Real Analysis GCE Name:__________________________

May, 2021

# 1:

Suppose that $\{x_n\}$ and $l \in \mathbb{R}$ are such that every Cauchy subsequence of $\{x_n\}$ converges to $l$.

a. Show that if $\{x_n\}$ is bounded, it must converge to $l$.

b. Show that if $\{x_n\}$ is unbounded, it is not true that it must converge to $l$. (Give a counterexample)

# 2:

Denote by $l(I)$ the length of any interval $I$. Define $m^*$ and $m^{**}$ from $\mathcal{P}(\mathbb{R})$ to $[0, \infty]$ by setting for any subset A of $\mathbb{R}$

$$m^*(A) = \inf\{\sum_{k=1}^{\infty} l(I_k) : A \subset \bigcup_{k=1}^{\infty} I_k, \quad I_k \text{ is an open and bounded interval}\},$$

$$m^{**}(A) = \inf\{\sum_{k=1}^{\infty} l(J_k) : A \subset \bigcup_{k=1}^{\infty} J_k, \quad J_k \text{ is any interval}\}.$$

Prove that $m^* = m^{**}$.

# 3:

Consider functions $f_n, g_n \in L^2([0,1])$ for all $n$. Assume $f_n \to 0$ almost everywhere on $[0,1]$ and $\sup_n \|g_n\|_{L^2} < \infty$.

a. Assume in addition that $|f_{n+1}| \leq |f_n|$ almost everywhere on [0,1] for all $n$. Show that $\int_{[0,1]} f_n g_n dx \to 0$.

b. Without the monotonicity assumption $|f_{n+1}| \leq |f_n|$, give a counterexample to the conclusion in part a.