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Exercise 1:

Consider the sequence of functions f_n defined on the non-negative reals by $f_n(x) = 2nxP(x)e^{-nx^2}$, where P is a polynomial function.

(i) Is f_n pointwise convergent on $[0,+\infty)$? Is f_n uniformly convergent on $[0,+\infty)$? Explain your answers to both questions.

(ii) Let g_n be a sequence of continuous functions defined on $[0,+\infty)$ and valued in $\mathbb{R}$. Assume that each g_n is in $L^1([0,+\infty))$ and that sequence g_n is uniformly convergent to zero. Prove or disprove: $\lim_{n\to\infty}\int_0^\infty g_n = 0$.

(iii) Determine (with proof) $\lim_{n\to\infty}\int_0^\infty f_n$.

Solution:

(i) When $x = 0$, $f_n(x) = 0$ for any $n \in \mathbb{N}$. When $x > 0$, since

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 2nxP(x)e^{-nx^2} = \lim_{n\to\infty} \frac{2nxP(x)}{e^{nx^x}} = 0,$$

then for any $\epsilon > 0$, there exist a $N \in \mathbb{N}$, such that $n > N$ we have

$$|2nxP(x)e^{-nx^2} - 0| < \epsilon,$$

thus we know that f_n converges to $f(x) = 0$ pointwise on $[0,\infty)$. But f_n is not uniformly convergent to $f(x) = 0$. We suppose $P(x) = 1$, then we have $f_n(x) = 2nxe^{-nx^2}$. When $x = \frac{1}{\sqrt{n}}$,

$$f_n(x) = 2n\frac{1}{\sqrt{n}}e^{-n\frac{1}{n}} = 2\sqrt{n}e^{-1},$$

so we have

$$\sup_{x\in[0,\infty)} |f_n(x) - 0| \geq 2\sqrt{n}e^{-1} \to \infty$$

when n goes to $+\infty$. Thus we know that f_n is not uniformly converges on $[0,+\infty)$.

(ii) The statement is not true. We suppose

$$g_n(x) = \begin{cases} \frac{4}{n^2}x, & x \in [0, \frac{n}{2}) \\ \frac{4}{n} - \frac{4}{n^2}x, & x \in [\frac{n}{2}, n] \\ 0, & x \in (n, +\infty), \end{cases}$$

then we know that for any $n \in \mathbb{N}$,

$$\int_{[0,\infty)} g_n(x)\,dx = \int_0^{\frac{n}{2}} \frac{4}{n^2}x\,dx + \int_{\frac{n}{2}}^n \frac{4}{n} - \frac{4}{n^2}x\,dx = 1,$$

so we know that $g_n(x) \in L^1([0,\infty))$. When $x \in [0, \frac{n}{2})$, $g_n(x) = \frac{4}{n^2}x \le \frac{2}{n}$ and when $x \in [\frac{n}{2}, n]$, $g_n(x) = \frac{4}{n} - \frac{4}{n^2}x \le \frac{2}{n}$, so we know that g_n uniformly converges to 0. But since for any $n \in \mathbb{N}$, $\int_0^\infty g_n(x)\,dx = 1$, then we have

$$\lim_{n\to+\infty} \int_{[0,\infty)} g_n(x)\,dx = \lim_{n\to\infty} 1 = 1.$$

Thus $\lim_{n\to\infty} \int_0^\infty g_n = 0$ can not hold.

(iii) We denote $y = nx^2$, then we have

$$\int_0^\infty 2nxP(x)e^{-nx^2}\,dx = \int_0^\infty e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy.$$

Since $P(x)$ is a polynomial function, for any fixed y, when $n \to \infty$, $P(\sqrt{\frac{y}{n}}) \to P(0)$ and then $e^{-y}P(\sqrt{\frac{y}{n}}) \to e^{-y}P(0)$. Since $P(x)$ is a polynomial function, there exist a $M > 0$, such that when $y \in [M,\infty)$, $e^{-y}P(\sqrt{\frac{y}{n}}) < \frac{1}{y^2}$, then we have

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty f_n &= \lim_{n\to\infty} \int_0^M f_n + \lim_{n\to\infty} \int_M^\infty f_n \\
&= \lim_{n\to\infty} \int_0^M e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy + \lim_{n\to\infty} \int_M^\infty e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy.
\end{aligned}$$

Since $P(x)$ is a polynomial function, then $P(\sqrt{\frac{y}{n}})$ is continuous on $y \in [0,M]$, then we have when $y \in [0,M]$,

$$\Big|e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\Big| \le e^{-y}\|P\|_\infty.$$

Since $e^{-y}\|P\|_\infty \in L^1([0,M])$ and $\frac{1}{y^2} \in L^1([M,+\infty))$, by the dominate convergence theorem, we have

$$\lim_{n\to\infty} \int_0^M e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy = \int_0^M e^{-y}P(0)\,dy = P(0)(1-e^{-M}),$$

and

$$\lim_{n\to\infty} \int_M^\infty e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy = \int_M^\infty e^{-y}P(0)\,dy = P(0)e^{-M}.$$

Thus we know that

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty f_n &= \lim_{n\to\infty} \int_0^M e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy + \lim_{n\to\infty} \int_M^\infty e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy \\
&= P(0)(1-e^{-M}) + P(0)e^{-M} \\
&= P(0).
\end{aligned}$$

Exercise 2(all answers require proofs:)
Let f_n be the sequence in $L^2(\mathbb{R})$ defined by $f_n = \mathbb{I}_{[n,n+1]}$.
(i) Let g be in $L^2(\mathbb{R})$. Does $\int f_n g$ have a limit as n tends to infinity?
(ii) Does the sequence f_n converge in $L^2(\mathbb{R})$?

