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Exercise 1:

Consider the sequence of functions f_n defined on the non-negative reals by $f_n(x) = 2nxP(x)e^{-nx^2}$, where P is a polynomial function.

(i) Is f_n pointwise convergent on $[0, +\infty)$? Is f_n uniformly convergent on $[0, +\infty)$? Explain your answers to both questions.

(ii) Let g_n be a sequence of continuous functions defined on $[0, +\infty)$ and valued in \mathbb{R} . Assume that each g_n is in $L^1([0, +\infty))$ and that sequence g_n is uniformly convergent to zero. Prove or disprove: $\lim_{n \rightarrow \infty} \int_0^\infty g_n = 0$.

(iii) Determine (with proof) $\lim_{n \rightarrow \infty} \int_0^\infty f_n$.

Solution:

(i) For any $x \in [0, \infty)$, as $P(x)$ is a polynomial function, by the L'Hospital's Rule, we have

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} 2nxP(x)e^{-nx^2} = \lim_{n \rightarrow \infty} \frac{2nxP(x)}{e^{nx^2}} = 0,$$

then for any $\epsilon > 0$, there exist a $N \in \mathbb{N}$, such that $n > N$ we have

$$|2nxP(x)e^{-nx^2} - 0| < \epsilon,$$

thus we know that f_n converges to $f(x) = 0$ pointwise on $[0, \infty)$. But f_n is not uniformly convergent to $f(x) = 0$. We suppose $P(x) = 1$, then we have $f_n(x) = 2nxe^{-nx^2}$. When $x = \frac{1}{\sqrt{n}}$,

$$f_n(x) = 2n \frac{1}{\sqrt{n}} e^{-n \frac{1}{n}} = 2\sqrt{n}e^{-1},$$

so we have

$$\sup_{x \in [0, \infty)} |f_n(x) - 0| \geq 2\sqrt{n}e^{-1} \rightarrow \infty$$

when n goes to $+\infty$. Thus we know that f_n is not uniformly converges on $[0, +\infty)$.

(ii) The statement is not true. Let

$$g_n(x) = \begin{cases} \frac{4}{n^2}x, & x \in [0, \frac{n}{2}) \\ \frac{4}{n} - \frac{4}{n^2}x, & x \in [\frac{n}{2}, n] \\ 0, & x \in (n, +\infty). \end{cases}$$

For each $n \in \mathbb{N}$, we have

$$\int_{[0, \infty)} g_n(x) dx = \int_0^{\frac{n}{2}} \frac{4}{n^2}x dx + \int_{\frac{n}{2}}^n \left(\frac{4}{n} - \frac{4}{n^2}x \right) dx = 1,$$

thus $g_n(x) \in L^1([0, \infty))$. When $x \in [0, \frac{n}{2})$, $g_n(x) = \frac{4}{n^2}x \leq \frac{2}{n}$ and when $x \in [\frac{n}{2}, n]$, $g_n(x) = \frac{4}{n} - \frac{4}{n^2}x \leq \frac{2}{n}$, thus g_n uniformly converges to 0. But since for each $n \in \mathbb{N}$, $\int_0^\infty g_n(x) dx = 1$, we have

$$\lim_{n \rightarrow +\infty} \int_{[0, \infty)} g_n(x) dx = \lim_{n \rightarrow \infty} 1 = 1.$$

Thus $\lim_{n \rightarrow \infty} \int_0^\infty g_n = 0$ can not hold.

(iii) We denote $y = nx^2$, then we have

$$\int_0^\infty 2nxP(x)e^{-nx^2} dx = \int_0^\infty e^{-y}P\left(\sqrt{\frac{y}{n}}\right) dy.$$

Since $P(x)$ is a polynomial function, for any fixed y , when $n \rightarrow \infty$, $P(\sqrt{\frac{y}{n}}) \rightarrow P(0)$ and then $e^{-y}P(\sqrt{\frac{y}{n}}) \rightarrow e^{-y}P(0)$. Since $P(x)$ is a polynomial function, there exist a $M > 0$, such that when $y \in [M, \infty)$, $e^{-y}P(\sqrt{\frac{y}{n}}) < \frac{1}{y^2}$, then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty f_n &= \lim_{n \rightarrow \infty} \int_0^M f_n + \lim_{n \rightarrow \infty} \int_M^\infty f_n \\ &= \lim_{n \rightarrow \infty} \int_0^M e^{-y}P\left(\sqrt{\frac{y}{n}}\right) dy + \lim_{n \rightarrow \infty} \int_M^\infty e^{-y}P\left(\sqrt{\frac{y}{n}}\right) dy. \end{aligned}$$

