mu(S) < 1$.