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Exercise 1:

A real-valued function f is increasing on a closed interval $[a, b] \subset \mathbb{R}$ if and only if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$.

(i) Using the definition of measurable, show that f is measurable on $[a, b]$.

(ii) Show that f is continuous, except perhaps a countable number of points.

Solution:

(i) For any $c \in \mathbb{R}$, we denote $S = f^{-1}([c, +\infty])$, by the definition of S, we have $S = \{x \in [a, b] | f(x) \geq c\}$. For any $x \in S$, if $y > x$ and $y \in [a, b]$, as f is increasing, we have $f(y) \geq f(x) \geq c$, thus $y \in S$. It is equivalent to that if $x \in S$, for any $y \in [a, b]$ and $y \geq x$, we have $y \in S$. This implies S can only be $\emptyset$, $[a, b]$, $(a, b]$, $[\inf S, b]$ and $(\inf S, b]$, all of the sets are measurable, thus f is measurable.

(ii) Let $f(x^-)$ and $f(x^+)$ denote the left and the right hand limits of f respectively. Let A be the set of points where f is not continuous. Then for any $x \in A \subset [a, b]$, we can find a rational number $f^*(x) \in \mathbb{Q}$, such that $f(x^-) < f(x^*) < f(x^+)$. Since f is increasing function, then for $x_1, x_2 \in A$ and $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$, also we have $f(x_1^+) \leq f(x_2^-)$. Thus we have $f(x_1^*) < f(x_1^+) \leq f(x_2^-) < f(x_2^*)$, then we know that $f(x_1^*) < f(x_2^*)$. Then there exists a injection between A and a subsets of rational number $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, then we know that A is also countable. Thus f is continuous except perhaps a countable number of points.

Exercise 2:

If f is Lebesgue integrable on $\mathbb{R}$, define

$$F(x) = \int_0^x f \, d\mu$$

where $\mu(E)$ is the Lebesgue measurable set $E \subset \mathbb{R}$. Show that

(i) F is continuous.

(ii) If $\mu(E) = 0$, then $\mu(F(E)) = 0$.

Solution:

(i) Let $\{x_n\}$ is a sequence in $\mathbb{R}$ and $x_n \to x_0$ as n goes to infinity. To show F is continuous, we need to show that $F(x_n)$ converges to $F(x_0)$, i.e.

$$\lim_{n\to+\infty} \int_0^{x_n} f \, d\mu = \int_0^{x_0} f \, d\mu.$$

Since we have

$$\lim_{n\to+\infty} \int_0^{x_n} f \, d\mu = \lim_{n\to+\infty} \int_0^{\infty} f \, \mathbb{I}_{[0,x_n]}(x) \, d\mu$$

and

$$|f \, \mathbb{I}_{[0,x_n]}(x)| \leq |f| \in L^1(\mathbb{R}),$$

by the dominate convergence theorem, we have

$$\lim_{n\to+\infty} \int_0^{\infty} f \, \mathbb{I}_{[0,x_n]}(x) \, d\mu = \int_0^{\infty} \lim_{n\to+\infty} f \, \mathbb{I}_{[0,x_n]}(x) \, d\mu.$$

Next we need to show that

$$\lim_{n\to+\infty} \mathbb{I}_{[0,x_n]}(x) = \mathbb{I}_{[0,x_0]}(x).$$

If $x_n \to x_0$, then for any $0 < t < x_0$, there exists a $N_1 \in \mathbb{N}$, such that $t < x_n$ for any $n > N_1$, and hence we have $\mathbb{I}_{[0,x_n]}(t) = 1$ for all $n > N_1$. Similarly, for $t > x_0$, there exists a $N_2 \in \mathbb{N}$ such that $\mathbb{I}_{[0,x_n]}(t) = 0$ for all $n > N_2$. Since $\{x_0\}$ is a singleton, it has zero measure. Thus let $\epsilon > 0$ be given, for any $x \in \mathbb{R}$, there exists a $N = \max\{N_1, N_2\} \in \mathbb{N}$ such that $|\mathbb{I}_{[0,x_n]}(x) - \mathbb{I}_{[0,x_0]}(x)| = 0$, thus

$$\lim_{n\to+\infty} \mathbb{I}_{[0,x_n]}(x) = \mathbb{I}_{[0,x_0]}(x) \, a.e.$$

Then we have

$$\lim_{n\to+\infty} \int_0^{\infty} f \, \mathbb{I}_{[0,x_n]}(x) \, d\mu = \int_0^{\infty} f \, \mathbb{I}_{[0,x_0]}(x) \, d\mu = \int_0^{x_0} f \, d\mu,$$

from which we know F is continuous.

