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<u>**Exercise 1:**</u>

Let K be compact subset of the metric space X. For a point $x \in X \setminus K$, show that there is a open set U containing K and an open set O containing x for which $U \cap O = \emptyset$.

Solution:

As $x \notin K$ and K is closed, we have $d(x, K) > 0$. Let $\alpha = d(x, K)$ and define

$$U = \{y \in X : d(y, K) < \frac{\alpha}{2}\}.$$

The function $f(y) = d(y, K)$ is a continuous function from X to $\mathbb{R}$, and since $(-\alpha/2, \alpha/2)$ is an open subset of $\mathbb{R}$, we have $U = f^{-1}((-\alpha/2, \alpha/2))$ is open in X. And for each $y \in K$, $d(y, K) = 0 < \alpha/2$, $K \subset U$. Set $O = B(x, \alpha/4)$, which is an open ball in X with the center x and radius $\alpha/4$.

Next we argue by contradiction to show that $U \cap O = \emptyset$. Assume that there exists $z \in U \cap O$, then $d(x, z) < \alpha/4$, and there exists $v \in K$ such that $d(z, v) < \alpha/4 + \alpha/2$. Thus we have

$$d(x, v) \leq d(x, z) + d(z, v) < \frac{\alpha}{4} + \frac{3\alpha}{4} = \alpha,$$

which contradicts with $\alpha = d(x, K)$.

<u>**Exercise 2:**</u>

Let A and B be subsets of a metric space (X, ρ). Define

$$\text{dist}(A, B) = \inf\{\rho(u, v) : u \in A, v \in B\}.$$

If A is compact and B is closed, show that $A \cap B = \emptyset$ if and only if $\text{dist}(A, B) > 0$.

Solution:

Suppose $\text{dist}(A, B) > 0$. If $A \cap B \neq \emptyset$, then there exists $x \in A \cap B$. Since $d(x, x) = 0$, we have $\text{dist}(A, B) = \inf\{\rho(u, v) : u \in A, v \in B\} = 0$, which contradicts with the fact $\text{dist}(A, B) > 0$.

Suppose that $A \cap B = \emptyset$. Argue by contradiction. If $\text{dist}(A, B) = 0$, there exists a sequence a_m in A and a sequence b_m in B such that $\rho(a_m, b_m)$ converges to 0. As A is compact, there exists a subsequence a_{m_k} of a_m converges to some $a \in A$. Since $\rho(a_{m_k}, b_{m_k})$ converges to 0 and $\rho(a_{m_k}, a)$ converges to 0, we have

$$\rho(b_{m_k}, a) \leq \rho(a_{m_k}, b_{m_k}) + \rho(a_{m_k}, a)$$

also converges to 0. Since B is closed, $a \in B$. We have $a \in A \cap B$, which contradicts with $A \cap B = \emptyset$.

Exercise 3:

Let K be a compact subset of a metric space X and O an open set containing K. Use the proceeding problem to show that there is an open set U for which $K \subset U \subset \bar{U} \subset O$.

Solution:

If $O = X$, let $U = O$, then U is both open and closed, we have $K \subset U \subset \bar{U} \subset O$.

Otherwise, we have $K \cap O^c = \emptyset$ and $O^c \neq \emptyset$. Since O^c is closed in X and K is a compact subset of X, by the result of exercise 2, for $\alpha = \mathrm{dist}(K, O^c)$, we have $\alpha > 0$. Define

$$U = \{x \in X : d(x, O^c) > \frac{\alpha}{2}\}.$$

Since for any subset S of X which is non-empty, the function $f(x) = d(x, S) = \inf\{d(x, s) : s \in S\}$ is continuous, we have U is open in X. But for all $x \in K$, $d(x, O^c) \geq \alpha > \alpha/2$, we can get $K \subset U$. Since $\bar{U} = \{x \in X : d(x, O^c) \geq \frac{\alpha}{2}\}$, we have $\bar{U} \cap O^c = \emptyset$. Thus $\bar{U} \subset O$ and then $K \subset U \subset \bar{U} \subset O$.

<u>Exercise 4:</u>

Show that $\ell^1 \subset \ell^2 \subset \ell^\infty$, and that these inclusions are strict. Also show that the identity functions from ℓ^1 to ℓ^2 and from ℓ^2 to ℓ^∞ are continuous.

