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Exercise 1:

Let E be measurable subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ a measurable function. For a in $\mathbb{R}$, set $\omega_f(a) = m(\{x \in E : f(x) > a\})$, where $m(\cdot)$ denotes the Lebesgue measure.

(i) If $f_k : E \to \mathbb{R}$ is a sequence of Lebesgue measurable, real-valued functions, such that $f_k \leq f_{k+1}$ and $f_k \to f$ almost everywhere, show that $\omega_f \leq \omega_{f_{k+1}}$ and $\omega_{f_k} \to \omega_f$.

(ii) Recall that f_k converges in measure to f if for all positive ϵ, $m(\{x \in E : |f_k(x) - f(x)| > \epsilon\})$ tends to zero as k tends to infinity. If f_k converges in measure to f then show that $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \epsilon)$, and $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a + \epsilon)$, for every $\epsilon > 0$.

(iii) If f_k converges in measure to f, show that $\omega_{f_k}(a) \to \omega_f(a)$ if ω_f is continuous at point a.

Solution:

(i) Firstly we show that $\omega_f \leq \omega_{f_{k+1}}$. Let $a \in \mathbb{R}$ be given. By the definition of $\omega_f(a) = m(\{x \in E : f(x) > a\})$, since $f_{k+1} \geq f_k$, if $x \in \{x \in E : f_k(x) > a\}$, then $x \in \{x \in E : f_{k+1}(x) > a\}$, thus $\{x \in E : f_k(x) > a\} \subset \{x \in E : f_{k+1}(x) > a\}$. Therefore we have

$$\omega_{f_k}(a) = m(\{x \in E : f_k(x) > a\}) \leq \omega_{f_{k+1}}(a) = m(\{x \in E : f_{k+1}(x) > a\}).$$

By the arbitrary of $a \in \mathbb{R}$, we have $\omega_f \leq \omega_{f_{k+1}}$.

Next we prove that $\omega_{f_k} \to \omega_f$. Let $a \in \mathbb{R}$ be given. To show that $\omega_{f_k}(a) \to \omega_f(a)$, we only need to show that

$$\int 1_{\{x\in E: f_k(x)>a\}} \to \int 1_{\{x\in E: f(x)>a\}}$$

as $k \to \infty$. Since $f_k \leq f_{k+1}$ and $f_k \to f$ a.e., we have $f_k \leq f$ a.e., $\forall k \in \mathbb{N}$. Thus $1_{\{x\in E: f_k(x)>a\}} \leq 1_{\{x\in E: f(x)>a\}}$ a.e. and $1_{\{x\in E: f_k(x)>a\}} \to 1_{\{x\in E: f(x)>a\}}$ a.e.. By the monotone convergence theorem, we have $\int 1_{\{x\in E: f_k(x)>a\}} \to \int 1_{\{x\in E: f(x)>a\}}$. Thus $\omega_{f_k}(a) \to \omega_f(a)$. Also by the arbitrary of $a \in \mathbb{R}$, $\omega_{f_k} \to \omega_f$.

(ii) Let $\epsilon > 0$ be given. Note that

$$\begin{aligned}
\{x \in E : f_k(x) > a\} &= \{x \in E : f_k(x) > a, |f_k(x) - f(x)| > \epsilon\} \cup \\
&\quad\ \{x \in E : f_k(x) > a, |f_k(x) - f(x)| \leq \epsilon\} \\
&\subset \{x \in E : |f_k(x) - f(x)| > \epsilon\} \cup \\
&\quad\ \{x \in E : f_k(x) > a, f_k(x) \leq f(x) + \epsilon\} \\
&\subset \{x \in E : |f_k(x) - f(x)| > \epsilon\} \cup \{x \in E : f(x) > a - \epsilon\},
\end{aligned}$$

by the subadditivity of Lebesgue measure, we have

$$m(\{x \in E : f_k(x) > a\}) \leq m(\{x \in E : |f_k(x) - f(x)| > \epsilon\}) + m(\{x \in E : f(x) > a - \epsilon\}).$$

Thus

$$\begin{aligned} \limsup_k \omega_{f_k}(a) &\leq \limsup_k m(\{x \in E : |f_k(x) - f(x)| > \epsilon\}) + \omega_f(a-\epsilon) \\ &= \omega_f(a-\epsilon) \end{aligned}$$

as f_k converges to f in measure.

