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Exercise 1:

Let f_n be a sequence of continuous functions from $[0,1]$ to $\mathbb{R}$ which is uniformly convergent. Let x_n be in $[0,1]$ such that $f_n(x_n) \geq f_n(x)$, for all x in $[0,1]$.

(i) Is the sequence x_n convergent?

(ii) Show that the sequence $f_n(x_n)$ is convergent.

Solution:

(i) No, the sequence x_n may not convergent. We assume that $f_n(x) = 0$ for all $x \in [0,1]$. And for any $k \in \mathbb{N}$ we set the sequence x_n is

$$x_n = \begin{cases} 0, & n = 2k \\ 1, & n = 2k-1, \end{cases}$$

Then we know that $x_n \in [0,1]$ and $f_n(x_n) = 0 = f_n(x)$ for any $x \in [0,1]$, but the sequence x_n is not convergent.

(ii) We suppose f_n is uniformly converges to f on $[0,1]$. Since f_n is continuous, then f is also a continuous function. For any $y \in [0,1]$, there exist a x, such that $f(y) \leq f(x)$. And since f_n is uniformly converges to f on $[0,1]$, for any $\epsilon > 0$, there exists a $N_1 \in \mathbb{N}$, when $n > N_1$, for any $y \in [0,1]$, we have

$$|f_n(y) - f(y)| < \epsilon,$$

which is equivalent to $f(y) - \epsilon < f_n(y) < f(y) + \epsilon$. We use the x_n to substitute the y, then we have $f_n(x_n) \leq f(x_n) + \epsilon \leq f(x) + \epsilon$.

On the other hand, for the above x, we have $f_n(x_n) \geq f_n(x)$. As f_n is uniformly converges to f on $[0,1]$, for the above $\epsilon > 0$, there exists a $N_2 \in \mathbb{N}$, when $n > N_2$, for the above x, we have $f_n(x) > f(x) - \epsilon$. And then we have $f_n(x_n) > f(x) - \epsilon$. Thus for the above ϵ and x, there exists a N^*, which is the biggest one we related, then when $n > N^*$, we have

$$f(x) - \epsilon < f_n(x_n) < f(x) + \epsilon.$$

So, we know that the sequence $f_n(x_n)$ is convergent.

Exercise 2:

Let $\mathbb{I}$ be the set of all irrational number ($\mathbb{I} \subset \mathbb{R}$).

(i) Using that $\mathbb{Q} = \mathbb{R} \setminus \mathbb{I}$ (the set of all rationals) is countable, show that given $\epsilon > 0$, there is a closed subset $F \subset \mathbb{I}$ such that $|\mathbb{I} \setminus F| < \epsilon$.

(ii) Is F compact? Please explain why or why not.

Solution:

(i) We rearrange the rational number and denote it as $\{a_n\}_{n=1}^{\infty}$. It is a countable set. For $\epsilon > 0$, and for each $a_n \in \mathbb{Q}$, we can find an open set

$$a_n \in (a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}),$$

then we know that $\cup_{n=1}^{\infty}(a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}})$ is an open coverage of $\mathbb{Q}$, and

$$\left| \cup_{n=1}^{\infty} (a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}) \right| \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

We denote $S = \cup_{n=1}^{\infty}(a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}})$, then $\mathbb{R} \setminus S \subset \mathbb{R} \setminus \mathbb{Q} = \mathbb{I}$. We set $F = \mathbb{R} \setminus S$, as S is an open set, then F is closed. And we have

$$|\mathbb{I} \setminus F| = |\mathbb{I}| - |\mathbb{R} \setminus S| = |\mathbb{I}| - |\mathbb{R}| + |S| < \epsilon.$$

(ii) No, $F$is not a compact set. Suppose F is compact, then F is closed and bounded, thus F has finite measure. Since we have $(\mathbb{I} \setminus F) \cup F$, then there exists a $M > 0$ such that

$$|\mathbb{I}| = |(\mathbb{I} \setminus F) \cup F| \leq |\mathbb{I} \setminus F| + |F| < \epsilon + M,$$

which is contradictory with $|\mathbb{I}| = \infty$. Thus F is not compact.

Exercise 3:

Find with proof:

$$\lim_{n \to \infty} \int_0^1 \frac{1 + nx^3}{(1 + x^2)^n} \, dx$$

Solution:

For $x \in (0, 1)$, we denote $f_n(x) = \frac{1+nx^3}{(1+x^2)^n}$. Firstly, for $x \in (0, 1)$, since $(1 + x^2)^n \geq 1 + nx^2$, then we have

$$f_n(x) \leq \frac{1 + nx^3}{1 + nx^2} \leq 1 \in L^1((0, 1)).$$

And for $x \in (0, 1)$, since $(1 + x^2)^n \geq \frac{1}{2}n(n - 1)x^4$, we have

$$f_n(x) = \frac{1 + nx^3}{(1 + x^2)^n} \leq \frac{2 + 2nx^3}{n(n - 1)x^4} = \frac{\frac{2}{x^4}}{n(n - 1)} + \frac{\frac{1}{x}}{n - 1},$$

so for any fixed $x \in (0, 1)$, we have $\lim_{n \to \infty} f_n(x) = 0$, thus we know that $f_n(x)$ converges to 0 pointwise. By the dominate convergence theorem, we have

$$\lim_{n \to \infty} \int_0^1 \frac{1 + nx^3}{(1 + x^2)^n} \, dx = \int_0^1 \lim_{n \to \infty} \frac{1 + nx^3}{(1 + x^2)^n} \, dx = 0.$$

Exercise 4:

Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) = 1$. Let f be in $L^1(X)$ such that $f \geq 0$ almost everywhere.

