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Jiamin JIAN

Exercise 1:

Suppose that u is a real-valued function defined on $[0,1]$, that $u \geq 0$ and that $u \in L^1([0,1])$. Define $E_n := \{x \in [0,1] : n-1 \leq u(x) \leq n\}$ for each positive integer n. Show that

$$\sum_{n=1}^{\infty} n|E_n| < +\infty.$$

Solution:

As $u \in L^1([0,1])$ and $u(x) \geq 0$, we have

$$\int_0^1 |u(x)|\,dx = \int_0^1 u(x)\,dx < +\infty.$$

And since

$$\begin{aligned}\int_0^1 u(x)\,dx &= \sum_{n=1}^{\infty}\int_{E_n} u(x)\,dx \\ &\geq \sum_{n=1}^{\infty}(n-1)|E_n| \\ &= \sum_{n=1}^{\infty} n|E_n| + \sum_{n=1}^{\infty}|E_n|,\end{aligned}$$

and $\sum_{n=1}^{\infty}|E_n| < +\infty$, then we have

$$\sum_{n=1}^{\infty} n|E_n| < +\infty.$$

Exercise 2:

Show that a subset E of a metric space X is open if and only if there is a continuous real-valued function f on X such that $E = \{x \in x : f(x) > 0\}$.

Solution:

If there is a continuous real-valued function f on X such that $E = \{x \in x : f(x) > 0\}$, we want to show that E is an open set. Since $(0, +\infty)$ is an open set, $E = \{x \in x : f(x) > 0\} = f^{-1}((0,+\infty))$ is also an open set as f is continuous on X. We can also verify the statement by definition. Suppose $y \in E$, since $E = \{x \in x : f(x) > 0\}$, we have $f(y) > 0$. Since f in continuous on X, we know that there exists a δ such that when $d(x,y) < \delta$, then $|f(x) - f(y)| < f(y)$, which implies $-f(y) < f(x) - f(y) < f(y)$, hence we have

$f(x) > 0$. Then we know that there exists a $\delta > 0$, when $x \in B_\delta(y)$, we have $f(x) > 0$. Thus for any $y \in E$, there exists a δ, and $B_\delta(y) \subset E$. So we know that E is an open set.

On the other direction, we want to show that if $E \subset X$ is open, there exists a continuous function f on X such that $E = \{x \in x : f(x) > 0\}$. For $E \in X$, we denote

$$f(x) = d(x, E^c) = \min_{y \in E^c} d(x, y).$$

Then we have when $x \in E^c$, $f(x) = 0$ and when $x \in E$, $f(x) > 0$, so we have $E = \{x \in x : f(x) > 0\}$. Next we need to show f is continuous on X . Let $x, y \in X$ and p is the any point in E^c, then

$$d(x, p) \leq d(x, y) + d(y, p),$$

and so

$$d(x, E^c) \leq d(x, y) + d(y, p)$$

as $d(x, A)$ is the minimum. Then we have $d(y, p) \geq d(x, E^c) - d(x, y)$ for all $p \in E^c$, thus we can get that $d(y, E^c) \geq d(x, E^c) - d(x, y)$, which is equivalent to

$$d(x, E^c) - d(y, E^c) \leq d(x, y).$$

Similarly, we can change the position of x and y then get

$$d(y, E^c) - d(x, E^c) \leq d(x, y),$$

so we have for any $x, y \in X$,

$$|d(x, E^c) - d(y, E^c)| \leq d(x, y).$$

Then for any $\epsilon > 0$, there exists a $\delta = \epsilon$, such that when $d(x, y) < \delta$, we have $|d(x, E^c) - d(y, E^c)| < d(x, y) = \epsilon$. So, we have showed that f is a continuous function on X.

Exercise 3:

Consider the sequence of functions $\{f_n\}$ defined on the non-negative reals: $[0, +\infty)$ where $f_n(x) = 2nxe^{-nx^2}$. Let g be a continuous and bounded function on $[0, +\infty)$ valued in $\mathbb{R}$.

(i) Find with proof

$$\lim_{n \to \infty} \int_0^\infty f_n(t) g(t)\, dt.$$

(ii) Define for x in $[0, +\infty)$,

$$g_n(x) = \int_0^\infty f_n(t) g(x + t)\, dt.$$

Assuming g is zero outside the interval $[0, M]$, where $M > 0$, does the sequence g_n have a limit in $L^1([0, +\infty))$?

