MA503 Lebesgue Measure and Integration   Date Darko Volkov

Metric Spaces

Introduction

Def: The sequence of real numbers a_n converges to the real number ℓ if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N$, $|a_n - \ell| < \varepsilon$

comment: $|a_n - \ell| < \varepsilon$

$\ell - \varepsilon < a_n < \ell + \varepsilon$ $\ell-\varepsilon$ ℓ $\ell+\varepsilon$

in other words, $\exists N \in \mathbb{N}$ such that $\forall n > N$, the distance between a_n and ℓ is less than ε

Generalize this notion of distance to be able to study questions of convergence (and continuity) are very general sets

Def: Let X be a set and $\rho: X \times X \longrightarrow \mathbb{R}^+$.

We say that ρ is distance on X (and (X, ρ) is a metric space) if for any x, y, z in X

(i) $\rho(x,y) = 0$ iff $x = y$ (definiteness)

(ii) $\rho(x,y) = \rho(y,x)$ (symmetry)

(iii) $\rho(x,y) \le \rho(x,z) + \rho(z,y)$ (triangle inequality)

Ex: 1. $\mathbb{R}$ $\rho(x,y) = |x-y|$

2. in $\mathbb{R}^d$ $x = (x_1, x_2, \cdots x_d)$ $y = (y_1, y_2, \cdots y_d)$

$$\rho(x,y) = \left(\sum_{i=1}^{d} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

This is the Euclidean distance in $\mathbb{R}^d$

Show that this does define a distance on $\mathbb{R}^d$

Introduce the usual inner product on $\mathbb{R}^d$:

for x, y in $\mathbb{R}^d$ $x = (x_1, x_2, \cdots x_d)$ $y = (y_1, y_2, \cdots y_d)$

$$\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i$$

Euclidean norm $|x| = \langle x, x \rangle^{\frac{1}{2}}$

From there $\rho(x,y) = |x - y|$

Pythogrean theorem

Let x, y in $\mathbb{R}^d$ if $\langle x,y\rangle = 0$ then $|x+y|^2 = |x|^2 + |y|^2$

Proof: $|x+y|^2 = \langle x+y, x+y\rangle$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

$$= |x|^2 + |y|^2 + \cancel{2\langle x,y\rangle} \to 0 \text{ by assumption}$$

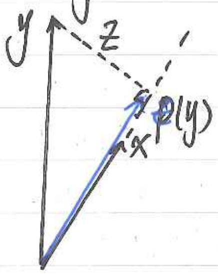
The Cauchy Schwartz Inequality

For any x, y in $\mathbb{R}^d$

$$|\langle x,y\rangle| \le |x||y|$$

proof: if $x=0$, this is clear

if x is nonzero, project y on the line defined by x



introduce $e = \frac{x}{|x|}$

$$p(y) = \langle y,e\rangle e$$

$$p(y) + z = y$$

$$\langle p(y)+z, e\rangle = \langle y,e\rangle$$

$$\langle p(y), e\rangle + \langle z,e\rangle = \langle y,e\rangle$$

note that $\langle p(y), e\rangle = \langle y,e\rangle\langle e,e\rangle = \langle y,e\rangle$

thus $\langle z,e\rangle = 0$ so. $\langle z, \langle y,e\rangle e\rangle = 0$

$$|y|^2 = |z + \langle y,e\rangle e|^2$$

Thus by the pythogrean theorem

$$|y|^2 = |z|^2 + |\langle y,e\rangle e|^2$$

In particular

$$|y|^2 \ge |\langle y,e\rangle e|^2 = |\langle y,e\rangle|^2 = |\langle y, \tfrac{x}{|x|}\rangle|^2$$

so, $|y|^2 \ge |\langle y, \frac{x}{|x|}\rangle|^2$

Multiply this inequality by $|x|^2$

$$|x|^2|y|^2 \ge |x|^2 |\langle y, \tfrac{x}{|x|}\rangle|^2$$

that is $|x|\,|y| \ge |\langle y,x\rangle|$

Prop. For any x and y in $\mathbb{R}^d$ $\quad |x+y| \leq |x|+|y|$

proof: $|x+y|^2 = \langle x+y, x+y\rangle$
$$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$
$$= |x|^2 + |y|^2 + 2\langle x,y\rangle$$
$$\leq |x|^2 + |y|^2 + 2|x||y|$$

so $|x+y|^2 \leq (|x|+|y|)^2$

we conclude that $|x+y| \leq |x|+|y|$

Finally we need to show that
for any x, y, z in $\mathbb{R}^d$
$$\rho(x,y) \leq \rho(y,z) + \rho(z,x)$$

Recall $\rho(x,y) = |x-y|$

Since $\rho(x,y) = |x-y| = |x-z+z-y|$
$$\leq |x-z| + |z-y|$$

We showed that $\quad = \rho(x,z) + \rho(z,y)$

Topology on metric spaces

An open set of a metric space (X,ρ) is the empty set or any union of open balls

Remark: The open ball with center x and positive radius r is the subset of X $\quad \{y : \rho(x,y) < r\}$
$$B(x,r) = \{y \in X : \rho(x,y) < r\}$$

Properties:

① $\emptyset$, X are open

② Any union of open subsets of X is an open subset of X

③ A finite intersection of open subsets of X is an open subset of X

proof:

① fix x in X

$X = \bigcup_{n=1}^{\infty} B(x,n)$ because if y is in X
let n be an integer greater than $\rho(x,y)$
$y \in B(x,n)$

② clear

③ **Lemma:** V is open in (X,ρ) if and only if

$\forall x \in V, \ \exists \alpha > 0 \quad B(x,\alpha) \subseteq V$

proof of lemma

assume V is open and not empty

let x be in V

By definition, there is an open ball $B(y,r)$ such that

$\{x\} \subset B(y,r) \subseteq V$

↑ x is not the center of the ball

set $\alpha = r - \rho(x,y)$ (Note that $r>0$)

Show $B(x,\alpha) \subseteq B(y,r)$

We can also denote $\alpha \leq r - \rho(x,y)$

$\rho(z,y) \leq \rho(z,x) + \rho(x,y)$

$< \alpha + \rho(x,y) \leq r$

$\rho(z,y) < r \Rightarrow z \in B(y,r)$

let z be in $B(x,\alpha)$

$$\rho(z,y) \leq \rho(z,x) + \rho(x,y) < \alpha + \rho(x,y) = r$$

so $z \in B(y,r)$, so $z \in V$

Thus, $B(x,\alpha) \subseteq V$

conversely

assume that V is a subset of X such that

$\forall x \in V, \ \exists \alpha_x \quad B(x,\alpha_x) \subseteq V$

$$V = \bigcup_{x\in V} B(x,\alpha_x)$$

(i) $V \subseteq \bigcup_{x\in V} B(x,\alpha_x)$ for any $x \in V$

$\Rightarrow x \in B^*(x,\alpha_x) \subset \bigcup_{x\in V} B(x,\alpha_x)$

(ii) $x \in \bigcup_{x\in V} B(x,\alpha_x)$

$\Rightarrow x \in B^*(x,\alpha_x) \subseteq V$

$\Rightarrow x \in V$

From there, let V_1 and V_2 be 2 open sets

Show that $V_1 \cap V_2$ is open

if $V_1 \cap V_2 = \phi$, clear, otherwise

let x be in $V_1 \cap V_2$

$\exists \alpha_1 > 0$ such that $B(x,\alpha_1) \subseteq V_1$

$\exists \alpha_2 > 0$ such that $B(x,\alpha_2) \subseteq V_2$

set $\alpha = \min\{\alpha_1, \alpha_2\}$

$B(x,\alpha) \subset V_1 \cap V_2$

this shows that $V_1 \cap V_2$ is open

Remark: An infinite intersection of open sets is not open, in general

Example. in $\mathbb{R}$

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)=\{0\}$$

Closure and Closed sets

(X,ρ) is a metric space $A\subset X$, a subset
We say that x is in the closure of A, if $\forall \alpha>0$ $A\cap B(x,\alpha)\neq\emptyset$
We define the set $\overline{A}$

$$\overline{A}=\{x\in X : x \text{ is in the closure of } A\}$$

Example in $\mathbb{R}$

$$\overline{(a,b]}=[a,b]$$

Closed subsets

By definition, E is a closed set in (X,ρ) if $E=\overline{E}$.
Note. for any subset A, $A\subset\overline{A}$.

Complements

if A is subset of X $A^c=\{x\in X : x\notin A\}$
$A^c=X\setminus A$

Prop. let E be a subset of (X,ρ), E is closed if and only if E^c is open

proof: assume E is closed
let x be in E^c
assume that $\forall\alpha>0$, $B(x,\alpha)\cap E\neq\emptyset$
then $x\in\overline{E}$
because E is closed $\overline{E}=E$
so x is in E. Contradiction
We showed that $\exists\alpha>0$ $B(x,\alpha)\cap E=\emptyset$
in other words $B(x,\alpha)\subseteq E^c$
We showed that E^c is open
conversely assume E^c is open

let x be in $\bar{E}$ show that $x \in E$

argue by contradiction

assume $x \in E^c$

so $\exists \alpha > 0$, s.t. $B(x,\alpha) \subset E^c$

thus $B(x,\alpha) \cap E = \emptyset$: contradicts $x \in \bar{E}$

Corollary Let (X,ρ) be a metric space

- $X, \emptyset$ are closed
- If $E_1, \cdots E_n$ are closed in X, then $\bigcup_{i=1}^{n} E_i$ is closed
- If F_i, $i \in I$ are closed in X, $\bigcap_{i \in I} F_i$ is closed.

proof: $\because$ $X^c = \emptyset$, $\emptyset^c = X$

- $\left(\bigcup_{i=1}^{n} E_i\right)^c = \underbrace{\bigcap_{i=1}^{n} E_i^c}_{\text{open}}$
- $\left(\bigcap_{i \in I} F_i\right)^c = \underbrace{\bigcup_{i \in I} F_i^c}_{\text{open}}$

Metric Subspaces

Let (X,ρ) be a metric space. Let A be a subset of X

Then (A,ρ) is also a metric space

Property let V be any subset of A, V is an open subset of A iff.

$\exists$ open subset W of X such that $V = A \cap W$

(similar statement for closed sets)

Example $A = (0,1]$ is a subset of $\mathbb{R}$

$(\frac{1}{2}, 1]$ is an open subset in A

$(\frac{1}{2}, 1] = A \cap (\frac{1}{2}, 3)$

For x in $(\frac{1}{2}, 1]$

$\exists \alpha > 0$. $B_A(x,\alpha) \subset (\frac{1}{2}, 1]$

$B_A(x,\alpha) = \{y \in A : |x-y| < \alpha\}$

$(0, \frac{1}{2}]$ is a closed set of A

$(0, \frac{1}{2}] = A \cap [-3, \frac{1}{2}]$

Sequence, Convergence, completeness

(X, ρ) is a metric space

Def: We say that the sequences x_n in X convergence to l in X
if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N, \; \rho(x_n, l) < \varepsilon$

Examples

- in $\mathbb{R}$, $a_n = \frac{1}{n}$ a_n converges to 0.

$X = (0, 1]$ $a_n = \frac{1}{n}, n \geq 1$ a_n is valued in X.
a_n diverges in X

Property: If the sequence a_n in (X, ρ) converges to l then l is unique
in other words
if a_n converges to l_1 and a_n converges to l_2 then $l_1 = l_2$

Prop. Every convergent sequence is bounded
Def: x_n is bounded in (X, ρ)
if $\exists M > 0$, $\rho(x_1, x_n) \leq M \quad \forall n \in \mathbb{N}$

proof: to $\varepsilon = 1$ $\exists N \in \mathbb{N}$ s.t $\forall n > N$. with the assumption that
x_n converges to l in (X, ρ)
$\rho(x_n, l) < 1$
for $n > N$ $\rho(x_1, x_n) \leq \rho(x_1, l) + \rho(l, x_n) < \rho(x_1, l) + 1$
pick $M = \max\{\rho(x_1, x_2), \cdots \rho(x_1, x_N), \rho(x_1, l) + 1\}$

Cauchy sequences

We say that the sequence x_n in (X, ρ) is Cauchy, if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$
$\forall p > q > N$, $\rho(x_p, x_q) < \varepsilon$

Prop: If $\{x_n\}$ is convergent sequence then x_n is Cauchy.
proof: Let l be the limit of x_n
Fix $\varepsilon > 0$
$\exists N \in \mathbb{N}, \forall n > N \quad \rho(x_n, l) < \varepsilon$
For $p > q > N$ $\rho(x_p, x_q) \leq \rho(x_p, l) + \rho(x_q, l) < 2\varepsilon$

Find a metric space (X, ρ) and Cauchy sequence x_n which diverges in X

$X = (0, 1]$

$x_n = \frac{1}{n}$

Definition: We say that (X, ρ) is complete if every Cauchy sequence in X convergences in X

Example: ① $\mathbb{R}$

② $\mathbb{R}^d$

Prop: Let (X, ρ) be a complete metric space and Y a subset of X

Y is complete if and only if Y is closed in X

proof: assume that Y is closed in X

let $\{y_n\}$ be a Cauchy sequence in Y

$\{y_n\}$ is also a Cauchy sequence in X

thus y_n converges to some l in X.

But Y is closed: thus l is in Y

$\overline{Y} = Y$

$y_N \in B(l, \varepsilon)$

$B(l, \varepsilon) \cap Y \neq \emptyset$

so $l \in \overline{Y} = Y$.

assume that Y is NOT closed in Y

$\overline{Y} \neq Y$

so $\exists\ l \in \overline{Y}$ such that $l \notin Y$

For every $n \in \mathbb{N}^*$ $B(l, \frac{1}{n}) \cap Y \neq \emptyset$

pick y_n in $B(l, \frac{1}{n}) \cap Y$

$\rho(y_n, l) < \frac{1}{n}$

thus, in the space X, $\lim_{n \to \infty} y_n = l$

Thus, y_n is Cauchy.

