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<u>Exercise 1:</u>

Let X and Y be two metric spaces and f a mapping from X to Y.

(i) Show that f is continuous if and only if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.

(ii) Prove or disprove: assume that f is injective. Then f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.

(iii) Prove or disprove: assume that X is compact. Then f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.

Solution:

(i) Firstly, we want to prove that if f is continuous, then for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$. Since $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is closed as f is continuous, where $f^{-1}(\overline{f(A)})$ is the inverse image of $\overline{f(A)}$. Since $A \subset f^{-1}(f(A))$, we have $A \subset f^{-1}(\overline{f(A)})$. Since the closure of A is contained in any closed set containing A, we can get that $\overline{A} \subset f^{-1}(\overline{f(A)})$. Thus we know that for any $x \in \overline{A}$, we have $f(x) \in \overline{f(A)}$, then we have $f(\overline{A}) \subset \overline{f(A)}$.

Secondly, we want to show that if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$, then f is continuous. To verify that f is continuous, we just need to show that for any closed set $C \subset Y$, the inverse image of the C under the function f is also a closed set. We denote $D = f^{-1}(C)$, then we need to show D is closed in X. Since $f(\overline{D}) \subset \overline{f(D)} = \overline{f(f^{-1}(C))} = \overline{C} = C$, we know that $f(\overline{D}) \subset C$. Thus we have $\overline{D} \subset f^{-1}(C) = D$, then we know that D is a closed subset of X. So, f is continuous.

(ii) The statement is not true. We can give a counter example as following. We suppose $X = \mathbb{R}^+, Y = \mathbb{R}$ and $\forall x \in X, f(x) = \frac{1}{x}$. Then $f(x)$ is continuous in X. Let $A = [1, +\infty)$, and we have $A \subset X$. So, $\overline{A} = [1, +\infty) = A$, and we know that $f(\overline{A}) = (0, 1]$. Since $f(A) = (0, 1]$, we have $\overline{f(A)} = [0, 1]$. Thus $f(\overline{A}) \subsetneqq \overline{f(A)}$, and we can not say $f(\overline{A}) = \overline{f(A)}$.

Another counter example is as follows. Let $X = [0, \infty), Y = \mathbb{R}$, and let $f(x) = e^{-x}$, $A = [0, \infty)$. f is a continuous function on X and f is injective. $A = \bar{A} = [0, \infty)$, thus $f(A) = f(\bar{A}) = (0, 1]$, which is not a closed subset of Y. And there are plenty of examples can show that $f(\overline{A}) \subsetneqq \overline{f(\overline{A})}$.

Note that the continuous function do not necessarily mapping closed sets to closed sets. This is the key point of (ii).

(iii) From the question (i), we know that if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$, we have f is continuous. Then, if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$, we have f is continuous.

Next we want to show if f is continuous, then for every subset A of X, $f(\overline{A}) = \overline{f(A)}$. By the result we get from question (i), we know that if f is continuous, then for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$. We just need to verify $\overline{f(A)} \subset f(\overline{A})$. Since $A \subset \overline{A}$, then $f(A) \subset f(\overline{A})$ and $\overline{f(A)} \subset \overline{f(\overline{A})}$. As $A \subset X$ and X is compact, then $\overline{A}$ is compact. As f is continuous, we have $\overline{f(\overline{A})} = f(\overline{A})$. So we can get $\overline{f(A)} \subset f(\overline{A})$. In summary, when f is continuous, we have $f(\overline{A}) \subset \overline{f(A)}$ and $\overline{f(A)} \subset f(\overline{A})$. Thus if f is continuous, for every subset A of X, we have $f(\overline{A}) = \overline{f(A)}$.

Note that continuous image of compact sets are compact, that why $\overline{f(\overline{A})} = f(\overline{A})$. In (ii), we only can get the conclusion that $f(\overline{A}) \subsetneqq \overline{f(A)}$.

Exercise 2:

Let $K \subset \mathbb{R}$ have finite measure and let $f \in L^\infty(\mathbb{R})$. Show that the function F defined by

$$F(x) := \int_K f(x+t)\,dt$$

is uniformly continuous on $\mathbb{R}$.

Solution:

We want to show that $\forall \epsilon > 0$, there exists a $\delta > 0$, such that when $|x-y| < \delta$, we have $|F(x) - F(y)| < \epsilon$. We verify the result by definition. Since

$$|F(x) - F(y)| = \Big| \int_K f(x+t)\,dt - \int_K f(y+t)\,dt \Big|,$$

by changing the variable and denoting $K_1 = \{k + x | k \in K\}$ and $K_2 = \{k + y | k \in K\}$, we have

$$|F(x) - F(y)| = \Big| \int_{K_1} f(t)\,dt - \int_{K_2} f(t)\,dt \Big|.$$

Denote $\operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| = C$. Since $f \in L^\infty(\mathbb{R})$, then $\forall \epsilon > 0$, there exist a positive number M such that

$$\int_{K_1 \bigcap [-M,M]^c} |f(t)|\,dt < \epsilon.$$

Otherwise, $\exists \epsilon > 0$, and $\forall M > 0$, we have $\int_{K_1 \bigcap [-M,M]^c} |f(t)|\,dt \geq \epsilon$. We set $M \to +\infty$, then $\int_{K_1 \bigcap [-M,M]^c} f(t)\,dt < C\mu\{K_1 \bigcap [-M,M]^c\} \to 0$. It is contradiction. So, for all $\epsilon > 0$, there exist a M, such that

