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Exercise 1:

Construct a subset $A \subset \mathbb{R}$ such that A is closed, contains no intervals, is uncountable, and has Lebesgue measure $\frac{1}{2}$ (i.e. $|A| = \frac{1}{2}$). Also explain why your set A has each of the above properties.

Hint: One possible approach here is to adjust the construction of the Cantor set to achieve a Cantor-like set with measure $\frac{1}{2}$, but you don't need to have seen the Cantor set to answer the question.

Solution:

We follow the construction of Cantor set by deleting the open middle forth from a set of line segment. We start by deleting the open middle $(\frac{3}{8}, \frac{5}{8})$ from the interval $[0, 1]$, leaving two line segments $A_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$. Next we do the same thing by deleting $(\frac{5}{32}, \frac{7}{32})$ and $(\frac{25}{32}, \frac{27}{32})$, then we have

$$A_2 = \left[0, \frac{5}{32}\right] \cup \left[\frac{7}{32}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{25}{32}\right] \cup \left[\frac{27}{32}, 1\right].$$

This process is continued as $n \to \infty$, we can get the Cantor-like set A.

Since we only delete the open interval from $[0, 1]$ each time, then the union of the intervals we deleted is an open set, thus the Cantor-like set A is closed. We denote $A^c = [0, 1] \setminus A$, then the Lebesgue measure of A^c is

$$|A^c| = \sum_{n=1}^{\infty} \frac{2^{n-1}}{4^n} = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2},$$

thus we know that the measure of Cantor-like set is $\frac{1}{2}$ and it is uncountable. Next we need to show the set A contains no intervals. Suppose the interval $(\alpha, \beta) \in A$. For the n-th time we delete the interval whose measure is $\frac{1}{4^n}$, so when $n \to \infty$, it is far smaller than $\beta - \alpha$, then we have to separate the interval (α, β). Thus similarly with the Cantor set, the Cantor-like set contains no intervals.

If we follow the construction of Cantor set by deleting the open middle with length a from a set of line segment, where $0 < a \leq \frac{1}{3}$, then what is left is a closed set with the measure

$$1 - \sum_{n=1}^{\infty} 2^{n-1} a^n = 1 - \frac{a}{1 - 2a} = \frac{1 - 3a}{1 - 2a} \in [0, 1)$$

as $0 < a \leq \frac{1}{3}$.

Exercise 2:

(i) Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n a sequence in $L^1(X)$. Let f be in $L^1(X)$. Assume that $\int f_n$ converges to $\int f$, f_n converges to f almost everywhere, and for each n, $f_n \geq 0$, almost everywhere. Show that f_n converges to f in $L^1(X)$.

Hint: Set $g_n = \min(f_n, f)$. Note that $|f_n - f| = f + f_n - 2g_n$.

(ii) Find a sequence f_n in $L^1(\mathbb{R})$ and f in $L^1(\mathbb{R})$ such that $\int f_n$ converges to $\int f$, f_n converges to f almost everywhere, but f_n does not converge to f in $L^1(\mathbb{R})$.

Solution:

(i) Let $g_n = \min(f_n, f)$, then $|f_n - f| = f + f_n - 2g_n$, thus we have

$$\int_X |f_n - f| \, d\mu = \int_X (f + f_n - 2g_n) \, d\mu.$$

Since $f \in L^1(X)$ and $f_n \in L^1(X)$, then $g_n = \min(f_n, f) \in L^1(X)$, we have

$$\int_X |f_n - f| \, d\mu = \int_X f \, d\mu + \int_X f_n \, d\mu - 2 \int_X g_n \, d\mu.$$

And by the definition of g_n, we know that g_n converges to f almost everywhere as f_n converges to f almost everywhere. As $f_n \geq 0$ almost everywhere, then $f \geq 0$ a.e. Since $|g_n| \leq |f|$ and $f \in L^1(X)$, by the dominate convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_X |f_n - f| \, d\mu &= \int_X f \, d\mu + \lim_{n\to\infty} \int_X f_n \, d\mu - 2 \lim_{n\to\infty} \int_X g_n \, d\mu \\
&= 2 \int_X f \, d\mu - 2 \int_X \lim_{n\to\infty} g_n \, d\mu \\
&= 2 \int_X f \, d\mu - 2 \int_X f \, d\mu = 0,
\end{aligned}$$

hence f_n converges to f in $L^1(X)$.

