Homework 1, 2019 Fall

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Exercise 1:

Let x and y be in $\mathbb{R}^d$. Assume that $x \neq 0$. Set $e = \frac{x}{\|x\|}$, $P(y) = \langle y, e\rangle e$, and $z = y - P(y)$.

(i) Show that $\langle z, P(y)\rangle = 0$.

(ii) Show that $\|P(y)\| \leq \|y\|$.

(iii) Infer the Cauchy Schwartz inequality.

Solution:

(i) By calculation, we have

$$\begin{aligned}
\langle z, P(y)\rangle &= \langle y - P(y), P(y)\rangle \\
&= \langle y, P(y)\rangle - \langle P(y), P(y)\rangle \\
&= \langle y, \langle y, e\rangle e\rangle - \langle \langle y, e\rangle e, \langle y, e\rangle e\rangle \\
&= (\langle y, e\rangle)^2 - (\langle y, e\rangle)^2 \langle e, e\rangle \\
&= (\langle y, e\rangle)^2 - (\langle y, e\rangle)^2 \\
&= 0.
\end{aligned}$$

(ii) Since $z = y - P(y)$, we have $y = z + P(y)$, then

$$\begin{aligned}
\|y\|^2 &= \|z + P(y)\|^2 \\
&= \|z\|^2 + \|P(y)\|^2 + 2\langle z, P(y)\rangle \\
&= \|z\|^2 + \|P(y)\|^2 \\
&\geq \|P(y)\|^2.
\end{aligned}$$

Thus $\|P(y)\| \leq \|y\|$.

(iii) Since $e = \frac{x}{\|x\|}$ and $P(y) = \langle y, e\rangle e$, we have

$$\|\langle y, \frac{x}{\|x\|}\rangle e\| = |\langle x, y\rangle| \frac{1}{\|x\|} \leq \|y\|.$$

Thus for any $x, y \in \mathbb{R}^d$ and $x \neq 0$,

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

If $x = 0$, we have $|\langle x, y\rangle| = 0 = \|x\|\|y\|$. Therefore for any $x, y \in \mathbb{R}^d$, $|\langle x, y\rangle| \leq \|x\|\|y\|$.

Exercise 2:

Set for $x = (x_1, x_2, \ldots, x_d)$ in $\mathbb{R}^d$,

$$\|x\|_1 = \sum_{i=1}^{d} |x_i|, \quad \|x\|_\infty = \max\{|x_i| : i = 1, \ldots, d\}.$$

Show that ρ_1 and ρ_∞ define two distances on $\mathbb{R}^d$, where

$$\rho_1(x, y) = \|x - y\|_1, \quad \rho_\infty(x, y) = \|x - y\|_\infty.$$

Solution:

Firstly we show that ρ_1 defines a distance on $\mathbb{R}^d$. Suppose that $x = (x_1, x_2, \ldots, x_d), y = (y_1, y_2, \ldots, y_d)$ and $z = (z_1, z_2, \ldots, z_d)$.

- $\rho_1(x, y) \geq 0$, for all $x, y \in \mathbb{R}^d$.
- $\rho_1(x, y) = \|x - y\|_1 = 0 \Leftrightarrow \sum_{i=1}^{d} |x_i - y_i| = 0 \Leftrightarrow x_i = y_i, i = 1, 2, \ldots, d \Leftrightarrow x = y$, for all $x, y \in \mathbb{R}^d$.
- $\rho_1(x, y) = \sum_{i=1}^{d} |x_i - y_i| = \sum_{i=1}^{d} |y_i - x_i| = \rho_1(y, x)$, for all $x, y \in \mathbb{R}^d$.
- For all $x, y, z \in \mathbb{R}^d$,

$$\begin{aligned}
\rho_1(x, y) + \rho_1(y, z) &= \sum_{i=1}^{d} |x_i - y_i| + \sum_{i=1}^{d} |y_i - z_i| \\
&= \sum_{i=1}^{d} (|x_i - y_i| + |y_i - z_i|) \\
&\geq \sum_{i=1}^{d} |x_i - z_i| \\
&= \rho_1(x, z).
\end{aligned}$$

Next we show that ρ_∞ defines a distance on $\mathbb{R}^d$. Suppose that $x = (x_1, x_2, \ldots, x_d), y = (y_1, y_2, \ldots, y_d)$ and $z = (z_1, z_2, \ldots, z_d)$.

- $\rho_\infty(x, y) \geq 0$, for all $x, y \in \mathbb{R}^d$.
- $\rho_\infty(x, y) = \|x - y\|_\infty = 0 \Leftrightarrow \max\{|x_i - y_i| : i = 1, 2, \ldots, d\} = 0 \Leftrightarrow |x_i - y_i| = 0, i = 1, 2, \ldots, d \Leftrightarrow x_i = y_i, i = 1, 2, \ldots, d \Leftrightarrow x = y$, for all $x, y \in \mathbb{R}^d$.
- $\rho_\infty(x, y) = \max\{|x_i - y_i| : i = 1, 2, \ldots, d\} = \max\{|y_i - x_i| : i = 1, 2, \ldots, d\} = \rho_\infty(y, x)$, for all $x, y \in \mathbb{R}^d$.