Solution:

Firstly we show that $\ell^1 \subset \ell^2 \subset \ell^\infty$. Let $x \in \ell^1$, for any $n \in \mathbb{N}$,

$$\begin{aligned} \Big(\sum_{i=1}^{n} |x_i|\Big)^2 &= \sum_{i=1}^{n} |x_i|^2 + \sum_{1\le i,j\le n} |x_i x_j| \\ &\ge \sum_{i=1}^{n} |x_i|^2, \end{aligned}$$

thus

$$\Big(\sum_{i=1}^{n} |x_i|^2\Big)^{\frac{1}{2}} \le \sum_{i=1}^{n} |x_i|.$$

Let $n \to \infty$, we obtain

$$\Big(\sum_{i=1}^{\infty} |x_i|^2\Big)^{\frac{1}{2}} \le \sum_{i=1}^{\infty} |x_i|.$$

Thus for all $x \in \ell^1$, $x \in \ell^2$ and $\|x\|_2 \le \|x\|_1$. It shows that the identity function from ℓ^1 to ℓ^2 is continuous. Similarly, let $x \in \ell^2$, since $\sum_{i=1}^{\infty} x_i^2$ converges, we have $\lim_{i\to\infty} x_i^2 = 0$, then $\lim_{i\to\infty} x_i = 0$. Thus x_i must be bounded. If $x \neq 0$, since $\|x\|_\infty > 0$, there exists $N \in \mathbb{N}$ such that $|x_i| < \|x\|_\infty/2$, for all $i > N$. But there exists $p \in \mathbb{N}$ such that $|x_p| > \|x\|_\infty/2$. Thus

$$\|x\|_\infty = \sup\{|x_i| : i \ge 1\} = \sup\{|x_m| : 1 \le i \le N\} = |x_q|$$

for some $q \in \{1, 2, \ldots, N\}$. Clearly $|x_q|^2 \le \sum_{i=1}^{\infty} x_i^2$, thus

$$\|x\|_\infty = |x_q| \le \Big(\sum_{i=1}^{\infty} x_i^2\Big)^{\frac{1}{2}} = \|x\|_2.$$

Thus $\ell^2 \subset \ell^\infty$ and it follows that the identity function from ℓ^2 to ℓ^∞ is continuous.

Next we give two examples. The constant sequence $a_i = 1, \forall i \in \mathbb{N}$ is in ℓ^∞, but it is not in ℓ^2. The sequence $a_i = 1/i, \forall i \in \mathbb{N}$ is in ℓ^2 as $\sum_{i=1}^{\infty} 1/i^2 < \infty$, but it is not in ℓ^1 as $\sum_{i=1}^{\infty} 1/i = \infty$.

Exercise 5:

Let X be a set and $\mathcal{A}$ be a subset of $\mathcal{P}(X)$.

(i) If $\mathcal{A}$ is closed under complementation and set difference, show that $\mathcal{A}$ is closed under finite union and finite intersection.

(ii) If $\mathcal{A}$ contains X and is closed under set difference and finite intersection, show that $\mathcal{A}$ is closed under finite union and set complementation.

Solution:

(i) If $A_1, A_2 \in \mathcal{A}$, as $\mathcal{A}$ is closed under set difference, we have $A_1 \setminus A_2 \in \mathcal{A}$. Thus

$$A_1 \cap A_2 = A_1 \setminus (A_1 \setminus A_2) \in \mathcal{A}.$$

Since $\mathcal{A}$ is closed under complementation, A_1^c and A_2^c are in $\mathcal{A}$, thus $A_1^c \cap A_2^c \in \mathcal{A}$ and

$$A_1 \cup A_2 = (A_1^c \cap A_2^c)^c \in \mathcal{A}.$$

By induction, we have $\mathcal{A}$ is closed under finite union and finite intersection.

(ii) If $X \in \mathcal{A}$ and $A \in \mathcal{A}$. since $\mathcal{A}$ closed under set difference, we have

$$A^c = X \setminus A \in \mathcal{A}.$$

For $A_1, A_2 \in \mathcal{A}$, we have A_1^c and A_2^c are in $\mathcal{A}$. And since $\mathcal{A}$ is closed under finite intersection, we have $A_1^c \cap A_2^c \in \mathcal{A}$. Therefore

$$A_1 \cup A_2 = X \setminus (A_1^c \cap A_2^c) \in \mathcal{A}.$$

Then by induction, we have $\mathcal{A}$ is closed under finite union and set complementation.

<u>Exercise 6:</u>

(i) Let V be an open subset of $\mathbb{R}$ which is neither empty nor equal to $\mathbb{R}$. Let V^c be the complement of V. Show that

$$V = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : |x| \leq n \text{ and } d(x, V^c) \geq \frac{1}{n}\}.$$

(ii) Infer that every open subset of $\mathbb{R}$ is a countable union of open intervals.

