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Exercise 1:

Let (X, ρ) be a metric space and S and T two non-empty subsets of X. Define

$$d(S,T) = \max\{\sup_{x\in S}\inf_{y\in T}\rho(x,y), \sup_{y\in T}\inf_{x\in S}\rho(x,y)\}.$$

Show that $d(S,T) = 0$ if and only if S and T have the same closure.

Solution:

Firstly we show that if $d(S,T) = 0$, S and T have the same closure. By the definition of $d(S,T)$, if $d(S,T) = 0$, then

$$\max\{\sup_{x\in S}\inf_{y\in T}\rho(x,y), \sup_{y\in T}\inf_{x\in S}\rho(x,y)\} = 0.$$

For any $x \in S, y \in T$, we have $\sup_{x\in S}\inf_{y\in T}\rho(x,y) \geq 0$ and $\sup_{y\in T}\inf_{x\in S}\rho(x,y) \geq 0$, then

$$\sup_{x\in S}\inf_{y\in T}\rho(x,y) = 0 \quad \text{and} \quad \sup_{y\in T}\inf_{x\in S}\rho(x,y) = 0.$$

Thus we have

$$\forall x \in S, \inf_{y\in T}\rho(x,y) = 0 \quad \text{and} \quad \forall y \in T, \inf_{x\in S}\rho(x,y) = 0.$$

Let $\epsilon > 0$ be given. For fixed $x \in S$, if $\inf_{y\in T}\rho(x,y) = 0$, there exists $y \in T$ such that $\rho(x,y) - \epsilon < 0$, which implies x is a point of closure of T, i.e. $x \in \bar{T}$, where $\bar{T}$ is the closure of T. By the arbitrary of $x \in S$, we have $S \subset \bar{T}$, thus $\bar{S} \subset \bar{T}$. Similarly, $\forall y \in T$, $\inf_{x\in S}\rho(x,y) = 0$, we also have $T \subset \bar{S}$, thus $\bar{T} \subset \bar{S}$. Therefore $\bar{S} = \bar{T}$, S and T have the same closure.

Next we want to prove that if S and T have the same closure, then $d(S,T) = 0$. If $\bar{S} = \bar{T}$, $\bar{S} \subset \bar{T}$, and then $S \subset \bar{S} \subset \bar{T}$. Let $\epsilon > 0$ be given. For any fixed $x \in S$, there exists $y \in T$ such that $\rho(x,y) < \epsilon$. By the arbitrary of ϵ, $\forall x \in S$, we have $\inf_{y\in T}\rho(x,y) = 0$. Thus $\sup_{x\in S}\inf_{y\in T}\rho(x,y) = 0$. Similarly, since $T \subset \bar{T} \subset \bar{S}$, we have $\sup_{y\in T}\inf_{x\in S}\rho(x,y) = 0$. Therefore,

$$d(S,T) = \max\{\sup_{x\in S}\inf_{y\in T}\rho(x,y), \sup_{y\in T}\inf_{x\in S}\rho(x,y)\} = 0.$$

Exercise 2:

Show that for every set $S \subset \mathbb{R}$ there exists a Borel set B such that $S \subset B$ and $m^*(S) = m^*(B)$, where m^* is the Lebesgue outer measure. Then show that for such S and B with $m^*(S) < \infty$, S is measurable if and only if $m^*(B \setminus S) = 0$.

Solution:

Firstly we show that for every set $S \subset \mathbb{R}$ there exists a Borel set B such that $S \subset B$ and $m^*(S) = m^*(B)$. If $m^*(S) = \infty$, choose $B = \mathbb{R}$. Then B is a Borel set, $S \subset B = \mathbb{R}$ and $m^*(S) = m^*(B) = \infty$. If $m^*(S) < \infty$. Let $\epsilon > 0$ be given. By the definition of $m^*(S)$, there exists a countable collection $\{I_k\}_{k=1}^{\infty}$ of open intervals such that

$$S \subset \bigcup_{k=1}^{\infty} I_k \quad \text{and} \quad \sum_{k=1}^{\infty} \ell(I_k) < m^*(S) + \epsilon,$$

where $\ell(I_k)$ is the length of open interval I_k for each $k \in \mathbb{N}$. Let $O = \bigcup_{k=1}^{\infty} I_k$, then O is an open set, $S \subset O$ and

$$m^*(O) = m^*\left(\bigcup_{k=1}^{\infty} I_k\right) \le \sum_{k=1}^{\infty} m^*(I_k) = \sum_{k=1}^{\infty} \ell(I_k) \le m^*(S) + \epsilon.$$

