GCE January, 2019

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Exercise 1:

Let $E := [0,1] - S_{\mathbb{Q}} = [0,1] \bigcap (S_{\mathbb{Q}})^c$ where $S_{\mathbb{Q}} := \{x \in [0,1] | x = \frac{\sqrt{p}}{q} \text{ for some } p, q \in \mathbb{Z}^+\}$. Prove or disprove: There exists a closed, uncountable subset $F \subset E$.

Solution:

We can prove that there exists a closed, uncountable subset $F \subset E$. Since $S_{\mathbb{Q}}$ is a countable set, there exists a bijection between $S_{\mathbb{Q}}$ and the positive integer number in the interval $[0,1]$, so we can enumerate the set $S_{\mathbb{Q}}$ as $\{a_n | n \in \mathbb{N}\}$. Thus we have $S_{\mathbb{Q}} = \{a_n | n \in \mathbb{N}\}$. And then we consider the union

$$\bigcup_{n=1}^{\infty} (a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}),$$

it is the union of countable many open intervals, thus it is an open set. Let

$$A = \Big\{ \bigcup_{n=1}^{\infty} (a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}) \Big\} \bigcap [0,1],$$

thus A is an open subset of $[0,1]$. And we have $S_{\mathbb{Q}} \subset A$.

Since A is an open subset of $[0,1]$, then $[0,1] \bigcap A^c$ is closed in $[0,1]$. We define $F = [0,1] \bigcap A^c$. Since the measure of set A is

$$m(A) = 2 \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \epsilon,$$

then we have $m(F) = 1 - \epsilon > 0$ for all $\epsilon < 1$. Thus the set F is uncountable. Since $F \subset E$ and it is both closed and uncountable, then the proposition is true.

Note that for any countable set S, $S \subset [0,1]$, let $E = [0,1] - S$, we can find a closed, uncountable subset $F \subset E$, and we have the supremum of the measure of F is 1.

Exercise 2:

For x in $[-1,1]$ set $P_n(x) = c_n(1-x^2)^n$ where c_n is such that $\int_{-1}^1 P_n = 1$.

(i) Show that there is a positive constant C such that $c_n \le C\sqrt{n}$.

(ii) Let f be a real valued continuous function on $[0,1]$ such that $f(0) = f(1) = 0$. Set for x in $[0,1]$

$$f_n(x) = \int_0^1 P_n(x-t)f(t)\,dt$$

Show that f_n is uniformly convergence to f.

(iii) Let g be in $L^1((0,1))$. Defining $g_n(x) = \int_0^1 P_n(x-t)g(t)\,dt$, is g_n uniformly convergence to g in $(0,1)$? Does g_n converge to g in $L^1((0,1))$?

Solution:

(i) Method 1:

Since $\int_{-1}^1 c_n(1-x^2)^n\,dx = 1$, we have

$$c_n = \frac{1}{2\int_0^1 (1-x^2)^n\,dx}.$$

Next we need to find a lower bound of the integral term $\int_0^1 (1-x^2)^n\,dx$. For $n > 1$,

$$\begin{aligned} \int_0^1 (1-x^2)^n\,dx &\ge \int_0^{\frac{1}{\sqrt{n}}} (1-x^2)^n\,dx \\ &\ge \frac{1}{\sqrt{n}}(1-\frac{1}{n})^n, \end{aligned}$$

then we have $c_n \le \frac{\sqrt{n}}{2(1-\frac{1}{n})^n}$. We just need to find a lower bound of $(1-\frac{1}{n})^n$. Since $(1-\frac{1}{n})^n = 1 - C_n^1\frac{1}{n} + C_n^2\frac{1}{n^2} + \cdots + (-\frac{1}{n})^n > \frac{1}{3} - \frac{2}{6n^2} > \frac{1}{4}$ when $n > 1$, choose $C = 2$, we have $c_n \le C\sqrt{n}$ for $n > 1$. And for $n = 1$, we get $c_1 = \frac{3}{4} < 2$, for $C = 2$, we also have $c_n \le C\sqrt{n}$ holds.