Since $P(x)$ is a polynomial function, then $P(\sqrt{\frac{y}{n}})$ is continuous on $y \in [0, M]$, then we have when $y \in [0, M]$,

$$\left| e^{-y}P\left(\sqrt{\frac{y}{n}}\right) \right| \leq e^{-y}\|P\|_\infty.$$

Since $e^{-y}\|P\|_\infty \in L^1([0, M])$ and $\frac{1}{y^2} \in L^1([M, +\infty))$, by the dominate convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_0^M e^{-y}P\left(\sqrt{\frac{y}{n}}\right) dy = \int_0^M e^{-y}P(0) dy = P(0)(1 - e^{-M}),$$

and

$$\lim_{n \rightarrow \infty} \int_M^\infty e^{-y}P\left(\sqrt{\frac{y}{n}}\right) dy = \int_M^\infty e^{-y}P(0) dy = P(0)e^{-M}.$$

Thus we know that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty f_n &= \lim_{n \rightarrow \infty} \int_0^M e^{-y}P\left(\sqrt{\frac{y}{n}}\right) dy + \lim_{n \rightarrow \infty} \int_M^\infty e^{-y}P\left(\sqrt{\frac{y}{n}}\right) dy \\ &= P(0)(1 - e^{-M}) + P(0)e^{-M} \\ &= P(0). \end{aligned}$$

Exercise 2: (all answers require proofs:)

Let f_n be the sequence in $L^2(\mathbb{R})$ defined by $f_n = \mathbb{I}_{[n, n+1]}$.

- (i) Let g be in $L^2(\mathbb{R})$. Does $\int f_n g$ have a limit as n tends to infinity?
(ii) Does the sequence f_n converge in $L^2(\mathbb{R})$?

Solution:

- (i) Firstly we show that $f_n = \mathbb{I}_{[n, n+1]}(x)$ converges to $f(x) = 0$ pointwise on \mathbb{R} . Since

$$|f_n - f| = |\mathbb{I}_{[n, n+1]}(x) - 0| = \mathbb{I}_{[n, n+1]}(x),$$

let $x \in \mathbb{R}$ be fixed, $\forall \epsilon > 0$, we can find a $N = [x] + 1$, such that for all $n > N$, we have

$$|f_n - f| = \mathbb{I}_{[n, n+1]}(x) = 0 < \epsilon.$$

Thus f_n converges to $f(x) = 0$ pointwisely on \mathbb{R} . Since

$$\left| \int_{\mathbb{R}} f_n g \, dx \right| \leq \int_{\mathbb{R}} |f_n g| \, dx = \int_n^{n+1} |g(x)| \, dx,$$

by Cauchy-Schwarz inequality (Hölder inequality), we have

$$\begin{aligned} \left| \int_{\mathbb{R}} f_n g \, dx \right| &\leq \int_n^{n+1} |g(x)| \, dx \\ &\leq \left(\int_n^{n+1} |g|^2 \, dx \right)^{\frac{1}{2}} \left(\int_n^{n+1} 1^2 \, dx \right)^{\frac{1}{2}} \\ &= \left(\int_n^{n+1} |g|^2 \, dx \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}} |g|^2 \mathbb{I}_{[n, n+1]}(x) \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

Since $|g|^2 \mathbb{I}_{[n, n+1]}(x) \leq |g(x)|^2$ and $g \in L^2(\mathbb{R})$, we have $\int_{\mathbb{R}} |g(x)|^2 \, dx < +\infty$. Thus $|g(x)|^2 \in L^1(\mathbb{R})$, by the dominate convergence theorem,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f_n g \, dx \right|^2 &\leq \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}} |g|^2 \mathbb{I}_{[n, n+1]}(x) \, dx \right) \\ &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} (|g|^2 \mathbb{I}_{[n, n+1]}(x)) \, dx. \end{aligned}$$

