Analysis Exercise

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Exercise 2:

Let (X, η) and (Y, ρ) be two Polish spaces. $C(X, Y)$ is the set of all continuous mappings $f : X \mapsto Y$. For $f, g \in C(X, Y)$, we define

$$d(f, g) = \sup_{x \in X} \rho(f(x), g(x)).$$

(i) Prove that $(C(X, Y), d)$ is a Polish space.

(ii) If $K \subset Y$ is compact, is $C(X, K)$ compact in $C(X, Y)$?

Solution:

(i) We need to show that $C(X, Y)$ is separable and completely metrizable. Firstly we show that $C(X, Y)$ is complete. Suppose $\{f_n\}$ is a Cauchy sequence in $C(X, Y)$, then $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall p > q > N$,

$$\sup_{x \in X} \rho(f_p(x), f_q(x)) < \epsilon.$$

For any $y \in X$, we have

$$\rho(f_p(y), f_q(y)) \leq \sup_{x \in X} \rho(f_p(x), f_q(x)) < \epsilon,$$

thus $f_n(y)$ is a Cauchy sequence in Y. As Y is a Polish space, Y is complete, then $f_n(y)$ converges to some $f(y)$ in Y. From this we can define a function

$$f : X \mapsto Y.$$

Next we show that f is also continuous. Since

$$\rho(f(x), f(y)) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y)),$$

and $\{f_n\}$ is a continuous function sequence, for the above ϵ, there exists a $N^* \in \mathbb{N}$ and $\delta > 0$, for any $x \in B(y, \delta)$ and $n > N^*$, we have

$$\rho(f(x), f(y)) < 3\epsilon.$$

Hence $f \in C(X, Y)$. And for the above ϵ and $p > q > N$, since $\rho(f_p(y), f_q(y)) < \epsilon$, let $p \to \infty$, we have $\rho(f(y), f_q(y)) \leq \epsilon$. By the arbitrary of $y \in X$, for $q > N$, we can get

$$\sup_{y \in X} \rho(f(y), f_q(y)) \leq \epsilon,$$

which shows that $f_n \to f$ in $C(X,Y)$. Thus $C(X,Y)$ is complete.

Next we need to show that $C(X,Y)$ is separable. Let

$$C_{m,n} = \{f \in C(X,Y) : \forall x, y \in X, \eta(x,y) < \frac{1}{m} \implies \rho(f(x), f(y)) < \frac{1}{n}\}.$$

As X is a Polish space, it is separable. Choose a countable set $X_m \subset X$ such that

$$X \subset \bigcup_{x \in X_m} B(x, \frac{1}{m}).$$

And let $D_{m,n} \subset C_{m,n}$ be countable such that for every $f \in C_{m,n}$ and for every $\epsilon > 0$, there is a $g \in D_{m,n}$ with

$$\rho(f(y), g(y)) < \epsilon$$

for $y \in X_m$. We claim that $D = \bigcup_{m,n \in \mathbb{N}} D_{m,n}$ is dense in $C(X,Y)$. If $f \in C(X,Y)$ and for every $\epsilon > 0$, let $n > \frac{3}{\epsilon}$ and let m satisfies that $f \in C_{m,n}$. Let $g \in C_{m,n}$ such that $\rho(f(y), g(y)) < \frac{1}{n}$ for all $y \in X_m$. For any fixed $x \in X$, let $y \in X_m$ such that $\eta(x,y) < \frac{1}{m}$, then

$$\rho(f(x), g(x)) \le \rho(f(x), f(y)) + \rho(f(y), g(y)) + \rho(g(y), g(x)) < \epsilon.$$

By the arbitrary of x, we have

$$d(f,g) = \sup_{x \in X} \rho(f(x), g(x)) \le \epsilon.$$

Hence D is dense in $C(X,Y)$ and $C(X,Y)$ is separable.

(ii) The statement is not true. We can give a counter example as follows. Set $K = [0,1], Y = \mathbb{R}$ and $X = [0,1]$, and we define a function sequence $f_n : X \mapsto K$ by

$$f_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{2} - \frac{1}{n}) \\ nx - \frac{n}{2} + 1, & x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

then we know that $K \subset Y$ and K is compact and $\{f_n\}$ is a continuous function sequence from X to K. And we define

$$f(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

thus when $n \to \infty$, $f_n(x)$ converges to $f(x)$ almost everywhere. But $f(x)$ is not a continuous function on X, $f(x) \notin C(X,K)$, thus for any subsequence $\{f_{n_k}\}$ of $\{f_n\}$, we know that $\{f_{n_k}\}$ is not converges in $C(X,K)$. Hence we know $C(X,K)$ is not compact in $C(X,Y)$.