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Exercise 1:

Let X be a measurable space $f_n : X \mapsto \mathbb{R}$ a sequence of measurable functions, and $f : X \mapsto \mathbb{R}$ a measurable function. By definition we say that f_n converges to f in measure if for all $\epsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}),$$

converges to zero, as $n \to \infty$, where μ is the measure on X.

(i) Find a measure space X, a sequence of measurable functions $f_n : X \to \mathbb{R}$, and $f : X \to \mathbb{R}$ a measurable function such that f_n converges to f almost everywhere but not in measure.

(ii) Find a measure space Y, a sequence of measurable functions $g_n : Y \to \mathbb{R}$, and $g : Y \to \mathbb{R}$ a measurable function such that g_n converges to g in measure but not almost everywhere.

Solution:

(i) Let $X = [1, \infty]$, $f_n(x) = 1_{[n,n+1]}(x), n \in \mathbb{N}$ and $f(x) = 0$. Firstly we prove that f_n converges to f almost everywhere. Let $\epsilon > 0$ be given, for each $x \in X$, choose $N = [x]+1$, where $[x]$ is the largest integer which is less than x, then

$$|f_n(x) - f(x)| = |1_{[n,n+1]}(x) - 0| = 0 < \epsilon, \quad \forall n \geq N.$$

Thus $f_n \to f$ pointwise on X, which implies $f_n \to f$ pointwise almost everywhere on X.

Next we show that f_n does not converge to f in measure. When $0 < \epsilon < 1$, for each $n \in \mathbb{N}$,

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) = \mu([n, n+1]) = 1,$$

which yields that f_n does not converge to f in measure.

(ii) Let $Y = [0, 1]$,

$$g_n(y) = 1_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(y), \ k \geq 0 \text{ and } 2^k \leq n < 2^{k+1},$$

and let $g(y) = 0, \forall y \in Y$. Firstly we prove that g_n converges to g in measure. Let $\epsilon > 0$ be given, then for each $n \in \mathbb{N}$,

$$\mu(\{y \in Y : |g_n(y) - g(y)| > \epsilon\}) \leq \mu\Big(\Big[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\Big]\Big) = \frac{1}{2^k} < \frac{2}{n},$$

since $2^k \leq n < 2^{k+1}$. Choose $N = [2/\epsilon] + 1$, then

$$\mu(\{y \in Y : |g_n(y) - g(y)| > \epsilon\}) < \frac{2}{n} < \epsilon, \quad \forall n \geq N,$$

thus g_n converges to g in measure.

Next we argue that g_n does not converges to g almost everywhere on Y. Let $0 < \epsilon < 1$, for any $y \in Y$, since

$$\sum_{k=0}^{\infty} \frac{2^k}{2^k} = \sum_{k=0}^{\infty} 1 = \infty,$$

for any $N \in \mathbb{N}$, there exists $n \geq N$ such that

$$|g_n(y) - g(y)| = 1 > \epsilon.$$

Therefore g_n is nowhere converge to g on $Y = [0, 1]$.

We can give another example. Let $Y = [0, 1]$ and the sequence of g_n as follows

$$g_1 = 1_{[0,1]}, g_2 = 1_{[0,\frac{1}{2}]}, g_3 = 1_{[\frac{1}{2},\frac{5}{6}]}, g_4 = 1_{[\frac{5}{6},1]} + 1_{[0,\frac{1}{12}]}, \cdots$$

where $\mu(\{y \in Y : |g_n(y) - 0| \neq 0\}) = \frac{1}{n}, \forall n \in \mathbb{N}$. Let $g(y) = 0$ for all $y \in Y$. By the definition of g_n and g, let $\epsilon > 0$ be given, choose $N = [1/\epsilon] + 1$, then

$$\mu(\{y \in Y : |g_n(y) - g(y)| > \epsilon\}) \leq \frac{1}{n} < \epsilon, \forall n \geq N.$$

Thus g_n converges to g in measure. Similarly, when $0 < \epsilon < 1$, for any $y \in Y$ and $N \in \mathbb{N}$, there exist $n \geq N$ such that

$$|g_n(y) - g(y)| = 1 > \epsilon$$

since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. Therefore g_n is nowhere converge to g on $Y = [0, 1]$.

Exercise 2:

Let X be a metric space such that X is a finite set.

(i) Show that any convergent sequence in X is eventually constant.

(ii) Find (with proof) all subsets of X that are compact.

Solution:

(i) Let (X, ρ) be the metric space. Denote $X = \{x_1, x_2, \cdots, x_m\}$, where m is a finite constant. Let

$$a = \inf\{\rho(x_i, x_j) : i, j \in 1, 2, \cdots, m, i \neq j\}.$$

Thus $a > 0$. Suppose $\{y_n\}$ is a convergent sequence in X, and $y \in X$ is the limit of y_n. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\rho(y_n, y) < \epsilon, \quad \forall n \geq N.$$

When $\epsilon < a$, by the definition of a, we have $y_n = y, \forall n \geq N$.