- For all $x, y, z \in \mathbb{R}^d$,

$$
\begin{aligned}
& \rho_\infty(x, y) + \rho_\infty(y, z) \\
= \;& \max\{|x_i - y_i| : i = 1, 2, \ldots, d\} + \max\{|y_i - z_i| : i = 1, 2, \ldots, d\} \\
\geq \;& \max\{|x_i - y_i| + |y_i - z_i| : i = 1, 2, \ldots, d\} \\
\geq \;& \max\{|x_i - z_i| : i = 1, 2, \ldots, d\} \\
= \;& \rho_\infty(x, z).
\end{aligned}
$$

Exercise 3:

Prove that if (X, ρ) is a metric space and we set for all x and y in X,

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)},$$

then d is a distance on X.

Solution:

We have following properties:

- $d(x, y) \geq 0$, for all $x, y \in X$.
- $d(x, y) = 0 \Leftrightarrow \rho(x, y) = 0 \Leftrightarrow x = y$, for all $x, y \in X$.
- Symmetric:
$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)} = \frac{\rho(y, x)}{1 + \rho(y, x)} = d(y, x).$$
- Triangle inequality: for all $x, y, z \in X$
$$\begin{aligned}
(x, y) + d(y, z) &= \frac{\rho(x, y)}{1 + \rho(x, y)} + \frac{\rho(y, z)}{1 + \rho(y, z)} \\
&= \frac{\rho(x, y) + \rho(y, z) + 2\rho(x, y)\rho(y, z)}{1 + \rho(x, y) + \rho(y, z) + \rho(x, y)\rho(y, z)} \\
&\geq \frac{\rho(x, y) + \rho(y, z) + \rho(x, y)\rho(y, z)}{1 + \rho(x, y) + \rho(y, z) + \rho(x, y)\rho(y, z)} \\
&\geq \frac{\rho(x, y) + \rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \\
&\geq \frac{\rho(x, z)}{1 + \rho(x, z)}
\end{aligned}$$

  since $f(x) = \frac{x}{1+x}$ is a monotone increasing function when $x > 0$.

Exercise 4:

Let x_n be a convergent sequence in the metric space (X, ρ). Show that the limit of x_n is unique.

Solution:

Suppose that x_n be a convergent sequence in the metric space (X, ρ) to x and y. Let $\epsilon > 0$ be given, there exists a $N \in \mathbb{N}$ such that

$$\rho(x_n, x) < \frac{\epsilon}{2}, \quad \forall n \geq N$$

and

$$\rho(x_n, y) < \frac{\epsilon}{2}, \quad \forall n \geq N.$$

Thus

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y) < \epsilon, \quad \forall n \geq N.$$

By the arbitrary of ϵ, we have $\rho(x, y) = 0$, thus $x = y$.

<u>Exercise 5:</u>

Let (X, ρ) be a metric space. Suppose that X is a finite set. Show that any subset of X is both open and closed.

Solution:

Suppose $X = \{x_1, x_2, \ldots, x_m\}$, where m is a finite constant. Let $\epsilon > 0$ be given. For each $x_i \in X$, $\{x_i\} \in B(x_i, \epsilon) \cap X$, where $B(x_i, \epsilon)$ is the open ball with the center x_i and radius ϵ, thus any singleton of X is closed. By taking finite union, any subset of X is closed.

Let $\delta = \min\{|x_i - x_j| : i, j = 1, 2, \ldots, m, i \neq j\}$. For each $x_i \in X$, $B(x_i, \delta) = \{x_i\} \subset X$. Thus any singleton of X is open. By taking finite union, any subset of X is open.

Exercise 6:

Let (X, ρ) be a metric space. Let A be a subset of X. Show that V is an open subset of A if and only if there is an open subset W of X such that $V = A \cap W$.

Solution:

Assume that V is an open subset of A. Then for some α_x,

$$V = \bigcup_{x \in V} B_A(x, \alpha_x) = \bigcup_{x \in V} \{y \in A : \rho(x, y) < \alpha_x\}.$$

We set $W = \bigcup_{x \in V} \{y \in X : \rho(x, y) < \alpha_x\}$, then W is an open subset of X such that $V = A \cap W$.

Conversely, assume that there is an open subset W of X such that $V = A \cap W$. Let $x \in V$, there exists $\alpha > 0$ such that $\{y \in X : \rho(x, y) < \alpha\} \subset W$. Thus

$$\{y \in A : \rho(x, y) < \alpha\} \subset W \cap A = V,$$

which implies that V is an open subset of A.