But $y_n \in Y$, y_n diverges in Y

Thus, Y is not complete

Application

Any closed subset of $\mathbb{R}$ $(\mathbb{R}^d)$ is complete

Vector space of bounded functions from $X \to \mathbb{R}$, where X is any set. $B(X)$

For $f, g \in B(X)$ define $\rho(f,g) = \sup\{|f(x)-g(x)|\}$ for $x \in X$

it is clear that

$\rho(f,g) = 0$ if and only if $f = g$

it is clear $\rho(f,g) = \rho(g,f)$

Triangle inequality let f, g, h be in $B(X)$

For any x in X

$$|f(x)-g(x)| \le |f(x)-h(x)| + |h(x)-g(x)|$$

For any x in X

$$|f(x)-g(x)| \le \rho(f,h) + \rho(h,g)$$

$$\sup_{x\in X}|f(x)-g(x)| \le \rho(f,h)+\rho(h,g)$$

$$\rho(f,g) \le \rho(f,h)+\rho(h,g)$$

Hwpb: Show that $B(X)$ is complete

Hint: Let f_n be a Cauchy sequence in $B(X)$

Show that all x in X $f_n(x)$ is Cauchy in $\mathbb{R}$

$\to l = f(x)$

Show that f is bounded on X $\rho(f_n, f) \to 0$

Continuity

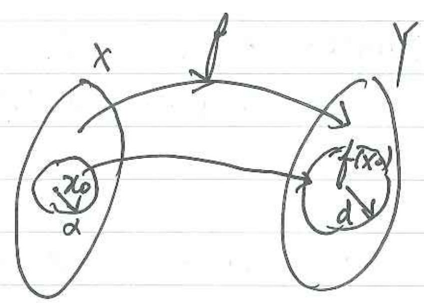
Definition: Let (X,ρ) and (Y,d) be metric spaces

Let $f: X \to Y$ and x_0 be in X

We say that f is continuous at x_0

if $\forall \varepsilon > 0$, $\exists \alpha > 0$ such that $\forall x \in X$

$$\rho(x, x_0) < \alpha \Rightarrow d(f(x), f(x_0)) < \varepsilon$$



Equivalently $\forall \varepsilon > 0$, $\exists \alpha > 0$,

$$f(B(x_0,\alpha)) \subseteq B(f(x_0),\varepsilon)$$

Examples

① $f: (X,\rho) \to (Y,d)$ Let f be a constant function

Then f is continuous at any x_0 in X

② Identity function $(X,\rho) \to (X,\rho)$ $Id(x) = x$
③ A linear function L from $\mathbb{R}^d$ to $\mathbb{R}^m$ is continuous at any x_0 in $\mathbb{R}^d$

Continuity and sequences

Prop: let (X,ρ) and (Y,d) are two metric spaces
let $f: X \to Y$ let x_0 be in X
f is continuous at x_0 if and only if
for every sequence $a_n \in X$ converging to x_0
$f(a_n)$ converges to $f(x_0)$

proof: Assume that f is continuous at x_0 and that a_n converges to x_0
fix $\varepsilon > 0$, $\exists \alpha > 0$, $\forall x \in X$
if $\rho(x, x_0) < \alpha$, $d(f(x), f(x_0)) < \varepsilon$
$\exists N \in \mathbb{N}$ $\forall n > N$ $\rho(a_n, x_0) < \alpha$
Thus $n > N$ $d(f(a_n), f(x_0)) < \varepsilon$, which proves that $f(a_n)$ converges to $f(x)$
assume that f is NOT continuous at x_0:
show that there is a sequence a_n in X converging to x_0 such that $f(a_n)$ does NOT converge to $f(x_0)$
$\exists \varepsilon > 0$, $\forall \alpha > 0$ $\exists x \in X$ s.t. $\rho(x, x_0) < \alpha$ ~~and~~ $d(f(x), f(x_0)) \geq \varepsilon$
to $\alpha = \frac{1}{n}$ $\exists a_n \in X$
such that $\rho(a_n, x_0) < \frac{1}{n}$ and $d(f(a_n), f(x_0)) \geq \varepsilon$.

Example: $f: \mathbb{R}^+ \to \mathbb{R}$

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x > 0 \\ c & \text{if } x = 0 \end{cases}$$

, where c is fixed constant in $\mathbb{R}$

Prove that f is not continuous at 0.

Global continuity

We say that $f(X,\rho) \longrightarrow (Y,d)$ is continuous if f is continuous at every point x_0 in X

prop: f $(X,\rho) \longrightarrow (Y,d)$ is continuous if and only if for every open subset V of Y. $f^{-1}(V)$ is open in X

Remark: $f^{-1}(V)$ "inverse image by f of V"

$$= \{x \in X,\ f(x) \in V\}$$

properties of inverse images

$$f^{-1}(\phi) = \phi$$

if $A \subset B \subseteq Y$, then $f^{-1}(A) \subseteq f^{-1}(B)$

$$f^{-1}(A^c) = (f^{-1}(A))^c$$

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

- if A_i, $i \in I$ is a collection of subsets of X,

$$f^{-1}\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} f^{-1}(A_i)$$

$$f^{-1}\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} f^{-1}(A_i)$$

proof of proposition

assume that f is continuous

let V is open subset in Y

if $f^{-1}(\phi) = \phi$, clear!

Otherwise let x_0 be in $f^{-1}(V)$: $f(x_0) \in V$. V is open

thus $\exists\, \varepsilon > 0$ $B(f(x_0), \varepsilon) \subseteq V$

use the f is continuous at x_0

$\exists\, \alpha > 0$ if $x \in X$ and $\|x - x_0\| < \alpha$, then

$$\|f(x) - f(x_0)\| < \varepsilon$$

that is $f(x) \in B(f(x_0), \varepsilon)$

$$f(B(x_0, \alpha)) \subseteq B(f(x_0), \varepsilon) \subset V$$

that is $B(x_0,\alpha) \subseteq f^{-1}(B(f(x_0),\varepsilon)) \subset f^{-1}(V)$

thus $f^{-1}(V)$ is open

assume that for every open subset V of Y, $f^{-1}(V)$ is open in X

Let x_0 be in X. Show that f is continuous at x_0

fixed $\varepsilon>0$ $\quad f^{-1}(B(f(x_0),\varepsilon))$ is open in X, and contains x_0

thus $\exists \alpha>0$ $B(x_0,\alpha) \subseteq f^{-1}(B(f(x_0),\varepsilon))$

that is $f(B(x_0,\alpha)) \subseteq B(f(x_0),\varepsilon)$

in other words

if $\rho(x,x_0)<\alpha$ then $d(f(x),f(x_0))<\varepsilon$

prop.

$f:(X,\rho)\longrightarrow(Y,d)$ is continuous if and only if for every closed subset F of Y, $f^{-1}(F)$ is closed in X

$f^{-1}(F)^c = f^{-1}(F^c)$

Application

Let X,Y,Z be metric spaces. $f:X\to Y$, $g:Y\to Z$

Assume that f and g are continuous

then $g\circ f: X\longrightarrow Z$ is also continuous

proof: let V be an open subset of Z

$g^{-1}(V)$ is open in Y

so $f^{-1}(g^{-1}(V))$ is open in X

$= (g\circ f)^{-1}(V)$

Def: $f:(X,\rho)\longrightarrow(Y,d)$ We say that f is uniformly continuous on X if $\forall \varepsilon>0$ $\exists \alpha>0$ $\forall x_1,x_2\in X$

$\rho(x_1,x_2)<\alpha \Rightarrow d(f(x_1),f(x_2))<\varepsilon$

We say that f is Lipschitz continuous on X

if $\exists M>0$ $\forall x_1,x_2\in X$ $\quad d(f(x_1),f(x_2)) \leq M\cdot\rho(x_1,x_2)$

Mean Value Theorem

Let $f:[a,b]\to\mathbb{R}$ be a continuous function on the closed interval $[a,b]$, and differentiable on the open interval (a,b), then there exits some c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Remark.

- It is clear that uniform continuity $\Rightarrow$ continuity
- Lipschitz continuity $\Rightarrow$ uniform continuity

Example: Let $L:\mathbb{R}^d\longrightarrow\mathbb{R}^m$ be a Linear function

Show that L is Lipschitz continuous

Let $e_1,\cdots e_d$ be the natural basis of $\mathbb{R}^d$

$x=(x_1e_1,\cdots x_de_d)$ for x in $\mathbb{R}^d$

$$\|Lx\|=\|x_1Le_1+\cdots x_dLe_d\|$$
$$\leq\|x_1Le_1\|+\cdots+\|x_dLe_d\|$$
$$=|x_1|\,\|Le_1\|+\cdots+|x_d|\,\|Le_d\|$$

let $C=\max\{\|Le_1\|,\cdots\|Le_d\|\}$

$$\leq C(|x_1|+|x_2|+\cdots|x_d|)$$

denote by $\|x\|_\infty=\max\{|x_1|,|x_2|,\cdots|x_d|\}$

$$\leq Cd\|x\|_\infty=Cd\sqrt{\|x\|_\infty^2}$$
$$\leq Cd\sqrt{x_1^2+\cdots x_d^2}=Cd\|x\|\quad\text{for all } x \text{ in } \mathbb{R}^d$$

thus

$\|L(x-y)\|\leq Cd\|x-y\|$ for x,y in $\mathbb{R}^d$

that is $\|Lx-Ly\|\leq Cd\|x-y\|$

Theorem: The contraction map principle

Let (X,ρ) be a complete metric space and $f:X\longrightarrow X$ be such that $\exists M\in[0,1)$ $\forall x,y\in X$

$$\rho(f(x),f(y))\leq M\rho(x,y)$$

The equation $x=f(x)$ has a unique solution x^* in X

If x_1 is any element X define the sequence $\begin{cases} x_1 \\ x_{n+1}=f(x_n) & \forall n\geq 1\end{cases}$

x_n converges to x^*

Additionally
$$\rho(x_n, x^*) \leq M^{n-1}\rho(x_1, x^*)$$

proof: step 1: uniqueness
step 2: existence
Show that x_n converges $\rho(x_{n+1}, x_n) \leq M^{n-1}\rho(x_2, x_1)$
x_n is Cauchy x_n converges to some l in X

Applications

① $f: [\sqrt{2}, +\infty) \longrightarrow [\sqrt{2}, +\infty)$
$$f(x) = \frac{x}{2} + \frac{1}{x}$$

$[\sqrt{2}, +\infty)$ is complete, f is a contraction
let x_1 be any number in $[\sqrt{2}, +\infty)$
$$x_{n+1} = f(x_n)$$
Then x_n converges to the unique solution of the equation
$$\begin{cases} x = f(x) \\ x \in [\sqrt{2}, +\infty) \end{cases}$$

$$x = \frac{x}{2} + \frac{1}{x} \qquad x = \sqrt{2}$$

② Newton iteration for solving the equation $g(x) = 0$

③ Theorem for the existence and uniqueness to the differential equation
$$\begin{cases} \dfrac{dx}{dt} = f(x, t) \\ x(0) = x_0 \end{cases}$$

④ Implicit function theorem and applications to differential geometry in $\mathbb{R}^3$: $F(x, y, z) = 0$, where F is continuously differentiable on an open set of $\mathbb{R}^3$ and $\|\nabla F\| \neq 0$.

Compact metric spaces

Def: We say that the metric space (X, ρ) is compact if every sequence valued in X has a convergent subsequence (in X).

Examples

① If X is finite then (X, ρ) is compact — prove in HW

② Any closed and bounded subset of $\mathbb{R}^d$

Prop: If (X, ρ) is compact then it is bounded.

$\exists M > 0 \quad \forall x, y \in X, \quad \rho(x, y) \leq M$

proof: Argue by contradiction

Assume (X, ρ) is compact and

$\forall M > 0 \quad \exists x, y \in X, \quad \rho(x, y) \leq M$

to $M = n \quad \exists x_n, y_n$ in X

s.t. $\rho(x_n, y_n) > n$

x_n has a convergent subsequence x_{n_k}

Denote by l its limit

y_{n_k} has a convergent subsequence $y_{n_{k_l}}$

Denote by m its limit

But $x_{n_{k_l}}$ converges to l

$$n_{k_l} < \rho(x_{n_{k_l}}, y_{n_{k_l}}) < \rho(x_{n_{k_l}}, l) + \rho(l, m) + \rho(m, y_{n_{k_l}})$$

as $l \longrightarrow +\infty$, contradiction

Prop: If (X, ρ) is compact, then it is complete.

proof: Let x_n be a Cauchy sequence in X

Fix $\varepsilon > 0 \quad \exists N_1 \in \mathbb{N} \quad \forall p > q > N_1$

$\rho(x_p, x_q) < \varepsilon$

let (x_{n_k}) be a convergent subsequence
let L be its limit.

$\exists N_2 \in \mathbb{N} \quad \forall k \geq N_2 \quad \rho(x_{n_k}, L) < \varepsilon$

Set $N = \max\{N_1, N_2\} + 1$

for $n > N \quad \rho(x_n, L) \leq \rho(x_{n_N}, L) + \rho(x_n, x_{n_N})$

as $n_N > N \quad \rho(x_n, L) < \varepsilon + \varepsilon = 2\varepsilon$

The Heine Borel Theorem

K is a compact subset of $\mathbb{R}^d$. if and only if K is closed and bounded in $\mathbb{R}^d$

Proof: Assume that K is closed and bounded in $\mathbb{R}^d$

$d=1$ Let x_n be a sequence in K

x_n is a bounded sequence in $\mathbb{R}$

by the Bolzano Weierstress theorem

x_n has a convergent subsequence x_{n_k} to l in $\mathbb{R}$

Show that $l \in K$ $\quad$ K is closed

x_{n_k} is valued in K. so its limit l is in $\overline{K}$

but K is closed, so $l \in K$

$d \geq 2$ let x_n be a sequence in K

x_n is bounded

$\exists M > 0 \quad \forall n \in \mathbb{N}, \quad \|x_n\| \leq M$

Introduce the coordinates of x_n in the natural basis of $\mathbb{R}^d$

$x_n = (x_{n,1}, \cdots x_{n,d})$

Clearly $|x_{n,1}| \leq M$

so there is a subsequence.

