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Exercise 1:

Let h_n be a sequence of non-negative, borel measurable functions on the interval $(0,1)$ such that $h_n \to 0$ in $L^1((0,1))$.

(i) Show $\sqrt{h_n} \to 0$ in $L^1((0,1))$.

(ii) Given an example to show that h_n^2 need not converge to zero in $L^1((0,1))$.

(iii) If g_n is in $L^1(\mathbb{R})$ such that $|g_n|^{\frac{1}{2}}$ is in $L^1(\mathbb{R})$, and g_n converges to zero in $L^1(\mathbb{R})$ as n tends to infinity, does $|g_n|^{\frac{1}{2}}$ converges to zero in $L^1(\mathbb{R})$?

Solution:

(i) We want to show that $\int_0^1 |\sqrt{h_n} - 0|\, d\mu \to 0$ as $n \to \infty$. Since $h_n \to 0$ in $L^1((0,1))$ and by the holder inequality, we have

$$\begin{aligned}
\int_0^1 |\sqrt{h_n} - 0|\, d\mu &\leq \Big(\int_0^1 |(\sqrt{h_n})^2|\, d\mu\Big)^{\frac{1}{2}} \Big(\int_0^1 1^2\, d\mu\Big)^{\frac{1}{2}} \\
&= \Big(\int_0^1 h_n\, d\mu\Big)^{\frac{1}{2}} \Big(\int_0^1 1\, d\mu\Big)^{\frac{1}{2}} \\
&= \Big(\int_0^1 |h_n - 0|\, d\mu\Big)^{\frac{1}{2}}.
\end{aligned}$$

So when n goes to infinity, we have $\int_0^1 |\sqrt{h_n} - 0|\, d\mu \to 0$. Thus we know that $\sqrt{h_n} \to 0$ in $L^1((0,1))$.

(ii) We suppose for $n \in \mathbb{N}$,

$$h_n(x) = n^{\frac{3}{2}} x \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x).$$

Then we have

$$\int_0^1 n^{\frac{3}{2}} x \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x)\, dx = n^{\frac{3}{2}} \int_{\frac{1}{n^2}}^{\frac{1}{n}} x\, dx = \frac{1}{2}\Big(\frac{1}{\sqrt{n}} - \frac{1}{n^{\frac{5}{2}}}\Big),$$

when $n \to +\infty$, we get $\|h_n\|_1 \to 0$, so we know that $h_n \to 0$ in $L^1((0,1))$. But for the $h_n^2(x)$, we have

$$\int_0^1 n^3 x^2 \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x)\, dx = n^3 \int_{\frac{1}{n^2}}^{\frac{1}{n}} x^2\, dx = \frac{1}{3} n^3 \Big(\frac{1}{n^3} - \frac{1}{n^6}\Big) = \frac{1}{3} - \frac{1}{3n^3}.$$

When n tends to infinity, $\int_0^1 n^3 x^2 \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x)\, dx \to \frac{1}{3}$, which is not goes to 0. So, we know that $h_n^2(x)$ don't converge to zero in $L^1((0,1))$.

(iii) No, $|g_n|^{\frac{1}{2}}$ need not converge to zero in $L^1(\mathbb{R})$. We can give a counter example. Suppose $g_n(x) = \frac{1}{x^2} \mathbb{I}_{[n,n^2]}(x)$, then we have

$$\int_{\mathbb{R}} |g_n(x)|\, dx = \int_n^{n^2} \frac{1}{x^2}\, dx = \frac{1}{n} - \frac{1}{n^2}.$$

When n goes to infinity, we have $\|g_n(x)\|_1 \to 0$, so $g_n(x)$ is in $L^1(\mathbb{R})$ and g_n converges to zero in $L^1(\mathbb{R})$. For the $|g_n|^{\frac{1}{2}} = \frac{1}{x}\mathbb{I}_{[n,n^2]}(x)$, for any $n \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} |g_n(x)|^{\frac{1}{2}}\, dx = \int_n^{n^2} \frac{1}{x}\, dx = \ln n.$$

When n goes to infinity, we have $\int_{\mathbb{R}} |g_n(x)|^{\frac{1}{2}}\, dx \to +\infty$, so $|g_n|^{\frac{1}{2}}$ is in $L^1(\mathbb{R})$ but g_n don't converges to zero in $L^1(\mathbb{R})$.