$$\begin{aligned}
|F(x) - F(y)| &= \Big| \int_{K_1} f(t)\,dt - \int_{K_2} f(t)\,dt \Big| \\
&= \Big| \int_{K_1 \bigcap [-M,M]} f(t)\,dt + \int_{K_1 \bigcap [-M,M]^c} f(t)\,dt \\
&\quad - \int_{K_2 \bigcap [-M,M]} f(t)\,dt - \int_{K_2 \bigcap [-M,M]^c} f(t)\,dt \Big| \\
&\leq \Big| \int_{K_1 \bigcap [-M,M]} f(t)\,dt - \int_{K_2 \bigcap [-M,M]} f(t)\,dt \Big| + 2\epsilon.
\end{aligned}$$

We denote $S = (K_1 \bigcap [-M,M]) \Delta (K_2 \bigcap [-M,M])$, then we have

$$|F(x) - F(y)| \leq \int_S |f(t)|\,dt + 2\epsilon \leq C\mu\{S\} + 2\epsilon.$$

As $K_1 \bigcap [-M,M]$ and $K_2 \bigcap [-M,M]$ are finite, and $K_1 = \{k + x | k \in K\}$, $K_2 = \{k + y | k \in K\}$, we can cover the set S by several open sets whose measure is $|y - x|$, then we have

$$|F(x) - F(y)| \leq Cm|y - x| + 2\epsilon,$$

where C is a positive number. We set $\delta = \frac{\epsilon}{Cm}$, then we have

$$|F(x) - F(y)| \leq 3\epsilon,$$

so, $F(x)$ is uniformly continuous on $\mathbb{R}$.

<u>Exercise 3:</u>

Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ such that $f_n \to 0$ a.e.

(i) Show that if $\{f_{2n}\}$ is increasing and $\{f_{2n+1}\}$ is decreasing, then

$$\int f_n \to 0.$$

(ii) Prove or disprove: if $\{f_{kn}\}$ is decreasing for every prime number k, then

$$\int f_n \to 0.$$

(Note on notation: e.g., if $k = 2$, then $\{f_{kn}\} = \{f_{2n}\}$. Note also that 1 is not prime).

Solution:

(i) Firstly, we consider the sequence $\{f_{2n} - f_2\}$. Since $\{f_{2n}\}$ is increasing, $f_{2n} \to 0$ and $\{f_n\} \in L^1(\mathbb{R})$ for all n, then $\{f_{2n} - f_2\}$ is increasing and $f_{2n} - f_2 \to -f_2$ a.e., then by the monotone convergence theorem, we have

$$\lim_{n\to+\infty} \int (f_{2n} - f_2) = \int \lim_{n\to+\infty} (f_{2n} - f_2) = \int -f_2,$$

thus

$$\lim_{n\to+\infty} \int f_{2n} = 0.$$

Similarly, as $\{f_{2n+1}\}$ is decreasing, we know that $\{f_1 - f_{2n-1}\}$ is an increasing sequence and $f_1 - f_{2n-1} \to f_1$ a.e., by the monotone convergence theorem, we have

$$\lim_{n\to+\infty} \int (f_1 - f_{2n-1}) = \int \lim_{n\to+\infty} (f_1 - f_{2n-1}) = \int f_1,$$

thus

$$\lim_{n\to+\infty} \int f_{2n-1} = 0.$$

Then we show that any subsequence of $\{\int f_n\}$, which denoted as $\{\int f_{n_k}\}$, has a convergent subsequence $\{f_{n_{k_l}}\}$ as follows

$$\lim_{n\to+\infty} \int f_{n_{k_l}} = 0.$$

For the subsequence $\{\int f_{n_k}\}$, we take the even number in the indicator set n_k if it is infinite, or we can take the odd number in the indicator set n_k if it is infinite, then we can get the subsequence of $\{\int f_{n_k}\}$, which is denoted as $\{\int f_{n_{k_l}}\}$. Since we have showed that $\lim_{n\to+\infty} \int f_{2n} = 0$ and $\lim_{n\to+\infty} \int f_{2n-1} = 0$, then we know that $\lim_{n\to+\infty} \int f_{n_{k_l}} = 0$. Thus all subsequence of $\{\int f_n\}$, which denoted as $\{\int f_{n_k}\}$, converges to 0. Therefore we know that

$$\int f_n \to 0.$$

(ii) The statement is not true. We can find a counter example as follows. Let

$$f_p(x) = p\,\mathbb{I}_{[0,\frac{1}{p}]}(x),$$

where p is a prime number and

$$f_m(x) = 2\,\mathbb{I}_{[0,\frac{1}{m}]}(x),$$

where m is a not prime number. Then we know that $\{f_{np}\}$ is decreasing for every prime number p. But we can find a subsequence of $\{f_n\}$, which is denoted as $\{f_p\}$, p is the prime number, and $\lim_{n\to+\infty}\int f_p \neq 0$ as

$$\lim_{p\to+\infty}\int f_p = \lim_{p\to+\infty}\int p\,\mathbb{I}_{[0,\frac{1}{p}]}(x)\,dx = 1.$$