Solution:
(i) Firstly we show that $f_n = \mathbb{I}_{[n,n+1]}(x)$ converges to $f(x) = 0$ pointwise on $\mathbb{R}$. Since

$$|f_n - f| = |\mathbb{I}_{[n,n+1]}(x) - 0| = \mathbb{I}_{[n,n+1]}(x),$$

for any fixed $x \in \mathbb{R}$, $\forall \epsilon > 0$,we can find a $N = [x] + 1$, such that $n > N$, we have

$$|f_n - f| = \mathbb{I}_{[n,n+1]}(x) = 0 < \epsilon.$$

Thus we know that f_n converges to $f(x) = 0$ pointwisely on $\mathbb{R}$. Since

$$\Big|\int_{\mathbb{R}} f_n g\,dx\Big| \le \int_{\mathbb{R}} |f_n g|\,dx = \int_n^{n+1} |g(x)|\,dx,$$

by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}} f_n g\,dx\Big| &\le \int_n^{n+1} |g(x)|\,dx \\
&\le \Big(\int_n^{n+1} |g|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_n^{n+1} 1^2\,dx\Big)^{\frac{1}{2}} \\
&= \Big(\int_n^{n+1} |g|^2\,dx\Big)^{\frac{1}{2}} \\
&= \Big(\int_{\mathbb{R}} |g|^2\mathbb{I}_{[n,n+1]}(x)\,dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since $|g|^2\mathbb{I}_{[n,n+1]}(x) \le |g(x)|^2$ and since $g \in L^2(\mathbb{R})$, we have $\int_{\mathbb{R}} |g(x)|^2\,dx < +\infty$, then we know that $|g(x)|^2 \in L^1(\mathbb{R})$, by the dominate convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\Big|\int_{\mathbb{R}} f_n g\,dx\Big|^2 &\le \lim_{n\to\infty}\Big(\int_{\mathbb{R}} |g|^2\mathbb{I}_{[n,n+1]}(x)\,dx\Big) \\
&= \int_{\mathbb{R}} \lim_{n\to\infty}\big(|g|^2\mathbb{I}_{[n,n+1]}(x)\big)\,dx.
\end{aligned}$$

Since $f_n = \mathbb{I}_{[n,n+1]}(x)$ converges to $f(x) = 0$ pointwisely on $\mathbb{R}$, we can show that $|g|^2\mathbb{I}_{[n,n+1]}(x)$ also converges to $f(x) = 0$ pointwisely on $\mathbb{R}$, then we have

$$\lim_{n\to\infty}\Big|\int_{\mathbb{R}} f_n g\,dx\Big|^2 \le \int_{\mathbb{R}} \lim_{n\to\infty}\big(|g|^2\mathbb{I}_{[n,n+1]}(x)\big)\,dx = 0.$$

Thus we know that $\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)g(x)\,dx = 0$.
(ii) Since $f_n = \mathbb{I}_{[n,n+1]}(x)$ converges to $f(x) = 0$ pointwisely on $\mathbb{R}$, but

$$\int_{\mathbb{R}} |f_n(x) - 0|^2\,dx = \int_{\mathbb{R}} f_n^2\,dx = \int_n^{n+1} 1\,dx = 1,$$

we know that f_n does not converges to $f(x) = 0$ in $L^2(\mathbb{R})$. Since $f_n = \mathbb{I}_{[n,n+1]}(x)$ converges to $f(x) = 0$ pointwisely on $\mathbb{R}$, then we can know that the sequence f_n does not converge in $L^2(\mathbb{R})$.

Exercise 3:

Let X be a matrix space. For any subset A of X, we denote by $\bar{A}$ the closure of A and $\mathring{A}$ the union of all open subsets contained in A. We set $\partial A = \bar{A} \setminus \mathring{A}$.

(i) Show that A is closed if and only if $\partial A \subset A$.

(ii) Show that A is open if and only if $\partial A \cap A = \emptyset$.

(iii) Is the identity $\partial(\partial B) = \partial B$ valid for all subsets B of X.

(iv) Show that if A is closed then $\partial(\partial A) = \partial A$.

Solution:

(i) When A is closed, we have $A = \bar{A}$, since $\mathring{A}$ the union of all open subsets contained in A, then $\mathring{A} \subset A$. Thus we have $\partial A = \bar{A} \setminus \mathring{A} = A \setminus \mathring{A} \subset A$ as $\mathring{A} \subset A$.