(ii) We claim that any subset of X is compact. Let Y be a nonempty subset of X, we want to prove that any open cover of Y has a finite subcover. Suppose $Y = \{y_1, y_2, \cdots, y_k\}$, where $y_i \in X, i = 1, 2, \cdots, k$. Let $O = \bigcup_{n=1}^{\infty} O_n$ is an open cover of Y. Then for each $y_i \in Y, i = 1, 2, \cdots, k$, since $Y \subset O$, there exists O_i such that $y_i \in O_i$. Thus $\bigcup_{i=1}^{k} O_i \subset O$ is an finite open cover of Y. Hence Y is compact. If $Y = \emptyset$, Y is also compact. Therefore any subset of X is compact.

<u>Exercise 3:</u>

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \text{ or } x \in [0,1] \setminus \mathbb{Q} \\ 1/q, & \text{if } x \in \mathbb{Q} \bigcap (0,1] \text{ and } x = p/q \text{ in lowest terms.} \end{cases}$$

For instance, $f(0.75) = 1/4$ due to $0.75 = 3/4$ in lowest term; $f(1/\sqrt{2}) = 0$ due to $1/\sqrt{2} \notin \mathbb{Q}$.

(i) Is f a Lebesgue measurable function? Justify your answer.

(ii) Find $\int_0^1 f(x)\,dx$.

(iii) Prove that $f(x) \le x$ for all $x \in [0, 1]$.

(iv) Find the set of points of discontinuity of f in $[0, 1]$.

Solution:

(i) f is a Lebesgue measurable function. Let $c \in \mathbb{R}$. If $c < 0$, since $f(x) \ge 0, \forall x \in [0, 1]$, then $\{x \in [0, 1] : f(x) > c\} = [0, 1]$ is measurable. If $c \ge 0$, by the definition of f, $\{x \in [0, 1] : f(x) > c\} \subset (0, 1] \bigcap \mathbb{Q}$. Thus $m^*(\{x \in [0, 1] : f(x) > c\}) \le m^*((0, 1] \bigcap \mathbb{Q}) = 0$, where m^* is the outer measure. We have $\{x \in [0, 1] : f(x) > c\}$ is measurable. Therefore for any $c \in \mathbb{R}$, the set $\{x \in [0, 1] : f(x) > c\}$ is measurable, we know that f is a Lebesgue measurable function.

(ii) Firstly, for $x \in (0, 1] \bigcap \mathbb{Q}$, $f(x) \le 1$, we have

$$\begin{aligned} \int_0^1 f(x)\,dx &= \int_{[0,1]\setminus\mathbb{Q}} f + \int_{(0,1]\bigcap\mathbb{Q}} f \\ &\le 0 + 1 \times m((0, 1] \bigcap \mathbb{Q}) = 0. \end{aligned}$$

And since $f(x) \ge 0, \forall x \in [0, 1]$, then $\int_0^1 f(x)\,dx \ge 0$. Therefore

$$\int_0^1 f(x)\,dx = 1.$$

(iii) When $x = 0$ or $x \in [0, 1] \setminus \mathbb{Q}$, $f(x) = 0 \le x$. When $x \in (0, 1] \bigcap \mathbb{Q}$, $x = p/q$ and $f(x) = 1/q$. Since $p \ge 1$, we have $f(x) \le x$. Thus for any $x \in [0, 1]$, $f(x) \le x$.

(iv) We claim that $(0, 1] \bigcap \mathbb{Q}$ is the set of points of discontinuity of f in $[0, 1]$. Suppose $x = p/q \in (0, 1] \bigcap \mathbb{Q}$, then $f(x) = 1/q > 0$. For every $\delta > 0$, the interval $(x - \delta, x + \delta)$ contains irrational point y such that $f(y) = 0$ and $|f(x) - f(y)| = 1/q > 0$. If $0 < \epsilon < 1/q$, then for every $\delta > 0$, we can choose $y \in (x - \delta, x + \delta)$ such that $|f(x) - f(y)| > \epsilon$. Therefore f is discontinuous at x. By the arbitrary of $x \in (0, 1] \bigcap \mathbb{Q}$, we know that $(0, 1] \bigcap \mathbb{Q}$ is the set of points of discontinuity of f in $[0, 1]$.