(iii) If h is in $L^1([0, +\infty))$, define for x in $[0, +\infty)$,

$$h_n(x) = \int_0^\infty f_n(t)h(x + t)\, dt.$$

Show that h_n is measurable on $[0, +\infty)$ and is in $L^1([0, +\infty))$.

(iv) Find, if it exists, with proof, the limit of h_n in $L^1([0, +\infty))$.

Solution:

(i) We denote $y = nt^2$, then we have

$$\int_0^\infty 2nte^{-nt^2} g(t)\, dt = \int_0^\infty e^{-y} g\Big(\sqrt{\frac{y}{n}}\Big)\, dy.$$

Since $g(x)$ is a continuous and bounded function on $[0, +\infty)$, we suppose that $|g(x)| \leq C$ for any $x \in [0, +\infty)$. Then we know that $|e^{-y}g(\sqrt{\frac{y}{n}})| \leq Ce^{-y}$ and $Ce^{-y} \in L^1([0, +\infty))$ as $\int_0^\infty |Ce^{-y}|\, dy = C < +\infty$. And for any fixed $y \in [0, +\infty)$, when $n \to \infty$, $g(\sqrt{\frac{y}{n}}) \to g(0)$ and then $e^{-y}g(\sqrt{\frac{y}{n}}) \to e^{-y}g(0)$. By the dominate convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty f_n(t)g(t)\, dt &= \int_0^\infty \lim_{n\to\infty} e^{-y} g\Big(\sqrt{\frac{y}{n}}\Big)\, dy \\
&= \int_0^\infty e^{-y} g(0)\, dy \\
&= g(0).
\end{aligned}$$

(ii) Since $f_n(x) = 2nxe^{-nx^2}$, we denote $y = nt^2$, then we have

$$g_n(x) = \int_0^\infty f_n(t)g(x + t)\, dt = \int_0^\infty e^{-y} g\Big(x + \sqrt{\frac{y}{n}}\Big)\, dy.$$

Next we want to show that g_n converges to g in $L^1([0, +\infty))$. Since

$$\begin{aligned}
\int_0^\infty |g_n(x) - g(x)|\, dx &= \int_0^\infty \Big| \int_0^\infty e^{-y} g\Big(x + \sqrt{\frac{y}{n}}\Big)\, dy - g(x) \Big|\, dx \\
&= \int_0^\infty \Big| \int_0^\infty e^{-y} g\Big(x + \sqrt{\frac{y}{n}}\Big)\, dy - \int_0^\infty g(x)e^{-y}\, dy \Big|\, dx \\
&= \int_0^\infty \Big| \int_0^\infty e^{-y} \Big(g\Big(x + \sqrt{\frac{y}{n}}\Big) - g(x) \Big)\, dy \Big|\, dx \\
&\leq \int_0^\infty \int_0^\infty e^{-y} \Big| g\Big(x + \sqrt{\frac{y}{n}}\Big) - g(x) \Big|\, dy\, dx,
\end{aligned}$$

and by Fubini theorem,

$$\begin{aligned}
\int_0^\infty \int_0^\infty e^{-y} \Big| g\Big(x + \sqrt{\frac{y}{n}}\Big) - g(x) \Big|\, dy\, dx &= \int_0^\infty \int_0^{M-\sqrt{\frac{y}{n}}} e^{-y} \Big| g\Big(x + \sqrt{\frac{y}{n}}\Big) - g(x) \Big|\, dx\, dy \\
&+ \int_0^\infty \int_{M-\sqrt{\frac{y}{n}}}^M e^{-y} |g(x)|\, dx\, dy,
\end{aligned}$$

when $n \to \infty$, we have

$$\int_0^\infty \int_{M-\sqrt{\frac{y}{n}}}^{M} e^{-y}|g(x)|\,dx\,dy \to 0,$$

thus we know that

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty |g_n(x)-g(x)|\,dx &\le \lim_{n\to\infty}\int_0^\infty\int_0^{M-\sqrt{\frac{y}{n}}} e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy\\ &\le \lim_{n\to\infty}\int_0^\infty\int_0^{M} e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy.\end{aligned}$$

Since $e^{-y}|g(x+\sqrt{\frac{y}{n}})-g(x)| \le 2Ce^{-y}$ and $2Ce^{-y} \in L^1([0,+\infty))$, then by the dominate convergence theorem we have

$$\lim_{n\to\infty}\int_0^\infty\int_0^{M} e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy = 0,$$

thus we know that

$$\lim_{n\to\infty}\int_0^\infty |g_n(x)-g(x)|\,dx = 0.$$

So, we have showed that g_n converges to g in $L^1([0,+\infty))$.