When $\partial A \subset A$, we have $\partial A \cup A \subset A \cup A = A$, then we know that $\bar{A} \subset A$. Since $A \subset \bar{A}$, we can get $\bar{A} = A$, thus A is closed.

(ii) When A is open, since $\mathring{A}$ the union of all open subsets contained in A, then we have $A \subset \mathring{A}$. And we know that $\mathring{A} \subset A$, then we can get $A = \mathring{A}$. As $\partial A = \bar{A} \setminus \mathring{A}$, we have $\partial A = \bar{A} \setminus A$, then it is obviously that $\partial A \cap A = \emptyset$.

When $\partial A \cap A = \emptyset$, we suppose A is not an open set, then there exists a element $x \in A$ such that no open set containing x is a subset of A. Since $\mathring{A}$ the union of all open subsets contained in A, we have $x \notin \mathring{A}$. And as $x \in A$, we know that $x \in \bar{A}$, then we have $x \in \bar{A} \setminus \mathring{A} = \partial A$. Then we can get $x \in \partial A \cap A$, it is contradict with the condition we have. So, the statement that A is not an open set is wrong. Thus we have A is an open set.

(iii) No, the statement is not true. We suppose $B = \mathbb{Q} \cap [0,1]$, which represents the rational number in the interval $[0,1]$. Then we have $\partial B = [0,1]$ and $\partial(\partial B) = \{0,1\}$, which is not equal to ∂B.

(iv) Since $\bar{A}$ is closed and $\mathring{A}$ is open, we have $\partial A = \bar{A} \setminus \mathring{A}$ is closed, then we can get $\overline{\partial A} = \partial A$. By the definition of ∂A, we have $\partial(\partial A) = \overline{\partial A} \setminus \mathring{\partial A} = \partial A \setminus \mathring{\partial A} \subset \partial A$. Next we need to show that $\partial A \subset \partial(\partial A) = \partial A \setminus \mathring{\partial A}$, then we just need to prove that $\mathring{\partial A} = \emptyset$ when A is closed.

When A is closed, since $\partial A = \bar{A} \setminus \mathring{A} = A \setminus \mathring{A}$. As $A \setminus \mathring{A} \subset A$, then we have $\mathring{\partial A} \subset \mathring{A}$. And since the union of subsets in $(A \setminus \mathring{A})$ is the subset of $A \setminus \mathring{A}$, we have $\mathring{\partial A} \subset A \setminus \mathring{A}$. Then we know that $\mathring{\partial A} \subset \mathring{A}$ and $\mathring{\partial A} \subset A \setminus \mathring{A}$. Thus we can get $\mathring{\partial A} \subset \mathring{A} \cap (A \setminus \mathring{A}) = \emptyset$. So, we have showed that $\mathring{\partial A} = \emptyset$. In conclusion, we have $\partial(\partial A) = \partial A$ when A is closed.

Exercise 4:

Let X be a measure space, f_n a sequence in $L^1(X)$ and f an element of $L^1(X)$ such that f_n converges to f almost everywhere and $\lim_{n\to\infty}\int|f_n| = \int|f|$. Show that $\lim_{n\to\infty}\int|f_n - f| = 0$.

Solution:

Since $|f_n - f| \leq |f_n| + |f|$ holds on X, we know that $|f_n| + |f| - |f_n - f|$ is a non-negative function. By the Fatou's lemma, we have

$$\int \lim_{n\to\infty}(|f_n| + |f| - |f_n - f|) \leq \liminf_{n\to\infty}\int(|f_n| + |f| - |f_n - f|).$$

Since f_n converges to f almost everywhere, then we know that $|f_n|$ converges to $|f|$ almost everywhere. Thus we have

$$\lim_{n\to\infty}(|f_n| + |f| - |f_n - f|) = 2|f|.$$

Then we can get that

$$\begin{aligned}
\int 2|f| &\leq \liminf_{n\to\infty}\int(|f_n| + |f| - |f_n - f|) \\
&\leq \liminf_{n\to\infty}\int(|f_n| + |f|) - \limsup_{n\to\infty}\int(|f_n - f|) \\
&= \int 2|f| - \limsup_{n\to\infty}\int(|f_n - f|),
\end{aligned}$$

so we have

$$\limsup_{n\to\infty}\int(|f_n - f|) \leq 0.$$

On the other hand, we have

$$0 \leq \liminf_{n\to\infty}\int(|f_n - f|)$$

as $|f_n - f| \geq 0$. Thus we know that

$$\limsup_{n\to\infty}\int(|f_n - f|) \leq 0 \leq \liminf_{n\to\infty}\int(|f_n - f|),$$

which is equivalent to

$$\limsup_{n\to\infty}\int(|f_n - f|) = \liminf_{n\to\infty}\int(|f_n - f|) = 0.$$

So we have

$$\lim_{n\to\infty}\int|f_n - f| = 0.$$