Next we show that $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a+\epsilon)$. Similarly,

$$\begin{aligned} \{x \in E : f(x) > a+\epsilon\} &= \{x \in E : f_(x) > a+\epsilon, |f_k(x) - f(x)| > \epsilon\} \cup \\ &\quad\ \{x \in E : f(x) > a+\epsilon, |f_k(x) - f(x)| \leq \epsilon\} \\ &\subset \{x \in E : |f_k(x) - f(x)| > \epsilon\} \cup \\ &\quad\ \{x \in E : f(x) > a+\epsilon, f(x) \leq f_k(x) + \epsilon\} \\ &\subset \{x \in E : |f_k(x) - f(x)| > \epsilon\} \cup \{x \in E : f_k(x) > a\}, \end{aligned}$$

by the subadditivity of Lebesgue measure, we have

$$m(\{x \in E : f(x) > a+\epsilon\}) \leq m(\{x \in E : |f_k(x) - f(x)| > \epsilon\}) + m(\{x \in E : f_k(x) > a\}).$$

Thus

$$\begin{aligned} \omega_f(a+\epsilon) &\leq \liminf_k m(\{x \in E : |f_k(x) - f(x)| > \epsilon\}) + \liminf_k \omega_{f_k}(a) \\ &= \liminf_k \omega_{f_k}(a) \end{aligned}$$

as f_k converges to f in measure.

(iii) Let $\epsilon > 0$ be given. Since ω_f is continuous at a, there exists $\delta > 0$ such that $|\omega_f(a) - \omega_f(a-\delta)| \leq \epsilon$. As f_k converges to f in measure, by the result of (ii), we have

$$\limsup_k \omega_{f_k}(a) \leq \omega_f(a-\delta) \leq \omega_f(a) + \epsilon,$$

and

$$\liminf_k \omega_{f_k}(a) \geq \omega_f(a+\delta) \geq \omega_f(a) - \epsilon.$$

Therefore

$$\omega_f(a) + \epsilon \geq \limsup_{k\to\infty} \omega_{f_k}(a) \geq \liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a) - \epsilon.$$

By the arbitrary of ϵ, let $\epsilon \to 0$, we have

$$\omega_f(a) = \limsup_{k\to\infty} \omega_{f_k}(a) = \liminf_{k\to\infty} \omega_{f_k}(a).$$

Hence $\omega_{f_k}(a) \to \omega_f(a)$.

Exercise 2:

(i) Define the sequence of functions $g_n : [0,1] \to \mathbb{R}$, $g_n(x) = nx^n$. Show that g_n converges almost everywhere to zero. Is there a function h in $L^1([0,1])$ such that $|g_n(x)| \leq h(x)$ for almost all x in $[0,1]$?

(ii) If f is in $L^\infty([0,1])$ and f is continuous at 1, show that $\int_0^1 nx^n f(x)\,dx$ converges to $f(1)$.

(iii) If we only assume that $f \in L^1([0,1])$ and f is continuous at 1, does $\int_0^1 nx^n f(x)\,dx$ converges to $f(1)$?

Solution:

(i) When $x = 0$, $nx^n = 0$ for any $n \in \mathbb{N}$. For the fixed $x \in (0,1)$, let $\epsilon > 0$ be given, there exists $N = \left[\frac{2}{(\frac{1}{x}-1)^2\epsilon}\right] + 2$ such that

$$\begin{aligned} |nx^n - 0| &= \frac{n}{(\frac{1}{x})^n} = \frac{n}{((\frac{1}{x}-1)+1)^n} \leq \frac{n}{(\frac{1}{x}-1)^2\frac{n(n-1)}{2}} \\ &= \frac{2}{(\frac{1}{x}-1)^2}\frac{1}{n-1} < \epsilon, \quad \forall n \geq N, \end{aligned}$$

where $[x]$ is the largest integer which less than or equal to x. Thus we have g_n converges to 0 when $x \in [0,1)$, which yields that g_n converges to 0 on $[0,1]$ as $\{1\}$ is a Lebesgue zero measure set in $[0,1]$.