$x_{f_1(x),1}$, of $x_{n,1}$ ($f_1 : \mathbb{N} \to \mathbb{N}$, strictly increasing)

convergent to $l_1 \in \mathbb{R}$

Date

$f_1(n),2$ has a convergent subsequent

$x_{f_2 \circ f_1(n),2}$ whose limit in $\mathbb{R}$ is l_2

$\vdots$

$x_{f_d \circ f_{d-1} \cdots \circ f_1(n),d}$ has a convergent subsequence

$x_{f_d \circ f_{d-1} \cdots f_1(n),d}$ whose limit in $\mathbb{R}$ is l_d

Set $g = f_d \circ f_{d-1} \circ \cdots f_1$

the sequence in $\mathbb{R}^d$

$x_{g(n)}$ converges to l in $\mathbb{R}^d$, such that $l = (l_1, l_2, \cdots l_d)$

As K is closed, $l \in K$

conversely,

Assume that K is a compact subset of $\mathbb{R}^d$

Then K is complete and bounded

As K is complete, it is closed in $\mathbb{R}^d$

Theorem: Let $f\ (X, \rho) \longrightarrow (Y, d)$ be a continuous function

If X is compact, then $f(X)$ is compact.

proof: Let $f(x_n)$ be a sequence in $f(X)$

x_n is a sequence in X, x_{n_k} has a convergent subsequence x_{n_k}

denote by l its limit in X,

As f is continuous, $f(x_{n_k})$ converges to $f(l)$

And f is uniformly continuous on X

proof: $\forall\, \varepsilon > 0,\ \exists\, \alpha > 0,\ \forall\, x, y \in X$

$\rho(x, y) < \alpha \Rightarrow d(f(x), f(y)) < \varepsilon$

Argue by contradiction

assume $\exists \varepsilon > 0$, $\forall \alpha > 0$ $\exists x, y \in X$
$\rho(x,y) < \alpha$ and $d(f(x), f(y)) \geq \varepsilon$

so to $\alpha = \frac{1}{n}$ $\exists x_n, y_n \in X$ s.t. $\rho(x_n, y_n) < \frac{1}{n}$
and $d(f(x_n), f(y_n)) \geq \varepsilon$

as X is compact, x_n, has a convergent subsequence
x_{n_k} denote by l its limit.

But $\rho(y_{n_k}, l) \leq \rho(x_{n_k}, y_{n_k}) + \rho(l, x_{n_k})$

$$\leq \frac{1}{n_k} + \rho(x_{n_k}, l)$$

thus $\lim\limits_{k \to +\infty} \rho(y_{n_k}, l) = 0$

y_{n_k} converges to l

as f is continuous on X

$f(x_{n_k})$ and $f(y_{n_k})$ both converge to $f(l)$

accordingly so $(f(x_{n_k}), f(y_{n_k})) \leq d(f(x_{n_k}), f(l)) + d(f(l), f(y_{n_k}))$

thus $\lim\limits_{k \to +\infty} d(f(x_{n_k}), f(y_{n_k})) = 0$ this contradicts $d(f(x_n), f(y_n)) \geq \varepsilon$ for all n.

Application $f : X \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}$
assume that X is closed and bounded (thus compact)
f is continuous
then f is uniformly continuous on X and $f(X)$ is compact
$f(X)$ is closed and bounded in $\mathbb{R}$

Define $m = \inf f(x)$ $M = \sup f(x)$

Date

We can claim that

$\exists x_1 \in X \quad f(x_1) = m$

$\exists x_2 \in X \quad f(x_2) = M$

because $m, M \in \overline{f(x)}$ as $f(x)$ is closed $m, M \in f(x)$

The open cover property

Definition: We say that metric space (X, ρ) is totally bounded.
if $\forall \varepsilon > 0 \ \exists x_1, x_2, \cdots x_n \in X$ s.t. $X = \bigcup_{i=1}^{n} B(x_i, \varepsilon)$ finite union

Proposition: If (X, ρ) is compact, then it is totally bounded

proof: Arguing by contradiction
assume that X is compact and NOT totally bounded

$\exists \varepsilon > 0$ s.t. $\forall_n x_1, \cdots x_n \in X$

$$X \neq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$$

Let y_1 be in X

$X \neq B(x_1, \varepsilon)$ so $\exists y \in X \setminus B(y_1, \varepsilon)$

$X \neq B(y_1, \varepsilon) \cup B(y_2, \varepsilon)$

so $\exists y_3 \in X \setminus (B(y_1, \varepsilon) \cup B(y_2, \varepsilon))$

$\vdots$

$$X \neq \bigcup_{i=1}^{n} B(y_i, \varepsilon) \quad \exists \ y_{n+1} \in X \setminus \bigcup_{i=1}^{n} B(y_i, \varepsilon)$$

by construction,

since $y_n \notin \bigcup_{i=1}^{n-1} B(y_i, \varepsilon)$

y_n is not in the ball

$\forall \ \ell \leq n-1 \quad \rho(y_n, y_\ell) \geq \varepsilon$

let y_{n_k} is any subsequence of y_n

$$\rho(y_{n_k}, y_{n_{k-1}}) \geq \varepsilon$$

Thus y_{n_k} does NOT converge, thus X is NOT compact.

Definition. We say that the metric space (X,ρ) satisfies the open cover property, if for any collection V_i of open subsets of X $i\in I$ such that $X=\bigcup_{i\in I} V_i$ has a finite subcover, that is there is $J\subseteq I$ such that J is finite and $X=\bigcup_{j\in J} V_j$

Theorem. Let (X,ρ) be a metric space. The following are equivalent

(i) X is compact

(ii) X satisfies the open cover property

(iii) X is complete and totally bounded

proof:

(i) $\Rightarrow$ (iii) done

The open cover property is very useful!

Let $f:(X,\rho)\longrightarrow(Y,d)$ be a continuous function
Show that if X is compact then f is uniformly continuous, by using the open cover property.

Fix $\varepsilon>0$ Let x be in X

As f is continuous at x

$\exists\, \alpha_x>0$ such that $\forall\, y\in B(x,\alpha_x)$ $\quad d(f(x),f(y))<\varepsilon$

$$X=\bigcup_{x\in X} B\left(x,\frac{\alpha_x}{2}\right)$$

As X is compact, apply the finite open cover property

$\exists\, x_1,\dots x_n$ such that $X=\bigcup_{i=1}^{n} B\left(x_i,\frac{\alpha_{x_i}}{2}\right)$

set $\alpha=\min\left\{\alpha\frac{x_1}{2},\dots,\alpha\frac{x_n}{2}\right\}$

Let u and v be in X such that $\rho(u,v)<\alpha$

$\exists\, j\in\{1,\dots n\}$ s.t. $u\in B\left(x_j,\frac{\alpha_{x_j}}{2}\right)$

but $\rho(u,v)<\alpha\leqslant\frac{\alpha_{x_j}}{2}$ $\quad v\in B(x_j,\alpha_{x_j})$

then $d(f(u), f(v))$

$\leq d(f(u), f(x_j)) + d(f(x_j), f(v))$
$< 2\varepsilon$

Normed spaces

Def: A vector space V is a normed space with norm N if N is a mapping from V to $\mathbb{R}^+$ such that

(i) if $x \in V$ and $N(x) = 0$ then $x = 0$ (definiteness)

(ii) if $x \in V$ and $\lambda \in \mathbb{R}$ $N(\lambda x) = |\lambda| N(x)$ (homogenity).

(iii) $\forall\, x, y \in V$ $N(x+y) \leq N(x) + N(y)$ (triangle inequality)

Examples / comments:

① $(\mathbb{R}, |\ |)$ is a normed space

② Any inner product space

$N(x) = \langle x, x\rangle^{\frac{1}{2}}$

$\mathbb{R}^d$

③ if (V, N) is a normed space set for x, y in V $\rho(x,y) = N(x-y)$ then ρ is a distance on V

④ if (V, N) is a normed space, and W is a vector subspace of V, then (W, N) is a normed space

⑤ X is any set.

$B(X) = \{$ bounded functions from X to $\mathbb{R}\}$

on $B(X)$ define the norm $\|\cdot\|_\infty$

$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$

for $x \in \mathbb{R}^d$, $x = \{x_1, x_2, \cdots x_d\}$

⑥ In $\mathbb{R}^d$ Euclidean norm

$$\|x\|_2 = (x_1^2 + \cdots x_d^2)^{\frac{1}{2}}$$

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_d| \qquad \|x\|_\infty = \max\{|x_j| : 1 \leq j \leq m\}$$

We showed that these 3 norms are equivalent

$\exists\, C_1, C_2, C_3, C_4 > 0$ s.t.

$\forall\, x \in \mathbb{R}^d$

$$C_1 \|x\|_\infty \leq \|x\|_2 \leq C_2 \|x\|_\infty$$

$$C_3 \|x\|_\infty \leq \|x\|_1 \leq C_4 \|x\|_\infty$$

Theorem: Let N be any norm on $\mathbb{R}^d$ $\exists\, A, B > 0$, s.t. $\forall x \in \mathbb{R}^d$

$$A\|x\|_\infty \leq N(x) \leq B\|x\|_\infty$$

In other words, on $\mathbb{R}^d$, all norms are equivalent, they all define the same open sets and the same closed sets.

proof: (HWPb) Let $e_1, \cdots e_d$ be the natural basis of $\mathbb{R}^d$

Let $x = (x_1, x_2, \cdots x_d)$ in $\mathbb{R}^d$

$$N(x_1 e_1 + x_2 e_2 + \cdots x_d e_d) \leq |x_1| N(e_1) + |x_2| N(e_2) + \cdots |x_d| N(e_d)$$

to determine A

$$S = \{ x \in \mathbb{R}^d, \ \|x\|_\infty = 1 \}$$

so S is compact

$$f: S \longrightarrow \mathbb{R}^+, \quad f(x) = N(x)$$

$\exists\, c > 0$, such that $\forall\, x \in S$, $f(x) \geq c$

① Define $\ell^1 = \{$ sequences u_n value in $\mathbb{R}$, such that $\sum_{n=1}^{\infty} |u_n| < +\infty \}$

for u in ℓ^1

$$\|u\|_1 = \sum_{n=1}^{\infty} |u_n|$$

$\ell^2 = \{$ sequences u_n valued in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} u_n^2 < +\infty \}$

$u \in \ell^2$ $\quad \|u\|_2 = \left(\sum_{n=1}^{\infty} u_n^2\right)^{\frac{1}{2}}$

$\ell^\infty = \{$ sequences u_n valued in $\mathbb{R}$ and bounded $\}$

For $u \in \ell^\infty$, $\|u\|_\infty = \sup\{|u_n|, n \geq 1\}$

Prop: $(\ell^1, \|\cdot\|_1)$, $(\ell^2, \|\cdot\|_2)$, $(\ell^\infty, \|\cdot\|_\infty)$ are complete normed spaces

$$\ell^1 \subsetneq \ell^2 \subsetneq \ell^\infty$$

Continuity of linear functions

Let (V, N_1) and (W, N_2) be 2 normed spaces and $L: V \to W$ a linear function

The following are equivalent

(i) L is continuous at 0

(ii) L is Lipschitz continuous

(iii) $\exists c > 0, \forall x \in V, N_2(Lx) \leq C N_1(x)$

prove in HW

Spaces of continuous functions

Let (X, ρ) be a metric space. $B(X)$ is complete

$C(X) = \{$ continuous functions from X to $\mathbb{R}\}$

Denote by $C_B(X) = B(X) \cap C(X)$

Proposition $C_B(X)$ is closed in $B(X)$

(thus $C_B(X)$ is complete)

proof: Let f_n be a sequence in $C_B(X)$ convergent to f in $B(X)$

Show that f is continuous in X

Let x_0 be in X, and $\varepsilon > 0$

$\exists N \in \mathbb{N}$, $\forall n > N$

$\|f_n - f\|_\infty < \varepsilon$

$\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$ for $n > N$

We know that f_{N+1} is continuous functions at x_0

$\exists \alpha > 0$ s.t. $\forall x \in B(x_0, \alpha)$

$|f_{N+1}(x) - f_{N+1}(x_0)| < \varepsilon$

Thus for $x \in B(x_0, \alpha)$

$$\begin{aligned} |f(x) - f(x_0)| &\le |f(x) - f_{N+1}(x)| \\ &+ |f_{N+1}(x) - f_{N+1}(x_0)| \\ &+ |f_{N+1}(x_0) - f(x_0)| < 3\varepsilon \end{aligned}$$

Remark: If X is compact metric space,

$C(X) = C_B(X)$

example:

