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Exercise 1:

Let (X, ρ) be a metric space and K_n a sequence of compact subsets of X such that $K_{n+1} \subset K_n$. Set

$$d_n = \sup\{\rho(x, y) : x \in K_n, y \in K_n\}$$

Assuming that d_n converges to zero show that $\bigcap_{n=1}^{\infty} K_n$ is a singleton.

Solution:

Since $\lim_{n\to+\infty} d_n = 0$, it means the diameter of the intersection of the K_n is zero. So, $\bigcap_{n=1}^{\infty} K_n$ is either empty or consists of a single point. For any $n \in \mathbb{N}$, we pick an element $a_n \in K_n$. So we can get a point sequence $\{a_n\}$, and we have $\{a_n : n \in \mathbb{N}\} \in K_1$. Since K_1 is compact, then we know there exists a sub-sequence of a_n, which is denoted as a_{n_k}, converges to a point a. For any $n \in \mathbb{N}$, since each K_n is compact, and a is the limit of a sequence in K_n, we have $a \in K_n$. Thus $a \in \bigcap_{n=1}^{\infty} K_n$. So we know that $\bigcap_{n=1}^{\infty} K_n$ is a singleton.

Exercise 2:

(i) Let $[a, b]$ be an interval in $\mathbb{R}$. If $\tilde{f}$ is continuous on $[a, b]$, g is differentiable on $[a, b]$ and monotonic, and g' is continuous on $[a, b]$, show that there is a c in $[a, b]$, such that

$$\int_a^b \tilde{f} g = g(a) \int_a^c \tilde{f} + g(b) \int_c^b \tilde{f}.$$

Hint: Introduce $F(x) = \int_a^x \tilde{f}$ and integral by parts.

(ii) Show that if g is as specified above and f is in $L^1([a, b])$, there is a c in $[a, b]$ such that

$$\int_a^b f g = g(a) \int_a^c f + g(b) \int_c^b f.$$

Solution:

(i) Since $\tilde{f}$ is continuous on $[a, b]$, we can introduce $F(x) = \int_a^x \tilde{f}$, so we know that

$F'(x) = \tilde{f}(x)$. Then through integral by parts, we have

$$\begin{aligned}\int_a^b f\tilde{(x)}g(x)\,dx &= \int_a^b g(x)\,dF(x)\\ &= g(b)F(b) - g(a)F(a) - \int_a^b g'(x)F(x)\,dx\\ &= g(b)\int_a^b \tilde{f}(x)\,dx - g(a)\int_a^a \tilde{f}(x)\,dx - \int_a^b g'(x)F(x)\,dx\\ &= g(b)\int_a^b \tilde{f}(x)\,dx - \int_a^b g'(x)F(x)\,dx.\end{aligned}$$

Since g is differentiable on $[a,b]$ and monotonic, and g' is continuous on $[a,b]$, we know that g' is integrable in $[a,b]$ and $g'(x) \geq 0$ for all $x \in [a,b]$. By the mean value theorem for integral, there exists $c \in [a,b]$, and

$$\int_a^b g'(x)F(x)\,dx = F(c)\int_a^b g'(x)\,dx = F(c)(g(b) - g(a)).$$

Thus for this $c \in [a,b]$, we have

$$\begin{aligned}\int_a^b f\tilde{(x)}g(x)\,dx &= g(b)\int_a^b \tilde{f}(x)\,dx - F(c)(g(b) - g(a))\\ &= g(b)\int_a^b \tilde{f}(x)\,dx - (g(b) - g(a))\int_a^c \tilde{f}(x)\,dx\\ &= g(b)\int_a^b \tilde{f}(x)\,dx - g(b)\int_a^c \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx\\ &= g(b)\int_c^b \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx.\end{aligned}$$

(ii) Since $C_c([a,b])$ is dense in $L^1([a,b])$, then we know that for any $f \in L^1([0,1])$, there exists a function sequence $\{f_n\} \subset C_c([a,b])$ and $\int_a^b |f_n - f| \to 0$ as $n \to +\infty$. Since g is differentiable on $[a,b]$ and monotonic, we know there exists $K > 0$, and $\forall x \in [a,b]$, we have $|g(x)| \leq K$. So, we have

$$\lim_{n\to+\infty}\int_a^b |gf - gf_n| \leq K \lim_{n\to+\infty}\int_a^b |f - f_n| = 0,$$

then by the conclusion we get from (i) we have

$$\int_a^b fg = \lim_{n\to+\infty}\int_a^b f_n g = \lim_{n\to+\infty}\Big(g(a)\int_a^{c_n} f_n + g(b)\int_{c_n}^b f_n\Big),$$

where c_n is depends on f_n for each n.

Since $\{c_n\} \subset [a,b]$ and $[a,b]$ is compact, there exists a subsequence of $\{c_n\}$, which is denoted as $\{c_{n_k}\}$, converges to c and $c \in [a,b]$. Thus we have

$$\begin{aligned}
\int_a^b fg &= \lim_{k\to+\infty}\Big(g(a)\int_a^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^b f_{n_k}\Big)\\
&= \lim_{k\to+\infty}\Big(g(a)\int_a^{c} f_{n_k} + g(a)\int_c^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^c f_{n_k} + g(b)\int_c^b f_{n_k}\Big)\\
&= g(a)\int_a^c f + g(b)\int_c^b f + \lim_{k\to+\infty}\Big(g(a)\int_c^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^c f_{n_k}\Big)\\
&= g(a)\int_a^c f + g(b)\int_c^b f.
\end{aligned}$$

Exercise 3:

Let $\{f_n\}$ be a sequence of functions $f_n : [0,1] \to \mathbb{R}$.