(i) show that

$$\lim_{p \to 0^+} \int f^p = \mu(\{x \in X : f(x) > 0\})$$

(ii) If $\mu(\{x \in X : f(x) > 0\}) < 1$, find

$$\lim_{p \to 0^+} \Big(\int f^p \Big)^{\frac{1}{p}}.$$

Solution:

(i) Since

$$\begin{aligned} \int_X f^p \, d\mu &= \int_{\{x \in X : f > 0\}} f^p \, d\mu + \int_{\{x \in X : f = 0\}} f^p \, d\mu \\ &= \int_{\{x \in X : f > 0\}} f^p \, d\mu, \end{aligned}$$

as f be in $L^1(X)$ and $f \geq 0$ almost everywhere, by the Fatou's lemma,

$$\mu(\{x \in X : f(x) > 0\}) = \int \mathbb{I}_{\{x \in X : f > 0\}}(x) \, d\mu \leq \liminf_{p \to 0^+} \int_{\{x \in X : f > 0\}} f^p \, d\mu.$$

On the other hand, we know that

$$\begin{aligned} \int_{\{x \in X : f > 0\}} f^p \, d\mu &= \int_{\{x \in X : 0 < f < n\}} f^p \, d\mu + \int_{\{x \in X : f \geq n\}} f^p \, d\mu \\ &\leq \int_{\{x \in X : f \geq n\}} f^p \, d\mu + n^p \mu(\{x \in X : f(x) > 0\}). \end{aligned}$$

For $0 < p < 1$, when $x \in \{x \in X : f(x) > n\}$, we have $f^p < f$, thus we have

$$\begin{aligned} \limsup_{p \to 0^+} \int_{\{x \in X : f > 0\}} f^p \, d\mu &\leq \mu(\{x \in X : f(x) > 0\}) + \limsup_{p \to 0^+} \int_{\{x \in X : f \geq n\}} f^p \, d\mu \\ &\leq \mu(\{x \in X : f(x) > 0\}) + \int_{\{x \in X : f \geq n\}} f \, d\mu \\ &\leq \mu(\{x \in X : f(x) > 0\}) + \int_X f \, \mathbb{I}_{\{x \in X : f \geq n\}}(x) \, d\mu \end{aligned}$$

Since $f \cdot \mathbb{I}_{\{x \in X : f \geq n\}}(x) \leq f \in L^1(X)$ and $\lim_{n \to \infty} f \, \mathbb{I}_{\{x \in X : f \geq n\}}(x) = 0$, by the dominate convergence theorem, we have

$$\limsup_{p \to 0^+} \int_{\{x \in X : f > 0\}} f^p \, d\mu \leq \mu(\{x \in X : f(x) > 0\}),$$

thus we know that

$$\lim_{p \to 0^+} \int f^p = \mu(\{x \in X : f(x) > 0\}).$$

(ii) We denote $S = \{x \in X : f > 0\}$, then

$$\begin{aligned}\int_X f^p \, d\mu &= \int_{\{x\in X: f>0\}} f^p \, d\mu + \int_{\{x\in X: f=0\}} f^p \, d\mu \\ &= \int_S f^p \, d\mu.\end{aligned}$$

And we denote that $F(p) = \log(\int_S f^p \, d\mu)$, then we know that

$$\lim_{p\to 0^+} \left(\int f^p \right)^{\frac{1}{p}} = \lim_{p\to 0^+} e^{\frac{F(p)}{p}}.$$

As $F(0) = \log(\mu(S))$, then we have

$$\begin{aligned}\lim_{p\to 0^+} \left(\int f^p \right)^{\frac{1}{p}} &= \lim_{p\to 0^+} \exp\left\{ \frac{F(p) - \log(\mu(S)) + \log(\mu(S))}{p} \right\} \\ &= \lim_{p\to 0^+} (\mu(S))^{\frac{1}{p}} \exp\left\{ \frac{F(p) - \log(\mu(S))}{p - 0} \right\}.\end{aligned}$$

As $F(p) = \log(\int_S f^p \, d\mu)$, we have

$$F^{'}(p) = \frac{\int_S f^p \cdot \log f \, d\mu}{\int_S f^p \, d\mu},$$

thus we have $F^{'}(0) = \frac{\int_S \log f \, d\mu}{\mu(S)}$. Then we know that

$$\begin{aligned}\lim_{p\to 0^+} \left(\int f^p \right)^{\frac{1}{p}} &= \lim_{p\to 0^+} (\mu(S))^{\frac{1}{p}} \exp\left\{ \lim_{p\to 0^+} \frac{F(p) - F(0)}{p - 0} \right\} \\ &= \lim_{p\to 0^+} (\mu(S))^{\frac{1}{p}} e^{F^{'}(0)} \\ &= 0\end{aligned}$$

as $\mu(S) < 1$.