$C([a,b])$, where a, b is in $\mathbb{R}$

the default norm on $C([a,b])$

$$\|f\|_\infty = \sup\{|f(x)|, x \in [a,b]\} = \sup_{[a,b]} |f(x)|$$

Date

$C([a,b])$ is complete for the $\|\cdot\|_\infty$ norm

Define another norm $C([a,b])$ by setting for $f \in C([a,b])$

$$\|f\|_1 = \int_a^b |f(x)|\,dx = \int_a^b |f|$$

Explain why this defines a norm on $C([a,b])$

clearly $\|f\|_1 \geq 0$

for $\lambda \in \mathbb{R}$ $\quad \|\lambda f\|_1 = \int_a^b |\lambda f| = |\lambda| \int_a^b |f| = |\lambda|\,\|f\|_1$

if $f, g \in C([a,b])$ for x in $[a,b]$

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|$$

thus $\int_a^b |f+g| \leq \int_a^b |f| + \int_a^b |g|$

so $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$

definitoness

lemma. Let $g \in C([a,b])$ s.t. $g \geq 0$

If $\int_a^b g = 0$, then $g = 0$

sketch of proof:

if $\exists x_0 \in [a,b]$ s.t. $g(x_0) > 0$ for $g \in C([a,b])$ s.t. $g \geq 0$

then $\int_a^b g > 0$

We have defined 2 norms on $C([a,b])$

$$\|f\|_\infty = \sup_{[a,b]} |f|$$

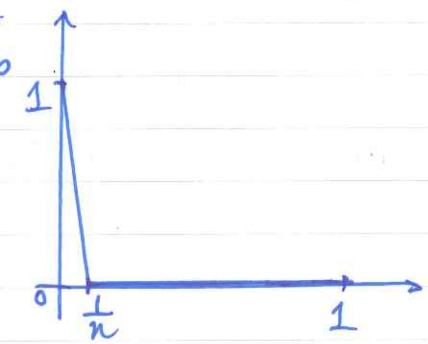
and $\|f\|_1 = \int_a^b |f|$

these 2 norms are NOT equivalent

We can find a sequence f_n in $C([0,1])$ such that

f_n converges to 0 for $\|\cdot\|_1$

f_n diverges for $\|\cdot\|_\infty$



$$f_n(x) = \begin{cases} 0 & \text{if } \frac{1}{n} \leq x \leq 1 \\ -n(x-\frac{1}{n}) & \text{if } 0 \leq x < \frac{1}{n} \end{cases}$$

$$\int_0^1 |f_n| = \frac{1}{2n} \longmapsto 0$$

$C([a,b])$ $\quad \|f\|_\infty = \sup_{[a,b]} |f|$ $\quad \|f\|_1 = \int_a^b |f|$

There is a sequence f_n such that $\|f_n\|_1 \to 0$ but f_n diverges in $(C([a,b]), \|\cdot\|_\infty)$

$(C[a,b], \|\cdot\|_\infty)$ is complete

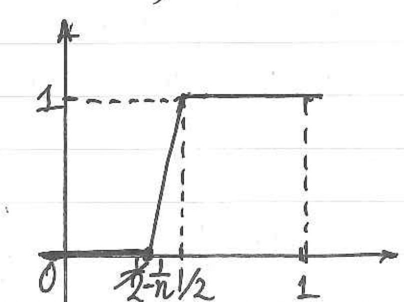
$(C[a,b], \|\cdot\|_1)$ is NOT complete

there is a Cauchy sequence in $C([a,b], \|\cdot\|_1)$ which diverges

f is Cauchy

if $p>q$, $\int_0^1 |f_p - f_q| = \int_{\frac{1}{2}-\frac{1}{q}}^{\frac{1}{2}} |f_p - f_q|$

$\leq \frac{2}{q}$



Argue by contradiction

Assume that $\exists f \in C[0,1]$ s.t. $\int_0^1 |f_n - f| \to 0$

$$\int_{\frac{1}{2}}^{1} |f_n - f| \leq \int_0^1 |f_n - f| \to 0.$$

$$\int_{\frac{1}{2}}^{1} |1 - f| = 0$$

$$f(x) = 1 \quad \forall x \in [\tfrac{1}{2}, 1]$$

next fix α in $(0, \frac{1}{2})$

show $f(x) = 0 \quad \forall x \in [0, \alpha]$

The Lebesgue space $L^1([0,1])$ is the completion of $(C[0,1], \|\cdot\|_1)$

Measure Theory

Let I be an interval of $\mathbb{R}$

Example: $\emptyset$, $\mathbb{R}$, $[a,b]$, $(a,b]$, $[a,+\infty)$, ...

Proposition The subset I of $\mathbb{R}$ is an interval if and only if for every $\forall a, b \in I$ such that $a < b$

$[a,b] \subset I$

Length of an interval

$\ell(\emptyset) = 0$

$\ell(\text{"unbounded interval"}) = +\infty$

if I is a bounded interval $I = [a,b]$, for some a and b in $\mathbb{R}$

we set $\ell(I) = b - a$

Define $\mathcal{R} = \{$ finite union of intervals in $\mathbb{R} \}$

Proposition $\mathcal{R}$ is closed under

(i) finite unions

(ii) finite intersections

(iii) taking complements

proof: (i) clear

(ii) let I and J be 2 intervals if $I \cap J = \phi$, then $I \cap J \in \mathcal{R}$

Otherwise let a and b be in $I \cap J$

as I and J are intervals

$[a, b] \subset I$ and $[a, b] \subset J$

thus $[a, b] \subseteq I \cap J$

thus $I \cap J$ is an interval

Let A, and B be in

$A = I_1 \cup \cdots \cup I_p$

$B = J_1 \cup \cdots \cup J_q$

when $I_1, \cdots I_p, J_1, \cdots J_q$ are intervals

but $A \cap B = \bigcup_{\substack{1 \le i \le p \\ 1 \le j \le q}} (I_i \cap J_j)$

since each $I_i \cap J_j$ is an intervals

$A \cap B \in \mathcal{R}$

(iii) If I is an interval

for example $I = [a, b]$

$I^c = (-\infty, a) \cup (b, +\infty) \in \mathcal{R}$

In general, I^c is in $\mathcal{R}$

let A be in $\mathcal{R}$ $A = I_1 \cup \cdots \cup I_p$ where $I_1, I_2, \cdots I_p$ are intervals

$A^c = (I_1^c) \cap \cdots (I_p^c)$, but each I_i^c is in $\mathcal{R}$

and use (ii)

thus $A^c \in \mathcal{R}$

Extended as a function $l: \mathcal{R} \to [0, +\infty]$

let A be in $\mathcal{R}$,

there is a unique way of writing $A = I_1 \cup \cdots \cup I_p$

where $I_1, I_2, \cdots I_p$ are intervals

p is minimum

and $I_1 < I_2 < \cdots I_p$
that is, say $A = [1,3] \cup [2,4] \cup [5,7) \cup [7,8)$
obtain $A = [1,4] \cup [5,8)$

define $\ell(A) = \ell(I_1) + \cdots + \ell(I_p)$

properties of ℓ on $\mathbb{R}$

① $\ell, \mathcal{R} \longrightarrow [0, +\infty)$
② $\ell(\phi) = 0$
③ if $A, B \in \mathcal{R}$ and $A \cap B = \phi$ then $\ell(A \cup B) = \ell(A) + \ell(B)$

proof. We can show by induction
that if $I_1, \cdots I_p$ are pairwise disjoint intervals,
then $\ell(I_1 \cup I_2 \cdots \cup I_p) = \ell(I_1) + \ell(I_2) + \cdots \ell(I_p)$
From there, the result follows easily

terminology by definition $I_1, I_2, \cdots I_p$ are pairwise disjoint
if for all i, j in $\{1, 2, \cdots p\}$
such that $i \neq j$ $\quad I_i \cap I_j = \phi$

Further properties

③ if the A_j's are pairwise disjoint in $\mathcal{R}$, then the $1 \leq j \leq n$
$\ell(A_1 \cup \cdots \cup A_n) = \ell(A_1) + \ell(A_2) + \cdots \ell(A_n)$

④ If A, B are in $\mathcal{R}$ and $A \subseteq B$
$\ell(B) = \ell(A) + \ell(B \setminus A)$
in particular, $\ell(A) \leq \ell(B)$

⑤ If $E, F \in \mathcal{R}$ $\quad \ell(E) + \ell(F) = \ell(E \cup F) + \ell(E \cap F)$
$\ell(E \cup F) \leq \ell(E) + \ell(F)$

⑥ If $E_1, E_2, \cdots E_n \in \mathcal{R}$
then $\ell\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \ell(E_i)$

③ clear

④ $B \setminus A = B \cap A^c$ is in $\mathcal{R}$

$B = (B\setminus A) \cup A$ clearly $(B\setminus A) \cap A = \phi$

$l(B) = \underbrace{l(B\setminus A)}_{\geq 0} + l(A)$

⑤ proof. $E \cup F = (E \setminus (E\cup F)) \cup (F \setminus (E\cup F)) \cup (E \cap F)$

note that this 3 sets are pairwise disjoint, apply ③

$l(E\cup F) = l(E\setminus(E\cup F))\ \ l(F\setminus(E\cup F)) + l(E\cap F)$

$l(E\cup F) = l(E) - l(E\cup F) + l(F) - l(E\cup F) + l(E\cap F)$

$\mathcal{R} = \{$ finite unions of intervals of $\mathbb{R}\}$

$l: \mathcal{R} \longrightarrow [0, +\infty]$

$l(I)$: "length of I" if I is an interval

if the A_i are pairwise disjoint and in $\mathcal{R}$

$$l\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{k} l(A_i)$$

if $A \subseteq B$ and are in $\mathcal{R}$ $\quad l(A) \leq l(B)$

if the $E_1, \cdots E_n$ are in $\mathcal{R}$ $\quad l\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} l(E_i)$

Note: addition in $[0, +\infty]$

$\forall\ x \in [0, +\infty]$, $\ x + (+\infty) = (+\infty) + x = +\infty$

if a sequence u_n is in $[0, +\infty)$ and $\sum_{n=1}^{\infty} u_n$ diverges in $\mathbb{R}$

write $\sum_{n=1}^{\infty} u_n = +\infty$

if v_n is in $[0, +\infty]$ and if one $v_k = +\infty$, $\ \sum_{n=1}^{\infty} v_n = +\infty$

<u>Countable Additivity</u>

Assume that the sequence $A_1, A_2, \cdots A_n$ is valued in $\mathcal{R}$, is pairwise disjoint, and $\bigcup_{n=1}^{\infty} A_n$ is also in $\mathcal{R}$

then $l\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} l(A_n)$

Date

Remark:

$$\bigcup_{n=1}^{\infty} [0, 2-\tfrac{1}{n}] = [0,2) \in \mathcal{R}$$

$$\bigcup_{n=1}^{\infty} \{n\} \notin \mathcal{R}$$

proof: We know that for any $k = 1, 2, \cdots$

$$l\left(\bigcup_{n=1}^{k} A_n\right) = \sum_{n=1}^{k} l(A_n)$$

$$\bigcup_{n=1}^{k} A_n \subseteq \underbrace{\bigcup_{n=1}^{\infty} A_n}_{\in \mathcal{R}}$$

thus $l\left(\bigcup_{n=1}^{k} A_n\right) \leq l\left(\bigcup_{n=1}^{\infty} A_n\right)$

so for any $k = 1, 2, \cdots$

$$\sum_{n=1}^{k} l(A_n) \leq l\left(\bigcup_{n=1}^{\infty} A_n\right)$$

let $k \to +\infty$ $\quad \sum_{n=1}^{\infty} l(A_n) \leq l\left(\bigcup_{n=1}^{\infty} A_n\right)$

conversely, set $A = \bigcup_{n=1}^{\infty} A_n$ we need to show that $l(A) \leq \sum_{n=1}^{\infty} l(A_n)$

by assumption $A \in \mathcal{R}$ $\quad A = I_1 \cup I_2 \cdots \cup I_p$, where $I_1, I_2, \cdots I_p$ are pairwise disjoint intervals.

case 1: Assume A is bounded, Fix $\varepsilon > 0$

I_j must contain a closed interval K_j such that

$$l(K_j) \geq l(I_j) - \frac{\varepsilon}{p}$$

example: (a, b) contains $\left[a + \frac{\varepsilon}{2p}, b - \frac{\varepsilon}{2p}\right]$

$K_1, K_2, \cdots \cup K_p$ is closed and bounded, thus compact

A_n is contained in an open interval V_n such that

$$l(V_n) \leq l(A_n) + \frac{\varepsilon}{2^n}$$

example: $[c,d] \subset \left(c-\frac{\varepsilon}{2^{n+1}},\ d+\frac{\varepsilon}{2^{n+1}}\right)$

Write $(K_1 \cup K_2, \cdots \cup K_p) \subset \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} V_n$

by the finite subcover property,

$(K_1 \cup \cdots \cup K_p) \subset \bigcup_{n=1}^{q} V_n$ for some q in $\mathbb{N}$

$V_n \subseteq \mathbb{R}$ $\quad \ell(K_1 \cup \cdots K_p) \leq \sum_{n=1}^{q} \ell(V_n)$

$$\ell(K_1)+\ell(K_2)+\cdots \ell(K_p) \geqslant \ell(I_1)+\cdots \ell(I_p)-\varepsilon = \ell(A)-\varepsilon$$

$$\sum_{n=1}^{q} \ell(V_n) \leq \sum_{n=1}^{q}\left(\ell(A_n)+\frac{\varepsilon}{2^n}\right)$$

$$\leq \sum_{n=1}^{\infty} \ell(A_n) + \varepsilon$$

Conclusion:

$$\ell(A) \leq \sum_{n=1}^{\infty} \ell(A_n) + 2\varepsilon$$

As $\varepsilon > 0$ is arbitrary, let $\varepsilon \to 0$

$$\ell(A) \leq \sum_{n=1}^{\infty} \ell(A_n)$$

case 2: A is unbounded $\quad A = I_1 \cup I_2 \cdots \cup I_p$

one of the I's say I_ℓ is unbounded so

$$\ell(A) = +\infty$$

Show that $\sum_{n=1}^{\infty} \ell(A_n) = +\infty$

set $B_r = A \cap [-r, r]$

$A_{n,r} = A_n \cap [-r, r]$

according to case 1

$$\ell(B_r) = \ell\left(\bigcup_{n=1}^{\infty} A_{n,r}\right)$$

$$= \sum_{n=1}^{\infty} \ell(A_{n,r}) < \sum_{n=1}^{\infty} \ell(A_n)$$

I_ℓ is unbounded

$$\ell(I_\ell) = +\infty = \lim_{r \to +\infty} \ell(I_\ell \cap [-r, r])$$

$$\ell(I_\ell \cap [-r, r]) \leq \ell(B_r)$$

as $\ell(I_\ell \cap [-r, r]) \leq \sum_{n=1}^{\infty} \ell(A_n)$

as $r \to +\infty$, we can find that $\sum_{n=1}^{\infty} \ell(A_n) = +\infty$

Def: Let X be any set. We say that $\mathcal{A}$ is a σ-algebra of subsets of X if

(i) $\emptyset \in \mathcal{A}$

(ii) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ (closed under taking complements)

(iii) if $A_1, A_2, \cdots A_n \cdots$ is a sequence in $\mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ (closed under countable unions)

If $\mathcal{A}$ is a σ-algebra we say that $\mu : \mathcal{A} \longrightarrow [0, +\infty]$ is a measure if $\mu(\emptyset) = 0$ and for all pairwise disjoint, a sequence A_n in $\mathcal{A}$, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

We say that $(X, \mathcal{A}, \mu)$ is a measure space.