Method 2:

We change the element and denote $x = \sin y$, then we have $\int_0^{\frac{\pi}{2}} c_n \cos^{2n+1} y\,dy = \frac{1}{2}$. Since

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1} y\,dy = 2n\int_0^{\frac{\pi}{2}} \cos^{2n-1} y\,dy - 2n\int_0^{\frac{\pi}{2}} \cos^{2n+1} y\,dy,$$

we denote $I_{2n+1} = \int_0^{\frac{\pi}{2}} \cos^{2n+1} y\,dy$, then we can get that $(2n+1)I_{2n+1} = 2nI_{2n-1}$. Since $I_1 = \int_0^{\frac{\pi}{2}} \cos y\,dy = 1$, we have $\int_0^{\frac{\pi}{2}} \cos^{2n+1} y\,dy = \frac{(2n)!!}{(2n+1)!!}$. And since

$$\begin{aligned} \frac{(2n)!!}{(2n+1)!!} &= \frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3} \\ &\ge \frac{\sqrt{2n+1}\sqrt{2n-1}\sqrt{2n-1}\sqrt{2n-3}\cdots\sqrt{3}\sqrt{1}}{(2n+1)(2n-1)\cdots 3} \\ &= \frac{1}{\sqrt{2n+1}}, \end{aligned}$$

then we have $c_n \le \frac{\sqrt{2n+1}}{2}$. Choose $C = 1$, we have $c_n \le C\sqrt{n}$.

(ii) Firstly we extend $f(x)$ to a function from $\mathbb{R}$ to $\mathbb{R}$ by zero. Then

$$f_n(x) = \int_0^1 P_n(x-t)f(t)\,dt = \int_{\mathbb{R}} P_n(x-t)f(t)\,dt.$$

Denote $x - t = y$, we have

$$f_n(x) = \int_{\mathbb{R}} P_n(y)f(x-y)\,dy.$$

Then we know that

$$\begin{aligned}
|f_n(x) - f(x)| &= \Big|\int_{\mathbb{R}} P_n(y)f(x-y)\,dy - \int_{-1}^1 P_n(y)f(x)\,dy\Big| \\
&= \Big|\int_{-1}^1 P_n(y)(f(x-y) - f(x))\,dy + \int_{([-1,1])^c} P_n(y)f(x-y)\,dy\Big| \\
&\le \int_{-1}^1 P_n(y)|(f(x-y) - f(x))|\,dy + \int_{([-1,1])^c} |P_n(y)f(x-y)|\,dy.
\end{aligned}$$

When $x \in [0,1]$ and $y \in ([-1,1])^c$, we have $x - y > 1$ or $x - y < 0$, then we know that $f(x-y) = 0$, thus

$$|f_n(x) - f(x)| \le \int_{-1}^1 P_n(y)|(f(x-y) - f(x))|\,dy.$$

As f is continuous on $\mathbb{R}$, $\forall \epsilon > 0$, there exists $\delta > 0$ such that when $|x - y - x| < \delta$, we have $|f(x-y) - f(x)| < \epsilon/2$. Denote $S = [-1,1]\bigcap(-\delta,\delta)^c$. Since $f(x)$ is continuous in $\mathbb{R}$, we denote $\sup_{x\in[0,1]} f(x) = \sup_{x\in\mathbb{R}} f(x) = M$, then $M < +\infty$ and

$$\begin{aligned}
|f_n(x) - f(x)| &\le \int_{-\delta}^{\delta} P_n(y)|(f(x-y) - f(x))|\,dy + \int_S P_n(y)|(f(x-y) - f(x))|\,dy \\
&\le \frac{\epsilon}{2}\int_{-\delta}^{\delta} P_n(y)\,dy + 2M\int_S P_n(y)\,dy \\
&\le \frac{\epsilon}{2} + 2M\int_S c_n(1-y^2)^n\,dy \\
&\le \frac{\epsilon}{2} + 4MC\sqrt{n}\int_{\delta}^1 (1-y^2)^n\,dy \\
&\le \frac{\epsilon}{2} + 4MC\sqrt{n}(1-\delta)(1-\delta^2)^n,
\end{aligned}$$

where C is a constant from the (i). Since $\lim_{n\to+\infty} 4MC\sqrt{n}(1-\delta)(1-\delta^2)^n = 0$, then there exists a $N \in \mathbb{N}$ such that when $n > N$, we have $4MC\sqrt{n}(1-\delta)(1-\delta^2)^n < \epsilon/2$. Overall, we know that $\forall\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n > N$ and $x \in [0,1]$, we have $|f_n(x) - f(x)| < \epsilon$, Thus f_n converges uniformly to f on $[0,1]$.

(iii) Firstly, the $g_n(x)$ is not uniformly convergent to g on $(0,1)$, we can give an counter example as following. Let

$$g(x) = \begin{cases} 1, & x = \frac{1}{2} \\ 0, & x \in (0,\frac{1}{2})\bigcup(\frac{1}{2},1), \end{cases}$$

obviously $g(x)$ is not continuous in $(0,1)$, but we have $g_n(x) = \int_0^1 P_n(x-t)g(t)\,dt = 0, \forall x \in (0,1)$. Thus $g_n(x)$ is continuous in $[0,1]$. Therefore $g_n(x)$ does not convergent uniformly to $g(x)$ on $(0,1)$.

