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Exercise 1:

Give an example of $f_n, f \in L^1(\mathbb{R})$ such that $f_n \to f$ uniformly, but $\|f_n\|_1$ does not converge to $\|f\|_1$.

Solution:

Example 1: For each $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{n}\mathbb{I}_{[1,n]}(x)$ and $f(x) = 0$. Since

$$|f_n(x) - 0| = |\frac{1}{n}\mathbb{I}_{[1,n]}(x) - 0| \leq \frac{1}{n},$$

we know that $f_n \to f$ uniformly. As $\|f(x)\|_1 = 0$ and

$$\|f_n\|_1 = \int_{\mathbb{R}} |f_n(x)|\, dx = \int_1^n \frac{1}{n}\, dx = \frac{n-1}{n} \to 1$$

as $n \to \infty$, we have $\|f_n\|_1$ does not converge to $\|f\|_1$.

Example 2: Let $f(x) = 0$ and

$$f_n(x) = \Big(-\frac{1}{2^{2n}} + \frac{1}{2^n} \Big) \cdot \mathbb{I}_{[0,2^{2n}]}(x), \quad n \in \mathbb{N}.$$

Since $|f_n(x) - f(x)| < \frac{1}{2^n}$, we know that $f_n \to f$ uniformly. And as

$$\|f_n(x)\|_1 = \int_0^{2^{2n}} -\frac{1}{2^{2n}} + \frac{1}{2^n}\, dx = 2^n - 1 \to +\infty,$$

we have $\|f_n\|_1$ does not converge to $\|f\|_1$.

Example 3: For each $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} \frac{1}{n^2}x, & x \in [0,n] \\ \frac{1}{n^2}(2n - x), & x \in [n, 2n] \\ 0, & \text{otherwise}, \end{cases}$$

and let $f(x) = 0$, for all $x \in \mathbb{R}$. As $|f_n(x) - f(x)| \leq \frac{1}{n}$, for all $x \in \mathbb{R}$, we have $f_n \to f$ uniformly. Since

$$\|f_n\| = \int_0^n \frac{1}{n^2}x\, dx + \int_n^{2n} \frac{1}{n^2}(2n - x)\, dx = 1, \quad ,\forall n \in \mathbb{N}$$

and $\|f\| = 0$, we have $\|f_n\|_1$ does not converge to $\|f\|_1$.

Exercise 2:

Show that for all $\epsilon > 0$ and all $f \in L^1(\mathbb{R}), \exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$ for some f_n with $|f_n| \leq n$ and $f_n = 0$ on $\mathbb{R} \setminus [-n, n]$.

Solution:

For each $n \in \mathbb{N}$, let

$$f_n(x) = f \cdot \mathbb{I}_{\{x:|f(x)|\leq n\}\cap\{x\in[-n,n]\}}(x),$$

then we know that $f_n = 0$ on $\mathbb{R} \setminus [-n, n]$ and $|f_n| \leq n$. Next we need to show that $\exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$. We know that

$$\begin{aligned} \|f_n - f\|_1 &= \int_{\mathbb{R}} |f_n - f|\, dx \\ &= \int_{\{|f|\geq n\}\cup\{x\in\mathbb{R}\setminus[-n,n]\}} |f|\, dx \\ &\leq \int_{\{|f|\geq n\}} |f|\, dx + \int_{-\infty}^{-n} |f|\, dx + \int_{n}^{+\infty} |f|\, dx. \end{aligned}$$

Let $g_n = |f|\mathbb{I}_{\{|f|>1\}} - |f|\mathbb{I}_{\{|f|>n\}}$ for each $n \in \mathbb{N}$, we have $g_n \geq 0$ and g_n is a sequence of non-decreasing functions, by the monotone convergence theorem, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f|\mathbb{I}_{\{|f|>1\}} - |f|\mathbb{I}_{\{|f|>n\}} = \int_{\mathbb{R}} \lim_{n\to\infty} \big(|f|\mathbb{I}_{\{|f|>1\}} - |f|\mathbb{I}_{\{|f|>n\}}\big).$$

Since $f \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} |f|\mathbb{I}_{\{|f|>1\}} < \infty$, thus

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f|\mathbb{I}_{\{|f|>n\}} = \int_{\mathbb{R}} \lim_{n\to\infty} |f|\mathbb{I}_{\{|f|>n\}} = 0.$$

Hence for all $\epsilon > 0$, there exists a N_1 such that $\int_{\{|f|\geq n\}} |f|\, dx = \int_{\mathbb{R}} |f|\mathbb{I}_{\{|f|>n\}}(x)\, dx < \epsilon/3$ for all $n \geq N_1$.

