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**Exercise 1:**

Use the Fubini theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = \pi^{\frac{n}{2}}$$

Here $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Hint: For $n = 2$, use polar coordinates.

**Solution:**

Firstly, for $a > 0$, we define

$$I(a) = \int_{-a}^{a} e^{-x^2}\, dx,$$

then we have

$$I^2(a) = \int_{-a}^{a} e^{-x^2}\, dx \int_{-a}^{a} e^{-y^2}\, dy.$$

As $(-a, a)$ is an interval with finite measure and $|e^{-x^2}| \leq 1$, by the Fubini theorem, we have

$$I^2(a) = \int_{-a}^{a} \int_{-a}^{a} e^{-(x^2+y^2)}\, dx\, dy.$$

Take the polar coordinates transformation as follows,

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases}$$

then we have

$$\int_0^{2\pi} \int_0^{a} re^{-r^2}\, dr\, d\theta < I^2(a) < \int_0^{2\pi} \int_0^{\sqrt{2}a} re^{-r^2}\, dr\, d\theta.$$

By calculation, we can get the inequalities

$$(1 - e^{-a^2})\pi < I^2(a) < (1 - e^{-2a^2})\pi.$$

Let $a \to \infty$, we have

$$\lim_{a\to\infty} I^2(a) = \int_{\mathbb{R}^2} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = \pi,$$

then we know that $\lim_{a\to\infty} I(a) = \int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}$. For the $n$ dimensional domain, we have

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2}\, d\mathbf{x} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-(x_1^2+x_2^2+\cdots x_n^2)}\, dx_1 dx_2 \cdots dx_n \\ &= \Big(\int_{\mathbb{R}} e^{-x_1^2}\, dx_1\Big)^n = \pi^{\frac{n}{2}}. \end{aligned}$$

**<u>Exercise 2:</u>**

Let $(X, \mathcal{A}, \mu)$ be a measure space, and $f$ be in $L^1(X)$. Let for all positive integers $n$ set $B_n = \{x \in X : n-1 \leq |f(x)| < n\}$.

(i) Show that $\mu(B_n) < \infty$ for all $n \geq 2$.

(ii) Show that $\sum_{n=2}^{\infty} n\mu(B_n) < \infty$.

(iii) Define $C_n = \{x \in X : n-1 \leq |f(x)| \leq n\}$. Is the sum $\sum_{n=2}^{\infty} n\mu(C_n)$ finite?

(iv) Show that

$$\sum_{n=2}^{\infty}\sum_{m=2}^{n} \frac{m^2}{n^2}\mu(B_m) < \infty.$$

(v) Show that for $n \geq 2$

$$\int |f|^2 1_{\{|f|<n\}} = \int |f|^2 1_{\{|f|<1\}} + \sum_{m=2}^{n} \int |f|^2 1_{B_m}$$

and infer that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<n\}} < \infty$$

**Solution:**

(i) Since $f \in L^1(X)$, we have $\int_X |f|\, d\mu < \infty$. For $n \geq 2$, we know that

$$\int_X |f|\, d\mu \geq \int_{B_n} |f|\, d\mu \geq (n-1)\int_{B_n} 1\, d\mu = (n-1)\mu(B_n).$$

When $n \geq 2$, we have $n - 1 \geq 1$, then we know that $\mu(B_n) < \infty$. So, for any $n \geq 2$, we have $\mu(B_n) < \infty$.

(ii) Since $B_n = \{x \in X : n-1 \leq |f(x)| < n\} = \{x \in X : n \leq |f(x)|+1 < n+1\}$, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n\mu(B_n) &= \sum_{n=2}^{\infty}\int_{B_n} n\, d\mu \\
&\leq \sum_{n=2}^{\infty}\int_{B_n} |f(x)|+1\, d\mu \\
&= \sum_{n=2}^{\infty}\int_{B_n} |f(x)|\, d\mu + \sum_{n=2}^{\infty}\int_{B_n} 1\, d\mu \\
&\leq 2\int_{\bigcup_{n=2}^{\infty} B_n} |f(x)|\, d\mu \\
&\leq 2\int_X |f(x)|\, d\mu < \infty.
\end{aligned}$$