Since $f_n = \mathbb{I}_{[n, n+1]}(x)$ converges to $f(x) = 0$ pointwise on \mathbb{R} , we also have $|g|^2 \mathbb{I}_{[n, n+1]}(x)$ converges to $f(x) = 0$ pointwise on \mathbb{R} , then

$$\lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f_n g \, dx \right|^2 \leq \int_{\mathbb{R}} \lim_{n \rightarrow \infty} (|g|^2 \mathbb{I}_{[n, n+1]}(x)) \, dx = 0.$$

Thus we know that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n(x) g(x) \, dx = 0$.

- (ii) Since $f_n = \mathbb{I}_{[n, n+1]}(x)$ converges to $f(x) = 0$ pointwise on \mathbb{R} , but

$$\int_{\mathbb{R}} |f_n(x) - 0|^2 \, dx = \int_{\mathbb{R}} f_n^2 \, dx = \int_n^{n+1} 1 \, dx = 1,$$

we know that f_n does not converges to $f(x) = 0$ in $L^2(\mathbb{R})$, then we can get that the sequence f_n does not converge in $L^2(\mathbb{R})$.

Exercise 3:

Let X be a matrix space. For any subset A of X , we denote by \bar{A} the closure of A and \mathring{A} the union of all open subsets contained in A . We set $\partial A = \bar{A} \setminus \mathring{A}$.

- (i) Show that A is closed if and only if $\partial A \subset A$.
- (ii) Show that A is open if and only if $\partial A \cap A = \emptyset$.
- (iii) Is the identity $\partial(\partial B) = \partial B$ valid for all subsets B of X ?
- (iv) Show that if A is closed then $\partial(\partial A) = \partial A$.

Solution:

(i) When A is closed, we have $A = \bar{A}$, since \mathring{A} the union of all open subsets contained in A , then $\mathring{A} \subset A$. Thus we have $\partial A = \bar{A} \setminus \mathring{A} = A \setminus \mathring{A} \subset A$ as $\mathring{A} \subset A$.

When $\partial A \subset A$, we have $\partial A \cup A \subset A \cup A = A$, thus we know that $\bar{A} \subset A$. Since $A \subset \bar{A}$, we have $\bar{A} = A$. Thus A is closed.

(ii) When A is open, since \mathring{A} the union of all open subsets contained in A , then we have $A \subset \mathring{A}$. And we know that $\mathring{A} \subset A$, then we can get $A = \mathring{A}$. As $\partial A = \bar{A} \setminus \mathring{A}$, we have $\partial A = \bar{A} \setminus A$, then it is obviously that $\partial A \cap A = \emptyset$.

When $\partial A \cap A = \emptyset$, we suppose A is not an open set, then there exists a element $x \in A$ such that no open set containing x is a subset of A . Since \mathring{A} the union of all open subsets contained in A , we have $x \notin \mathring{A}$. And as $x \in A$, we know that $x \in \bar{A}$, then we have $x \in \bar{A} \setminus \mathring{A} = \partial A$. Then we can get $x \in \partial A \cap A$, it is contradict with the condition we have. So, the statement that A is not an open set is wrong. Thus we have A is an open set.

(iii) No, the statement is not true. We suppose $B = \mathbb{Q} \cap [0, 1]$, which represents the rational number in the interval $[0, 1]$. Then we have $\partial B = [0, 1]$ and $\partial(\partial B) = \{0, 1\}$, which is not equal to ∂B .

(iv) Since \bar{A} is closed and \mathring{A} is open, we have $\partial A = \bar{A} \setminus \mathring{A}$ is closed, then we can get $\overline{\partial A} = \partial A$. By the definition of ∂A , we have $\partial(\partial A) = \overline{\partial A} \setminus \mathring{\partial A} = \partial A \setminus \mathring{\partial A} \subset \partial A$. Next we need to show that $\partial A \subset \partial(\partial A) = \partial A \setminus \mathring{\partial A}$, then we just need to prove that $\mathring{\partial A} = \emptyset$ when A is closed.