Examples

① X is any set $\{\emptyset, X\}$ is a σ-algebra.

$P(X) = \{\text{subsets of } X\}$ is also a σ-algebra.

② Let X be a countable set.

Consequently $X = \{a_1, a_2, \cdots a_n, \cdots\}$

typically $\mathbb{N}$, $\mathbb{N}^*$, $\mathbb{Z}$ $\mathbb{Q}$

$\mathbb{R}$ is NOT countable

$P(X)$ is the "default" σ-algebra.

define a measure μ on $P(X)$ by setting:

for $A \in P(X)$, if A is infinite, $\mu(A) = +\infty$,

otherwise $\mu(A)$ is # of elements in A

③ $R = \{$ finite unions of intervals in $\mathbb{R}\}$
is NOT a σ-algebra as $\bigcup_{n=1}^{\infty}\{n\} \notin R$

Properties: Let $(X, \mathcal{A}, \mu)$ be a measure space

① if $A, B \in \mathcal{A}$ and $A \subseteq B$ $B\setminus A \in \mathcal{A}$
then $\mu(B) = \mu(A) + \mu(B\setminus A)$
and $\mu(A) \leq \mu(B)$

② if $E, F \in \mathcal{A}$ $\mu(E\cup F) + \mu(E\cap F) = \mu(E) + \mu(F)$

③ if $B_1, B_2, \cdots B_n, \cdots \in \mathcal{A}$
then $\bigcap_{n=1}^{\infty} B_n \in \mathcal{A}$

④ If $A_1, A_2, \cdots A_n, \cdots \in \mathcal{A}$ countable subadditivity
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

proof: ① and ②

① $B = A \cup (B\setminus A)$
$\mu(B) = \mu(A) + \mu(B\setminus A)$
$$B\setminus A = \underset{\in \mathcal{A}}{B\cap A^c} = (B^c \cup A)^c$$

③ $$\left(\bigcap_{n=1}^{\infty} B_n\right)^c = \bigcup_{n=1}^{\infty} B_n^c$$

By definition of σ-algebra
each $B_n^c \in \mathcal{A}$ so $\bigcup_{n=1}^{\infty} B_n^c \in \mathcal{A}$

thus $\bigcap_{n=1}^{\infty} B_n \in \mathcal{A}$

④ from A_n construct a pairwise disjoint sequence B_n
$B_1 = A_1$
$B_2 = A_2 \setminus A_1$
$B_3 = A_3 \setminus (A_1 \cup A_2)$

Date

$\therefore \quad B_n = A_n \setminus (A_1 \cup \cdots A_{n-1})$

Note that $(A_1 \cup \cdots \cup A_{n-1}) \in \mathcal{A}$

so $(A_1 \cup \cdots \cup A_{n-1})^c \in \mathcal{A}$

thus $A_n \setminus (A_1 \cup \cdots \cup A_{n-1}) \in \mathcal{A}$

B_n is a pairwise disjoint sequence

let $k < l$

$B_l = A_l \setminus (A_1 \cup \cdots \cup A_{l-1})$ this union contains A_k

$\vdots$

$B_k = A_k \setminus (A_1 \cup \cdots \cup A_{k-1}) \subseteq A_k$

so $B_l \cap B_k = \phi$

$\bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{k} B_n$ (proof by induction)

so $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$

Apply σ-additivity (e.g. countable additivity) to the B_n's

$$\sum_{n=1}^{\infty} \mu(B_n) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$\rotatebox{90}{$\geq$}$$

$$\sum_{n=1}^{\infty} \mu(A_n)$$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Smallest σ-algebra containing a given subset of $P(X)$

if $\mathcal{A}_1$. and $\mathcal{A}_2$ are two σ-algebras in $P(X)$ so is $\mathcal{A}_1 \cap \mathcal{A}_2$

Similarly $\mathcal{A}_i$, $i \in I$ is a collection of σ-algebras in $P(X)$ so is $\bigcap_{i \in I} \mathcal{A}_i$.

Let $\mathcal{S}$ be any subset of $P(X)$, the smallest σ-algebra containing $\mathcal{S}$ is the intersection of all σ-algebras containing $\mathcal{S}$.

in $\mathbb{R}$

$\mathcal{R} = \{$ finite union of intervals in $\mathbb{R} \}$

The smallest σ-algebra containing in $\mathcal{R}$ is called the σ-algebra of Borel sets in $\mathbb{R}$, denoted by $\mathcal{B}$.

Prop. $\mathcal{B}$ contains all countable subsets of $\mathbb{R}$
all open subsets of $\mathbb{R}$
all closed subsets of $\mathbb{R}$

proof: if a_n is a sequence in $\mathbb{R}$

$\{a_n\} \in \mathcal{R}$

$\bigcup_{n=1}^{\infty} \{a_n\} \in \mathcal{B}$

lemma: Every open subsets V of $\mathbb{R}$ is a countable union of intervals in $\mathbb{R}$

proof: $\mathbb{Q}$ is countable

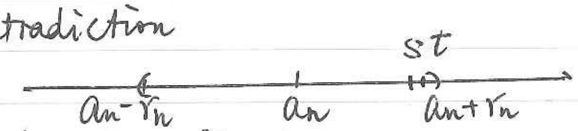
$V \cap \mathbb{Q}$ is countable

write $V \cap \mathbb{Q} = \bigcup_{n=1}^{\infty} \{a_n\}$

define $r_n = \sup\{ r \in (0, +\infty) : (a_n - r, a_n + r) \subseteq V \}$

show that $(a_n - r_n, a_n + r_n) \subseteq V$

Argue by contradiction



finally show that $V = \bigcup_{n=1}^{\infty} (a_n - r_n, a_n + r_n)$

Thanks to. this lemma, it is clear that every open subset in $\mathbb{R}$ is in $\mathcal{B}$

(iii) use (ii) to take complement.

The extension theorem

$l : R \longrightarrow [0, +\infty]$

l is countably additive on R

l can be extended to a measure on the σ-algebra $\mathcal{B}$

this extension is unique

this is the Lebesgue measure on Borel sets of $\mathbb{R}$

Brief outline of proof.

done by defining first the outer measure of any subset A of R $m^*(A) = \inf\left\{ \sum_{k=1}^{\infty} l(I_k) : I_k \text{ is an interval and } A \subset \bigcup_{k=1}^{\infty} I_k \right\}$

Problem: this fails to be a measure on $P(\mathbb{R})$

there are 2 ~~sub~~subsets A and B of $\mathbb{R}$ such that $A \cap B = \emptyset$ and $m^*(A \cup B) < m^*(A) + m^*(B)$

thus

Definition A subset E of $\mathbb{R}$ is measurable if any subset A of $\mathbb{R}$, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

Thanks to this definition, if E and F are 2 disjoints and measurable subsets of R

$$m^*(E \cup F) = m^*((E \cup F) \cap E) + m^*((E \cup F) \cap E^c)$$
$$= m^*(E) + m^*(F)$$

Theorem, the set of measurable subsets defined in this fashion is a σ-algebra. this is called the σ-algebra of lebesgue measurable sets of $\mathbb{R}$

m^* defines a measure of this σ-algebra, called the Lebesgue measure denote by m

Properties.

1. Every interval of $\mathbb{R}$ is Lebesgue measurable and its measure is its length
   Consequently every borel set is a Lebesgue measurable set.
2. Every subset of $\mathbb{R}$ with outer measure 0 is Lebesgue measurable with Lebesgue measure 0

Lebesgue measurable subsets of $\mathbb{R}$

Properties:

1. Every interval of $\mathbb{R}$ is Lebesgue measurable, so are open subsets, so are closed subsets of $\mathbb{R}$
2. Every subsets of $\mathbb{R}$ with outer measure zero is Lebesgue measurable
3. The Lebesgue measure of interval is its length
4. Translation invariance
   If S is Lebesgue measurable and x in $\mathbb{R}$
   $S+x=\{s+x,\ s\in S\}$ is also Lebesgue measurable
   and $m(S+x)=m(S)$
5. Approximation by open and by closed sets
   let S be measurable subset of $\mathbb{R}$
   $\forall \varepsilon>0$, $\exists$ open subset V of $\mathbb{R}$
   $\exists$ closed subset F of $\mathbb{R}$
   such that $F\subseteq S\subseteq V$
   and $m(V\setminus S)<\varepsilon$, $m(S\setminus F)<\varepsilon$

proof: ④ Let T be any subset of $\mathbb{R}$

$$m^*(T)=\inf\left\{\sum_{k=1}^{\infty} l(I_k):\ I_k \text{ is an interval, } T\subset\bigcup_{k=1}^{\infty} I_k\right\}$$

I_k is an interval $\Leftrightarrow$ I_k+x is an interval.

$$l(I_k+x)=l(I_k)$$

it follows that $m^*(T) = m^*(T+x)$

Assume S is Lebesgue measurable

then $m(S) = m^*(S)$

$\forall \ A \in \mathcal{P}(\mathbb{R})$

$m^*(A) = m^*(A \cap S^c) + m^*(A \cap S)$

$$A \cap (S+x) = \{ a \in A \ \ \exists s \in S, \ a = s+x \}$$

$$= (A-x) \cap S + x$$

$(S+x)^c = S^c + x$

$A \cap (S+x)^c = A \cap (S^c + x)$

$= (A-x) \cap S^c + x$

$m^*(A-x) = m^*((A-x) \cap S^c) + m^*((A-x) \cap S)$

since A is measurable

but $m^*(A-x) = m^*(A)$

$m^*((A-x) \cap S^c) = m^*((A-x) \cap S^c + x)$

$= m^*(A \cap (S+x)^c)$

$m^*((A-x) \cap S) = m^*((A-x) \cap S + x)$

$= m^*(A \cap (S+x))$

Conclusion

$m^*(A) = m^*(A \cap (S+x)^c) + m^*(A \cap (S+x))$. for any A in $\mathcal{P}(\mathbb{R})$

thus $S+x$ is measurable

$m(S+x) = m^*(S+x) = m^*(S) = m(S)$

proof of ⑤ Let S be a Lebesgue measurable set

set let $\varepsilon > 0$.

$$m(S) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k); \ I_k \text{ is an interval and } S \in \bigcup_{k=1}^{\infty} I_k \right\}$$

there is a sequence I_k, such that $S \subset \bigcup_{k=1}^{\infty} I_k$

$$\sum_{k=1}^{\infty} \ell(I_k) - \varepsilon \le m(S)$$

to each k, there is an open interval J_k such that $I_k \subseteq J_k$ and $m(J_k) = m(I_k) + \frac{\varepsilon}{2^{k+1}}$

set $V = \bigcup_{k=1}^{\infty} J_k$ is open in $\mathbb{R}$

$S \subseteq V$ $\quad m(V) \le \sum_{k=1}^{\infty} \ell(J_k)$

$$= \sum_{k=1}^{\infty} \left(\ell(I_k) + \frac{\varepsilon}{2^{k+1}}\right) = \sum_{k=1}^{\infty} \ell(I_k) + \varepsilon$$

$m(V) \le m(S) + 2\varepsilon$

$S \subseteq V$

Case 1: $\quad m(V) = m(V \setminus S) + m(S)$

if $m(S) < +\infty$ $\quad m(V \setminus S) = m(V) - m(S)$

this shows that $m(V \setminus S) < 2\varepsilon$

Case 2: if $m(S) = +\infty$

Set $S_p = S \cap [-p, p]$ for any $p = 1, 2, \cdots$

according to case 1

$(m(S_p) \le 2p)$

fixing $\varepsilon > 0$ $\exists$ open subset V_p of $\mathbb{R}$ such that $S_p \subset V_p$

$m(V_p \setminus S_p) \le \frac{\varepsilon}{2^p}$

Set $V = \bigcup_{p=1}^{\infty} V_p$ is open in $\mathbb{R}$

$S \subset V$ is clear by conduction

evaluate $m(V \setminus S)$

$$V \setminus S = V \cap S^c = \bigcup_{p=1}^{\infty} (V_p \cap S^c)$$

$S_p \subset S$ so $S^c \subset S_p$

this shows that $V\setminus S \subset \bigcup_{p=1}^{\infty} (V_p \cap S_p^c)$

$$= \bigcup_{p=1}^{\infty} (V_p \setminus S_p)$$

$$m(V\setminus S) \leqslant m\left(\bigcup_{p=1}^{\infty}(V_p\setminus S_p)\right)$$

$$\leq \sum_{p=1}^{\infty} m(V_p \setminus S_p) \leqslant \sum_{p=1}^{\infty} \frac{\varepsilon}{2^p} = \varepsilon$$

To the measurable set S and $\varepsilon>0$. find a closed subset F of $\mathbb{R}$ such that $F \subseteq S$ and $m(S\setminus F) < \varepsilon$

S^c is measurable

$\exists$ open subset W of $\mathbb{R}$ such that $S^c \subset W$

$m(W\setminus S^c) < \varepsilon$ (using approximation by open subsets)

$W^c \subset S$ and W^c is closed in $\mathbb{R}$

$S\setminus W^c = S\cap W = W\cap S = W\setminus S^c$

thus $m(S\setminus W^c) < \varepsilon$

Theorem: The increasing sequence of subsets property

Let $(X, \mathcal{A}, \mu)$ be a measure space

and A_n is an increasing sequence in $\mathcal{A}$ $A_n \subseteq A_{n+1}$

then $\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

Similarly if B_n is decreasing in $\mathcal{A}$, then $\lim_{n\to+\infty} \mu(B_n) = \mu\left(\bigcap_{n=1}^{\infty} B_n\right)$

and $\mu(B_1) < +\infty$

proof: as $A_n \subseteq A_{n+1}$ $\mu(A_n) < \mu(A_{n+1})$

so $\mu(A_n)$ is an increasing sequence in $[0,+\infty]$

it thus convergences in $[0,+\infty]$

for any n $A_n \subset \bigcup_{n=1}^{\infty} A_n$

thus $\mu(A_n) \leq \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$ so $\lim_{n\to+\infty} \mu(A_n) \leq \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

Date

set $B_1 = A_1 \quad B_2 = A_2 \setminus A_1 \quad \cdots \quad B_n = A_n \setminus A_{n-1}$.