Exercise 2:

Let f be in $L^\infty((0,1))$. Show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

Solution:

Since $f \in L^\infty((0,1))$ and $\mu((0,1)) = 1 < \infty$, then we know that for any $p \geq 1$, $f \in L^p((0,1))$. We denote $t \in [0, \|f\|_\infty)$, then the set

$$A = \{x \in (0,1) : |f(x)| \geq t\}$$

has positive and bounded measure. Since

$$\begin{aligned}
\|f\|_p &= \Big(\int_{(0,1)} |f|^p\, d\mu\Big)^{\frac{1}{p}} \geq \Big(\int_A |f|^p\, d\mu\Big)^{\frac{1}{p}} \\
&\geq \Big(t^p \mu(A)\Big)^{\frac{1}{p}} = t(\mu(A))^{\frac{1}{p}},
\end{aligned}$$

and $\mu(A)$ is finite, then when $p \to +\infty$, we have $(\mu(A))^{\frac{1}{p}} \to 1$ and

$$\liminf_{p \to +\infty} \|f\|_p \geq t.$$

Since t is arbitrary and $t \in [0, \|f\|_\infty)$, we have

$$\liminf_{p \to +\infty} \|f\|_p \geq \|f\|_\infty.$$

On the other hand, as $|f(x)| \leq \|f\|_\infty$ for almost every $x \in (0,1)$, then for $1 \leq q < p$, since $f(x)$ is in $L^p((0,1))$ and $f(x)$ is in $L^q((0,1))$, we have

$$\begin{aligned}
\|f\|_p &= \Big(\int_{(0,1)} |f|^p\, d\mu\Big)^{\frac{1}{p}} \\
&= \Big(\int_{(0,1)} |f|^q |f|^{p-q}\, d\mu\Big)^{\frac{1}{p}} \\
&\leq (\|f\|_\infty)^{\frac{p-q}{p}} (\|f\|_q)^{\frac{q}{p}}.
\end{aligned}$$

Since $\|f\|_q < +\infty$, then when $p \to +\infty$, we know that

$$\limsup_{p \to +\infty} \|f\|_p \leq \|f\|_\infty.$$

Thus we have

$$\limsup_{p\to+\infty}\|f\|_p \le \|f\|_\infty \le \liminf_{p\to+\infty}\|f\|_p,$$

then we know that $\|f\|_p \to \|f\|_\infty$ as $p\to\infty$.

Exercise 3:

Let a_n be a sequence in $[0,1]$ such that the set $S=\{a_n : n=1,2,\dots\}$ is dense in $[0,1]$. Set

$$f(x)=\sum_{n=1}^{\infty}\frac{|x-a_n|^{-\frac{1}{2}}}{n^2}.$$

(i) Show that f is in $L^1([0,1])$.

(ii) Is f in $L^2([0,1])$?

(iii) Is there a continuous function

$$g:[0,1]\setminus S\to\mathbb{R}$$

such that $f=g$ almost everywhere?

