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<u>**Exercise 1:**</u>

Let h_n be a sequence of non-negative, borel measurable functions on the interval $(0,1)$ such that $h_n \to 0$ in $L^1((0,1))$.

(i) Show $\sqrt{h_n} \to 0$ in $L^1((0,1))$.

(ii) Given an example to show that h_n^2 need not converge to zero in $L^1((0,1))$.

(iii) If g_n is in $L^1(\mathbb{R})$ such that $|g_n|^{\frac{1}{2}}$ is in $L^1(\mathbb{R})$, and g_n converges to zero in $L^1(\mathbb{R})$ as n tends to infinity, does $|g_n|^{\frac{1}{2}}$ converges to zero in $L^1(\mathbb{R})$?

Solution:

(i) We want to show that $\int_0^1 |\sqrt{h_n} - 0|\, d\mu \to 0$ as $n \to \infty$. Since $h_n \to 0$ in $L^1((0,1))$ and by the Hölder inequality, we have

$$\begin{aligned}
\int_0^1 |\sqrt{h_n} - 0|\, d\mu &\le \Big(\int_0^1 |(\sqrt{h_n})^2|\, d\mu\Big)^{\frac{1}{2}} \Big(\int_0^1 1^2\, d\mu\Big)^{\frac{1}{2}} \\
&= \Big(\int_0^1 h_n\, d\mu\Big)^{\frac{1}{2}} \Big(\int_0^1 1\, d\mu\Big)^{\frac{1}{2}} \\
&= \Big(\int_0^1 |h_n - 0|\, d\mu\Big)^{\frac{1}{2}}.
\end{aligned}$$

So when n goes to infinity, we have $\int_0^1 |\sqrt{h_n} - 0|\, d\mu \to 0$. Thus we know that $\sqrt{h_n} \to 0$ in $L^1((0,1))$.

(ii) For $n \in \mathbb{N}$, let

$$h_n(x) = n^{\frac{3}{2}} x \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x).$$

Then we have

$$\int_0^1 n^{\frac{3}{2}} x \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x)\, dx = n^{\frac{3}{2}} \int_{\frac{1}{n^2}}^{\frac{1}{n}} x\, dx = \frac{1}{2}\Big(\frac{1}{\sqrt{n}} - \frac{1}{n^{\frac{5}{2}}}\Big),$$

when $n \to +\infty$, we get $\|h_n\|_1 \to 0$, so we know that $h_n \to 0$ in $L^1((0,1))$. But for the $h_n^2(x)$, we have

$$\int_0^1 n^3 x^2 \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x)\, dx = n^3 \int_{\frac{1}{n^2}}^{\frac{1}{n}} x^2\, dx = \frac{1}{3} n^3 \Big(\frac{1}{n^3} - \frac{1}{n^6}\Big) = \frac{1}{3} - \frac{1}{3n^3}.$$

When n tends to infinity, $\int_0^1 n^3 x^2 \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x)\, dx \to \frac{1}{3}$, which is not goes to 0. So, we know that $h_n^2(x)$ don't converge to zero in $L^1((0,1))$.

The counter example on above is hard and not elegant. For all $n \in \mathbb{N}$, let $h_n(x) = n 1_{(0,1/n^2)}$, then h_n be a sequence of non-negative, borel measurable functions on the interval $(0,1)$ and

$$\|h_n\|_1 = \int_0^1 n 1_{(0,\frac{1}{n^2})}(x)\, dx = n \frac{1}{n^2} = \frac{1}{n} \to 0$$

as $n \to \infty$, thus $h_n \to 0$ in $L^1((0,1))$. But for each $n \in \mathbb{N}$,

$$\int_0^1 h_n^2(x)\,dx = \int_0^1 n^2 1_{(0,\frac{1}{n^2})}\,dx = n^2\frac{1}{n^2} = 1.$$

Therefore h_n^2 does not converges to 0 in $L^1((0,1))$.