(ii) For each $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} \frac{1}{n}, & x \in [-n, 0] \\ -\frac{1}{n}, & x \in (0, n] \end{cases}$$

and let $f(x) = 0$, since $|f_n(x)| \leq \frac{1}{n}$ for all $x \in \mathbb{R}$, f_n converges to f almost everywhere. As

$$\int_{\mathbb{R}} f_n \, d\mu = \int_{-n}^{0} \frac{1}{n} \, d\mu + \int_0^n \Big(-\frac{1}{n} \Big) \, d\mu = 1 - 1 = 0,$$

we have f_n in $L^1(\mathbb{R})$ and $\int f_n$ converges to $\int f$. But since

$$\int_{\mathbb{R}} |f_n - f| \, d\mu = \int_{-n}^{n} \frac{1}{n} \, d\mu = 2, \quad \forall n \in \mathbb{N},$$

f_n does not converge to f in $L^1(\mathbb{R})$.

<u>**Exercise 3:**</u>

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(i) Let f be in $L^1([0,\infty))$. Show that

$$\lim_{x\to 0^+}\int_0^\infty f(t)e^{-xt}\,dt = \int_0^\infty f(t)\,dt$$

(ii) Let $[a,b]$ be an interval in $\mathbb{R}$. If $\tilde{f}$ is continuous on $[a,b]$ and monotonic, and g' is continuous on $[a,b]$, we can prove that there is a c in $[a,b]$ such that

$$\int_a^b \tilde{f}g = g(a)\int_a^c \tilde{f} + g(b)\int_c^b \tilde{f}.$$

Using this result, show that if g is as specified above and f is in $L^1([a,b])$, there is a c in $[a,b]$ such that

$$\int_a^b fg = g(a)\int_a^c f + g(b)\int_c^b f.$$

(iii) Let f be in $L^\infty([0,\infty))$. Assume that there is a constant L in $\mathbb{R}$ such that $\lim_{x\to\infty}\int_0^x f = L$. Show that

$$\lim_{x\to 0^+}\int_0^\infty f(t)e^{-xt}\,dt = L.$$

Solution:

(i) When $x \geq 0$ and $t \geq 0$, we have $|f(t)e^{-xt}| \leq |f(t)|$. As $f \in L^1([0,\infty))$ and for any fixed t, $\lim_{x\to 0^+} f(t)e^{-xt} = f(t)$. By the dominate convergence theorem, we have

$$\lim_{x\to 0^+}\int_0^\infty f(t)e^{-xt}\,dt = \int_0^\infty \lim_{x\to 0^+} f(t)e^{-xt}\,dt = \int_0^\infty f(t)\,dt.$$

(ii) Since $\tilde{f}$ is continuous on $[a,b]$, introduce $F(x) = \int_a^x \tilde{f}$, we know F is continuous and $F'(x) = \tilde{f}(x)$. Apply integral by parts,

$$\begin{aligned}
\int_a^b \tilde{f(x)}g(x)\,dx &= \int_a^b g(x)\,dF(x) \\
&= g(b)F(b) - g(a)F(a) - \int_a^b g'(x)F(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - g(a)\int_a^a \tilde{f}(x)\,dx - \int_a^b g'(x)F(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - \int_a^b g'(x)F(x)\,dx.
\end{aligned}$$

Since g is differentiable on $[a,b]$ and monotonic, and g' is continuous on $[a,b]$, we know that g' is integrable in $[a,b]$ and $g'(x) \geq 0$ for all $x \in [a,b]$. By the mean value theorem for integral, there exists $c \in [a,b]$ such that

$$\int_a^b g'(x)F(x)\,dx = F(c)\int_a^b g'(x)\,dx = F(c)(g(b) - g(a)).$$

Thus for this $c \in [a,b]$, we have

$$\begin{aligned}
\int_a^b f(\tilde{x})g(x)\,dx &= g(b)\int_a^b \tilde{f}(x)\,dx - F(c)(g(b)-g(a)) \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - (g(b)-g(a))\int_a^c \tilde{f}(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - g(b)\int_a^c \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx \\
&= g(b)\int_c^b \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx.
\end{aligned}$$