We claim that there is no such h in $L^1([0,1])$ with $|g_n(x)| \leq h(x)$ for almost all x in $[0,1]$. We argue it by contradiction. Suppose there is a $h \in L^1([0,1])$ such that $|g_n(x)| \leq h(x)$ for almost all x in $[0,1]$, by dominate convergence theorem, since $g_n \to 0$ a.e. on $[0,1]$, we have

$$\lim_{n\to\infty}\int_{[0,1]} g_n = \lim_{n\to\infty}\int_{[0,1]} nx^n = \int_{[0,1]} 0 = 0.$$

But for each $n \in \mathbb{N}$,

$$\int_{[0,1]} g_n = \int_{[0,1]} nx^n\,dx = \frac{n}{n+1},$$

thus $\int_{[0,1]} g_n$ converges to 1 rather than 0, which is a contradiction.

(ii) Note that

$$\begin{aligned} \Big|\int_0^1 nx^n f(x)\,dx - f(1)\Big| &= \Big|\int_0^1 nx^n f(x)\,dx - \int_0^1 (n+1)x^n f(1)\,dx\Big| \\ &= \Big|\int_0^1 nx^n (f(x)-f(1))\,dx - \int_0^1 x^n f(1)\,dx\Big| \\ &\leq \int_0^1 nx^n |f(x)-f(1)|\,dx + \int_0^1 x^n |f(1)|\,dx \\ &= \int_0^1 nx^n |f(x)-f(1)|\,dx + \frac{|f(1)|}{n+1}. \end{aligned}$$

Since $f(x)$ continuous at 1, let $\epsilon > 0$ be given, there exists $\delta > 0$ such that for any $x \in (1-\delta, 1)$, we have $|f(x) - f(1)| < \epsilon/3$. And since $f(x) \in L^\infty([0,1])$, we know that

$$\begin{aligned}
\Big| \int_0^1 nx^n f(x)\, dx - f(1) \Big| &\leq \int_0^1 nx^n |f(x) - f(1)|\, dx + \frac{|f(1)|}{n+1} \\
&= \int_0^{1-\delta} nx^n |f(x) - f(1)|\, dx + \int_{1-\delta}^1 nx^n |f(x) - f(1)|\, dx + \frac{|f(1)|}{n+1} \\
&< 2\|f\|_\infty \int_0^{1-\delta} nx^n\, dx + \frac{\epsilon}{3} \int_{1-\delta}^1 nx^n\, dx + \frac{|f(1)|}{n+1} \\
&= 2\|f\|_\infty \frac{n}{n+1} (1-\delta)^{n+1} + \frac{\epsilon}{3} \frac{n}{n+1} \big(1 - (1-\delta)^{n+1}\big) + \frac{|f(1)|}{n+1} \\
&\leq 2\|f\|_\infty (1-\delta)^{n+1} + \frac{\epsilon}{3} + \frac{|f(1)|}{n+1}.
\end{aligned}$$

Since

$$\lim_{n\to\infty} 2\|f\|_\infty (1-\delta)^{n+1} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{|f(1)|}{n+1} = 0,$$

there exists $N \in \mathbb{N}$ such that

$$2\|f\|_\infty (1-\delta)^{n+1} < \frac{\epsilon}{3} \quad \text{and} \quad \frac{|f(1)|}{n+1} < \frac{\epsilon}{3}, \quad \forall n \geq N.$$

Thus

$$\Big| \int_0^1 nx^n f(x)\, dx - f(1) \Big| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \quad \forall n \geq N,$$

which implies that $\int_0^1 nx^n f(x)\, dx \to f(1)$.

(iii) Similarly with (ii), since $f(x)$ continuous at 1, let $\epsilon > 0$ be given, there exists $\delta > 0$ such that for any $x \in (1-\delta, 1)$, we have $|f(x) - f(1)| < \epsilon/4$.

$$\begin{aligned}
\Big| \int_0^1 nx^n f(x)\, dx - f(1) \Big| &\leq \int_0^1 nx^n |f(x) - f(1)|\, dx + \frac{|f(1)|}{n+1} \\
&= \int_0^{1-\delta} nx^n |f(x) - f(1)|\, dx + \int_{1-\delta}^1 nx^n |f(x) - f(1)|\, dx + \frac{|f(1)|}{n+1} \\
&< \int_0^{1-\delta} nx^n |f(x)|\, dx + \int_0^{1-\delta} nx^n |f(1)|\, dx + \frac{\epsilon}{4} + \frac{|f(1)|}{n+1} \\
&= \int_0^{1-\delta} nx^n |f(x)|\, dx + \frac{n}{n+1} |f(1)| (1-\delta)^{n+1} + \frac{\epsilon}{4} + \frac{|f(1)|}{n+1}.
\end{aligned}$$