$B_n \in \mathcal{A}$

$A_1 \subset A_2 \subset \cdots A_{n-1}$

The sequence B_n is pairwise disjoint

$$\bigcup_{n=1}^{k} B_n = \bigcup_{n=1}^{k} A_n \text{ and } \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

Using σ-additivity,

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \qquad \sum_{n=2}^{\infty}\left(\mu(A_n) - \mu(A_{n-1})\right) + \mu(A_1)$$

if for all n $\mu(A_n) < +\infty$

$$\sum_{n=2}^{\infty}\left(\mu(A_n) - \mu(A_{n-1})\right) + \mu(A_1)$$

$$= \mu(A_2) - \mu(A_1) + \mu(A_3) - \mu(A_2) + \cdots + \mu(A_1)$$

$$= \lim_{n \to +\infty} \mu(A_n)$$

if $\exists p \quad \mu(A_p) = +\infty$

this case is in fact trivial.

Let A and S be 2 subsets of $\mathbb{R}$ and x be in $\mathbb{R}$

$$S + x = \{s + x,\ s \in S\}$$

$$A \cap (S + x) = ((A - x) + x) \cap (S + x)$$

$f: \mathbb{R} \to \mathbb{R}$

$t_x(y) = y + x$

t_x is bijective : $(t_x)^{-1} = t_x$

Let C and D be 2 subsets of $\mathbb{R}$

$$t_x(C \cap D) = t_x(C) \cap t_x(D)$$

$$\big((A-x)+x\big)\cap(S+x)=t_x\big((A-x)\cap S\big)$$
$$=\big((A-x)\cap S\big)+x$$

Countable subsets of $\mathbb{R}$

$A=\{x_1, x_2, \cdots x_n, \cdots\}$

countable subsets of $\mathbb{R}$ are measurable and the measure is 0.

$\{x_i\}$ is measurable

$A=\bigcup_{i=1}^{\infty}\{x_i\}$ is measurable

$$m(A)=m\Big(\bigcup_{i=1}^{\infty}\{x_i\}\Big)\leq\sum_{i=1}^{\infty}m(\{x_i\})=0$$

If a measurable subset S of $\mathbb{R}$ has measure 0, is it countable?

NO

The Cantor Set

Start from $C_1=[0,1]$

$C_2=[0,\frac{1}{3}]\cup[\frac{2}{3},1]$

$C_3=[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{3}{9}]\cup[\frac{6}{9},\frac{7}{9}]\cup[\frac{8}{9},\frac{9}{9}]$

Keep iterating by taking out the middle open $\frac{1}{3}$ of each interval between 2 consecutive steps.

That way we construct a sequence C_n of subsets

- C_n is decreasing $C_{n+1}\subseteq C_n$
- C_n is a finite union of closed subintervals of $[0,1]$

  thus C_n is closed and included in $[0,1]$

  C_n is compact
- $m(C_{n+1})=\frac{2}{3}m(C_n)$

  $m(C_n)=\sum\left(\frac{2}{3}\right)^{n-1}$

The Cantor Set K is defined by $K=\bigcap_{n=1}^{\infty}C_n$

(i) K is compact
(ii) $m(K)=0$
(iii) K is nonempty and is uncountable

(i) K is closed and bounded
(ii) $C_{n+1} \subseteq C_n$ decreasing sequence of measurable subsets of $\mathbb{R}$, $m(C_1) < +\infty$

thus $m\left(\bigcap_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} m(C_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^{n-1} = 0$

(iii)

lemma: Let X be a compact metric space and F_i, $i \in I$ a collection of closed subsets such that $\forall J \subset I$ if J is finite then $\bigcap_{j\in J} F_j \neq \phi$

then $\bigcap_{i\in I} F_i \neq \phi$

proof: Argue by contradiction
assume that $\bigcap_{i\in I} F_i = \phi$

$$\bigcup_{i\in I}(F_i^c) = X$$

as F_i^c are open in X, there is a finite subcover
$\bigcup_{j\in J}(F_j^c) = X$, where $J \subseteq I$ and J is finite

thus $\bigcap_{j\in J} F_j = \phi$: contradiction

to show (iii) $[0,1]$ is compact
$C_n \subseteq [0,1]$, C_n is closed in $[0,1]$

for any p, $\bigcap_{n=1}^{p} C_n = C_p \neq \phi$ (by construction)

thus $\bigcap_{n=1}^{\infty} C_n \neq \phi$

show that K is uncountable

Argue by contradiction

assume that $K = \{x_2, x_3, \cdots x_n, \cdots\}$

Construct a sequence of subsets F_n:

$$C_2 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$

one of these 2 intervals does NOT contain x_2

denote it by F_2

F_2 is either $[0, \frac{1}{9}]$ or $[\frac{2}{9}, \frac{3}{9}]$

does not contain x_3

denote by F_2

or F_2 is $[\frac{6}{9}, \frac{7}{9}] \cup [\cancel{\frac{7}{9}, \frac{8}{9}}] \cup [\frac{8}{9}, 1]$

That way the sequence F_n satisfies $F_n \subset C_n$

$$\{x_1, \cdots x_n, \cdots\} \cap F_n = \phi$$

$$\bigcap_{k=1}^{n} F_k = F_n \neq \phi$$

thus $\bigcap_{n=1}^{\infty} F_n \neq \phi$ by the lemma, since each F_n is closed and included in $[0, 1]$.

Let x be in $\bigcap_{n=1}^{\infty} F_n$

as $F_n \subseteq C_n$ $\quad \bigcap_{n=1}^{\infty} F_n \subset \bigcap_{n=1}^{\infty} C_n = K$

so $x \in K = \{x_2, \cdots x_n, \cdots\}$

thus $x = x_r$

But by construction $x_r \notin F_r$

thus $x_r \notin \bigcap_{k=1}^{\infty} F_k$

$x \in [0, 1]$ $\quad x = \sum_{n=1}^{\infty} a_n \left(\frac{1}{3}\right)^n$ $\quad a_n = \{0, 1, 2\}$

$$x = \sum_{n=1}^{\infty} b_n \left(\frac{1}{2}\right)^n \quad , \quad b_n = \{0, 1\}$$

Reading there exists a non Lebesgue measurable subset of $\mathbb{R}$

The extended real line

$\overline{\mathbb{R}}$: $\mathbb{R} \cup \{-\infty, +\infty\}$

operations

- if $x \neq -\infty$ and $x \in \overline{\mathbb{R}}$ $\quad x + (+\infty) = +\infty$
- if $x \neq +\infty$ and is in $\overline{\mathbb{R}}$ $\quad x + (-\infty) = -\infty$
- $(+\infty) + (-\infty)$ NOT allowed
- we set $0 \cdot (+\infty) = 0$
  $0 \cdot (-\infty) = 0$

Measurable functions

Definition. $f : (X, \mathcal{A}, \mu) \longrightarrow \overline{\mathbb{R}}$ is measurable if $\forall$ V open subset of $\mathbb{R}$, $f^{-1}(V) \in \mathcal{A}$

$$f^{-1}(\{+\infty\}) \in \mathcal{A} \ , \ f^{-1}(\{-\infty\}) \in \mathcal{A}$$

Proposition Let $f : (X, \mathcal{A}, \mu) \longrightarrow \overline{\mathbb{R}}$
the following are equivalent:

(i) f is measurable
(ii) $\forall c \in \mathbb{R}$ $\quad f^{-1}((c, +\infty]) \in \mathcal{A}$
(iii) $\forall c \in \mathbb{R}$ $\quad f^{-1}([-\infty, c]) \in \mathcal{A}$
(iv) $\forall c \in \mathbb{R}$ $\quad f^{-1}([-\infty, c)) \in \mathcal{A}$
(v) $\forall c \in \mathbb{R}$ $\quad f^{-1}([c, +\infty]) \in \mathcal{A}$

$f : X \to Y$
$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

proof: (i) $\Rightarrow$ (ii)

$$(c, +\infty] = (c, +\infty) \cup \{+\infty\}$$

$$f^{-1}((c, +\infty]) = f^{-1}((c, +\infty) \cup \{+\infty\})$$
$$= \underbrace{f^{-1}((c, +\infty))}_{\in \mathcal{A}} \cup \underbrace{f^{-1}(\{+\infty\})}_{\in \mathcal{A}}$$

(ii) $\Rightarrow$ (iii) $\quad f^{-1}([-\infty, c])^c = \underbrace{f^{-1}((c, +\infty])}_{\in \mathcal{A}}$

(iii) $\Rightarrow$ (iv) $\quad [-\infty, c) = \bigcup_{n=1}^{\infty} [-\infty, c-\frac{1}{n}]$

$$f^{-1}([-\infty, c)) = \bigcup_{n=1}^{\infty} \underbrace{f^{-1}([-\infty, c-\frac{1}{n}])}_{\in \mathcal{A} \text{ by (iii)}}$$

but $\mathcal{A}$ is closed under countable unions

Corollary: Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function, f is a Lebsegue measurable function.

Let A be any subset of $\mathbb{R}$

Let $g: A \to \mathbb{R}$ be a continuous function, g is Lebsegue measurable

Lemma: Let $f: (X, \mathcal{A}, \mu) \longrightarrow \overline{\mathbb{R}}$ let $A \in \mathcal{A}$

Assume that f is measurable on A and on A^c

Then f is measurable on X

proof: let c be in $\mathbb{R}$

$$f^{-1}((c, +\infty]) = \left(f^{-1}((c, +\infty]) \cap A\right) \cup \left(f^{-1}((c, +\infty]) \cap A^c\right)$$

Lemma: Let f, g be measurable functions from $(X, \mathcal{A}, \mu)$ to $\overline{\mathbb{R}}$

Then $|f|$, $f+g$, fg are measurable

Remark: set $X' = X \setminus (\{x \in X, f(x) = +\infty, g(x) = -\infty\} \cup \{x \in X, f(x) = -\infty, g(x) = +\infty\})$

$f+g$ is measurable on X'

proof:

$\{x : |f(x)| < t\} = \{x : -t < f(x) < t\}$. if $t > 0$

$= \{x : -t < f(x)\} \cap \{x : f(x) < t\} \in \mathcal{A}$

let $t \leq 0$ $\quad \{x : |f(x)| < t\} = \phi \in \mathcal{A}$

- $t \in \mathbb{R}$

$$\{x : (f+g)(x) < t\} = \bigcup_{q \in \mathbb{Q}} \{x : f(x) < t-q\} \cap \{x : g(x) < q\}$$

It is clear that if $f(x) < t-q$ and $g(x) < q$
then $f(x) + g(x) < t$

Conversely if $f(x) + g(x) < t$
if $g(x) = f(x) = -\infty$ then $f(x) < t$ and $g(x) < 0$.
if $f(x) > -\infty$ $f(x) + g(x) < t$
$g(x) < t - f(x)$
$\exists q \in \mathbb{Q}$ s.t. $g(x) < q < t - f(x)$
each $\{x : f(x) < t-q\} \in \mathcal{A}$ and $\{x : g(x) < q\} \in \mathcal{A}$
thus $\bigcup_{q \in \mathbb{Q}} \{x : f(x) < t-q\} \{x : g(x) < q\} \in \mathcal{A}$

prove that f^2 is measurable
$\{x : f^2(x) < t\} = \emptyset$ if $t \leq 0$

if $t > 0$, $\{x : f^2(x) < t\} = \{x : f(x) < \sqrt{t}\} \cap \{x : f(x) > -\sqrt{t}\}$ $\in \mathcal{A}$

Set $X'' = \{x \in X : |f(x)| < +\infty, |g(x)| < +\infty\}$
Show that fg is measurable on X''

$$fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$$

Limsup's and liminf's
Let u_n be a sequence valued in $\overline{\mathbb{R}}$.
We say that u_n is convergent if
- $\exists p \in \mathbb{N}$ $\forall n > p$, $u_n \in \mathbb{R}$ and u_n converges in $\mathbb{R}$ or diverges to $+\infty$