(i) Define equicontinuity for this sequence.

(ii) Show that if each f_n is differentiable on $[0,1]$ and $|f_n'(x)| \le 1$ for all x in $[0,1]$ and $n \in \mathbb{N}$, the sequence is equicontinuous.

(iii) Suppose the sequence is uniformly bounded and that (ii) holds. Show that f_n has a subsequence which converges uniformly to a continuous function.

(iv) Show through an example that the limit may not be differentiable.

Solution:

(i) The definition of equicontinuity of sequence $\{f_n\}$ at point x is as follows: $\forall \epsilon > 0, \exists \delta > 0$, such that $|x-y| < \delta$ and $\forall n \in \mathbb{N}$, we have $|f_n(x)-f_n(y)| < \epsilon$. And the definition of uniformly equicontinuity of sequence $\{f_n\}$ is as follows: $\forall x \in [0,1], \forall \epsilon > 0, \exists \delta > 0$, such that $|x-y| < \delta$ and $\forall n \in \mathbb{N}$, we have $|f_n(x) - f_n(y)| < \epsilon$.

(ii) Since f_n is differentiable on $[0,1]$, by the mean value theorem, we know that $\forall x, y \in [0,1]$, there exists a $c \in [x,y]$ and we have

$$|f_n(y) - f_n(x)| = |f_n'(c)||y-x|.$$

Since $|f_n'(x)| \le 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}$, then we have

$$|f_n(y) - f_n(x)| \le |y-x|.$$

We set $\delta = \epsilon$, then for all $\epsilon > 0$, there exists $\delta = \epsilon$, such that when $|y-x| < \delta$, $\forall n \in \mathbb{N}$, we have $|f_n(y) - f_n(x)| < \epsilon$. So we know the sequence $\{f_n\}$ is equicontinuous.

(iii) By the Arzel$\grave{a}$-Ascoli theorem, we can get f_n has a subsequence which converges uniformly to a continuous function directly. Next we can show the proof of Arzel$\grave{a}$-Ascoli theorem.

We enumerate $\{x_i\}_{i\in\mathbb{N}}$ as the rational number in $[0,1]$. Since the sequence $\{f_n\}$ is uniformly bounded, then the set of points $\{f_n(x_1)\}$ is bounded, by the Bolzano-Weierstrass theorem, there is a subsequence $\{f_{n1}(x_1)\}$ converges. Repeating the same argument for the sequence points $\{f_{n1}(x_2)\}$, there is a subsequence $\{f_{n2}\}$ of $\{f_{n1}\}$ such that $\{f_{n2}(x_2)\}$ converges. By induction this process can be continued forever, and so there is a chain of subsequences

$$\{f_n\} \supset \{f_{n1}\} \supset \{f_{n2}\} \supset \cdots$$

Such that for each $k \in \mathbb{N}$, the subsequence$\{f_{nk}\}$ converges at point x_k. We choose the diagonal subsequence $\{f_{kk}\}$. Except for the first n functions, $\{f_{kk}\}$ is a subsequence of the nth row $\{f_{nk}\}$. Therefore, the sequence $\{f_{kk}\}$ converges simultaneously on all x_n.

Next we need to show that $\{f_{kk}\}$ is converges uniformly on $[a, b]$. We just need to prove the uniform Cauchy criterion holds. Given any $\epsilon > 0$ and rational $x_k \in [0, 1]$, there is an integer $N(\epsilon, x_k)$ such that when $n, m > N$, we have

$$|f_{nn}(x_k) - f_{mm}(x_k)| < \frac{\epsilon}{3}.$$

Since $\bigcap(x_k - \frac{1}{n}, x_k + \frac{1}{n})$ covers the compact interval [0, 1], then by the Heine-Borel theorem there is a finite subcover, we denote the finite subcover as $U_1, \ldots, U_J$. There exists an integer K such that each open interval U_j, $1 \le j \le J$, contains a rational number x_k with $1 \le k \le K$. Finally, for any $x \in [0, 1]$, there are j and k so that x and x_k belong to the same interval U_j. For this k, we have

$$\begin{aligned} |f_{nn}(x) - f_{mm}(x)| &\le |f_{nn}(x) - f_{nn}(x_k)| + |f_{nn}(x_k) - f_{mm}(x_k)| + |f_{mm}(x_k) - f_{mm}(x)| \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

for all $N = \max\{N(\epsilon, x_1), \ldots, N(\epsilon, x_K)\}$ as f_n is equicontinuous. So, for the subsequence $\{f_{kk}\}$, the uniform Cauchy criterion holds. Thus we know that $\{f_{kk}\}$ converges to a continuous function.

(iv) We denote $f_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}}$, $x \in [0, 1]$. Since for all $n \in \mathbb{N}$ and $x \in [0, 1]$,

$$|f_n'(x)| = \left| \frac{x - \frac{1}{2}}{\sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}}} \right| < 1$$

and $f_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}} < 2$, by the conclusion we get from (ii) and (iii), we know that the sequence $\{f_n\}$ is equicontinuous and it has a subsequence which converges uniformly to a continuous function. When $n \to +\infty$, we have $f_n(x) \to f(x) = |x - \frac{1}{2}|$, which is not differentiable. So, we know that the limit of this type sequence may not be differentiable.

Exercise 4:

Let f be a lebesgue measurable function such that

$$\int_0^1 f(x) e^{Kx}\, dx = 0$$

for all $K = 1, 2, 3, \ldots$. Show that necessarily $f(x) = 0$ for almost every $0 \le x \le 1$.