Solution:

(i) Let

$$V_n = \{x \in \mathbb{R} : |x| \leq n \text{ and } d(x, V^c) \geq \frac{1}{n}\}.$$

For each $n \in \mathbb{N}$, if $x \in V_n$, assume $x \notin V$, then $x \in V^c$. Since V is an open subset of $\mathbb{R}$, V^c is closed in $\mathbb{R}$, we have $d(x, V^c) = 0$, which is a contradiction. Thus $V_n \subset V$ for each $n \in \mathbb{N}$. Conversely, let $x \in V$, since V is open in $\mathbb{R}$, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset V$, thus $d(x, V^c) > \delta$. Let p be in $\mathbb{N}$ such that $1/p < \delta$ and q be in $\mathbb{N}$ such that $|x| \leq q$. Let $n = \max\{p, q\}$, we have $x \in V_n$. Therefore, we have

$$V = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : |x| \leq n \text{ and } d(x, V^c) \geq \frac{1}{n}\}.$$

(ii) For $\emptyset$ and $\mathbb{R}$, the statement is straightforward. Otherwise, we denote V is the open subset of $\mathbb{R}$. $\forall x \in V$, $\exists \delta_x > 0$ such that $B(x, \delta_x) \subset V$, thus

$$V = \bigcup_{x \in V} B(x, \delta_x).$$

Note that the function $f(x) = d(x, V^c)$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Thus $\{x \in \mathbb{R} : d(x, V^c) \geq 1/n\}$ is closed. As V_n is closed and bounded, it is compact in $\mathbb{R}$. Since $V \subset \bigcup_{x \in V} B(x, \delta_x)$, there exists finite subset J_m of V such that $V_m \subset \bigcup_{x \in J_m} B(x, \delta_x)$. It follows that

$$V = \bigcup_{m=1}^{\infty} \bigcup_{x \in J_m} B(x, \delta_x),$$

which is countable union.

Exercise 7:

Let X be a set and $\mathcal{A}$ be a σ-algebra of subsets of X. Let Y be a subset of X. Define a subset $\mathcal{B}$ of $\mathcal{P}(Y)$ by setting $\mathcal{B} := \{A \cap Y : A \in \mathcal{A}\}$. Show that $\mathcal{B}$ is a σ-algebra of subsets of Y.

Solution:

(i) Since $\emptyset$ and X are in $\mathcal{A}$, we have

$$\emptyset \cap Y = \emptyset \in \mathcal{B}, \quad X \cap Y = Y \in \mathcal{B}.$$

(ii) Let $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $B = A \cap Y$. Since $A^c = X \setminus A \in \mathcal{A}$,

$$B^c = Y \setminus B = Y \setminus (A \cap Y) = A^c \cap Y \in \mathcal{B},$$

thus $\mathcal{B}$ is closed under taking complements in Y.

(iii) Let $\{B_m\}_m$ be a sequence in B, thus there is a sequence $\{A_m\}_m$ such that $B_m = A_m \cap Y$ for each $m \in \mathbb{N}$. Thus

$$\bigcup_{m=1}^{\infty} B_m = \bigcup_{m=1}^{\infty} (A_m \cap Y) = \Big(\bigcup_{m=1}^{\infty} A_m\Big) \cap Y \in \mathcal{B}$$

as $\bigcup_{m=1}^{\infty} A_m$ is in $\mathcal{A}$. Therefore $\mathcal{B}$ is closed under countable union in Y.

Exercise 8:

Let X be a set and $\mathcal{A}_i$ be a σ-algebra of subsets of X for each i in I. Show that $\bigcap_{i \in I} \mathcal{A}_i$ is σ-algebra of subsets of X.

Solution:

(i) As $\emptyset$ and X are in $\mathcal{A}_i$ for each $i \in I$, then $\emptyset \in \bigcap_{i \in I} \mathcal{A}_i$ and $X \in \bigcap_{i \in I} \mathcal{A}_i$.

(ii) Let $A \in \bigcap_{i \in I} \mathcal{A}_i$, then $A \in \mathcal{A}_i, \forall i \in I$. We have $A^c \in \mathcal{A}_i, \forall i \in I$, thus $A^c \in \bigcap_{i \in I} \mathcal{A}_i$. $\bigcap_{i \in I} \mathcal{A}_i$ is closed under taking complements in X.

(iii) Let $\{A_m\}_m$ be a sequence in $\bigcap_{i \in I} \mathcal{A}_i$, thus $\{A_m\}_m$ is a sequence in $\mathcal{A}_i$, $\forall i \in I$. Then $\bigcup_{m=1}^{\infty} A_m \in \mathcal{A}_i, \forall i \in I$. Thus we have $\bigcup_{m=1}^{\infty} A_m \in \bigcap_{i \in I} \mathcal{A}_i$. It shows that $\bigcap_{i \in I} \mathcal{A}_i$ is closed under countable union in X.

