MA - 503 Midterm - Fall 2019
NO DOCUMENTS ALLOWED

Exercise 1

Let X be a metric space, and A be closed and non empty in X.
a. Show that $d(x, A) = 0$ if and only if $x \in A$.
b. Let B be an open set such that $A \subset B$. Assume $B \neq X$. Show that for all $x \in X$, $d(x, A) + d(x, B^c) > 0$.
c. Show that the function $f : X \to \mathbb{R}$, $f(x) = \dfrac{d(x, B^c)}{d(x, A) + d(x, B^c)}$ is continuous. Find $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$.

Exercise 2

Let X be a metric space and x_n a convergent sequence in X. Denote l its limit. Let

$$K = \{x_n : n \geq 1\} \cup \{l\}.$$

Show that K is compact. **Hint**: use the open subcover property.

Exercise 3

We proved in class that for any measurable subset S of $\mathbb{R}$ and any $\epsilon > 0$, there is an open subset V of $\mathbb{R}$ such that $S \subset V$ and $m(V \setminus S) < \epsilon$. Infer that there is a closed subset F of $\mathbb{R}$ such that $F \subset S$ and $m(S \setminus F) < \epsilon$.

Exercise 4

Find (with proof) a function $f : [0, 1] \to \mathbb{R}$ which is differentiable but not Lipschitz continuous. **Hint**: you may use something like $x^a \sin \dfrac{1}{x}$.