Note that

$$\int_{n}^{+\infty} |f|\, dx = \int_{\mathbb{R}} |f|\mathbb{I}_{[n,+\infty)}(x)\, dx,$$

and let $h_n = |f|\mathbb{I}_{[1,+\infty)} - |f|\mathbb{I}_{[n,+\infty)}$, we also have $h_n \geq 0$ and h_n is a sequence of non-decreasing functions, by the monotone convergence theorem, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f|\mathbb{I}_{[1,+\infty)} - |f|\mathbb{I}_{[n,+\infty)} = \int_{\mathbb{R}} \lim_{n\to\infty} \big(|f|\mathbb{I}_{[1,+\infty)} - |f|\mathbb{I}_{[n,+\infty)}\big).$$

Since $f \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} |f|\mathbb{I}_{[1,\infty)} < \infty$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f|\mathbb{I}_{[n,+\infty)} = \int_{\mathbb{R}} \lim_{n\to\infty} |f|\mathbb{I}_{[n,+\infty)} = 0.$$

Thus for the above $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that $\int_{\mathbb{R}} |f|\mathbb{I}_{[n,+\infty)} < \epsilon/3$ for all $n \geq N_2$.

Similarly, for the above $\epsilon > 0$, there exists $N_3 \in \mathbb{N}$ such that $\int_{\mathbb{R}} |f|\mathbb{I}_{(-\infty,-n]} < \epsilon/3$ for all $n \geq N_3$. Choose $N = \max\{N_1, N_2, N_3\}$, for $n \geq N$, we have

$$\|f_n - f\|_1 \leq \int_{\{|f|\geq n\}} |f|\, dx + \int_{-\infty}^{-n} |f|\, dx + \int_{n}^{+\infty} |f|\, dx < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Hence we know that $\exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$.

Exercise 3:

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(i) If f is in $L^1(X) \cap L^\infty(X)$, show that $|f|^p \in L^1(X)$ for all p in $(1, \infty)$.

(ii) If f is in $L^1(X) \cap L^\infty(X)$, show that

$$\lim_{p\to\infty} \left(\int |f|^p \right)^{\frac{1}{p}} = \|f\|_\infty.$$

(iii) Set $A = \{x \in X : |f(x)| > 0\}$. If f is in $L^\infty(X), \mu(A) < \infty$, and $\mu(A) \neq 1$, find

$$\lim_{p\to 0^+} \left(\int |f|^p \right)^{\frac{1}{p}}.$$

(iv) We now assume that the set A defined in (iii) satisfies $\mu(A) = 1$, that f is in $L^\infty(X)$, and $\ln |f|$ is in $L^1(X)$, find

$$\lim_{p\to 0^+} \left(\int |f|^p \right)^{\frac{1}{p}}.$$

Solution:

(i) To show $|f|^p \in L^1(X)$, we only need to show that for any $p \in (1, \infty)$, $f \in L^p(X)$. For any $p \in (1, \infty)$, since $f \in L^1(X) \cap L^\infty(X)$, we have

$$\begin{aligned} \|f\|_p &= \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \\ &= \left(\int_X |f||f|^{p-1} \, d\mu \right)^{\frac{1}{p}} \\ &\leq (\|f\|_\infty)^{\frac{p-1}{p}} (\|f\|_1)^{\frac{1}{p}} < \infty, \end{aligned}$$

thus we know that $f \in L^p(X)$.