When A is closed, since $\partial A = \bar{A} \setminus \mathring{A} = A \setminus \mathring{A}$. As $A \setminus \mathring{A} \subset A$, then we have $\mathring{\partial A} \subset \mathring{A}$. And since the union of subsets in $(A \setminus \mathring{A})$ is the subset of $A \setminus \mathring{A}$, we have $\mathring{\partial A} \subset A \setminus \mathring{A}$. Then we know that $\mathring{\partial A} \subset \mathring{A}$ and $\mathring{\partial A} \subset A \setminus \mathring{A}$. Thus we can get $\mathring{\partial A} \subset \mathring{A} \cap (A \setminus \mathring{A}) = \emptyset$. So, we have showed that $\mathring{\partial A} = \emptyset$. In conclusion, we have $\partial(\partial A) = \partial A$ when A is closed.

Exercise 4:

Let X be a measure space, f_n a sequence in $L^1(X)$ and f an element of $L^1(X)$ such that f_n converges to f almost everywhere and $\lim_{n \rightarrow \infty} \int |f_n| = \int |f|$. Show that $\lim_{n \rightarrow \infty} \int |f_n - f| = 0$.

Solution:

Method 1: Since $|f_n - f| \leq |f_n| + |f|$ holds on X , we know that $|f_n| + |f| - |f_n - f|$ is a non-negative function. By the Fatou's lemma,

$$\int \lim_{n \rightarrow \infty} (|f_n| + |f| - |f_n - f|) \leq \liminf_{n \rightarrow \infty} \int (|f_n| + |f| - |f_n - f|).$$

Since f_n converges to f almost everywhere, then $|f_n|$ converges to $|f|$ almost everywhere. Thus we have

$$\lim_{n \rightarrow \infty} (|f_n| + |f| - |f_n - f|) = 2|f|.$$

Then we can get that

$$\begin{aligned} \int 2|f| &\leq \liminf_{n \rightarrow \infty} \int (|f_n| + |f| - |f_n - f|) \\ &\leq \liminf_{n \rightarrow \infty} \int (|f_n| + |f|) - \limsup_{n \rightarrow \infty} \int (|f_n - f|) \\ &= \int 2|f| - \limsup_{n \rightarrow \infty} \int (|f_n - f|), \end{aligned}$$

as $f \in L^1(X)$, then $\int |f| < +\infty$, we have

$$\limsup_{n \rightarrow \infty} \int (|f_n - f|) \leq 0.$$

On the other hand, we have

$$0 \leq \liminf_{n \rightarrow \infty} \int (|f_n - f|)$$

as $|f_n - f| \geq 0$. Thus we know that

$$\limsup_{n \rightarrow \infty} \int (|f_n - f|) \leq 0 \leq \liminf_{n \rightarrow \infty} \int (|f_n - f|),$$

which is equivalent to

$$\limsup_{n \rightarrow \infty} \int (|f_n - f|) = \liminf_{n \rightarrow \infty} \int (|f_n - f|) = 0.$$

So we have

$$\lim_{n \rightarrow \infty} \int |f_n - f| = 0.$$

Method 2: By the triangle inequality, $|f_n - f| \leq |f_n| + |f|$ for all $n \in \mathbb{N}$. Let $g_n = |f_n| + |f|$, $\forall n \in \mathbb{N}$, as $f \in L^1(X)$ and $f_n \in L^1(X)$, we have

$$\int_X |g_n| = \int_X |f_n| + |f| = \int_X |f_n| + \int_X |f| < \infty,$$

thus $g_n \in L^1(X)$ for all $n \in \mathbb{N}$. Since f_n converges to f almost everywhere, we have $g_n \rightarrow 2|f|$ almost everywhere. Denote $g = 2|f|$, since $\lim_{n \rightarrow \infty} \int |f_n| = \int |f|$, we have

$$\lim_{n \rightarrow \infty} \int_X g_n = \lim_{n \rightarrow \infty} \int_X |f| + |f_n| = \int_X 2|f| = \int_X g.$$

Since $|f_n - f| \rightarrow 0$ almost everywhere, by the general dominate convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int |f_n - f| = 0.$$