Solution:

(i) We check $f\in L^1([0,1])$ by definition, since

$$\begin{aligned}
\int_0^1\sum_{n=1}^{\infty}\frac{|x-a_n|^{-\frac{1}{2}}}{n^2}\,dx &= \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1|x-a_n|^{-\frac{1}{2}}\,dx\\
&= \sum_{n=1}^{\infty}\frac{1}{n^2}\Big[\int_0^{a_n}(a_n-x)^{-\frac{1}{2}}\,dx+\int_{a_n}^1(x-a_n)^{-\frac{1}{2}}\,dx\Big]\\
&= \sum_{n=1}^{\infty}\frac{1}{n^2}\Big[2(a_n)^{\frac{1}{2}}+2(1-a_n)^{\frac{1}{2}}\Big]
\end{aligned}$$

and $a_n\in[0,1]$, then we know that

$$\int_0^1\sum_{n=1}^{\infty}\frac{|x-a_n|^{-\frac{1}{2}}}{n^2}\,dx\le 4\sum_{n=1}^{\infty}\frac{1}{n^2}<+\infty$$

as $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. Thus we know that $f\in L^1([0,1])$.

(ii) No, we can show that $f\notin L^2([0,1])$. For $x\in[0,1]$, we have

$$\begin{aligned}
\|f\|_2 &= \int_0^1(\sum_{n=1}^{\infty}\frac{|x-a_n|^{-\frac{1}{2}}}{n^2})^2\,dx\\
&\ge \int_0^1\sum_{n=1}^{\infty}(\frac{|x-a_n|^{-\frac{1}{2}}}{n^2})^2\,dx\\
&= \sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^1|x-a_n|^{-1}\,dx.
\end{aligned}$$

To show $f \notin L^2([0,1])$, we just need to prove that $\int_0^1 |x-a_n|^{-1}\,dx = +\infty$. We denote $y = x - a_n$, then we have

$$\int_0^1 |x-a_n|^{-1}\,dx = \int_{-a_n}^{1-a_n} |y|^{-1}\,dy.$$

Since there exists $k > 0$ such that $\frac{1}{k} < a_n$, then we have $-\frac{1}{k} < 0 < 1 - a_n$ and

$$\int_0^1 |x-a_n|^{-1}\,dx \geq \int_{-a_n}^{-\frac{1}{k}} |y|^{-1}\,dy = \int_{\frac{1}{k}}^{a_n} y^{-1}\,dy = \ln a_n + \ln k.$$

When $k \to +\infty$, we have $\ln k + \ln a_n \to \infty$. So, we know that $\int_0^1 |x-a_n|^{-1}\,dx = +\infty$. Thus $\|f\|_2 = +\infty$, then we have $f \notin L^2([0,1])$.

(iii) To show that there is a continuous function $g : [0,1] \setminus S \to \mathbb{R}$ such that $f = g$ almost everywhere, we just need to prove that f is continuous in $[0,1] \setminus S$. So for $x \in [0,1] \setminus S$, we want to show that: $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall y \in [0,1] \setminus S$ such that $|x-y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Firstly, we deal with $f(x) - f(y)$, and then we can get

$$\begin{aligned}
|f(x) - f(y)| &= \Big|\sum_{n=1}^{\infty} \frac{|x-a_n|^{-\frac{1}{2}}}{n^2} - \sum_{n=1}^{\infty} \frac{|y-a_n|^{-\frac{1}{2}}}{n^2}\Big| \\
&= \Big|\sum_{n=1}^{\infty} \frac{1}{n^2}(|x-a_n|^{-\frac{1}{2}} - |y-a_n|^{-\frac{1}{2}})\Big| \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n^2}\Big||x-a_n|^{-\frac{1}{2}} - |y-a_n|^{-\frac{1}{2}}\Big|.
\end{aligned}$$

Since $g(x) = |x-a_n|^{-\frac{1}{2}}$ is continuous on $(0,1]$, then $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall y \in (0,1]$ such that $|x-y| < \delta$, we have

$$\Big||x-a_n|^{-\frac{1}{2}} - |y-a_n|^{-\frac{1}{2}}\Big| < \frac{6}{\pi^2}\epsilon.$$

Since S is countable and dense in $[0,1]$, then for the above ϵ and δ, $\forall y \in [0,1] \setminus S$ such that $|x-y| < \delta$, we have

$$|f(x) - f(y)| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}\Big||x-a_n|^{-\frac{1}{2}} - |y-a_n|^{-\frac{1}{2}}\Big| < \frac{\pi^2}{6} \times \frac{6}{\pi^2}\epsilon = \epsilon.$$

Thus we know that $f(x)$ is continuous in $[0,1]\setminus S$, which is equivalent to $f(x)$ is continuous almost everywhere in $[0,1]$. So, there exists a continuous function $g : [0,1] \setminus S \to \mathbb{R}$ such that $f = g$ almost everywhere.

