GCE May, 2018

Jiamin JIAN

Exercise 1:

Let (X,ρ) be a metric space and K_n a sequence of compact subsets of X such that $K_{n+1} \subset K_n$. Set

$$d_n = \sup\{\rho(x,y) : x \in K_n, y \in K_n\}$$

Assuming that d_n converges to zero show that $\bigcap_{n=1}^{\infty} K_n$ is a singleton.

Solution:

We can show that for each $n \in \mathbb{N}$, there exists $x_n, y_n \in K_n$ such that $d_n = \rho(x_n, y_n)$. By the definition of d_n, there exists sequences $\{u_m\}$ and $\{v_m\}$ in K_n such that $d_n = \lim_{m\to\infty} \rho(u_m, v_m)$. As K_n is compact, there exists a convergent subsequence $\{u_{m_k}\}$ of $\{u_m\}$ in K_n and a convergent subsequence $\{v_{m_k}\}$ of $\{v_m\}$ in K_n. Suppose the limit of $\{u_{m_k}\}$ is $x_n \in K_n$ and the limit of $\{v_{m_k}\}$ is $y_n \in K_n$, by the triangle inequality,

$$\rho(u_{m_k}, v_{m_k}) \leq \rho(x_n, u_{m_k}) + \rho(x_n, y_n) + \rho(v_{m_k}, y_n),$$

let $k \to \infty$, then

$$\lim_{k\to\infty} \rho(u_{m_k}, v_{m_k}) = d_n \leq \rho(x_n, y_n).$$

By the definition of d_n, for all $x_n, y_n \in K_n$, we have $\rho(x_n, y_n) \leq d_n$, thus $\rho(x_n, y_n) = d_n$.

Since $\lim_{n\to+\infty} d_n = 0$, it means the diameter of the intersection of the K_n is zero. So, $\bigcap_{n=1}^{\infty} K_n$ is either empty or consists of a single point. For any $n \in \mathbb{N}$, we pick an element $a_n \in K_n$. So we can get a point sequence $\{a_n\}$, and we have $\{a_n : n \in \mathbb{N}\} \in K_1$. Since K_1 is compact, then we know there exists a sub-sequence of a_n, which is denoted as a_{n_k}, converges to a point a. For any $n \in \mathbb{N}$, since each K_n is compact, and a is the limit of a sequence in K_n, we have $a \in K_n$. Thus $a \in \bigcap_{n=1}^{\infty} K_n$. So we know that $\bigcap_{n=1}^{\infty} K_n$ is a singleton.

Exercise 2:

(i) Let $[a,b]$ be an interval in $\mathbb{R}$. If $\tilde{f}$ is continuous on $[a,b]$, g is differentiable on $[a,b]$ and monotonic, and g' is continuous on $[a,b]$, show that there is a c in $[a,b]$, such that

$$\int_a^b \tilde{f}g = g(a)\int_a^c \tilde{f} + g(b)\int_c^b \tilde{f}.$$

Hint: Introduce $F(x) = \int_a^x \tilde{f}$ and integral by parts.

(ii) Show that if g is as specified above and f is in $L^1([a,b])$, there is a c in $[a,b]$ such that

$$\int_a^b fg = g(a)\int_a^c f + g(b)\int_c^b f.$$

Solution:

(i) Since $\tilde{f}$ is continuous on $[a,b]$, we can introduce $F(x) = \int_a^x \tilde{f}$, and we have $F'(x) = \tilde{f}(x)$. Then by using integral by parts, we have

$$\begin{aligned}
\int_a^b f(\tilde{x})g(x)\,dx &= \int_a^b g(x)\,dF(x) \\
&= g(b)F(b) - g(a)F(a) - \int_a^b g'(x)F(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - g(a)\int_a^a \tilde{f}(x)\,dx - \int_a^b g'(x)F(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - \int_a^b g'(x)F(x)\,dx.
\end{aligned}$$

Since g is differentiable on $[a,b]$ and monotonic, and g' is continuous on $[a,b]$, we know that g' is integrable in $[a,b]$ and $g'(x) \geq 0$ for all $x \in [a,b]$. And since $F(x)$ is continuous, by the mean value theorem for integral, there exists $c \in [a,b]$, and

$$\int_a^b g'(x)F(x)\,dx = F(c)\int_a^b g'(x)\,dx = F(c)(g(b) - g(a)).$$

Thus for this $c \in [a,b]$, we have

$$\begin{aligned}
\int_a^b f(\tilde{x})g(x)\,dx &= g(b)\int_a^b \tilde{f}(x)\,dx - F(c)(g(b) - g(a)) \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - (g(b) - g(a))\int_a^c \tilde{f}(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - g(b)\int_a^c \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx \\
&= g(b)\int_c^b \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx.
\end{aligned}$$

(ii) Since $C_c([a,b])$ is dense in $L^1([a,b])$, then we know that for any $f \in L^1([0,1])$, there exists a function sequence $\{f_n\} \subset C_c([a,b])$ such that $\int_a^b |f_n - f| \to 0$ as $n \to +\infty$.