Thus for each $n \in \mathbb{N}$, there exists an open set O_n such that

$$S \subset O_n \quad \text{and} \quad m^*(O_n) < m^*(S) + \frac{1}{n}.$$

Let $B = \bigcap_{n=1}^{\infty} O_n$, then $S \subset B$, B is a Borel set and

$$m^*(B) - m^*(S) = m^*\left(\bigcap_{n=1}^{\infty} O_n\right) - m^*(S) \le m^*(O_m) - m^*(S) < \frac{1}{m},$$

for all $m \in \mathbb{N}$. Let $m \to \infty$, then $m^*(B) - m^*(S) = 0$.

Next we show that for such S and B with $m^*(S) < \infty$, S is measurable if and only if $m^*(B \setminus S) = 0$. If S is measurable and $m^*(S) < \infty$, by the construction of B and the arguement on the above, we have

$$m^*(B \setminus S) = m(B \setminus S) = m(B) - m(S) = m^*(B) - m^*(S) = 0$$

as B is a Borel set and $S \subset B$. Thus $m^*(B \setminus S) = 0$.

If $m^*(S) < \infty$ and $m^*(B \setminus S) = 0$, we want to prove that S is measurable. B is a Borel set, thus B is measurable. Since $m^*(B \setminus S) = 0$, we have $B \setminus S$ is measurable. Therefore $S = B \setminus (B \setminus S)$ is measurable as the family of measurable sets is a σ-algebra.

<u>Exercise 3:</u>

Suppose f_n, g_n are Lebesgue measurable function on $\mathbb{R}$, with $f_n, g_n \geq 0, \forall n \in \mathbb{N}$. Suppose also that $f_n \to f$ a.e., $g_n \to g$ a.e.,

$$\int f_n \to \int f < \infty,$$

and

$$\int g_n \to \int g < \infty.$$

Prove or give a counterexample: if $\{f_n g_n\}$ is bounded in L^1, then

$$\int f_n g_n \to \int fg.$$

Solution:

We can give a counterexample as follows: for each $n \in \mathbb{N}$, let

$$f_n(x) = g_n(x) = \sqrt{n} 1_{[0,\frac{1}{n}]}(x)$$

and let $f(x) = g(x) = 0$. Then we have f_n, g_n are Lebesgue measurable function on $\mathbb{R}$, with $f_n, g_n \geq 0, \forall n \in \mathbb{N}$. And we have $f_n \to f$ a.e., $g_n \to g$ a.e.. Since

$$\int f_n = \int g_n = \int_0^1 \left(\sqrt{n} 1_{[0,\frac{1}{n}]}(x)\right) dx = \frac{1}{\sqrt{n}} \to 0$$

as $n \to \infty$ and

$$\int f = \int g = \int_{\mathbb{R}} 0 \, dx = 0,$$

then $\int f_n \to \int f < \infty$ and $\int g_n \to \int g < \infty$. For each $n \in \mathbb{N}$,

$$\begin{aligned} \int f_n g_n &= \int_0^1 f_n^2(x) \, dx \\ &= \int_0^1 \left(n 1_{[0,\frac{1}{n}]}(x)\right) dx \\ &= 1 < \infty, \end{aligned}$$

thus $\{f_n g_n\}$ is bounded in L^1. But

$$\int fg = \int_{\mathbb{R}} 0 \, dx = 0,$$

then we have $\int f_n g_n$ does not converge to $\int fg$.