GCE: 503, Analysis and measure theory

August 2019

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Exercise 1:

(i). Prove or disprove: the function f defined by $f(x) = \sqrt{x}$ is uniformly continuous on the internal $(0, 1)$.

(ii). Prove or disprove: the function g defined by $g(x) = \dfrac{1}{\sqrt{x}}$ is uniformly continuous on the internal $(0, 1)$. by contradiction.

Exercise 2:

Let f_n be a sequence in $L^1(\mathbb{R})$ which converges to f in L^1 norm. Let a_n be a sequence of real numbers which converges to 0. Show that

Lebgne Dominated.

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x + a_n) - f(x)| dx = 0.$$

$|f_n(x+a_n) - f_n(\cancel{x+}x)| + |f_n(x) - f(x)|$

Exercise 3:

Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7.

(i). Is E compact?

(ii). Is E dense in $[0, 1]$?

(iii). Is E countable?

(iv). Is E Lebesgue measurable? If yes, find its measure.

Exercise 4:

Let (X, d) be a metric space with the following property: for every infinite subset S of X, $\inf\{d(x, y) : x \neq y, x, y \in S\} = 0$. Show that for every $\epsilon > 0$, X can be covered by finitely many open balls of radius ϵ.