Since g is differentiable on $[a,b]$ and monotonic, we know there exists $K>0$, and $\forall x\in[a,b]$, we have $|g(x)|\le K$. So, we have

$$\lim_{n\to+\infty}\int_a^b |gf-gf_n|\le K\lim_{n\to+\infty}\int_a^b |f-f_n|=0,$$

then by the conclusion we get from (i) we have

$$\int_a^b fg=\lim_{n\to+\infty}\int_a^b f_n g=\lim_{n\to+\infty}\Big(g(a)\int_a^{c_n} f_n+g(b)\int_{c_n}^b f_n\Big),$$

where c_n is depends on f_n for each n.

Since $\{c_n\}\subset[a,b]$ and $[a,b]$ is compact, there exists a subsequence of $\{c_n\}$, which is denoted as $\{c_{n_k}\}$, converges to c and $c\in[a,b]$. Thus we have

$$\begin{aligned}\int_a^b fg &= \lim_{k\to+\infty}\Big(g(a)\int_a^{c_{n_k}} f_{n_k}+g(b)\int_{c_{n_k}}^b f_{n_k}\Big)\\ &= \lim_{k\to+\infty}\Big(g(a)\int_a^{c} f_{n_k}+g(a)\int_c^{c_{n_k}} f_{n_k}+g(b)\int_{c_{n_k}}^c f_{n_k}+g(b)\int_c^b f_{n_k}\Big)\\ &= g(a)\int_a^c f+g(b)\int_c^b f+\lim_{k\to+\infty}\Big(g(a)\int_c^{c_{n_k}} f_{n_k}+g(b)\int_{c_{n_k}}^c f_{n_k}\Big)\\ &= g(a)\int_a^c f+g(b)\int_c^b f.\end{aligned}$$

Exercise 3:

Let $\{f_n\}$ be a sequence of functions $f_n : [0,1] \to \mathbb{R}$.

(i) Define equicontinuity for this sequence.

(ii) Show that if each f_n is differentiable on $[0,1]$ and $|f_n'(x)| \leq 1$ for all x in $[0,1]$ and $n \in \mathbb{N}$, the sequence is equicontinuous.

(iii) Suppose the sequence is uniformly bounded and that (ii) holds. Show that f_n has a subsequence which converges uniformly to a continuous function.

(iv) Show through an example that the limit may not be differentiable.

Solution:

(i) The definition of equicontinuity of sequence $\{f_n\}$ at point x is as follows: $\forall \epsilon > 0, \exists \delta > 0$, such that whenever $|x - y| < \delta$ and $\forall n \in \mathbb{N}$, we have $|f_n(x) - f_n(y)| < \epsilon$.

(ii) Since f_n is differentiable on $[0,1]$, by the mean value theorem, we know that $\forall x, y \in [0,1]$, there exists a $c \in [x, y]$ and we have

$$|f_n(y) - f_n(x)| = |f_n'(c)||y - x|.$$

Since $|f_n'(x)| \leq 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}$, then we have

$$|f_n(y) - f_n(x)| \leq |y - x|.$$

Let $\epsilon > 0$ be given, choose $\delta = \epsilon$, then when $|y-x| < \delta$, $\forall n \in \mathbb{N}$, we have $|f_n(y)-f_n(x)| < \epsilon$. So we know the sequence $\{f_n\}$ is equicontinuous.

(iii) By the Arzel*à*-Ascoli theorem, we can get f_n has a subsequence which converges uniformly to a continuous function directly. Next we can show the proof of Arzel*à*-Ascoli theorem.