Since $C_c([a,b])$ is dense in $L^1([a,b])$, then we know that for any $f \in L^1([0,1])$, there exists a function sequence $\{f_n\} \subset C_c([a,b])$ such that $\int_a^b |f_n - f| \to 0$ as $n \to +\infty$. Since g is differentiable on $[a,b]$ and monotonic, we know there exists $K > 0$, and $\forall x \in [a,b]$, we have $|g(x)| \leq K$. So, we have

$$\lim_{n\to+\infty}\int_a^b |gf - gf_n| \leq K \lim_{n\to+\infty}\int_a^b |f-f_n| = 0,$$

then by the conclusion we get from (i) we have

$$\int_a^b fg = \lim_{n\to+\infty}\int_a^b f_n g = \lim_{n\to+\infty}\Big(g(a)\int_a^{c_n} f_n + g(b)\int_{c_n}^b f_n\Big),$$

where c_n is depends on f_n for each n.

Since $\{c_n\} \subset [a,b]$ and $[a,b]$ is compact, there exists a subsequence of $\{c_n\}$, which is denoted as $\{c_{n_k}\}$, converges to c and $c \in [a,b]$. Thus we have

$$\begin{aligned}
\int_a^b fg &= \lim_{k\to+\infty}\Big(g(a)\int_a^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^b f_{n_k}\Big) \\
&= \lim_{k\to+\infty}\Big(g(a)\int_a^c f_{n_k} + g(a)\int_c^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^c f_{n_k} + g(b)\int_c^b f_{n_k}\Big) \\
&= g(a)\int_a^c f + g(b)\int_c^b f + \lim_{k\to+\infty}\Big(g(a)\int_c^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^c f_{n_k}\Big) \\
&= g(a)\int_a^c f + g(b)\int_c^b f.
\end{aligned}$$

(iii) For all $K > 0$, we have

$$\lim_{x\to0^+}\int_0^\infty f(t)e^{-xt}\,dt = \lim_{x\to0^+}\Big(\int_0^K f(t)e^{-xt}\,dt + \int_K^\infty f(t)e^{-xt}\,dt\Big).$$

Since $f \in L^\infty([0,\infty))$, let $K \to \infty$, we can get

$$\lim_{x\to0^+}\int_0^\infty f(t)e^{-xt}\,dt = \lim_{x\to0^+}\lim_{K\to\infty}\int_0^K f(t)e^{-xt}\,dt,$$

then we have

$$\begin{aligned}
\lim_{x\to 0^+}\int_0^\infty f(t)e^{-xt}\,dt &= \lim_{x\to 0^+}\lim_{K\to\infty}\Big(\int_0^K f(t)\,dt+\int_0^K f(t)(e^{-xt}-1)\,dt\Big)\\
&= L+\lim_{x\to 0^+}\lim_{K\to\infty}\int_0^K f(t)(e^{-xt}-1)\,dt\\
&= L+\lim_{K\to\infty}\lim_{x\to 0^+}\int_0^K f(t)(e^{-xt}-1)\,dt
\end{aligned}$$

as $\int_0^K f(t)e^{-xt}\,dt$ is continuous with x and K. As $f(t)\in L^\infty([0,\infty))$, we have

$$\int_0^K |f(t)|\,dt \le K\|f\|_\infty < \infty,$$

then we know that $f(t)\in L^1([0,K])$. And since $|f(t)(e^{-xt}-1)|\le|f(t)|$ when $x\ge 0, t\ge 0$, by the dominate convergence theorem, we have

$$\lim_{x\to 0^+}\int_0^K f(t)(e^{-xt}-1)\,dt=\int_0^K f(t)\lim_{x\to 0^+}(e^{-xt}-1)\,dt=0,$$

hence we can get

$$\lim_{x\to 0^+}\int_0^\infty f(t)e^{-xt}\,dt = L.$$