(ii) We need to show that the continuous image of a zero measure set is also a zero measure set. For $E \in \mathbb{R}$ and $\mu(E) = 0$, we can find a disjoint sequence E_n such that

$E \subset \cup_{n=1}^{\infty} E_n$ and for any $\epsilon > 0$ we have $\mu(\cup_{n=1}^{\infty} E_n) < \epsilon$. Since $F(E) \subset F(\cup_{n=1}^{\infty} E_n)$, we know that

$$\mu(F(E)) \leq \mu(F(\cup_{n=1}^{\infty} E_n)).$$

Since F is continuous, if f is lipchitz continuous or f is absolutely continuous, then there exists a constant $K > 0$ and we have $\mu(F(\cup_{n=1}^{\infty} E_n)) \leq K\mu(\cup_{n=1}^{\infty} E_n) < K\epsilon$. So, we know that $\mu(F(E)) = 0$.

Exercise 3:

Let f be in $L^1(\mathbb{R})$ such that $f \geq 0$ almost everywhere and $\int_{\mathbb{R}} f = 1$. Set $f_n(x) = nf(nx)$. Let g be in $L^\infty(\mathbb{R})$.

(i) Let x_0 be in $\mathbb{R}$. Assume that g is continuous at x_0. show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x_0 - y)g(y)\, dy = g(x_0).$$

(ii) If g is uniformly continuous, is this limit uniformly in x_0?

(iii) If h is in $L^1(\mathbb{R})$ show that the function in x

$$\int_{\mathbb{R}} f_n(x-y)h(y)\, dy$$

converges to h in $L^1(\mathbb{R})$.

Solution:

(i) We denote $z = x_0 - y$, so we have

$$\int_{\mathbb{R}} f_n(x_0 - y)g(y)\, dy = \int_{\mathbb{R}} f_n(z)g(x_0 - z)\, dz = \int_{\mathbb{R}} nf(nz)g(x_0 - z)\, dz,$$

and then we denote $u = nz$,

$$\int_{\mathbb{R}} nf(nz)g(x_0 - z)\, dz = \int_{\mathbb{R}} f(u)g(x_0 - \frac{u}{n})\, du.$$

Since $f \in L^1(\mathbb{R})$ and $g(x) \in L^\infty(\mathbb{R})$, there exists a $M > 0$ such that

$$|f(u)g(x_0 - \frac{u}{n})| \leq Mf(u) \in L^1(\mathbb{R}),$$

by the dominate convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x_0 - y)g(y)\, dy &= \lim_{n\to\infty} \int_{\mathbb{R}} f(u)g(x_0 - \frac{u}{n})\, du \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} f(u)g(x_0 - \frac{u}{n})\, du \\
&= \int_{\mathbb{R}} f(u)g(x_0)\, du \\
&= g(x_0)
\end{aligned}$$

as g is continuous at x_0.

(ii) We need to show that $\int_{\mathbb{R}} f_n(x-y)g(y)\, dy$ is uniformly converges to $g(x)$ when g is uniformly continuous on $\mathbb{R}$. By the definition of $f_n(x)$, we have

$$\int_{\mathbb{R}} f_n(x)\, dx = \int_{\mathbb{R}} nf(nx)\, dx = \int_{\mathbb{R}} f(nx)\, d(nx) = 1.$$

For any $x \in \mathbb{R}$,

$$\begin{aligned}\Big|\int_{\mathbb{R}} f_n(x-y)g(y)\,dy - g(x)\Big| &= \Big|\int_{\mathbb{R}} f_n(z)g(x-z)\,dz - g(x)\Big| \\ &= \Big|\int_{\mathbb{R}} f_n(z)g(x-z)\,dz - \int_{\mathbb{R}} f_n(z)g(x)\,dz\Big| \\ &\le \int_{\mathbb{R}} f_n(z)|g(x-z)-g(x)|\,dz \\ &= \int_{\mathbb{R}} nf(nz)|g(x-z)-g(x)|\,dz,\end{aligned}$$

we denote $u = nz$, then we have

$$\Big|\int_{\mathbb{R}} f_n(x-y)g(y)\,dy - g(x)\Big| \le \int_{\mathbb{R}} f(u)\Big|g\Big(x-\frac{u}{n}\Big) - g(x)\Big|\,du.$$