(iii) We claim that the sum $\sum_{n=2}^{\infty} n\mu(C_n)$ is finite. As $C_n = \{x \in X : n-1 \leq |f(x)| \leq n\} \subset B_n \cup B_{n+1}$, we have

$$\mu(C_n) \leq \mu(B_n \cup B_{n+1}) \leq \mu(B_n) + \mu(B_{n+1}),$$

thus

$$\sum_{n=2}^{\infty} n\mu(C_n) \leq \sum_{n=2}^{\infty} n\mu(B_n) + \sum_{n=2}^{\infty} n\mu(B_{n+1}).$$

Since $\int_{B_{n+1}} |f|\,d\mu \geq n \int_{B_{n+1}} 1\,d\mu = n\mu(B_{n+1})$, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n\mu(B_{n+1}) &\leq \sum_{n=2}^{\infty} \int_{B_{n+1}} |f|\,d\mu \\
&= \int_{\bigcup_{n=2}^{\infty} B_{n+1}} |f|\,d\mu \\
&< \int_X |f|\,d\mu < \infty.
\end{aligned}$$

As we showed $\sum_{n=2}^{\infty} n\mu(B_n) < \infty$ in (ii), hence we have

$$\sum_{n=2}^{\infty} n\mu(C_n) \leq \sum_{n=2}^{\infty} n\mu(B_n) + \sum_{n=2}^{\infty} n\mu(B_{n+1}) < \infty.$$

(iv) We can rewrite the $\sum_{n=2}^{\infty}\sum_{m=2}^{n} \frac{m^2}{n^2}\mu(B_m)$ and get

$$\begin{aligned}
\sum_{n=2}^{\infty}\sum_{m=2}^{n} \frac{m^2}{n^2}\mu(B_m) &= \sum_{m=2}^{\infty} \mu(B_m)m^2 \sum_{n=m}^{\infty} \frac{1}{n^2} \\
&= \sum_{m=2}^{\infty} m\mu(B_m) \sum_{n=m}^{\infty} \frac{m}{n^2}.
\end{aligned}$$

Next we need to show that $\sum_{n=m}^{\infty} \frac{m}{n^2}$ is bounded. When $m \geq 2$, we have

$$\sum_{n=m}^{\infty} \frac{m}{n^2} < m \int_{m-1}^{\infty} \frac{1}{x^2}\,dx = \frac{m}{m-1} \leq 2,$$

then we know that

$$\sum_{n=2}^{\infty}\sum_{m=2}^{n} \frac{m^2}{n^2}\mu(B_m) < 2\sum_{m=2}^{\infty} m\mu(B_m) < \infty.$$

Or by the inequalities as follows,

$$\begin{aligned}
\sum_{n=2}^{\infty}\sum_{m=2}^{n} \frac{m^2}{n^2}\mu(B_m) &= \sum_{m=2}^{\infty} \mu(B_m)m^2 \sum_{n=m}^{\infty} \frac{1}{n^2} \\
&< \sum_{m=2}^{\infty} m^2\mu(B_m) \sum_{n=m}^{\infty} \frac{1}{n(n-1)} \\
&= \sum_{m=2}^{\infty} m^2\mu(B_m) \frac{1}{m-1} \\
&= \sum_{m=2}^{\infty} m\mu(B_m) \frac{m}{m-1} \\
&< \sum_{m=2}^{\infty} 2m\mu(B_m) < \infty.
\end{aligned}$$

(v) Firstly, we show that

$$\int |f|^2 1_{\{|f|<n\}}\, d\mu = \int |f|^2 1_{\{|f|<1\}}\, d\mu + \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu.$$

By calculation, we have

$$\begin{aligned}
\int |f|^2 1_{\{|f|<n\}}\, d\mu &= \int |f|^2 1_{\{|f|<1\}}\, d\mu + \int |f|^2 1_{\{1\le|f|<n\}}\, d\mu \\
&= \int |f|^2 1_{\{|f|<1\}}\, d\mu + \int |f|^2 \sum_{m=2}^{n} 1_{\{m-1\le|f|<m\}}\, d\mu \\
&= \int |f|^2 1_{\{|f|<1\}}\, d\mu + \sum_{m=2}^{n} \int |f|^2 1_{\{m-1\le|f|<m\}}\, d\mu \\
&= \int |f|^2 1_{\{|f|<1\}}\, d\mu + \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu,
\end{aligned}$$

then we get the equation we wanted. Next we show that $\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<n\}} < \infty$. Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<n\}}\, d\mu \\
=\ &\sum_{n=1}^{\infty} \frac{1}{n^2} \Big( \int |f|^2 1_{\{|f|<1\}}\, d\mu + \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu \Big) \\
=\ &\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<1\}}\, d\mu + \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu.
\end{aligned}$$