Therefore $\bigcap_{i \in I} \mathcal{A}_i$ is σ-algebra of subsets of X.

Exercise 9:

Show from the definition of the outer measure m^* on $\mathbb{R}$ that for any two subsets A and B of $\mathbb{R}$, $m^*(A \cup B) \leq m^*(A) + m^*(B)$.

Solution:

By the definition of outer measure, let $\epsilon > 0$ be given, there exist a countable union of open interval sequence $\{I_n\}_n$ and a countable union of open interval sequence $\{J_n\}_n$ such that $A \subset \bigcup_{n=1}^{\infty} I_n$, $B \subset \bigcup_{n=1}^{\infty} J_n$ and

$$\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \frac{\epsilon}{2}$$

$$\sum_{n=1}^{\infty} \ell(J_n) \leq m^*(B) + \frac{\epsilon}{2}.$$

We denote I'_n is one of the open interval I_n or J_n, then

$$\Big(\bigcup_{n=1}^{\infty} I_n\Big) \bigcup \Big(\bigcup_{n=1}^{\infty} J_n\Big) \subset \bigcup_{n=1}^{\infty} I'_n.$$

It follows that $A \cup B \subset \bigcup_{n=1}^{\infty} I'_n$. And by the definition of outer measure,

$$\begin{aligned} m^*(A \cup B) &\leq \sum_{n=1}^{\infty} \ell(I'_n) \\ &\leq \sum_{n=1}^{\infty} \ell(I_n) + \sum_{n=1}^{\infty} \ell(J_n) \\ &\leq m^*(A) + m^*(B) + \epsilon. \end{aligned}$$

By the arbitrary of ϵ, we have

$$m^*(A \cup B) \leq m^*(A) + m^*(B).$$

<u>Exercise 10:</u>

(i) Let $(X, \mathcal{A}, \mu)$ be a measure space, B_n a decreasing sequence of $\mathcal{A}$ such that $\mu(B_1) < \infty$. Show that $\lim_{n\to\infty} \mu(B_n) = \mu(\bigcap_{n=1}^{\infty} B_n)$.

(ii) Find a measure space $(X, \mathcal{A}, \mu)$ and a deceasing sequence B_n of $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(B_n) \neq \mu(\bigcap_{n=1}^{\infty} B_n)$.

Solution:

(i) Since B_n is a decreasing sequence of $\mathcal{A}$, $B_1 \setminus B_n$ is a increasing sequence of $\mathcal{A}$. By the increasing set property,

$$\lim_{n\to\infty} \mu(B_1 \setminus B_n) = \mu\Big(\bigcup_{n=1}^{\infty}(B_1 \setminus B_n)\Big) = \mu\Big(\bigcup_{n=1}^{\infty}(B_1 \bigcap B_n^c)\Big).$$

Since

$$\bigcup_{n=1}^{\infty}(B_1 \bigcap B_n^c) = B_1 \bigcap \Big(\bigcup_{n=1}^{\infty} B_n^c\Big) = B_1 \bigcap \Big(\bigcap_{n=1}^{\infty} B_n\Big)^c = B_1 \setminus \Big(\bigcap_{n=1}^{\infty} B_n\Big),$$

we have

$$\mu\Big(\bigcup_{n=1}^{\infty}(B_1 \bigcap B_n^c)\Big) = \mu(B_1) - \mu\Big(\bigcap_{n=1}^{\infty} B_n\Big).$$

And since $\mu(B_1 \setminus B_n) = \mu(B_1) - \mu(B_n)$,

$$\lim_{n\to\infty} (\mu(B_1) - \mu(B_n)) = \mu(B_1) - \mu\Big(\bigcap_{n=1}^{\infty} B_n\Big).$$

Since $\mu(B_1) < \infty$, we know that

$$\lim_{n\to\infty} \mu(B_n) = \mu\Big(\bigcap_{n=1}^{\infty} B_n\Big).$$

(ii) Set $X = \mathbb{R}$ and $b_n = [n, \infty)$, then $B_{n+1} \subset B_n$ for each $n \in \mathbb{N}$. Since $\mu(B_n) = \infty$ for each $n \in \mathbb{N}$, $\lim_{n\to\infty} \mu(B_n) = \infty$. But as $\bigcap_{n=1}^{\infty} B_n = \emptyset$, we have $\mu\Big(\bigcap_{n=1}^{\infty} B_n\Big) = 0$.