MA - 503 Midterm - Fall 2018
NO DOCUMENTS ALLOWED

Exercise 1

Let (X, d) be a metric space. Let K be a compact subset of X and V an open subset of X such that $K \subset V$ and $V \neq X$.
(i). Does $K \neq V$ hold in general?
(ii). We set for x in X

$$f(x) = \frac{d(x, V')}{d(x, K) + d(x, V')}.$$

Show that f is continuous and find, with proof, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$.

Exercise 2

Let $f(x) = |x|$ for x in $(-\pi, \pi)$.
(i). Use f to show that $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$.
(ii). Is the Fourier series of f uniformly convergent? Derive the formula $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

Exercise 3

Let f be in $L^1(\mathbb{R}^d)$ such that $\int f = 1$. Set $f_n(x) = n^d f(nx)$.
(i). Let g in $C_c(\mathbb{R}^d)$. Show that $f_n * g \to g$, pointwise.
(ii). For h in $L^1(\mathbb{R}^d)$, show that $f_n * h \to h$, in L^1 norm.

Exercise 4

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let A_n be a sequence in $\mathcal{A}$ such that $\mu(A_n)$ converges to zero.
(i). Prove or disprove:
if $f : X \to [0, \infty)$ is a measurable function and $\mu(X) < \infty$ then $\int_{A_n} f$ converges to zero.
(ii). Let g be in $L^1(X)$. Show that $\int_{A_n} g$ converges to zero.