GCE May, 2017

Jiamin JIAN

__Exercise 1:__

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let A_n be a sequence in $\mathcal{A}$ such that $\mu(A_n)$ converges to zero.

(i) Prove or disprove: if $f : X \to [0, +\infty)$ is a measurable function and $\mu(X) < +\infty$, then $\int_{A_n} f$ converges to zero.

(ii) Let g be in $L^1(X)$. Show that $\int_{A_n} g$ converges to zero.

Solution:

(i) The statement is not true. We suppose $X = (0, 1]$ and $f(x) = \frac{1}{x^2}$, then we know that $\mu(X) < +\infty$ and $f(x)$ is measurable on X. We set $A_n = [\frac{1}{n^2}, \frac{1}{n}], n \in \mathbb{N}$. Thus we have for all $n \in \mathbb{N}$, $A_n \subset X$. And

$$\mu(A_n) = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} \to 0$$

as n goes to infinity. But for the $\int_{A_n} f$, we have

$$\int_{A_n} f \, d\mu = \int_{\frac{1}{n^2}}^{\frac{1}{n}} \frac{1}{x^2} \, dx = n^2 - n \to +\infty$$

as $n \to +\infty$. So, we know that $\int_{A_n} f$ does not converges to zero.

Another counter example is as follows. Let $X = (0, 2]$, $A_n = [1/n, 2/n]$ for all $n \in \mathbb{N}$. Let $f(x) = \frac{1}{x}$. Then $\mu(X) = 2 < \infty$ and $\mu(A_n) = \frac{1}{n} \to 0$ as n goes to infinity. But

$$\int_{A_n} f = \int_{\frac{1}{n}}^{\frac{2}{n}} \frac{1}{n} = \ln 2, \quad \forall n \in \mathbb{N}.$$

Thus $\int_{A_n} f$ does not converges to 0.

(ii) We denote

$$g_n(x) = g(x)\mathbb{I}_{A_n}(x),$$

where $\mathbb{I}_{A_n}(\cdot)$ is a indicator function on A_n. Since A_n is a sequence in $\mathcal{A}$ such that $\mu(A_n) \to 0$ as $n \to +\infty$, then we know that $g_n(x)$ converges to 0 almost everywhere. As

$$|g_n(x)| = |g(x)\mathbb{I}_{A_n}(x)| \leq |g(x)|$$

and $g \in L^1(X)$, we know that g is a dominate function of g_n. By the dominate convergence theorem, we have

$$\lim_{n \to \infty} \int_X g_n(x) \, d\mu = \int_X 0 \, d\mu = 0,$$

thus we have

$$\lim_{n \to \infty} \int_X g_n(x) \, d\mu = \lim_{n \to \infty} \int_{A_n} g \, d\mu = 0.$$

So, we know that $\int_{A_n} g$ converges to zero.

<u>Exercise 2:</u>

Let (x, d) be a bounded metric space. For any non empty subset S of X and x in X we define:

$$d(x, S) = \inf\{d(x, s) : s \in S\}.$$

If A and B are two non empty subsets of X we define:

$$d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}.$$

(i) Prove or disprove: If $d_H(A, B) = 0$, are A and B necessarily equal?

(ii) Let $\mathcal{C}$ be the set of all non empty closed subsets of X. Show that d_H defines a metric on $\mathcal{C}$.

Solution:

(i) The statement is not true. By the definition of $d_H(A, B)$, since $d_H(A, B) = 0$, we have

$$\max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\} = 0,$$

then we have $\sup_{x \in A} d(x, B) = \sup_{x \in B} d(x, A) = 0$, so we know that $\forall x \in A, d(x, B) = 0$ and $\forall x \in B, d(x, A) = 0$. For any $x \in A$, since $d(x, B) = \inf\{d(x, y) : y \in B\} = 0$, we can find a sequence $\{y_n\}$, and for any $x \in A$ this sequence converges to x. So we have $B \subset \bar{A}$, where $\bar{A}$ is the closure of A. Similarly, we have $A \subset \bar{B}$.

We suppose $A = [0, 1)$ and $B = [0, 1]$, thus $A \neq B$. Since $A \subset B$, $\forall x \in A$, $\exists y \in B$ such that $x = y$ and $d(x, y) = 0$, we have $\sup_{x \in A} d(x, B) = 0$. On the other hand, when $x \in B$ and $x \in [0, 1)$, since $A = [0, 1)$, we know that for any $x \in [0, 1)$, there exists a $y \in A$ such that $x = y$ and then $d(x, y) = 0$. And when $x \in B$ and $x = \{1\}$, since $y \in A = [0, 1)$, we have $d(x, A) = \inf\{d(x, y) : y \in A\} = 0$. Thus it is also holds that $\sup_{x \in B} d(x, A) = 0$. Then we know that $d_H(A, B) = 0$ but $A \neq B$. So, A and B is not necessarily equal.

(ii) Since $\mathcal{C}$ is the set of all non empty closed subsets of X, for $A \in \mathcal{C}$ and $B \in \mathcal{C}$, A, B are both closed sets. Next we need to verify the definition of the metric.

(a) $d_H(A, B) \geq 0$: since (X, d) is a metric space, then $d(x, B) \geq 0$ and $d(x, A) \geq 0$, thus we have $d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\} \geq 0$.

(b) $d_H(A, B) = 0 \iff A = B$: if A = B, then we have $d(x, B) = 0$ for any $x \in A$ and $d(x, A) = 0$ for any $x \in B$, thus we know that $d_H(A, B) = 0$. If $d_H(A, B) = 0$, by the result we get from (i), we know that $A \subset \bar{B}$ and $B \subset \bar{A}$. Since A and B are both closed sets, then we have $A \subset B$ and $B \subset A$, thus we can get $A = B$.

(c) $d_H(A, B) = d_H(B, A)$: since $d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}$ and $d_H(B, A) = \max\{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\}$, thus we have $d_H(A, B) = d_H(B, A)$.

(d) For $A, B, C \in \mathcal{C}$, $d_H(A, B) \leq d_H(A, C) + d_H(C, B)$: since $d_H(A, C) + d_H(C, B) \geq \sup_{x \in A} d(x, C) + \sup_{x \in C} d(x, B)$, then we know that $d_H(A, C) + d_H(C, B) \geq \sup_{x \in A} d(x, B)$.

Similarly, we have $d_H(A,C) + d_H(C,B) \geq \sup_{x \in B} d(x,A)$, thus we can get $d_H(A,C) + d_H(C,B) \geq \max\{\sup_{x \in A} d(x,B), \sup_{x \in B} d(x,A)\} = d_H(A,B)$.

Exercise 3:

Let $(X, \mathcal{A}, \mu)$ be a measure space and $\{f_k\}$ a sequence in $L^p(X)$ where $1 \leq p \leq +\infty$. Suppose that $\{f_k\}$ converges in $L^p(X)$ to f. Show that f_k converges in measure to f on X.

Hint: According to the definition f convergence in measure, you need to show that for any positive ϵ, $\mu(\{x \in X : |f_k(x) - f(x)| \geq \epsilon\})$ converges to zero as k tends to infinity.

Solution:

When $p = +\infty$, since the sequence $\{f_k\}$ converges to f in $L^\infty(X)$, then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, when $n > N$, we have $\|f_n - f\|_\infty < \epsilon$. It means that $|f_n - f|$ is less than ϵ almost everywhere. Thus we have $\mu(|f_n - f| > \epsilon) = 0$ when $n \to \infty$. So we get that $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\})$ converges to zero as n tends to infinity.

When $1 \leq p < \infty$, for any $\epsilon > 0$, we have

$$\begin{aligned} \|f_n - f\|_p^p &= \int_X |f_n - f|^p \, d\mu \\ &\geq \int_{\{x \in X : |f_n - f|^p \geq \epsilon^p\}} |f_n - f|^p \, d\mu \\ &\geq \epsilon^p \mu(\{x \in X : |f_n - f|^p \geq \epsilon^p\}) \\ &= \epsilon^p \mu(\{x \in X : |f_n - f| \geq \epsilon\}), \end{aligned}$$

thus we know that

$$\mu(\{x \in X : |f_n - f| \geq \epsilon\}) \leq \frac{1}{\epsilon^p} \|f_n - f\|_p^p.$$

Since $\{f_n\}$ converges in $L^p(X)$ to f, we have $\|f_n - f\|_p^p \to 0$ as $n \to \infty$. So, for all $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\})$ converges to zero as n tends to infinity.

Exercise 4:

Suppose $g_n, g \in L^1(\mathbb{R})$, g_n converges to g almost everywhere, and $\int g_n$ converges to $\int g$. Define $f_n(x) := g_n(x+n)$.

(i) Prove or disprove: there exists an f in $L^1(\mathbb{R})$ such that f_n converges to f almost everywhere.

(ii) Prove or disprove: if there is an f as in (i), then $\int f_n$ converges to $\int f$.

Solution:

(i) The statement is not true. We suppose $g_n(x) = (x + \frac{1}{n})\mathbb{I}_{[0,1]}(x)$ and $g(x) = x\mathbb{I}_{[0,1]}(x)$, then we have

$$|g_n(x) - g(x)| = |(x + \frac{1}{n})\mathbb{I}_{[0,1]}(x) - x\mathbb{I}_{[0,1]}(x)| = \frac{1}{n} \to 0$$

when n tends to infinity. So, g_n converges to g almost everywhere. Since

$$\int_{\mathbb{R}} g_n(x)\,dx = \int_0^1 (x + \frac{1}{n})\,dx = \frac{1}{2} + \frac{1}{n} \to \frac{1}{2}$$

as $n \to +\infty$ and

$$\int_{\mathbb{R}} g(x)\,dx = \int_0^1 x\,dx = \frac{1}{2},$$

we know that $\int g_n$ converges to $\int g$. As $f_n(x) := g_n(x+n)$, then $f_n(x) = (x+n+\frac{1}{n})\mathbb{I}_{[0,1]}(x)$, it is diverges as $f_n(x) > n$ for any $x \in [0,1]$.

(ii) The statement is not true. We set $g_n(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ for all $n \in \mathbb{N}$ and $g(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ too. Since $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx = 1$, then we have $g_n(x) \in L^1(\mathbb{R})$ and $g(x) \in L^1(\mathbb{R})$. As $g_n(x) = g(x)$, we know that g_n converges to g almost everywhere. By the definition of $f_n(x)$, we know that $f_n(x) = g_n(x+n) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x+n)^2}{2}}$ and when $f(x) = 0$, for any fix $x \in \mathbb{R}$ we have,

$$|f_n(x) - f(x)| = |\frac{1}{\sqrt{2\pi}}e^{-\frac{(x+n)^2}{2}} - 0| \to 0$$

as $n \to +\infty$. So, we know that f is in $L^1(\mathbb{R})$ and f_n converges to f almost everywhere. But for any $n \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x+n)^2}{2}}\,dx = 1,$$

and we know that

$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} 0\,dx = 0,$$

thus $\int f_n$ does not converges to $\int f$.