We enumerate $\{x_i\}_{i \in \mathbb{N}}$ as the rational number in $[0,1]$. Since the sequence $\{f_n\}$ is uniformly bounded, then the set of points $\{f_n(x_1)\}$ is bounded, by the Bolzano-Weierstrass theorem, there is a subsequence $\{f_{n1}(x_1)\}$ converges. Repeating the same argument for the sequence points $\{f_{n1}(x_2)\}$, there is a subsequence $\{f_{n2}\}$ of $\{f_{n1}\}$ such that $\{f_{n2}(x_2)\}$ converges. By induction this process can be continued forever, and so there is a chain of subsequences

$$\{f_n\} \supset \{f_{n1}\} \supset \{f_{n2}\} \supset \cdots$$

Such that for each $k \in \mathbb{N}$, the subsequence$\{f_{nk}\}$ converges at point x_k. We choose the diagonal subsequence $\{f_{kk}\}$. Except for the first n functions, $\{f_{kk}\}$ is a subsequence of the nth row $\{f_{nk}\}$. Therefore, the sequence $\{f_{kk}\}$ converges simultaneously on all x_n.

Next we need to show that $\{f_{kk}\}$ is converges uniformly on $[a, b]$. We just need to prove the uniform Cauchy criterion holds. Given any $\epsilon > 0$ and rational $x_k \in [0,1]$, there is an integer $N(\epsilon, x_k)$ such that when $n, m > N$, we have

$$|f_{nn}(x_k) - f_{mm}(x_k)| < \frac{\epsilon}{3}.$$

Since $\bigcap(x_k - \frac{1}{n}, x_k + \frac{1}{n})$ covers the compact interval [0, 1], then by the Heine-Borel theorem there is a finite subcover, we denote the finite subcover as $U_1, \ldots, U_J$. There exists an integer K such that each open interval U_j, $1 \leq j \leq J$, contains a rational number x_k with $1 \leq k \leq K$. Finally, for any $x \in [0,1]$, there are j and k so that x and x_k belong to the same interval U_j. For this k, we have

$$\begin{aligned}|f_{nn}(x) - f_{mm}(x)| &\leq |f_{nn}(x) - f_{nn}(x_k)| + |f_{nn}(x_k) - f_{mm}(x_k)| + |f_{mm}(x_k) - f_{mm}(x)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

for all $N = \max\{N(\epsilon, x_1), \ldots, N(\epsilon, x_K)\}$ as f_n is equicontinuous. So, for the subsequence $\{f_{kk}\}$, the uniform Cauchy criterion holds. Thus we know that $\{f_{kk}\}$ converges to a continuous function.

(iv) We denote $f_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}}$, $x \in [0,1]$. Since for all $n \in \mathbb{N}$ and $x \in [0,1]$,

$$|f_n'(x)| = \left|\frac{x - \frac{1}{2}}{\sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}}}\right| < 1$$

and $f_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}} < 2$, by the conclusion we get from (ii) and (iii), we know that the sequence $\{f_n\}$ is equicontinuous and it has a subsequence which converges uniformly to a continuous function. When $n \to +\infty$, we have $f_n(x) \to f(x) = |x - \frac{1}{2}|$, which is not differentiable. So, we know that the limit of this type sequence may not be differentiable.

<u>Exercise 4:</u>

Let f be a lebesgue measurable function such that

$$\int_0^1 f(x)e^{Kx}\,dx = 0$$

for all $K = 1, 2, 3, \ldots$. Show that necessarily $f(x) = 0$ for almost every $0 \leq x \leq 1$.

Solution:

Let $A = \{x \in [0,1] : f(x) \neq 0\}$, we want to show that $m(A) = 0$, where $m(\cdot)$ is the Lebesgue measure. For each $n \in \mathbb{N}$, let $A_n = \{x \in [0,1] : f_n(x) \geq 1/n\}$. Then if f is nonnegative on $[0,1]$, then

$$A = \bigcup_{n=1}^{\infty} A_n.$$

And for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \int_0^1 f(x)e^{Kx}\,dx &\geq \int_{A_n} f(x)e^{Kx}\,dx \\ &\geq \int_{A_n} \frac{1}{n}e^{Kx}\,dx \\ &\geq \frac{1}{n}m(A_n). \end{aligned}$$

Thus $0 \geq 1/n \cdot m(A_n)$ for each $n \in \mathbb{N}$. We have $m(A_n) = 0$ for all $n \in \mathbb{N}$. Therefore

$$m(A) = m\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} m(A_n) = 0.$$

Thus we have $m(A) = 0$, which follows that $f(x) = 0$ for almost every $0 \leq x \leq 1$.