For the first term in the right hand side of the above equation, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<1\}}\, d\mu < \sum_{n=1}^{\infty} \frac{1}{n^2} \int_X |f|\, d\mu < \infty.$$

And for the second term in the right hand side of the above equation, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu &< \sum_{n=2}^{\infty} \frac{1}{n^2} \sum_{m=2}^{n} \int m^2 1_{B_m}\, d\mu \\
&= \sum_{n=2}^{\infty} \sum_{m=2}^{n} \frac{m^2}{n^2} \mu(B_m) < \infty.
\end{aligned}$$

Thus we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<n\}}\, d\mu < \infty.$$

**Exercise 3:**

Prove or disprove: suppose that $f, g : \mathbb{R} \to \mathbb{R}$, with $f$ being a measurable function, and $g$ being a continuous function. Then $f \circ g$ is measurable. By definition, $(f \circ g)(x) = f(g(x))$, that is, it is the composition of the two functions.

**Solution:**

No, the statement is not true and we can find a counter example as follows. Suppose that $C$ is the Cantor set and we define a mapping $\phi$: for any $x \in C$, let $0.c_1c_2c_3 \cdots$ be its ternary expansion, where $c_n = 0$ or $c_n = 2$, $n = 1, 2, \cdots$ and let

$$\phi(x) = 0.\frac{c_1}{2}\frac{c_2}{2}\frac{c_3}{2}\cdots ,$$

where the expansion on the right is now interpreted as a binary expansion in terms of digits 0 and 1. It is clear that the image of $C$, under $\phi$, is a subset of $[0, 1]$. And next we extend the domain to the entire unit interval $[0, 1]$. If $x \in [0, 1] \setminus C$, then $x$ is a member of one of the open intervals $(a, b)$ removed from $[0, 1]$ in the construction of $C$, and therefore $\phi(a) = \phi(b)$. And we define $\phi(x) = \phi(a) = \phi(b)$. Since $\phi(\cdot)$ is increasing on $[0, 1]$, and since the range of $\phi(\cdot)$ is the entire interval $[0, 1]$, $\phi(\cdot)$ has no jump discontinuities. Since a monotonic function can have no discontinuities other than jump discontinuities, we know that $\phi(\cdot)$ is continuous. Then we define

$$\varphi(x) = x + \phi(x),\ x \in [0, 1]$$

with range $[0, 2]$. Since $\phi(\cdot)$ is increasing on $[0, 1]$ and continuous there, $\varphi$ is strictly increasing and topological there (continuous and one-to-one with a continuous inverse on the range $\varphi$). Since each open interval removed from $[0, 1]$ in the construction of the Cantor set $C$ is mapped by $\varphi$ onto an interval of $[0, 2]$ of the equal length, $\mu(\varphi(I \setminus C)) = \mu(I \setminus C) = 1$. Since $C$ is a set of measure zero, $\varphi$ is an example of a topological mapping that maps a set of measure zero onto a set of positive measure.

Now let $D$ is a non-measurable subset of $\varphi(C)$ and let $E = \varphi^{-1}(D)$. Then the characteristic function $f = 1_E(x)$ of the set $E$ is measurable and $g = \varphi^{-1}$ is continuous, but the composite function $f(g(x))$ is non-measurable characteristic function of the non-measurable set $D$.

**Claim:** suppose that $f, g : \mathbb{R} \to \mathbb{R}$, with $f$ being a measurable function, and $g$ being a continuous function. Then $g \circ f$ is measurable.

**Proof:** Since $f : (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is Lebesgue-measurable and as $g : \mathbb{R} \to \mathbb{R}$ is continuous, it is Borel-measurable. Take any $B \in \mathcal{B}_{\mathbb{R}}$, we want to show that $(g \circ f)^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$. By measurability of $g$, since $B \in \mathcal{B}_{\mathbb{R}}$, we have $B' = g^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$. By the measurability of $f$, this implies that $f^{-1}(B') \in \mathcal{B}_{\mathbb{R}}$. This shows that $g \circ f$ is measurable for the $\sigma$-algebras $\mathcal{B}_{\mathbb{R}}$.