(iii) Since $C_c([0,+\infty))$ is dense in $L^1([0,+\infty))$ and $h(x) \in L^1([0,+\infty))$, we can find a sequence $\{h^k\}_{k=1}^\infty$ such that $h^k \to h$ in $L^1([0,+\infty))$. We want to show h_n is measurable by showing it is the limit of a sequence of measurable functions. By the result we got from (ii), for any $k \in \mathbb{N}$, we have $h_n^k = \int_0^\infty f_n(t)g(t)\,dt$ converges to $h^k(x)$ in $L^1([0,+\infty))$. Firstly we show that $h_n^k(x)$ converges to $h_n(x)$ almost everywhere. For any $x \in [0,+\infty)$, we have

$$\begin{aligned}|h_n(x)-h_n^k(x)| &= \Big|\int_0^\infty f_n(t)h(x+t)\,dt - \int_0^\infty f_n(t)h^k(x+t)\,dt\Big|\\ &= \Big|\int_0^\infty f_n(t)(h(x+t)-h^k(x+t))\,dt\Big|\\ &\le \int_0^\infty f_n(t)|h(x+t)-h^k(x+t)|\,dt,\end{aligned}$$

we denote $z = x+t$, then

$$|h_n(x)-h_n^k(x)| \le \int_x^\infty f_n(z-x)|h(z)-h^k(z)|\,dz.$$

Since $f_n(x) = 2nxe^{-nx^2}$, when $x = \frac{1}{\sqrt{2n}}$, the $f_n(x)$ gets the maximum value as $\sqrt{2n}e^{-\frac{1}{2}}$, thus we have

$$\begin{aligned}|h_n(x)-h_n^k(x)| &\le \int_x^\infty f_n(z-x)|h(z)-h^k(z)|\,dz\\ &\le \|f_n\|_\infty \int_x^\infty |h(z)-h^k(z)|\,dz\\ &\le \|f_n\|_\infty \int_0^\infty |h(z)-h^k(z)|\,dz\\ &= \|f_n\|_\infty \|h-h^k\|_1 \to 0\end{aligned}$$

as $k \to +\infty$. Then we show that h_n^k is continuous. This means we want to show that for $x \in [0, +\infty)$, let $x_j \to x$, then $h_n^k(x_j) \to h_n^k(x)$. By the definition of $h_n^k(x_j)$, we have

$$h_n^k(x_j) = \int_0^\infty f_n(t) h^k(x_j + t)\, dt = \int_0^\infty e^{-y} h^k\Big(x_j + \sqrt{\frac{y}{n}}\Big)\, dy.$$

And since $h^k \in C_c([0, +\infty))$, $|e^{-y} h^k(x_j + \sqrt{\frac{y}{n}}| \leq \|h^k\|_\infty e^{-y} \in L^1([0, +\infty))$, by the dominate convergence theorem, we have

$$\lim_{j\to\infty} h_n^k(x_j) = \int_0^\infty \lim_{j\to\infty} e^{-y} h^k\Big(x_j + \sqrt{\frac{y}{n}}\Big)\, dy = \int_0^\infty e^{-y} h^k\Big(x + \sqrt{\frac{y}{n}}\Big)\, dy = h_n^k(x),$$

thus we know that h_n^k is uniformly continuous. From above, we have $h_n^k \to h_n$ almost everywhere and h_n^k is uniformly continuous, then we have h_n is the limit of a sequence of measurable functions. So, we get that h_n is measurable on $[0, +\infty)$.

Next we show that h_n is in $L^1([0, +\infty))$. Since

$$\begin{aligned}
\|h_n\|_1 &= \int_0^\infty |h_n(x)|\, dx \\
&= \int_0^\infty \Big| \int_0^\infty f_n(t) h(x+t)\, dt \Big|\, dx \\
&\leq \int_0^\infty \int_0^\infty |f_n(t) h(x+t)|\, dt\, dx,
\end{aligned}$$

by Fubini theorem, we have

$$\begin{aligned}
\|h_n\|_1 &\leq \int_0^\infty \int_0^\infty |f_n(t) h(x+t)|\, dx\, dt \\
&= \int_0^\infty f_n(t) \Big(\int_0^\infty |h(x+t)|\, dx \Big)\, dt \\
&= \int_0^\infty f_n(t) \Big(\int_t^\infty |h(z)|\, dz \Big)\, dt \\
&\leq \int_0^\infty f_n(t) \Big(\int_0^\infty |h(z)|\, dz \Big)\, dt \\
&= \|h\|_1 \int_0^\infty f_n(t)\, dt \\
&= \|h\|_1 < +\infty.
\end{aligned}$$

Thus we know that h_n is in $L^1([0, +\infty))$.

