exercise 1:
Problem 9.5.71.

exercise 2:
Problem 9.5.72.

exercise 3:
Problem 9.5.73.

exercise 4:
Show that $l^1 \subset l^2 \subset l^\infty$, and that these inclusions are strict. Also show that the identity functions from l^1 to l^2 and from l^2 to l^∞ are continuous.

exercise 5:
Let X be a set and $\mathcal{A}$ be a subset of $\mathcal{P}(X)$.
(i). If $\mathcal{A}$ is closed under complementation and set difference, show that $\mathcal{A}$ is closed under finite union and finite intersection.
(ii). If $\mathcal{A}$ contains X and is closed under set difference and finite intersection, show that $\mathcal{A}$ is closed under finite union and set complementation.

exercise 6:
(i). Let V be an open subset of $\mathbb{R}$ which is neither empty nor equal to $\mathbb{R}$. Let V^c be the complement of V. Show that

$$V = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : |x| \leq n \text{ and } d(x, V^c) \geq \frac{1}{n}\}$$

(ii). Infer that every non empty open subset of $\mathbb{R}$ is a countable union of compact subsets.
(iii). Infer that every open subset of $\mathbb{R}$ is a countable union of open intervals.

exercise 7:
Let X be a set and $\mathcal{A}$ be a σ- algebra of subsets of X. Let Y be a subset of X. Define a subset $\mathcal{B}$ of $\mathcal{P}(Y)$ by setting $\mathcal{B} := \{A \cap Y : A \in \mathcal{A}\}$. Show that $\mathcal{B}$ is a σ- algebra of subsets of Y.

exercise 8:
Let X be a set and $\mathcal{A}_i$ be a σ- algebra of subsets of X for each i in I. Show that $\bigcap_{i \in I} \mathcal{A}_i$ is σ-

algebra of subsets of X.

exercise 9:
Show from the definition of the outer measure m^* on $\mathbb{R}$ that for any two subsets A and B of $\mathbb{R}$, $m^*(A \cup B) \leq m^*(A) + m^*(B)$.

exercise 10:
(i). Let $(X, \mathcal{A}, \mu)$ be a measure space, B_n a decreasing sequence of $\mathcal{A}$ such that $\mu(B_1) < \infty$. Show that $\lim\limits_{n\to\infty} \mu(B_n) = \mu(\bigcap\limits_{n=1}^{\infty} B_n)$.
(ii). Find a measure space $(X, \mathcal{A}, \mu)$ and a decreasing sequence B_n of $\mathcal{A}$ such that $\lim\limits_{n\to\infty} \mu(B_n) \neq \mu(\bigcap\limits_{n=1}^{\infty} B_n)$.

exercise 11:
Let S be the subset of $[0, 1]$ defined by all the sums of the series $\sum\limits_{n=1}^{\infty} \frac{a_n}{10^n}$ where a_n is in the set $\{0, 2, 4, 6, 8\}$.
(i). Assume that $\sum\limits_{n=1}^{\infty} \frac{a_n}{10^n} = \sum\limits_{n=1}^{\infty} \frac{b_n}{10^n}$ where a_n and b_n are in $\{0, 2, 4, 6, 8\}$. Show that $a_n = b_n$ for all interger $n \geq 1$.
Hint: Show that

$$|\sum_{n=1}^{p} \frac{a_n}{10^n} - \sum_{n=1}^{p} \frac{b_n}{10^n}| \leq \frac{8}{9} \frac{1}{10^p}$$

(ii). Show that S is uncountable.
Hint: Define $f : S \to \mathcal{P}(\mathbb{N}^*)$: let $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$, $a_n \in \{0, 2, 4, 6, 8\}$. $p \notin f(x)$ if $a_p = 0$, $p \in f(x)$ otherwise. It suffices to show that f is surjective.
(iii). Show that S has Lebesgue measure zero.
Hint: Use the set S_p defined by all the sums of the series $\sum\limits_{n=1}^{p} \frac{a_n}{10^n}$ where a_n is in the set $\{0, 2, 4, 6, 8\}$.