OR: $\exists p \in \mathbb{N}$ $\forall n > p$ $u_n = +\infty$
OR $\exists p \in \mathbb{N}$ $\forall n > p$ $u_n = -\infty$

Let u_n be any sequence valued in $[-\infty, +\infty]$

define $s_n = \sup\{u_k ; k \geq n\}$ $\quad i_n = \inf\{u_k : k \geq n\}$

s_n and i_n are sequences valued in $[-\infty, +\infty]$

Notice that $s_{n+1} \leq s_n$, $i_n \leq i_{n+1}$

s_n and i_n converge in $[-\infty, +\infty]$

Definition: $\limsup u_n = \lim s_n$

$\liminf u_n = \lim i_n$

Examples

$u_n = (-1)^n$ $\quad \limsup u_n = 1$

$\liminf u_n = -1$

$v_n = \sin n$ $\quad \limsup v_n = 1$

$\liminf v_n = -1$

$w_n = n(-1)^n$ $\quad \limsup w_n = +\infty$

$\liminf w_n = -\infty$

$x_n = -n$ $\quad \limsup x_n = -\infty$

$\liminf x_n = -\infty$

Property: Let u_n be a sequence in $\overline{\mathbb{R}}$ (i) Then $\liminf u_n \leq \limsup u_n$

(ii) $\limsup u_n \leq \liminf u_n$ if and only if u_n converges in $\overline{\mathbb{R}}$

In that case $\lim u_n = \limsup u_n = \liminf u_n$

Lemma: Let f_n be a sequence of measurable functions on $(X, \mathcal{A}, \mu)$

then $\limsup f_n$ and $\liminf f_n$ are measurable

$(\limsup f_n)(x) = \limsup (f_n(x))$

proof: Set $s_n(x) = \sup\{f_k(x), k \geq n\}$

$= \sup\{f_n(x), f_{n+1}(x), \dots\}$

Show that $\underbrace{\{x : s_n(x) > t\}}_{A} = \underbrace{\bigcup_{k=n}^{+\infty} \{x : f_k(x) > t\}}_{B}$

Let x be in A $\quad s_n(x) > t$

for some $\varepsilon > 0$ $\quad s_n(x) > t + \varepsilon$

By definition of $S_n(x)$

$\exists\, l \geq n$. s.t. $f_l(x) > S_n(x) - \frac{\varepsilon}{2}$

so $f_l(x) > t + \frac{\varepsilon}{2}$ so $f(x) > t$: $x \in B$

let $x \in B$ $\exists\, p \geq n$ s.t. $f_p(x) > t$

but $S_n(x) \geq f_p(x)$

so $S_n(x) > t$. $x \in A$

This shows that S_n is measurable

introduce $i_n = \inf\{f_k : k \geq n\} = \inf\{f_n, f_{n+1}, \cdots\}$

The same proof technique will show that i_n is measurable

$\limsup f_n = \lim S_n = \inf\{S_1, S_2, \cdots S_n, \cdots\}$

Since S_n is decreasing

Thus $\limsup S_n$ is measurable. Similarly $\liminf f_n = \lim i_n = \sup\{i_1, \cdots i_n, \cdots\}$

Corollary:

① Let $f_n : (X, \mathcal{A}, \mu) \longrightarrow [+\infty, +\infty]$ be a sequence of measurable functions such that $\forall\, x \in X$. $f_n(x)$ converges in $\overline{\mathbb{R}}$

We say that f_n is pointwise convergent

Define $f : X \to \overline{\mathbb{R}}$ by $f(x) = \lim f_n(x)$. Then f is measurable

② $g_n : (X, \mathcal{A}, \mu) \longrightarrow \overline{\mathbb{R}}$; assume that there is a set $A \subseteq \mathcal{A}$ such that (i) $\mu(A) = 0$

(ii) $\forall\, x \in X \setminus A$, $g_n(x)$ converges in $\overline{\mathbb{R}}$

Define g by
$$g(x) = \begin{cases} \lim g_n(x) & \text{if } x \in X \setminus A \\ 0 & \text{if } x \in A \end{cases}$$

Then g is measurable

Simple functions

$(X, \mathcal{A}, \mu)$ measure space Let A be in $\mathcal{A}$

Define the function $X \to \mathbb{R}$
$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

(the indicator function for the set A)

1_A is measurable: for $t \in \mathbb{R}$

$$\{x \in X : f(x) < t\} = \begin{cases} \text{if } t \leq 1 & A \\ \text{if } t > 1 & X \end{cases}$$

Simple functions are functions that can be written in the form

$$\sum_{j=1}^{n} c_j \mathbb{1}_{A_j}, \text{ where } A_j \in \mathcal{A} \quad c_j \in \overline{\mathbb{R}}$$

and if $c_k = +\infty$, $c_\ell = -\infty$

$A_\ell \cap A_k = \phi$

Proposition Let $f : (X, \mathcal{A}, \mu) \longrightarrow [0, +\infty]$ be measurable.

There is an increasing sequence of simple functions f_n, which is pointwise convergent to f

divide it into 2^n pieces

proof:

Define $A_{n,k} = \left\{x \in X : \frac{k-1}{2^n} \leq f(x) \leq \frac{k}{2^n}\right\}$, for $k = 1, 2, \cdots \underline{n 2^n}$
reach n

$$B_n = \{x \in X : f(x) > n\}$$

Set $$f_n(x) = \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \mathbb{1}_{A_{n,k}} + n \cdot \mathbb{1}_{B_n}$$

We can prove that $f_n(x) \leq f_{n+1}(x)$ and f_n is pointwise convergent to f.

$$x \in A_{n,k} \Leftrightarrow \underset{\displaystyle \frac{2k-2}{2^{n+1}}}{\underset{\shortparallel}{\frac{k-1}{2^n}}} \leq f(x) \leq \frac{k}{2^n} = \frac{2k}{2^{n+1}}$$

$$\Leftrightarrow \quad x \in A_{n+1, 2k-1} \qquad\qquad x \in A_{n+1, 2k}$$

Lebesgue integrals of simple functions valued in $[0, +\infty]$

Let $f : (X, \mathcal{A}, \mu) \longrightarrow \overline{\mathbb{R}}$ be a simple functions

$f(X)$ is a finite subset of $\overline{\mathbb{R}}$. order it.

$$c_1 < c_2 < \cdots < c_n$$

set $A_j = f^{-1}(c_j)$

$f = \sum_{j=1}^{n} c_j \mathbb{1}_{A_j}$ the A_j's are pairwise disjoint, and $\bigcup_{j=1}^{m} A_j = X$

If we assume that f is valued in $[0, +\infty]$, then $c_1 \geq 0$

Define $\int f = \sum_{j=1}^{n} c_j \mu(A_j)$

This uses the rule $0(+\infty) = (+\infty)\cdot 0 = 0$

$\left(\int_{\mathbb{R}} 0 = 0 , \int_A (+\infty) = 0 , \text{ if } \mu(A) = 0 \right)$

If B is in $\mathcal{A}$ $\int_B f = \sum_{j=1}^{n} c_j \mu(A_j \cap B)$

lemma: Assume that $A_1, \cdots A_n$ are in $\mathcal{A}$ and pairwise disjoint

Let $f = \sum_{j=1} c_j \mathbb{1}_{A_j}$, where $c_j \in [0, +\infty]$,

then $\int f = \sum_{j=1} c_j \mu(A_j)$

This can be proved by induction on n.

lemma: Let f, g be two simple functions on $(X, \mathcal{A}, \mu)$ valued on $[0, +\infty]$

(i) if $f \leq g$, then $\int f \leq \int g$

(ii) $\int f + g = \int f + \int g$

(iii) if $t \in [0, +\infty]$, $\int tf = t \int f$

proof: Write f and g

$$f = \sum_{j=1}^{m} a_j \mathbb{1}_{A_j} \qquad g = \sum_{k=1}^{n} b_k \mathbb{1}_{B_k}$$

where $a_1 < a_2 < \cdots < a_m$

$b_1 < b_2 < \cdots < b_n$

Date

$X = \bigcup_{j=1}^{m} A_j = \bigcup_{k=1}^{n} B_k$ these unions are pairwise disjoint.

$$\int f = \sum_{j=1}^{m} a_j \mu(A_j)$$
$$= \sum_{j=1}^{m} \sum_{k=1}^{n} a_j \quad \mu(A_j \cap B_k)$$
$$\int g = \sum_{j=1}^{m} \sum_{k=1}^{n} b_k \mu(A_j \cap B_k)$$

(i) assume $f \leq g$.

if $A_j \cap B_k = \phi$, then $\mu(A_j \cap B_k) = 0$

if $A_j \cap B_k \neq \phi$, then $a_j \leq b_k$

so $\int f \leq \int g$

(ii) first assume $n=1$ so $g = b \mathbb{1}_B$

$$(f+g) = \sum_{j=1}^{m} (a_j + b) \mathbb{1}_{A_j \cap B} + \sum_{j=1}^{m} a_j \mathbb{1}_{A_j \setminus B}.$$

the $2m$ sets $A_j \cap B$ and $A_j \setminus B$ are pairwise disjoint

According to the previous lemma,

$$\int (f+g) = \sum_{j=1}^{m} (a_j + b) \mu(A_j \cap B) + \sum_{j=1}^{m} a_j \mu(A_j \setminus B)$$
$$= \sum_{j=1}^{m} a_j [\mu(A_j \cap B) + \mu(A_j \setminus B)] + \sum_{j=1}^{m} b \mu(A_j \cap B)$$
$$= \sum_{j=1}^{m} a_j \mu(A_j) + \quad b\mu(B) = \int f + \int g$$

from there, the proof can be completed by induction

(iii) clear

Lemma: $(X, \mathcal{A}, \mu)$ is a measure space
$f: X \to [0, +\infty]$ simple function
Define $\nu: \mathcal{A} \longrightarrow [0, +\infty]$ by setting for B in $\mathcal{A}$

$$\nu(B) = \int_B f \text{, then } \nu \text{ is a measure on } X.$$
$$= \int_X f \mathbb{1}_B$$

B_n $\nu\left(\bigcup_{p=1}^{\infty} B_p\right)$ $f = \sum_{j=1}^{n} c_j \mathbb{1}_{A_j}$

Summary:

$(X, \mathcal{A}, \mu)$ $f = \sum_{j=1}^{n} c_j \mathbb{1}_{A_j}$, $A_j \in \mathcal{A}$ $c_j \in [0, +\infty]$

$$\int f = \sum_{j=1}^{n} c_j \mu(A_j)$$

any simple function f on X can be written as $\sum_{j=1}^{m} d_j \mathbb{1}_{B_j}$ $d_j \in [0, +\infty]$
the B_j's are pairwise disjoint.

We proved that if $f, g: X \longrightarrow [0, +\infty]$ are measurable

(i) $\int f + g = \int f + \int g$

(ii) $\int cf = c \int f$, $c \in [0, +\infty]$

(iii) if $f \le g$, then $\int f \le \int g$

proof of the lemma above:

$$f = \sum_{j=1}^{n} c_j \mathbb{1}_{A_j}$$

where $c_j \in [0, +\infty]$, $A_j \in \mathcal{A}$
A_j's are pairwise disjoint.

$$f \mathbb{1}_B = \sum_{j=1}^{n} c_j \mathbb{1}_{A_j} \mathbb{1}_B = \sum_{j=1}^{n} c_j \mathbb{1}_{A_j \cap B}.$$

the sets $A_j \cap B$ are in $\mathcal{A}$, are disjoint

$$\nu(B) = \int f \mathbb{1}_B = \sum_{j=1}^{n} c_j \mu(A_j \cap B)$$

Def: let $f\ (X, \mathcal{A}, \mu) \to [0,+\infty]$ be measurable
We define $\int f = \sup\left\{ \int s : 0 \leq s \leq f \text{ and } s: X \to [0,+\infty] \text{ is simple} \right\}.$

A simple argument will show that
if $A \in \mathcal{A}$ $\quad \int_A f = \int_X f 1_A$

If $f, g : X \to [0,+\infty]$ are measurable and $f \leq g$
$\{\text{simple functions } s: X \to [0,+\infty] : 0 \leq s \leq f\}$
$\subseteq \{\text{simple functions} : s: X \to [0,+\infty] : 0 \leq s \leq g\}$
so $\int f \leq \int g$

In particular, if $A, B \in \mathcal{A}$ and $A \subseteq B$
$$f 1_A \leq f 1_B$$
$$\int_A f \leq \int_B f$$

Prop: Let $f : (X, \mathcal{A}, \mu) \to [0,+\infty]$ be a measurable function
$\int f = 0$ if and only if f is zero almost everywhere in X

proof: assume f is zero a.e. in X
this means that $\exists A \in \mathcal{A}$ s.t. $\mu(A) = 0$
and $x \in X \setminus A$, $f(x) = 0$
let s be a simple function such that $0 \leq s \leq f$
$0 \leq s \leq (+\infty) 1_A$

so $0 \leq \int s \leq 0$
this shows that $\int f = 0$.

Conversely if $\int f = 0$ set $A_n = \left\{ x \in X : f(x) \geq \frac{1}{n} \right\}$
$A_n \in \mathcal{A}$
A_n is an increasing sequence
$A_n \subseteq A_{n+1}$ $\quad \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

$$0 = \int f \geqslant \int_{A_n} f \geqslant \int_{A_n} \frac{1}{n} = \frac{1}{n}\mu(A_n) \geqslant 0$$

Thus $\mu(A_n) = 0$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

because the sequence A_n is increasing in $\mathcal{A}$

Set $A = \bigcup_{n=1}^{\infty} A_n$, $\mu(A) = 0$

$A^c = \{x \in X : f(x) = 0\}$ thus f is zero a.e.