(iv) We want to show that h_n converges to h in $L^1([0, +\infty))$. Let $\epsilon > 0$, since $C_c([0, +\infty))$ is dense in $L^1([0, +\infty))$, then there exists a $g \in C_c([0, +\infty))$ such that $\|h - g\|_1 < \epsilon$. So we have

$$\begin{aligned}
\|h_n - h\|_1 &= \|h_n - g_n + g_n - g + g - f\|_1 \\
&\leq \|h_n - g_n\|_1 + \|g_n - g\|_1 + \|g - f\|_1,
\end{aligned}$$

where the definition of g_n is as question (ii). By the result we get form (ii), for the ϵ above, we have $\|g_n - g\| < \epsilon$, then we know that

$$\|h_n - h\|_1 < \|h_n - g_n\|_1 + 2\epsilon.$$

Next we need to deal with $\|h_n - g_n\|_1$. Since

$$\begin{aligned}
\|h_n - g_n\|_1 &= \int_0^\infty |h_n(x) - g_n(x)|\, dx \\
&\leq \int_0^\infty \int_0^\infty f_n(t)|h(x+t) - g(x+t)|\, dt\, dx,
\end{aligned}$$

we denote $z = x + t$ and by Fubini theorem we have

$$\begin{aligned}
\|h_n - g_n\|_1 &\leq \int_0^\infty \int_0^\infty f_n(t)|h(x+t) - g(x+t)|\, dt\, dx \\
&= \int_0^\infty f_n(t) \int_t^\infty |h(z) - g(z)|\, dz\, dt \\
&\leq \int_0^\infty f_n(t) \int_0^\infty |h(z) - g(z)|\, dz\, dt \\
&= \int_0^\infty f_n(t)\|h - g\|_1 dt \\
&= \|h - g\|_1 \int_0^\infty f_n(t) dt \\
&= \|h - g\|_1 < \epsilon.
\end{aligned}$$

Thus we know that

$$\|h_n - h\|_1 < \|h_n - g_n\|_1 + 2\epsilon < 3\epsilon$$

for any $\epsilon > 0$. So, we have showed that h_n converges to h in $L^1([0, +\infty))$.

Exercise 4:

Show that a set $E \subset \mathbb{R}$ is Lebesgue measurable if and only if $E = H \cup Z$ where H is a countable union of closed sets and Z has measure zero. You may use the following property: for any Lebesgue measurable subset A of $\mathbb{R}$ and any $\epsilon > 0$, there is a closed subset F of $\mathbb{R}$ such that $F \subset A$ and the measure of $A \setminus F$ is less than ϵ.

Solution:

If $E \subset \mathbb{R}$ is Lebesgue measurable, then we know that $\forall \epsilon > 0$, there is a closed subset H of $\mathbb{R}$ such that $H \subset E$ and the measure of $E \setminus H$ is less than ϵ. We denote $Z = E \setminus H$, then we have $m(Z) = 0$ and $Z \cup H = (E \setminus H) \cup H = E$.

Since H is a countable union of closed sets, then H is a $\mathcal{F}_\sigma$ set and it is measurable. And as Z is a zero measure set, it is also Lebesgue measurable. Thus we know that $E = H \cup Z$ is Lebesgue measurable.

Exercise 5:

Give an example of a sequence f_n in $L^1((0,1))$ such that $f_n \to 0$ in $L^1((0,1))$ but f_n does not converge to zero almost everywhere.

Solution:

We suppose that

$$f_n(x) = \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x),$$

whenever $k \geq 0, 2^k \leq n < 2^{k+1}$. For any $n \in \mathbb{N}$, we have

$$\int_0^1 |f_n(x)|\, dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x)\, dx = \frac{1}{2^k} < +\infty,$$

so we know that $f_n \in L^1((0,1))$. And similarly we have

$$\int_0^1 |f_n(x) - 0|\, dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x)\, dx = \frac{1}{2^k} < \frac{2}{n},$$

then when $n \to +\infty$, $\int_0^1 |f_n(x) - 0|\, dx \to 0$, thus we get $f_n \to 0$ in $L^1((0,1))$. But for any $x \in (0,1)$, and for any $N \in \mathbb{N}$, we can find a $n > N$ with $f_n(x) = 1$. Thus f_n can not converges to 0 anywhere for $x \in (0,1)$.