<u>**Exercise 4:**</u>

Find (with proof)

$$\lim_{n\to\infty} \int_0^1 \frac{\sin(x^n)}{x^n}\, dx.$$

Solution:

For any $x \in (0,1)$, and for each $n \in \mathbb{N}$, we have $x^n \in (0,1)$, then

$$0 < \frac{\sin(x^n)}{x^n} < 1.$$

For the fixed $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{\sin(x^n)}{x^n} = 1.$$

Thus $\frac{\sin(x^n)}{x^n} \to 1$ almost everywhere on $[0,1]$. And since $1 \in L^1([0,1])$, by the dominate convergence theorem, we have

$$\lim_{n\to\infty} \int_0^1 \frac{\sin(x^n)}{x^n}\, dx = \int_0^1 \lim_{n\to\infty} \frac{\sin(x^n)}{x^n}\, dx = \int_0^1 1\, dx = 1.$$

Exercise 5:

Let $f_n \in L^2([0,1])$ and $f \in L^2([0,1])$.

(i) Prove that $\|f_n - f\|_2 \to 0$ implies that $\|f_n\|_2 \to \|f\|_2$.

(ii) Does $\|f_n\|_2 \to \|f\|_2$ imply $\|f_n - f\|_2 \to 0$? Justify your answer.

(iii) Suppose $\|f_n\|_2 \to \|f\|_2$ and $f_n \to f$ almost everywhere on $[0,1]$. Show that $\|f_n - f\|_2 \to 0$.

Solution:

(i) For each $n \in \mathbb{N}$, as $f_n \in L^2([0,1]), f \in L^2([0,1])$, then $\|f_n\|_2 < \infty$ and $\|f\|_2 < \infty$. By the Minkowski inequality, we have

$$\|f_n - f\|_2 + \|f\|_2 \geq \|f_n\|_2 \Rightarrow \|f_n - f\|_2 \geq \|f_n\|_2 - \|f\|_2,$$

and

$$\|f_n - f\|_2 + \|f_n\|_2 \geq \|f\|_2 \Rightarrow \|f_n - f\|_2 \geq \|f\|_2 - \|f_n\|_2.$$

Thus $|\|f\|_2 - \|f_n\|_2| \leq \|f_n - f\|_2$. Therefore $\|f_n - f\|_2 \to 0$ implies that $\|f_n\|_2 \to \|f\|_2$.

(ii) No, we can give a counter example. Let

$$f_n(x) = \sqrt{n} 1_{[0,\frac{1}{n}]}(x) + 1, \quad f(x) = \sqrt{2}.$$

Then

$$\begin{aligned}
\|f_n\|_2^2 &= \int_0^1 f_n^2(x)\,dx \\
&= \int_0^1 \left(n 1_{[0,\frac{1}{n}]}(x) + 2\sqrt{n} 1_{[0,\frac{1}{n}]}(x) + 1\right) dx \\
&= 2 + \frac{2}{\sqrt{n}} \to 2
\end{aligned}$$

as $n \to \infty$. And since

$$\|f\|_2^2 = \int_0^1 f^2 x\,dx = \int_0^1 2\,dx = 2,$$

we have $\|f_n\|_2 \to \|f\|_2$. But

$$\begin{aligned}
\|f_n - f\|_2^2 &= \int_0^1 (f_n(x) - f(x))^2\,dx \\
&= \int_0^1 \left(n 1_{[0,\frac{1}{n}]}(x) + 2(1-\sqrt{2})\sqrt{n} 1_{[0,\frac{1}{n}]}(x) + (\sqrt{2}-1)^2\right) dx \\
&= 1 + (\sqrt{2}-1)^2 + \frac{2(1-\sqrt{2})}{\sqrt{n}} \\
&\to 1 + (\sqrt{2}-1)^2
\end{aligned}$$

as $n \to \infty$. Thus $\|f_n - f\|_2$ does not converge to 0.

(iii) Let $g_n = 2(f_n^2 + f^2) - |f_n - f|^2$. Then we have $g_n = (f + f_n)^2 \geq 0$. By Fatou's lemma,

$$\int \liminf_n g_n \leq \liminf_n \int 2(f_n^2 + f^2) - |f_n - f|^2.$$

Since $f_n \to f$ almost everywhere on $[0,1]$, $g_n \to 4f^2$ almost everywhere on $[0,1]$. Thus

$$4 \int f^2 \leq \liminf_n \int 2(f_n^2 + f^2) - |f_n - f|^2.$$

As $f_n \in L^2([0,1]), f \in L^2([0,1])$ and $\|f_n\|_2 \to \|f\|_2$, then

$$\lim_{n\to\infty} \int 2f_n^2 = \int 2f^2.$$

Thus

$$4 \int f^2 = 4\|f\|_2^2 \leq 4\|f\|_2^2 - \limsup_n \int |f_n - f|^2,$$

which yields

$$\limsup_n \int |f_n - f|^2 = \limsup_n \|f_n - f\|_2^2 \leq 0.$$

Since $\liminf_n \|f_n - f\|_2^2 \geq 0$, we have

$$\limsup_n \|f_n - f\|_2^2 = \liminf_n \|f_n - f\|_2^2 = 0.$$

Therefore $\|f_n - f\|_2 \to 0$ as $n \to \infty$.