MA - 503 Final test - Fall 2016

Number all pages that you are turning in and make sure to write your name on each page.

Exercise 1

Let X be a compact metric space. Show that there is a subset D of X which is countable and dense.

Exercise 2

For any positive ϵ find a subset V of $\mathbb{R}$ which is open, dense, and whose Lebesgue measure is less than ϵ.
Hint: You may denote by $\{a_n : n \in \mathbb{N}, n \geq 1\}$ the set of all rational numbers.

Exercise 3

For all α in $\mathbb{R}$ find, with proof, $\displaystyle\int_0^\infty x^\alpha \, dx$. (in this problem you are only allowed to apply anti-derivative formulas for continuous functions in closed and bounded intervals).

Exercise 4

Let a and b be two real numbers such that $a < b$. Assume that f is in $L^2([a, b])$ and such that

$$\int_a^b fP = 0,$$

for any polynomial function P. Show that f is zero in $L^2([a, b])$.
Hint: You may use that for any continuous function g on $[a, b]$, there is a sequence of polynomial functions P_n which converges uniformly to g.

Exercise 5

(i). Let f be a function in $L^1([0, 1])$. Assume that f is continuous at 0. Find $\displaystyle\lim_{n\to\infty} \int_0^1 ne^{-nx} f(x)dx$.

(ii). Find $\displaystyle\lim_{n\to\infty} \int_{\frac{1}{n}}^1 ne^{-nx} x^{-\frac{1}{2}} dx$.