1. (Exercise 1/Fall 2019 Final)
Let (a, b) be an interval of $\mathbb{R}$ and f a real valued function on (a, b) which is uniformly zero outside a compact subset K of (a, b). Show that there is an $\alpha > 0$ such that $f(x) = 0$ if x is $(a, a + \alpha)$ or $(b - \alpha, b)$.
Solution:

2. (Exercise 2/Fall 2019 Final)
Let (X, ρ) be a metric space.
(a) Let K be a compact subset of X. Define
$$diam K = \sup\{\rho(x, y) : x \in K, y \in K\}.$$
Show that there is a u and v in K such that $diam K = \rho(u, v)$.
(b) Let K_n be a sequence of compact subsets of X such that $K_{n+1} \subset K_n$ and $diam K_n$ converges to 0. Show that there is an l in X such that $\displaystyle\bigcap_{n=1}^{\infty} K_n = \{l\}$. Recall that by definition a compact set is non-empty.
Solution:

3. (Exercise 3/Fall 2019 Final)
Let V be an open subset of $\mathbb{R}^d$ and $||.||_\infty$ the usual norm on $L^\infty(V)$. If f is continuous and bounded in V, show that $||f||_\infty = \sup\{|f(x)| : x \in V\}$.
Solution:

4. (Exercise 4/Fall 2019 Final)
Find an a in $\mathbb{R}$ such that defining $f(x, t) = x^a e^{-tx}$, for $(x, t) \in [0, 1] \times [0, \infty)$, f is continuous, the function $t \to f(x, t)$ is in $L^1([0, \infty))$ for all x in $[0, 1]$, but the function $g(x) = \displaystyle\int_0^\infty f(x, t)dt$ is not continuous in $[0, 1]$. Is there a function h in $L^1([0, \infty))$ such that $f(x, t) \leq h(t)$ for all x in $[0, 1]$ and almost all t in $[0, \infty)$?
Solution:

5. (Exercise 5/Fall 2019 Final)
Let f be in $L^1(\mathbb{R})$. Show that $\displaystyle\lim_{x \to +\infty} \int_{-\infty}^{\infty} f(t) \cos(xt)dt = 0$.
Solution: