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<u>**Exercise 1:**</u>

Let V be a normed vector space and S a subset of V. Let S^c be the complement of S. Let x be in S and y be in S^c. The line segment $[x, y]$ is by definition the set

$$\{(1-t)x + ty : t \in [0,1]\}.$$

Show that the intersection of $[x, y]$ and ∂S is non empty, where ∂S is the boundary of S (by definition the boundary of S is the set of points that are in the closure of S and that are not in the interior of S).

Solution:

<u>Method 1</u>: To show that $\{(1-t)x + ty : t \in [0,1]\} \bigcap \partial S \neq \emptyset$, we need to find a $t^* \in [0,1]$ such that $(1-t^*)x + t^* y \in \partial S$. Let

$$t^* = \sup\{t \in [0,1] : (1-t)x + ty \in S\}.$$

By the definition of t^*, there exists a sequence of $t_n \in [0,1]$ such that $(1-t_n)x + t_n y \in S$ and t_n converges to t^* in S. Thus $(1-t^*)x + t^* y$ is in the closure of S. Next we show that $(1-t^*)x + t^* y$ is not in the interior of S. Argue by contradiction. Suppose that $(1-t^*)x+t^*y$ is in the interior of S, then there exists a $\delta > 0$ such that $B((1-t^*)x + t^*y, \delta)$ is a subset of S, where $B((1-t^*)x + t^*y, \delta) = \{(1-t)x + ty : |t - t^*| < \delta, t \in [0,1]\}$. Thus we have $(1-(t^*+\delta/2))x + (t^*+\delta/2)y \in S$, which contradicts with the definition of t^*. Therefore $(1-t^*)x+t^*y$ is in the boundary S and then $(1-t^*)x+t^*y \in \{(1-t)x+ty : t \in [0,1]\} \bigcap \partial S$.

<u>Method 2</u>: We want to prove that the intersection of $[x, y]$ and ∂S is non empty, which implies that we need to find a $t^* \in [0,1]$, $\forall \delta > 0$, $B((1-t^*)x + t^*y, \delta) \bigcap S \neq \emptyset$ and $B((1-t^*)x + t^*y, \delta) \bigcap S^c \neq \emptyset$, where $B((1-t^*)x + t^*y, \delta) = \{(1-t)x + ty : |t - t^*| < \delta, t \in [0,1]\}$. Let

$$Z = \{t \in [0,1] : (1-t)x + ty \in S\},$$

and let $t^* = \sup Z$. And we denote $B((1-t^*)x + t^*y, \delta) = B_{t^*,\delta}$.

Firstly, we show that $B_{t^*,\delta} \bigcap S \neq \emptyset$. By the definition of t^*, $\forall \delta > 0, \exists \epsilon = \frac{\delta}{2}$, $\big(1-(t^*-\epsilon)x\big) + (t^*-\epsilon)y \in S$. And since $|t^*-\epsilon-t^*| = \epsilon < \delta$, we have $\big(1-(t^*-\epsilon)x\big)+(t^*-\epsilon)y \in B_{t^*,\delta}$, thus $B_{t^*,\delta} \bigcap S \neq \emptyset$.

Secondly, we need verify that $B_{t^*,\delta} \bigcap S^c \neq \emptyset$. Suppose $B_{t^*,\delta} \bigcap S^c = \emptyset$, then we have that $B_{t^*,\delta} \subset S$. By the definition of t^*, $\forall \delta > 0, \exists \epsilon = \frac{\delta}{2}$, $\big(1-(t^*+\epsilon)x\big) + (t^*+\epsilon)y \notin S$. And since $|t^*-\epsilon-t^*| = \epsilon < \delta$, we have $\big(1-(t^*+\epsilon)x\big) + (t^*+\epsilon)y \in B_{t^*,\delta}$. It is contradict with $B_{t^*,\delta} \subset S$, then we know that $B_{t^*,\delta} \bigcap S^c \neq \emptyset$.

Overall, we find $t^* \in [0,1]$, $(1-t^*)x + t^*y \in [x,y]$, and for all $\delta > 0$, we have $B_{t^*,\delta} \bigcap S \neq \emptyset$ and $B_{t^*,\delta} \bigcap S^c \neq \emptyset$, thus $(1-t^*)x + t^*y \in \partial S$. So, we conclude that the intersection of $[x,y]$ and ∂S is non empty.

<u>**Exercise 2:**</u>

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let g be a measurable function defined on X. Set

$$p_g(t) = \mu(x \in X : |g(x)| > t).$$

(i) If f is in $L^1(X)$ show that there is a constant $C > 0$ such that $p_f(t) \leq \frac{C}{t}$.

(ii) Find a measurable function h defined almost everywhere on $\mathbb{R}$ such that $\exists C > 0$, $p_h(t) \leq \frac{C}{t}$ and h is not in $L^1(\mathbb{R})$.

Solution:

(i) Since $f \in L^1(X)$, there exists $C > 0$ such that $\int_X |f| \, d\mu \leq C < +\infty$. We can decompose the integral as following:

$$\begin{aligned}
\int_X |f| \, d\mu &= \int_X |f| \mathbb{I}_{\{|f|>t\}} \, d\mu + |f| \mathbb{I}_{\{|f|\leq t\}} \, d\mu \\
&= \int_X |f| \mathbb{I}_{\{|f|>t\}} \, d\mu + \int_X |f| \mathbb{I}_{\{|f|\leq t\}} \, d\mu \\
&\geq \int_X |f| \mathbb{I}_{\{|f|>t\}} \, d\mu \\
&\geq t \int_X \mathbb{I}_{\{|f|>t\}} \, d\mu \\
&= t p_f(t).
\end{aligned}$$

Then we have $tp_f(t) \leq C$, thus $p_f(t) \leq \frac{C}{t}$. Or directly

$$\begin{aligned}
\int_X |f| \, d\mu &\geq \int_{\{x \in X : |f(x)|>t\}} |f| \, d\mu \\
&\geq t \int_{\{x \in X : |f(x)|>t\}} d\mu \\
&= t p_f(t).
\end{aligned}$$

(ii) We suppose that

$$h(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{|x|}, & x \neq 0, \end{cases}$$

then $h(x) \notin L^1(\mathbb{R})$ as

$$\int_{\mathbb{R}} |h(x)| \, dx = 2 \int_0^\infty \frac{1}{x} \, dx = \infty.$$

And note that

$$p_h(t) = \int_{\mathbb{R}} \mathbb{I}_{\{|h|>t\}} \, d\mu = \int_{\mathbb{R}} \mathbb{I}_{\{|x|<\frac{1}{t}\}} \, d\mu = \int_{\{|x|<\frac{1}{t}\}} d\mu = \frac{2}{t},$$

hence choose $C = 2$ then we have $p_h(t) \leq \frac{C}{t}$.

<u>Exercise 3:</u>

Let $\{f_n\} : [0,1] \mapsto [0,\infty)$ be a sequence of functions, each of which is non-decreasing on the interval $[0,1]$. Suppose the sequence is uniformly bounded in $L^2([0,1])$. Show that there exists a sub sequence that converges in $L^1([0,1])$.

Solution:

Since f_n is non-decreasing, then for $x \in [0,1]$, we have $\int_x^1 f_n(y)\,dy \geq (1-x)f_n(x)$. On the other hand, since the sequence is uniformly bounded in $L^2([0,1])$, we have $\forall n \in \mathbb{N}$, $\exists C > 0$, and $\|f_n\|_2 \leq C$. And then we have

$$\begin{aligned}
\int_x^1 f_n(y)\,dy &= \int_0^1 f_n(y)\mathbb{I}_{[x,1]}(y)\,dy \\
&\leq \Big(\int_0^1 f_n^2(y)\,dy\Big)^{\frac{1}{2}}\Big(\int_0^1 \mathbb{I}_{[x,1]}^2(y)\,dy\Big)^{\frac{1}{2}} \\
&\leq C(1-x)^{\frac{1}{2}}.
\end{aligned}$$

Such that we have $(1-x)f_n(x) \leq C(1-x)^{\frac{1}{2}}$, then $f_n(x) \leq C(1-x)^{-\frac{1}{2}}$. Until now we find a type of function $f(x) = C(1-x)^{-\frac{1}{2}}$ that can control the sequence f_n, where C is from the bound of f_n in the $L^2([0,1])$.

Exercise 4:

Consider the sequence of functions $f_n : [0,1] \mapsto \mathbb{R}$ where $f_1(x) = \sqrt{x}, f_2(x) = \sqrt{x+\sqrt{x}}, f_3(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$, and in general $f_n(x) = \sqrt{x+\sqrt{x+\sqrt{\cdots+\sqrt{x}}}}$ with n roots.

(i) Show that this sequence converges pointwise on $[0,1]$ and find the limit function f such that $f_n \to f$.

(ii) Does this sequence converge uniformly on $[0,1]$? Prove or disprove uniform convergence.

Solution:

(i) Firstly, we show that the sequence $f_n(x)$ is non-decreasing for the fixed x. We use the mathematical induction. For the fixed $x \in [0,1]$, when $k = 1$, since $f_k(x) = \sqrt{x}$ and $f_{k+1}(x) = \sqrt{x+\sqrt{x}}$, we have $f_k(x) \leq f_{k+1}(x)$. We suppose when $k = n-1$, the formula $f_k(x) \leq f_{k+1}(x)$ holds. When $k = n$, since $f_n(x) = \sqrt{x + f_{n-1}(x)}$ and $f_{n+1}(x) = \sqrt{x+f_n(x)}$, and since $f_{n-1}(x) \leq f_n(x)$, we have $\sqrt{x+f_{n-1}(x)} \leq \sqrt{x+f_n(x)}$, thus $f_n(x) \leq f_{n+1}(x)$. Thus by induction the sequence $f_n(x)$ is non-decreasing for the fixed $x \in [0,1]$.

Next we show that the function sequence f_n is uniformly bounded on $[0,1]$. We also use the mathematical induction. When $k = 1$, $f_k(x) = \sqrt{x} < 2$. We suppose that when $k = n-1$, we have $f_k(x) < 2$. When $k = n$, $f_n(x) = \sqrt{f_{n-1}(x) + x} < \sqrt{2+1} < 2$. Thus the sequence $f_n(x)$ is uniformly bounded by 2 for $x \in [0,1]$.

Since the sequence $f_n(x)$ is non-decreasing for the fixed x, and the sequence $f_n(x)$ uniformly bounded by 2, by monotone convergence theorem, the sequence f_n converges pointwise to some function f on $[0,1]$. When $x \in (0,1]$, since $f_{n+1}(x) = \sqrt{x+f_n(x)}$, when $n \to \infty$, we have $f(x) = \sqrt{x+f(x)}$. So we can get $f^2(x) - f(x) - x = 0$, thus $f(x) = \frac{1+\sqrt{1+4x}}{2}$ as $f(x) \geq 0$. When $x = 0$, we have $f_n(x) = 0, \forall n$. Thus we can get

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1+\sqrt{1+4x}}{2}, & x \in (0,1]. \end{cases}$$

and f_n converges pointwise to f on $[0,1]$.

(ii) For all $n \in \mathbb{N}$, f_n is continuous on $[0,1]$, but $f(x)$ we get in (i) is not a continuous function on $[0,1]$, thus the sequence f_n is not converge uniformly to f on $[0,1]$.

Exercise 5:

S is a normed space, and we define $B_1 = \{x \in S : \|x\| \leq 1\}$. Prove or disprove: B_1 is compact.

Solution:

The B_1 is not compact, we can give a counter example. We consider $S = l^2$ and $B_1 = \{x \in l^2 : \|x\| = 1\}$.

Firstly, we can show that B_1 is bounded and closed. By the definition of B_1, we know that B_1 is bounded by 1. $\forall x, y \in B_1$, since $\|x\| \leq \|x - y\| + \|y\|$ and $\|y\| \leq \|x - y\| + \|x\|$, we have $|\|x\| - \|y\|| \leq \|x - y\|$, thus the norm is continuous from l^2 to $\mathbb{R}$. Since the image set $\{1\}$ is closed, then we know the inverse image of $\{1\}$ is also closed, which is actually B_1. Thus B_1 is bounded and closed.

Next, we verify that there exists $\epsilon > 0$ such that B_1 cannot be covered by finitely many balls with radius ϵ. We define e_i as follows:

$$e_{i,m} = \begin{cases} 1, & m = i \\ 0, & m \neq i \end{cases},$$

thus $e_i \in B_1$. Clearly, $\forall i, j$, if $i \neq j$, we have $\|e_i - e_j\| = \sqrt{2}$. Choose $\epsilon = \sqrt{2}/2$, suppose B_1 can be covered by the finite many balls with radius $\sqrt{2}/2$. Since the number of the elements in B_1 is infinity, at least one of such ball contains at least e_j and e_k with $j \neq k$. Let x be the center of this ball, then we have $\|e_j - e_k\| \leq \|e_j - x\| + \|e_k - x\| < \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2}$. It contradicts with the fact that $\forall k, j$, if $k \neq j$, $\|e_i - e_j\| = \sqrt{2}$. Hence $\exists \epsilon = \sqrt{2}/2$, B_1 cannot be covered by finitely many balls with radius ϵ. Thus B_1 is not compact.