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**Exercise 1:**

Give an example of $f_n, f \in L^1(\mathbb{R})$ such that $f_n \to f$ uniformly, but $\|f_n\|_1$ does not converge to $\|f\|_1$.

**Solution:**

Example 1: We suppose that $f_n(x) = \frac{1}{n}\mathbb{I}_{[1,n]}(x)$ and $f(x) = 0$. Since

$$|f_n(x) - 0| = |\frac{1}{n}\mathbb{I}_{[1,n]}(x) - 0| \leq \frac{1}{n},$$

we know that $f_n \to f$ uniformly. As $\|f(x)\|_1 = 0$ and

$$\|f_n\|_1 = \int_{\mathbb{R}} |f_n(x)|\, dx = \int_1^n \frac{1}{n}\, dx = \frac{n-1}{n} \to 1$$

as $n \to \infty$. So we have $\|f_n\|_1$ does not converge to $\|f\|_1$.

Example 2: We set $f(x) = 0$ and

$$f_n(x) = \left( -\frac{1}{2^{2n}} + \frac{1}{2^n} \right) \cdot \mathbb{I}_{[0,2^{2n}]}(x).$$

Since $|f_n(x) - f(x)| < \frac{1}{2^n}$, we know that $f_n \to f$ uniformly. And as

$$\|f_n(x)\|_1 = \int_0^{2^{2n}} -\frac{1}{2^{2n}} + \frac{1}{2^n}\, dx = 2^n - 1 \to +\infty,$$

we have $\|f_n\|_1$ does not converge to $\|f\|_1$.

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**Exercise 2:**

Show that for all $\epsilon > 0$ and all $f \in L^1(\mathbb{R}), \exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$ for some $f_n$ with $|f_n| \leq n$ and $f_n = 0$ on $\mathbb{R} \setminus [-n, n]$.

**Solution:**

We suppose that

$$f_n(x) = f \cdot \mathbb{I}_{\{x:|f(x)|\leq n\}\cap\{x\in[-n,n]\}}(x),$$

so we know that $f_n = 0$ on $\mathbb{R} \setminus [-n, n]$ and $|f_n| \leq n$. Next we need to show that $\exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$. We know that

$$
\begin{aligned}
\|f_n - f\|_1 &= \int_{\mathbb{R}} |f_n - f|\, dx \\
&= \int_{\{|f|\geq n\}\cup\{x\in\mathbb{R}\setminus[-n,n]\}} |f|\, dx \\
&\leq \int_{\{|f|\geq n\}} |f|\, dx + \int_{-\infty}^{-n} |f|\, dx + \int_n^{+\infty} |f|\, dx.
\end{aligned}
$$

Since

$$\int_{\{|f|\geq n\}} |f|\, dx = \int_{\mathbb{R}} |f|\mathbb{I}_{\{|f|>n\}}(x)\, dx$$

and $|f|\mathbb{I}_{\{|f|>n\}}(x)$ goes to 0 pointwise and $|f|\mathbb{I}_{|f|>n}(x) < |f| \in L^1(\mathbb{R})$, by the dominate convergence theorem, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f|\mathbb{I}_{|f|>n}\, dx = \int_{\mathbb{R}} \lim_{n\to\infty} |f|\mathbb{I}_{|f|>n}(x)\, dx = 0.$$

Similarly, since

$$\int_{n}^{+\infty} |f|\, dx = \int_{\mathbb{R}} |f|\mathbb{I}_{[n,+\infty)}(x)\, dx,$$

and $|f|\mathbb{I}_{[n,+\infty)}(x) \to 0$ as $n \to \infty$ pointwise and $|f|\mathbb{I}_{[n,+\infty)}(x) \leq |f| \in L^1(\mathbb{R})$, by the dominate convergence theorem, we can get

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f|\mathbb{I}_{[n,+\infty)}(x)\, dx = \int_{\mathbb{R}} \lim_{n\to\infty} |f|\mathbb{I}_{[n,+\infty)}(x)\, dx = 0.$$

Then we also can get

$$\lim_{n\to\infty} \int_{-\infty}^{-n} |f|\, dx = 0.$$

Thus we have

$$\lim_{n\to\infty} \|f_n - f\|_1 \leq \lim_{n\to\infty} \left( \int_{\{|f|\geq n\}} |f|\, dx + \int_{-\infty}^{-n} |f|\, dx + \int_{n}^{+\infty} |f|\, dx \right) = 0,$$

hence we know that $\exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$.

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**Exercise 3:**

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(i) If $f$ is in $L^1(X) \cap L^\infty(X)$, show that $|f|^p \in L^1(X)$ for all $p$ in $(1, \infty)$.