(iii) No, $|g_n|^{\frac{1}{2}}$ need not converge to zero in $L^1(\mathbb{R})$. We can give a counter example. Suppose $g_n(x) = \frac{1}{x^2}\mathbb{I}_{[n,n^2]}(x)$, then we have

$$\int_{\mathbb{R}} |g_n(x)|\,dx = \int_n^{n^2} \frac{1}{x^2}\,dx = \frac{1}{n} - \frac{1}{n^2}.$$

When n goes to infinity, we have $\|g_n(x)\|_1 \to 0$, so $g_n(x)$ is in $L^1(\mathbb{R})$ and g_n converges to zero in $L^1(\mathbb{R})$. For the $|g_n|^{\frac{1}{2}} = \frac{1}{x}\mathbb{I}_{[n,n^2]}(x)$, for any $n \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} |g_n(x)|^{\frac{1}{2}}\,dx = \int_n^{n^2} \frac{1}{x}\,dx = \ln n.$$

When n goes to infinity, we have $\int_{\mathbb{R}} |g_n(x)|^{\frac{1}{2}}\,dx \to +\infty$, so $|g_n|^{\frac{1}{2}}$ is in $L^1(\mathbb{R})$ for each $n \in \mathbb{N}$, but g_n don't converges to zero in $L^1(\mathbb{R})$.

Another counter example is as follows. For each $n \in \mathbb{N}$, let

$$g_n(x) = \frac{1}{n^2}1_{[0,n]},$$

then

$$\int_{\mathbb{R}} |g_n| = \int_{\mathbb{R}} \frac{1}{n^2}1_{[0,n]}(x)\,dx = \frac{1}{n} \to 0$$

as $n \to \infty$. Thus $g_n \in L^1(\mathbb{R})$ and g_n converges to 0 in $L^1(\mathbb{R})$ as n goes to infinity. And note that

$$\int_{\mathbb{R}} |g_n|^{\frac{1}{2}} = \int_{\mathbb{R}} \frac{1}{n}1_{[0,n]}(x)\,dx = 1 < \infty,$$

thus $|g_n|^{1/2} \in L^1(\mathbb{R})$. But we have that $|g_n|^{1/2}$ does not converges to 0 in $L^1(\mathbb{R})$.

Exercise 2:

Let f be in $L^\infty((0,1))$. Show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

Solution:

Since $f \in L^\infty((0,1))$ and $\mu((0,1)) = 1 < \infty$, then for all $p \geq 1$,

$$\int_{(0,1)} |f|^p \, d\mu \geq \|f\|_\infty^p \mu((0,1)) < \infty,$$

thus $f \in L^p((0,1))$. Let

$$A = \{x \in (0,1) : f(x) > \|f\|_\infty - \epsilon\},$$

by the definition of $\|f\|_\infty$, we know that $\mu(A) > 0$. For all $p \in [1, \infty)$, since

$$\begin{aligned}\|f\|_p &= \Big(\int_{(0,1)} |f|^p \, d\mu\Big)^{\frac{1}{p}} \geq \Big(\int_A |f|^p \, d\mu\Big)^{\frac{1}{p}} \\ &\geq \Big((\|f\|_\infty - \epsilon)^p \mu(A)\Big)^{\frac{1}{p}} = (\|f\|_\infty - \epsilon)(\mu(A))^{\frac{1}{p}},\end{aligned}$$

and since $\mu(A) \leq 1$, we have

$$\liminf_{p \to +\infty} \|f\|_p \geq \liminf_{p \to +\infty} (\|f\|_\infty - \epsilon)(\mu(A))^{\frac{1}{p}} = \|f\|_\infty - \epsilon.$$

By the arbitrary of $\epsilon > 0$, we have

$$\liminf_{p \to +\infty} \|f\|_p \geq \|f\|_\infty.$$

On the other hand, as $|f(x)| \leq \|f\|_\infty$ for almost every $x \in (0,1)$, then for $1 \leq q < p$, since $f(x)$ is in $L^p((0,1))$ and $f(x)$ is in $L^q((0,1))$, we have

$$\begin{aligned}\|f\|_p &= \Big(\int_{(0,1)} |f|^p \, d\mu\Big)^{\frac{1}{p}} \\ &= \Big(\int_{(0,1)} |f|^q |f|^{p-q} \, d\mu\Big)^{\frac{1}{p}} \\ &\leq (\|f\|_\infty)^{\frac{p-q}{p}} (\|f\|_q)^{\frac{q}{p}}.\end{aligned}$$

Since $\|f\|_q < +\infty$, then when $p \to +\infty$, we know that

$$\limsup_{p \to +\infty} \|f\|_p \leq \|f\|_\infty.$$

We also can get $\limsup_{p \to +\infty} \|f\|_p \leq \|f\|_\infty$ directly as follows

$$\begin{aligned}\|f\|_p &= \Big(\int_{(0,1)} |f|^p \, d\mu\Big)^{\frac{1}{p}} \\ &\leq \Big(\int_{(0,1)} \|f\|_\infty^p \, d\mu\Big)^{\frac{1}{p}} \\ &\leq \|f\|_\infty (\mu((0,1)))^{\frac{1}{p}}.\end{aligned}$$