Secondly, we can show that $g_n(x)$ convergent to $g(x)$ in $L^1((0,1))$. Since the continuous functions with compact support are dense in L^1 space, then for all $\epsilon > 0$, there exist a continuous function $f(x) \in C_c([0,1])$ such that $\|f-g\|_1 < \epsilon$. We define the $f_n(x)$ as (ii), then by triangle inequality for the norm,

$$\|g-g_n\|_1 \leq \|g-f\|_1 + \|f-f_n\|_1 + \|f_n - g_n\|_1.$$

Since f_n is uniformly converges to f, for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n > N$ and $x \in (0,1)$, we have $|f(x) - f_n(x)| < \epsilon/2$. Thus for the above $\epsilon > 0$, for all $n \geq N$,

$$\|f-f_n\|_1 = \int_{(0,1)} |f(x) - f_n(x)|\,dx \leq \frac{\epsilon}{2} m((0,1)) = \frac{\epsilon}{2}.$$

Since $P_n(x-t)$ is continuous for $t \in [0,1]$, we can find the upper bound for $P_n(x-t)$, which is denoted as C. And since

$$\begin{aligned}
\|f_n - g_n\|_1 &= \int_0^1 \Big| \int_0^1 P_n(x-t)g(t) - \int_0^1 P_n(x-t)f(t)\,dt \Big| dx \\
&= \int_0^1 \Big| \int_0^1 P_n(x-t)(g(t)-f(t))\,dt \Big| dx \\
&\leq \int_0^1 \int_0^1 P_n(x-t)|g(t)-f(t)|\,dtdx,
\end{aligned}$$

we have

$$\begin{aligned}
\|f_n - g_n\|_1 &\leq \int_0^1 \int_0^1 P_n(x-t)|g(t)-f(t)|\,dtdx \\
&\leq C \int_0^1 \int_0^1 |g(t)-f(t)|\,dtdx \\
&= C \int_0^1 |g(t)-f(t)|\,dt \\
&= C\|g-f\|_1.
\end{aligned}$$

For the above $\epsilon > 0$, by the property that continuous function is dense in L^1 space, we have $\|f-g\|_1 < \epsilon/(2(C+1))$. Thus for all $n \geq N$,

$$\begin{aligned}
\|g_n - g\|_1 &\leq \|g-f\|_1 + \|f-f_n\|_1 + \|f_n-g_n\|_1 \\
&\leq \|g-f\|_1 + \frac{\epsilon}{2} + C\|g-f\|_1 \\
&= (C+1)\|g-f\|_1 + \frac{\epsilon}{2} \\
&\leq \epsilon.
\end{aligned}$$

Therefore we have g_n convergent to g in $L^1((0,1))$.

Exercise 3:

Give an example of $f_n, f : \mathbb{R} \mapsto [0, \infty)$ such that $f_n \in L^1(\mathbb{R})$ for every $n \in \mathbb{N}$, $f \in L^2(\mathbb{R})$, $f_n \leq f$ for every $n \in \mathbb{N}$, $f_n \to 0$ a.e., and $\int f_n \nrightarrow 0$.

Solution:

Let $f(x) = \frac{1}{x}\mathbb{I}_{[1,\infty)}$ and $f_n(x) = \frac{1}{x}\mathbb{I}_{[n,2n]}$. For each $n \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} |f_n(x)|\, dx = \int_n^{2n} \frac{1}{x}\, dx = \ln 2,$$

thus $f_n \in L^1(\mathbb{R})$ for every $n \in \mathbb{N}$. And since

$$\int_{\mathbb{R}} |f(x)|^2\, dx = \int_1^{\infty} \frac{1}{x^2}\, dx = 1,$$

we have $f \in L^2(\mathbb{R})$. For all $n \in \mathbb{N}$, f_n is just the restriction of f on the interval $[n, 2n]$, then $f_n \leq f$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have $f_n(x) \leq \frac{1}{n}$ for all $x \in [n, 2n]$, thus $f_n \to 0$ almost everywhere. But we have

$$\int_{\mathbb{R}} f_n(x)\, dx = \int_n^{2n} \frac{1}{x}\, dx = \ln 2,$$

for all $n \in \mathbb{N}$, thus $\int f_n \nrightarrow 0$.