(ii) If $f$ is in $L^1(X) \cap L^\infty(X)$, show that

$$\lim_{p\to\infty} \left( \int |f|^p \right)^{\frac{1}{p}} = \|f\|_\infty.$$

(iii) Set $A = \{x \in X : |f(x)| > 0\}$. If $f$ is in $L^\infty(X), \mu(A) < \infty$, and $\mu(A) \neq 1$, find

$$\lim_{p\to 0^+} \left( \int |f|^p \right)^{\frac{1}{p}}.$$

(iv) We now assume that the set $A$ defined in (iii) satisfies $\mu(A) = 1$, that $f$ is in $L^\infty(X)$, and $\ln|f|$ is in $L^1(X)$, find

$$\lim_{p\to 0^+} \left( \int |f|^p \right)^{\frac{1}{p}}.$$

**Solution:**

(i) We need to show $|f|^p \in L^1(X)$, so we just need to show that for any $p \in (1, \infty)$, $f \in L^p(X)$. For any $p \in (1, \infty)$, since $f \in L^1(X) \cap L^\infty(X)$, we have

$$
\begin{aligned}
\|f\|_p &= \left( \int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \\
&= \left( \int_X |f| |f|^{p-1} \, d\mu \right)^{\frac{1}{p}} \\
&\leq (\|f\|_\infty)^{\frac{p-1}{p}} (\|f\|_1)^{\frac{1}{p}} < \infty,
\end{aligned}
$$

thus we know that $f \in L^p(X)$. So, we know that $|f|^p \in L^1(X)$ for all $p$ in $(1, \infty)$.

(ii) We denote $t \in [0, \|f\|_\infty)$, then the set

$$
A = \{ x \in X : |f(x)| \geq t \}
$$

has positive and bounded measure. Since

$$
\begin{aligned}
\|f\|_p &= \left( \int_{(0,1)} |f|^p \, d\mu \right)^{\frac{1}{p}} \geq \left( \int_A |f|^p \, d\mu \right)^{\frac{1}{p}} \\
&\geq \left( t^p \mu(A) \right)^{\frac{1}{p}} = t(\mu(A))^{\frac{1}{p}},
\end{aligned}
$$

if $\mu(A)$ is finite, then when $p \to +\infty$, we have $(\mu(A))^{\frac{1}{p}} \to 1$ and if $\mu(A) = \infty$, then $(\mu(A)^{\frac{1}{p}}) = \infty$, in both cases we have

$$
\liminf_{p \to +\infty} \|f\|_p \geq t.
$$

Since $t$ is arbitrary and $t \in [0, \|f\|_\infty)$, we have

$$
\liminf_{p \to +\infty} \|f\|_p \geq \|f\|_\infty.
$$

On the other hand, as $f(x)$ is in $L^1(X)$, we have

$$
\begin{aligned}
\|f\|_p &= \left( \int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \\
&= \left( \int_X |f| |f|^{p-1} \, d\mu \right)^{\frac{1}{p}} \\
&\leq (\|f\|_\infty)^{\frac{p-1}{p}} (\|f\|_1)^{\frac{1}{p}}.
\end{aligned}
$$

Since $\|f\|_1 < +\infty$, then when $p \to +\infty$, we know that

$$
\limsup_{p \to +\infty} \|f\|_p \leq \|f\|_\infty.
$$

Thus we have

$$
\limsup_{p \to +\infty} \|f\|_p \leq \|f\|_\infty \leq \liminf_{p \to +\infty} \|f\|_p,
$$

then we know that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

(iii) When $\mu(A) < 1$, we have

$$\int_X |f|^p \, d\mu = \int_A |f|^p \, d\mu$$
$$\leq \|f\|_\infty^p \mu(A).$$

Since $f \in L^\infty(X)$ and $\mu(A) < 1$, we know that

$$\lim_{p \to 0^+} \left( \int |f|^p \right)^{\frac{1}{p}} \leq \lim_{p \to 0^+} \|f\|_\infty (\mu(A))^{\frac{1}{p}} = 0$$

But if we set $f = 1$ and $\mu(X) < \infty$, we know that $f \in L^\infty(X)$, if $\mu(A) > 1$, we have

$$\lim_{p \to 0^+} \left( \int |f|^p \right)^{\frac{1}{p}} = \lim_{p \to 0^+} (\mu(A))^{\frac{1}{p}} = \infty.$$

Thus the limit is not exist.

(iv) Since we have $A = \{x \in X : |f| > 0\}$, then

$$\int_X |f|^p \, d\mu = \int_{\{x \in X : |f| > 0\}} |f|^p \, d\mu + \int_{\{x \in X : |f| = 0\}} |f|^p \, d\mu$$
$$= \int_A |f|^p \, d\mu.$$

And we denote that $F(p) = \log(\int_A |f|^p \, d\mu)$, then we know that

$$\lim_{p \to 0^+} \left( \int |f|^p \right)^{\frac{1}{p}} = \lim_{p \to 0^+} e^{\frac{F(p)}{p}}.$$

As $F(0) = \log(\mu(A)) = 0$ and $e^x$ is continuous, then we have

$$\lim_{p \to 0^+} \left( \int |f|^p \right)^{\frac{1}{p}} = \lim_{p \to 0^+} \exp\left\{ \frac{F(p) - F(0)}{p - 0} \right\}$$
$$= \exp\left\{ \lim_{p \to 0^+} \frac{F(p) - F(0)}{p - 0} \right\}$$
$$= e^{F'(0)}.$$

As $F(p) = \log(\int_A |f|^p \, d\mu)$ and $\ln|f|$ is in $L^1(X)$, we have

$$F'(p) = \frac{\int_A |f|^p \cdot \log|f| \, d\mu}{\int_A |f|^p \, d\mu},$$

thus we have $F'(0) = \frac{\int_A \log|f| \, d\mu}{\mu(A)} = \int_A \log|f| \, d\mu$ . Then we know that

$$\lim_{p \to 0^+} \left( \int f^p \right)^{\frac{1}{p}} = e^{F'(0)}$$
$$= \exp(\int_A \log|f| \, d\mu).$$