Thus we have

$$\limsup_{p\to+\infty} \|f\|_p \leq \|f\|_\infty \leq \liminf_{p\to+\infty} \|f\|_p,$$

then we know that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

<u>Exercise 3:</u>

Let a_n be a sequence in $[0,1]$ such that the set $S = \{a_n : n = 1, 2, \dots\}$ is dense in $[0,1]$. Set

$$f(x) = \sum_{n=1}^{\infty} \frac{|x-a_n|^{-\frac{1}{2}}}{n^2}.$$

(i) Show that f is in $L^1([0,1])$.

(ii) Is f in $L^2([0,1])$?

(iii) Is there a continuous function

$$g : [0,1] \setminus S \to \mathbb{R}$$

such that $f = g$ almost everywhere?

Solution:

(i) We check $f \in L^1([0,1])$ by definition, since

$$\begin{aligned}
\int_0^1 \sum_{n=1}^{\infty} \frac{|x-a_n|^{-\frac{1}{2}}}{n^2}\, dx &= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 |x-a_n|^{-\frac{1}{2}}\, dx \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \Big[\int_0^{a_n} (a_n - x)^{-\frac{1}{2}}\, dx + \int_{a_n}^1 (x-a_n)^{-\frac{1}{2}}\, dx \Big] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \Big[2(a_n)^{\frac{1}{2}} + 2(1-a_n)^{\frac{1}{2}} \Big]
\end{aligned}$$

and $a_n \in [0,1]$, then we know that

$$\int_0^1 \sum_{n=1}^{\infty} \frac{|x-a_n|^{-\frac{1}{2}}}{n^2}\, dx \le 4 \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$$

as $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Thus we know that $f \in L^1([0,1])$.

(ii) No, we can show that $f \notin L^2([0,1])$. For $x \in [0,1]$, we have

$$\begin{aligned}
\|f\|_2 &= \int_0^1 \Big(\sum_{n=1}^{\infty} \frac{|x-a_n|^{-\frac{1}{2}}}{n^2}\Big)^2\, dx \\
&\ge \int_0^1 \sum_{n=1}^{\infty} \Big(\frac{|x-a_n|^{-\frac{1}{2}}}{n^2}\Big)^2\, dx \\
&= \sum_{n=1}^{\infty} \frac{1}{n^4} \int_0^1 |x-a_n|^{-1}\, dx.
\end{aligned}$$

To show $f \notin L^2([0,1])$, we just need to prove that $\int_0^1 |x-a_n|^{-1}\, dx = +\infty$. We denote $y = x - a_n$, then we have

$$\int_0^1 |x-a_n|^{-1}\, dx = \int_{-a_n}^{1-a_n} |y|^{-1}\, dy.$$

Since there exists $k > 0$ such that $\frac{1}{k} < a_n$, then we have $-\frac{1}{k} < 0 < 1 - a_n$ and

$$\int_0^1 |x - a_n|^{-1}\,dx \geq \int_{-a_n}^{-\frac{1}{k}} |y|^{-1}\,dy = \int_{\frac{1}{k}}^{a_n} y^{-1}\,dy = \ln a_n + \ln k.$$

When $k \to +\infty$, we have $\ln k + \ln a_n \to \infty$. So, we know that $\int_0^1 |x - a_n|^{-1}\,dx = +\infty$. Thus $\|f\|_2 = +\infty$, then we have $f \notin L^2([0,1])$.

(iii) To show that there is a continuous function $g : [0,1] \setminus S \to \mathbb{R}$ such that $f = g$ almost everywhere, we just need to prove that f is continuous in $[0,1] \setminus S$. So for $x \in [0,1] \setminus S$, we want to show that: $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall y \in [0,1] \setminus S$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Note that

$$\begin{aligned}
|f(x) - f(y)| &= \Big|\sum_{n=1}^{\infty} \frac{|x - a_n|^{-\frac{1}{2}}}{n^2} - \sum_{n=1}^{\infty} \frac{|y - a_n|^{-\frac{1}{2}}}{n^2}\Big| \\
&= \Big|\sum_{n=1}^{\infty} \frac{1}{n^2}(|x - a_n|^{-\frac{1}{2}} - |y - a_n|^{-\frac{1}{2}})\Big| \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n^2}\Big||x - a_n|^{-\frac{1}{2}} - |y - a_n|^{-\frac{1}{2}}\Big|.
\end{aligned}$$