Exercise 4:

Let $\mathcal{R}$ be the set of all rectangles $(a_1,b_1) \times (a_2,b_2)$ in $\mathbb{R}^2$ such that a_1, b_1, a_2, b_2 are rational numbers.

(i) Let V be an open set in $\mathbb{R}^2$. Show that

$$V = \bigcup_{R \in \mathcal{R}, R \subset V} R.$$

(ii) Recall that the Borel sets of $\mathbb{R}^2$ are the sets in the smallest sigma algebra of $\mathbb{R}^2$ containing all open sets. Show that the smallest sigma algebra of $\mathbb{R}^2$ containing $\mathcal{R}$ is equal to the set set of Borel sets of $\mathbb{R}^2$.

Solution:

(i) Since $\bigcup_{R \in \mathcal{R}, R \subset V} R \subset V$, to prove $V = \bigcup_{R \in \mathcal{R}, R \subset V} R$, we just need to show that $V \subset \bigcup_{R \in \mathcal{R}, R \subset V} R$. Suppose that $\vec{x} = (x_1, x_2) \in V$, since V is an open set, then there exists an open ball such that $B(\vec{x}, r) \subset V$, where r is a positive constant and it is called the radius of the ball. So we can find a rectangle $R = (a_1, b_1) \times (a_2, b_2)$, whose center is exactly $\vec{x}$. We denote $d((a_1, b_1), (a_2, b_2))$ is the distance between (a_1, b_1) and (a_2, b_2). Suppose $d((a_1, b_1), (a_2, b_2)) < r$, then when know that $\vec{x} \in R$, $R \subset V$ and $R \in \mathcal{R}$. For any $x \in V$ we can do same thing, so we have $V \subset \bigcup_{R \in \mathcal{R}, R \subset V} R$. Thus we know that $V = \bigcup_{R \in \mathcal{R}, R \subset V} R$.

(ii) We denote $\sigma(\mathcal{R})$ is the sigma algebra on $\mathbb{R}^2$ generated by sets in $\mathcal{R}$. And we denote $\mathcal{B}(\mathbb{R}^2)$ as the Borel sets of $\mathbb{R}^2$. Since R is open rectangle in $\mathbb{R}^2$ and $\mathcal{R} = \{(a_1, b_1) \times (a_2, b_2) | a_i, b_i \in \mathbb{Q}, i = 1, 2\}$, so $\mathcal{R}$ is the open set in $\mathbb{R}^2$. Then we know that $\sigma(\mathbb{R}) \subset \mathcal{B}(\mathbb{R}^2)$. On the other hand, V is open set and by the result we get in (i), we have $V = \bigcup_{R \in \mathcal{R}, R \subset V} R$. Since the number of set R is countable, then we have $V \in \sigma(\mathcal{R})$. Thus the open sets in $\mathbb{R}^2$ is subset of $\sigma(\mathcal{R})$. Since $\mathcal{B}(\mathbb{R}^2)$ is generated by the open sets in $\mathbb{R}^2$, then we have $\mathcal{B}(\mathbb{R}^2) \subset \sigma(\mathcal{R})$. So we can get $\mathcal{B}(\mathbb{R}^2) = \sigma(\mathcal{R})$. Then we know that the smallest sigma algebra of $\mathbb{R}^2$ containing $\mathcal{R}$ is equal to the set set of Borel sets of $\mathbb{R}^2$.