As $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, there exists a $M > 0$ such that

$$\Big|f(u)\Big(g(x-\frac{u}{n}) - g(x)\Big)\Big| \le 2Mf(u) \in L^1(\mathbb{R}),$$

by the dominate convergence theorem, we have

$$\lim_{n\to\infty}\Big|\int_{\mathbb{R}} f_n(x-y)g(y)\,dy - g(x)\Big| \le \int_{\mathbb{R}} \lim_{n\to\infty} f(u)\Big|g\Big(x-\frac{u}{n}\Big) - g(x)\Big|\,du.$$

Since g is uniformly continuous on $\mathbb{R}$, let $\epsilon > 0$ be given, there exists a $N \in \mathbb{N}$ such that for any $x \in \mathbb{R}$ and whenever $n > N$, we have $|g(x-\frac{u}{n}) - g(x)| < \epsilon$. So, for the above ϵ and N, for any $x \in \mathbb{R}$, when $n > N$ we have

$$\int_{\mathbb{R}} f(u)\Big|g\Big(x-\frac{u}{n}\Big) - g(x)\Big|\,du \le \int_{\mathbb{R}} f(u)\epsilon\,du = \epsilon$$

thus we know that $\int_{\mathbb{R}} f_n(x-y)g(y)\,dy$ is uniformly converges to $g(x)$.

(iii) As $h \in L^1(\mathbb{R})$ and $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, for any $\epsilon > 0$, there exists a function $g \in C_c(\mathbb{R})$, such that

$$\|g-h\|_1 < \epsilon.$$

We denote $\int_{\mathbb{R}} f_n(x-y)h(y)\,dy = h_n(x)$ and $\int_{\mathbb{R}} f_n(x-y)g(y)\,dy = g(x)$, then we have

$$\|h(x)-h_n(x)\|_1 \le \|h(x)-g(x)\| + \|g(x)-g_n(x)\| + \|g_n(x)-h_n(x)\|.$$

For the above ϵ, as $\|g-h\|_1 < \epsilon$ and by the result we get from (ii), $g_n(x)$ is uniformly converges to $g(x)$, we have $\|g_n(x)-g(x)\| < \epsilon$, then we have

$$\lim_{n\to\infty}\|h(x)-h_n(x)\|_1 = \lim_{n\to\infty}\|g_n(x)-h_n(x)\|.$$

Next we need to verify the term $\|g_n(x) - h_n(x)\|$, since

$$\begin{aligned}
\|g_n(x) - h_n(x)\| &= \Big\| \int_{\mathbb{R}} f_n(x-y)h(y)\,dy - \int_{\mathbb{R}} f_n(x-y)g(y)\,dy \Big\| \\
&= \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} f_n(x-y)(h(y)-g(y))\,dy \Big|\,dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} f_n(x-y)|h(y)-g(y)|\,dy\,dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) \Big| h\Big(x-\frac{u}{n}\Big) - g\Big(x-\frac{u}{n}\Big) \Big|\,du\,dx,
\end{aligned}$$

by Fubini's theorem, we have

$$\begin{aligned}
\|g_n(x) - h_n(x)\| &\leq \int_{\mathbb{R}} f(u) \int_{\mathbb{R}} \Big| h\Big(x-\frac{u}{n}\Big) - g\Big(x-\frac{u}{n}\Big) \Big|\,dx\,du \\
&= \int_{\mathbb{R}} f(u) \int_{\mathbb{R}} |h(y)-g(y)|\,dy\,du \\
&= \int_{\mathbb{R}} f(u)\|h-g\|_1\,du \\
&= \|h-g\|_1
\end{aligned}$$

as $\int_{\mathbb{R}} f = 1$. Thus we know that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|h(x) - h_n(x)\|_1 \leq \|h(x) - g(x)\| + \|g(x) - g_n(x)\| + \|g_n(x) - h_n(x)\| < 3\epsilon$$

for all $n \geq N$. Therefore

$$\lim_{n\to\infty} \|h(x) - h_n(x)\|_1 = 0,$$

which means $\int_{\mathbb{R}} f_n(x-y)h(y)\,dy$ converges to h in $L^1(\mathbb{R})$.