Since $g(x) = |x - a_n|^{-\frac{1}{2}}$ is continuous on $(0,1]$, then $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall y \in (0,1]$ with $|x - y| < \delta$, we have

$$\Big||x - a_n|^{-\frac{1}{2}} - |y - a_n|^{-\frac{1}{2}}\Big| < \frac{6}{\pi^2}\epsilon.$$

Since S is countable and dense in $[0,1]$, then for the above ϵ and δ, $\forall y \in [0,1] \setminus S$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}\Big||x - a_n|^{-\frac{1}{2}} - |y - a_n|^{-\frac{1}{2}}\Big| < \frac{\pi^2}{6} \times \frac{6}{\pi^2}\epsilon = \epsilon.$$

Thus we know that $f(x)$ is continuous in $[0,1] \setminus S$, then $f(x)$ is continuous almost everywhere in $[0,1]$. So, there exists a continuous function $g : [0,1] \setminus S \to \mathbb{R}$ such that $f = g$ almost everywhere.

<u>Exercise 4:</u>

Let $\mathcal{R}$ be the set of all rectangles $(a_1, b_1) \times (a_2, b_2)$ in $\mathbb{R}^2$ such that a_1, b_1, a_2, b_2 are rational numbers.

(i) Let V be an open set in $\mathbb{R}^2$. Show that

$$V = \bigcup_{R \in \mathcal{R}, R \subset V} R.$$

(ii) Recall that the Borel sets of $\mathbb{R}^2$ are the sets in the smallest sigma algebra of $\mathbb{R}^2$ containing all open sets. Show that the smallest sigma algebra of $\mathbb{R}^2$ containing $\mathcal{R}$ is equal to the set set of Borel sets of $\mathbb{R}^2$.

Solution:

(i) Since $\bigcup_{R \in \mathcal{R}, R \subset V} R \subset V$, to prove $V = \bigcup_{R \in \mathcal{R}, R \subset V} R$, we just need to show that $V \subset \bigcup_{R \in \mathcal{R}, R \subset V} R$. Suppose that $\vec{x} = (x_1, x_2) \in V$, since V is an open set, then there exists an open ball such that $B(\vec{x}, r) \subset V$, where r is a positive constant and it is called the radius of the ball. So we can find a rectangle $R = (a_1, b_1) \times (a_2, b_2)$, whose center is exactly $\vec{x}$. We denote $d((a_1, b_1), (a_2, b_2))$ is the distance between (a_1, b_1) and (a_2, b_2). Suppose $d((a_1, b_1), (a_2, b_2)) < r$, then when know that $\vec{x} \in R$, $R \subset V$ and $R \in \mathcal{R}$. For any $x \in V$ we can do same thing, so we have $V \subset \bigcup_{R \in \mathcal{R}, R \subset V} R$. Thus we know that $V = \bigcup_{R \in \mathcal{R}, R \subset V} R$.

(ii) We denote $\sigma(\mathcal{R})$ is the sigma algebra on $\mathbb{R}^2$ generated by sets in $\mathcal{R}$. And we denote $\mathcal{B}(\mathbb{R}^2)$ as the Borel sets of $\mathbb{R}^2$. Since R is open rectangle in $\mathbb{R}^2$ and $\mathcal{R} = \{(a_1, b_1) \times (a_2, b_2) | a_i, b_i \in \mathbb{Q}, i = 1, 2\}$, so $\mathcal{R}$ is the open set in $\mathbb{R}^2$. Then we know that $\sigma(\mathcal{R}) \subset \mathcal{B}(\mathbb{R}^2)$. On the other hand, V is open set and by the result we get in (i), we have $V = \bigcup_{R \in \mathcal{R}, R \subset V} R$. Since the number of set R is countable, then we have $V \in \sigma(\mathcal{R})$. Thus the open sets in $\mathbb{R}^2$ is subset of $\sigma(\mathcal{R})$. Since $\mathcal{B}(\mathbb{R}^2)$ is generated by the open sets in $\mathbb{R}^2$, then we have $\mathcal{B}(\mathbb{R}^2) \subset \sigma(\mathcal{R})$. So we can get $\mathcal{B}(\mathbb{R}^2) = \sigma(\mathcal{R})$. Then we know that the smallest sigma algebra of $\mathbb{R}^2$ containing $\mathcal{R}$ is equal to the set set of Borel sets of $\mathbb{R}^2$.