(ii) We denote $t \in [0, \|f\|_\infty)$, then the set

$$A = \{x \in X : |f(x)| \geq t\}$$

has positive and bounded measure. (Or let $\epsilon > 0$ be given, let $A = \{x \in X : |f(x)| > \|f\|_\infty - \epsilon\}$.) Since

$$\begin{aligned} \|f\|_p &= \left(\int_{(0,1)} |f|^p \, d\mu \right)^{\frac{1}{p}} \geq \left(\int_A |f|^p \, d\mu \right)^{\frac{1}{p}} \\ &\geq \left(t^p \mu(A) \right)^{\frac{1}{p}} = t(\mu(A))^{\frac{1}{p}}, \end{aligned}$$

if $\mu(A)$ is finite, then when $p \to +\infty$, we have $(\mu(A))^{\frac{1}{p}} \to 1$ and if $\mu(A) = \infty$, then $(\mu(A)^{\frac{1}{p}}) = \infty$, in both cases we have

$$\liminf_{p\to+\infty} \|f\|_p \geq t.$$

Since t is arbitrary and $t \in [0, \|f\|_\infty)$, we have

$$\liminf_{p\to+\infty} \|f\|_p \geq \|f\|_\infty.$$

On the other hand, as $f(x)$ is in $L^1(X)$, we have

$$\begin{aligned} \|f\|_p &= \Big(\int_X |f|^p \, d\mu\Big)^{\frac{1}{p}} \\ &= \Big(\int_X |f||f|^{p-1} \, d\mu\Big)^{\frac{1}{p}} \\ &\leq (\|f\|_\infty)^{\frac{p-1}{p}} (\|f\|_1)^{\frac{1}{p}}. \end{aligned}$$

Since $\|f\|_1 < +\infty$, then when $p \to +\infty$, we know that

$$\limsup_{p\to+\infty} \|f\|_p \leq \|f\|_\infty.$$

Thus we have

$$\limsup_{p\to+\infty} \|f\|_p \leq \|f\|_\infty \leq \liminf_{p\to+\infty} \|f\|_p,$$

then we know that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

(iii) When $\mu(A) < 1$, we have

$$\begin{aligned} \int_X |f|^p \, d\mu &= \int_A |f|^p \, d\mu \\ &\leq \|f\|_\infty^p \mu(A). \end{aligned}$$

Since $f \in L^\infty(X)$ and $\mu(A) < 1$, we know that

$$\lim_{p\to 0^+} \Big(\int |f|^p\Big)^{\frac{1}{p}} \leq \lim_{p\to 0^+} \|f\|_\infty (\mu(A))^{\frac{1}{p}} = 0$$

But if we set $f = 1$ and $\mu(X) < \infty$, we know that $f \in L^\infty(X)$, if $\mu(A) > 1$, we have

$$\lim_{p\to 0^+} \Big(\int |f|^p\Big)^{\frac{1}{p}} = \lim_{p\to 0^+} (\mu(A))^{\frac{1}{p}} = \infty.$$

Thus the limit is not exist.

(iv) Since we have $A = \{x \in X : |f| > 0\}$, then

$$\begin{aligned} \int_X |f|^p \, d\mu &= \int_{\{x\in X:|f|>0\}} |f|^p \, d\mu + \int_{\{x\in X:|f|=0\}} |f|^p \, d\mu \\ &= \int_A |f|^p \, d\mu. \end{aligned}$$

And denote that $F(p) = \log(\int_A |f|^p \, d\mu)$, then

$$\lim_{p\to 0^+} \Big(\int |f|^p\Big)^{\frac{1}{p}} = \lim_{p\to 0^+} e^{\frac{F(p)}{p}}.$$

As $F(0) = \log(\mu(A)) = 0$ and e^x is continuous, we have

$$\begin{aligned}\lim_{p\to 0^+}\left(\int |f|^p\right)^{\frac{1}{p}} &= \lim_{p\to 0^+}\exp\left\{\frac{F(p)-F(0)}{p-0}\right\}\\ &= \exp\left\{\lim_{p\to 0^+}\frac{F(p)-F(0)}{p-0}\right\}\\ &= e^{F^{'}(0)}.\end{aligned}$$

Since $F(p) = \log(\int_A |f|^p\, d\mu)$ and $\ln|f|$ is in $L^1(X)$, we can get that

$$F^{'}(p) = \frac{\int_A |f|^p \cdot \log|f|\, d\mu}{\int_A |f|^p\, d\mu},$$

thus $F^{'}(0) = \frac{\int_A \log|f|\, d\mu}{\mu(A)} = \int_A \log|f|\, d\mu$. Therefore by calculation

$$\begin{aligned}\lim_{p\to 0^+}\left(\int f^p\right)^{\frac{1}{p}} &= e^{F^{'}(0)}\\ &= \exp(\int_A \log|f|\, d\mu).\end{aligned}$$