Since $nx^n \to 0$ as $n \to \infty$ when $x \in (0, 1-\delta)$, there exists a $N_1 \in \mathbb{N}$ such that $nx^n < 1$ and $nx^n|f(x)| < |f(x)|$, $\forall n \geq N_1$. Since $f(x) \in L^1([0,1])$, then $|f(x)| \in L^1([0,1])$, and since $nx^n|f(x)| \to 0$ as $n \to \infty$ for each $x \in (0, 1-\delta)$, by dominate convergence theorem, we have

$$\lim_{n\to\infty} \int_0^{1-\delta} nx^n |f(x)|\, dx = \int_0^{1-\delta} 0\, dx = 0.$$

And by the result that

$$\lim_{n\to\infty} \frac{n}{n+1} |f(1)| (1-\delta)^{n+1} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{|f(1)|}{n+1} = 0,$$

there exists $N \in \mathbb{N}$ such that

$$\frac{n}{n+1}|f(1)|(1-\delta)^{n+1} < \frac{\epsilon}{4} \quad \text{and} \quad \frac{|f(1)|}{n+1} < \frac{\epsilon}{4}, \quad \forall n \geq N.$$

Therefore

$$\Big| \int_0^1 nx^n f(x)\, dx - f(1) \Big| < \epsilon, \quad \forall n \geq N.$$

Thus we also have $\int_0^1 nx^n f(x)\, dx \to f(1)$.

Exercise 3:

Let X be a metric space and A and B two subsets of X such that $A \cap B = \emptyset$ and $A \cup B = X$. Show that the following statements are equivalent:

(1) Any function $f : X \to \mathbb{R}$ is continuous if and only if the restriction of f to A and the restriction of f to B are continuous.

(2) A and B are both open and closed in X.

Solution:

Firstly we show that (2) implies (1). Suppose A and B are both open and closed in X. If $f : X \to \mathbb{R}$ is continuous, then $f|_A$ is continuous and $f|_B$ is also continuous, where $f|_A$ is the restriction of f to A. If $f|_A$ is continuous and $f|_B$ is continuous. Firstly we show that for all $x \in A$, $y \in B$, there exists a $\delta > 0$ such that

$$B(x, \delta) \cap B = \emptyset, B(y, \delta) \cap A = \emptyset,$$

where $B(x, \delta)$ is a open ball with center x and radius δ. Otherwise, there exists $x \in A$, $y \in B$, for all $\delta > 0$, $B(x, \delta) \cap B \neq \emptyset$ and $B(y, \delta) \cap A \neq \emptyset$, which imply that $x \in \bar{B}$ and $y \in \bar{A}$, where $\bar{B}$ is the closure of B. Since A and B are both closed in X, then we have $x \in \bar{B} = B$, $y \in \bar{A} = A$, thus $x \in A \cap B, y \in A \cap B$, which contradicts with $A \cap B = \emptyset$.

For each $x \in X$, since $A \cap B = \emptyset$ and $A \cup B = X$, we suppose without lose of generality that $x \in A$. $f|_A$ is continuous and there exists $\delta > 0$ such that $B(x, \delta) \cap B = \emptyset$. Thus f is continuous at x. By the arbitrary of x, we have f is continuous on X.

Next we prove that (1) implies (2). Suppose that any function $f : X \to \mathbb{R}$ is continuous if and only if the restriction of f to A and the restriction of f to B are continuous. Let

$$f(x) = 1_A(x) - 1_B(x), \quad x \in X.$$

Then we have

$$f|_A(x) = 1_A(x) \quad \text{and} \quad f|_B(x) = -1_B(x).$$

Thus both of $f|_A$ and $f|_B$ are continuous. We have $f(x) = 1_A(x) - 1_B(x)$ is continuous on X. $\{1\}$ is a closed subset of $\mathbb{R}$, then $f^{-1}(\{1\}) = A$ is closed in X because f is continuous on X. Thus $X \setminus A = B$ is open in X. Similarly, $\{-1\}$ is a closed subset of $\mathbb{R}$, then $f^{-1}(\{-1\}) = B$ is closed in X because f is continuous on X. Thus $X \setminus B = A$ is open in X. Therefore A and B are both open and closed in X.