The monotone convergence theorem

Let f_n be a sequence measurable functions on $(X, \mathcal{A}, \mu)$

such that $0 \leqslant f_n \leqslant f_{n+1}$

Define $f : X \to [0, +\infty]$ by $f(x) = \lim_{n\to+\infty} f_n(x)$ pointwise convergence

(We know that f is measurable)

$$\lim \int f_n = \int f$$

proof: $0 \leqslant \int f_n \leqslant \int f_{n+1}$

thus $\lim_{n\to\infty} \int f_n$ exits in $[0, +\infty]$

as $f_n \leqslant f$ $\quad \int f_n \leqslant \int f$

$\lim \int f_n \leqslant \int f$

conversely, show that $\int f \leqslant \lim \int f_n$

Let s be a simple function such that $0 \leqslant s \leqslant f$

Fix c in $(0, 1)$

Set $A_n = \{x \in X : c\,s(x) \leqslant f_n(x)\}$

$A_n \in \mathcal{A}$

Since $f_n \leqslant f_{n+1}$ $\quad A_n \subseteq A_{n+1}$

$$\bigcup_{n=1}^{\infty} A_n = X$$

(Introduce $\nu: \mathcal{A} \longrightarrow [0,+\infty]$)

$$\nu(B) = \int_B s$$

We know that ν is a measure

Apply the increasing sequence of measurable sets property

$$\nu(X) = \lim_{n\to+\infty} \nu(A_n)$$

$$\nu(A_n) = \int_{A_n} s \leqslant \frac{1}{c}\int_{A_n} f_n \leqslant \frac{1}{c}\int f_n$$

let $n \to +\infty$:

$$\int s = \nu(X) \leqslant \frac{1}{c}\lim_{n\to+\infty}\int f_n$$

As s is any simple function such as $0 \leqslant s \leqslant f$

We find that

$$\int f \leqslant \frac{1}{c}\lim\int f_n$$

As c is arbitrary in $(0,1)$

let $c \to 1$

$$\int f \leqslant \lim \int f_n$$

Corollary: Let f, g $X: \longrightarrow [0,+\infty]$ be measurable functions

$$\int f+g = \int f + \int g$$

proof:

There is a sequence f_n (resp. g_n) of simple functions such that $0 \leqslant f_n \leqslant f_{n+1}$ and $f_n \longrightarrow f$ pointwise (resp $0 \leqslant g_n \leqslant g_{n+1}$ and $g_n \to g$ pointwise)

We know that $\int f_n + g_n = \int f_n + \int g_n$

apply the monotone convergence theorem to find

$$\int f+g = \int f + \int g$$

Remark:

if $f: X \to [0,+\infty]$ is measurable and $c \in [0,+\infty]$

$$\int cf = c\int f$$

Fatou's lemma nonnegative sequence of measurable functions

let $f_n: (X,\mathcal{A},\mu) \to [0,+\infty]$ be a sequence of measurable functions. then $\int \liminf f_n \le \liminf \int f_n$

proof: set $i_n = \inf\limits_{k\ge n} f_k$ then $0 \le i_n \le i_{n+1}$

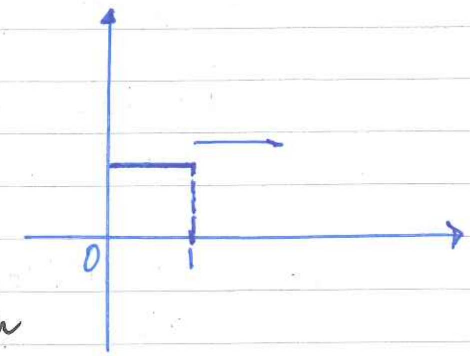
and $\lim i_n = \liminf f_n$

Apply the monotone convergence theorem to i_n

$$\int \lim i_n = \lim \int i_n$$

$$\parallel$$

$$\int \liminf f_n$$



$$i_n \le f_n$$

so $\int i_n \le \int f_n$

thus $\underbrace{\liminf \int i_n}_{\parallel \; \lim\int f_n} \le \liminf \int f_n$

Integration

$(X,\mathcal{A},\mu)$ measure space

① simple functions from X to $[0,+\infty]$

② measurable functions f from X to $[0,+\infty]$

$$\int f = \sup\{\int s : s \text{ is simple function on } X \text{ and } 0 \le s \le f\}$$

Monotone convergence theorem (MCT) $f_n \le f_{n+1}$: $\int \lim f_n = \lim \int f_n$

Fatou's lemma $\int \liminf f_n \le \liminf \int f_n$

Lebesgue integrals

Let $f : (X, \mathcal{A}, \mu) \longrightarrow \overline{\mathbb{R}}$

Define its positive and its negative point.

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that $f^+ \geq 0$ $f^- \geq 0$ $f = f^+ - f^-$

$|f| = f^+ + f^-$

Assume now that f is measurable, both f^+ and f^- are measurable

If $\int f^+ < +\infty$ $\int f^- < +\infty$ then we say that f is integrable

$\mathcal{L}^1(X) \stackrel{\Delta}{=} \{$ set of integrable functions from X to $\overline{\mathbb{R}}\}$

We define the $\int f$ to be $\int f^+ - \int f^-$

Remark: Let $f : X \to \overline{\mathbb{R}}$ be measurable

The following are equivalent:

(i) $f \in \mathcal{L}^1(X)$

(ii) $|f| \in \mathcal{L}^1(X)$

(iii) $\exists\, g \in \mathcal{L}^1(X)$, s.t $|f| \leq g$

proof: $f \in \mathcal{L}^1(X) \Leftrightarrow \int f^+ < +\infty$ and $\int f^- < +\infty$

$\int f^+ + \int f^- < +\infty$

$\int (f^+ + f^-) < +\infty$

$\int |f| < +\infty$

(i) and (ii) implies (iii) clear

assuming (iii)

Since $|f| \leq g \Rightarrow \int |f| \leq \int g < +\infty$

thus (ii) proved

Date

linearity

Let f and g be in $L^1(X)$ and c be in $\mathbb{R}$, then $f+g$, cf are $L^1(X)$ and $\int cf = c\int f$, $\int f+g = \int f + \int g$

Remark: how do we interpret $f+g$, since f and g are valued in $\overline{\mathbb{R}}$

$\int |f| < +\infty$ and $\int |g| < +\infty$

Set $A = \{x \in X : |f(x)| = +\infty\} \cup \{x \in X : |g(x)| = +\infty\}$
A is measurable and $\mu(A) = 0$

$$\int f+g = \int_{X \setminus A} f+g$$

next prove the linearity formula

$c \geq 0$ $\quad (cf) = (cf)^+ - (cf)^- = cf^+ - cf^-$

thus, $\int cf = \int cf^+ - \int cf^- = c\int f^+ - c\int f^- = c(\int f^+ - \int f^-)$
$= c\int f$

if $c \leq 0$ set $d = -c$

addition formula

$$(f+g) = (f+g)^+ - (f+g)^-, \quad f = f^+ - f^-, \quad g = g^+ - g^-$$

$$= f^+ + g^+ - f^- - g^- =$$

$$(f+g)^+ + f^- + g^- = f^+ + g^+ + (f+g)^-$$

but additivity was shown for measurable functions valued on $[0, +\infty]$

$$\int (f+g)^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int (f+g)^-$$

$$\int f+g = \int (f+g)^+ - \int (f+g)^- = (\int f^+ - \int f^-) + (\int g^+ - \int g^-) = \int f + \int g$$

Remark
If $f \in \mathcal{L}^1(X)$, $\left|\int f\right| \leq \int |f|$ (HWpb)

Lebesgue dominated convergence theorem

Let f_n be a sequence of measurable functions from $(X, \mathcal{A}, \mu) \to \overline{\mathbb{R}}$
Assume for all x in X, $f_n(x)$ converges to some limit $f(x)$ in $\overline{\mathbb{R}}$
($f: X \to \overline{\mathbb{R}}$ is known to be measurable)
Dominance condition: $\exists g \in \mathcal{L}^1(X)$ s.t. $\forall n \in \mathbb{N}$, $|f_n| \leq g$
Then (i) $f \in \mathcal{L}^1(X)$
(ii) $\lim \int |f_n - f| = 0$.
(iii) $\lim \int f_n = \int f$

proof: (i) $\forall x \in X$ $\forall n \in \mathbb{N}$

$$|f_n(x)| \leq g(x)$$

let $n \to +\infty$ $|f(x)| \leq g(x)$ $\forall x \in X$

as f is measurable and $g \in \mathcal{L}^1(X)$ this shows that $f \in \mathcal{L}^1(X)$

(ii) Results from Fatou's lemma:
if $h_n: X \to [0, +\infty]$ is measurable, then

$$\int \liminf h_n \leq \liminf \int h_n$$

Set $h_n = 2g - |f_n - f|$
Note $|f_n - f| \leq |f_n| + |f| \leq 2g$
so $h_n \geq 0$ and h_n is measurable — construct a nonnegative sequence of functions

as $\int \liminf h_n \leq \liminf \int h_n$

$$\int 2g \leq \liminf\left(\int 2g - \int |f_n - f|\right)$$

Does not depend on n

$$0 \leq \liminf\left(-\int |f_n - f|\right)$$

$$0 \leq -\limsup\left(\int |f_n - f|\right)$$

$$\limsup\left(\int |f_n - f|\right) \leq 0 \leq \liminf\left(\int |f_n - f|\right)$$

$$\lim \int |f_n - f| = 0$$

(iii) Need to show $\lim \int f_n = \int f$

$$\left|\int (f_n - f)\right| \leq \int |f_n - f|$$

Lebesgue dominated convergence theorem 'stronger form'

Let $f_n\ (X, \mathcal{A}, \mu) \longrightarrow \overline{\mathbb{R}}$ be a sequence measurable functions

Assume that for almost all x in X

$f_n(x) \to f(x)$ (converges to a limit $f(x)$)

Assume that $\exists\, g \in \mathcal{L}^1(X)$ such that

$\forall\, n \in \mathbb{N}$ $|f_n(x)| \leq g(x)$ almost everywhere

(extended to X by 0)

Then (i) $f \in \mathcal{L}^1(X)$

(ii) $\int |f_n - f| \longrightarrow 0$.

(iii) $\int f_n \to \int f$

proof: Denote by A_n the set $\{x \in X : |f_n(x)| > g(x)\}$

Denote by B the set $\{x \in X : f_n(x)$ does not converge to $0)$

By assumption $\mu(A_n) = 0$ and $\mu(B) = 0$

Define $A = \left(\bigcup_{n=1}^{\infty} A_n\right) \cup B$ $\mu(A) = 0$

apply the previous theorem on $X \setminus A$

Finally note that $\int_X |f_n - f| = \int_{X\setminus A} |f_n - f|$

Dominated convergence

Proposition: Let f be in $C([a,b])$, then the Riemann integral

$(R)\int_a^b f$ is equal to $\int_a^b f$

proof: Introduce the following partition of $[a,b]$

$x_{0,n} = a$ $\quad x_{1,n} = a + \frac{(b-a)}{n}$, $\ldots$ $x_{j,n} = a + j\frac{(b-a)}{n}$, $\ldots x_{n,n} = b$

Define the simple function on $[a,b]$

$$f_n = \sum_{k=0}^{n} f(x_{k,n}) \cdot \mathbb{1}_{[x_{k,n};\, x_{k+1,n})}$$

$$\int_a^b f_n = \sum_{k=0}^{n} f(x_{k,n})\, \mu(x_{k+1,n} - x_{k,n}) = \sum_{k=0}^{n} f(x_{k,n})\left(\frac{b-a}{n}\right)$$

This is a Riemman sum for f. we know that as $n \to +\infty$

$$\int_a^b f_n \longrightarrow (R\int)_a^b f$$

Use dominated convergence to show that

$$\int_a^b f_n \to \int_a^b f$$

Show that $f_n(x) \longrightarrow f(x)$, a.e.

As $[a,b]$ is compact, f is uniformly continuous on $[a,b]$

Fix $\varepsilon > 0$, $\exists \alpha > 0$, $\forall u, v \in [a,b]$

$$|u - v| < \alpha \Rightarrow |f(u) - f(v)| < \varepsilon$$

For all n such that $\frac{1}{n} < \alpha$

$\forall x \in [x_{k,n}, x_{k+1,n})$

$$|f(x) - f(x_{k,n})| < \varepsilon$$

so $$|f(x) - f_n(x)| < \varepsilon$$

This shows that $f_n(x) \longrightarrow f(x)$ $\forall x \in [a,b)$ pointwise convergence

so $f_n(x) \longrightarrow f(x)$ a.e. on $[a,b]$

<u>Dominance</u> Set g ~~to~~ be the constant function $\max_{[a,b]} |f|$

$$|f_n| \leq g \quad \text{and} \quad g \in L([a,b])$$

By dominated convergence

$$\int_a^b f_n \longrightarrow \int_a^b f$$

② Show that $\lim_{n\to+\infty}\int_0^n \ln x\left(1-\frac{x}{n}\right)^n dx = \int_0^{+\infty}(\ln x)(e^{-x})\,dx$

$$\int_0^n \left(1-\frac{x}{n}\right)\ln x\,dx = \int_0^{+\infty}(\ln x)\cdot\left(1-\frac{x}{n}\right)^n \mathbb{1}_{[0,n]}(x)\,dx$$

pointwise convergence $f_n,\ f : [0,+\infty) \to \mathbb{R}$

$$f_n(x) = (\ln x)\left(1-\frac{x}{n}\right)^n \mathbb{1}_{[0,n]}(x)$$

$$f(x) = \ln x\, e^{-x}$$

Fix x in $[0,+\infty)$ for all n such that $n > x$

$$\mathbb{1}_{[0,n]}(x) = 1$$

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}$$

if $u\in[0,1]$, $\ln(1-u) \leq -u$

if $0\leq x\leq n$ $\quad 0\leq \frac{x}{n}\leq 1$

$$\ln\left(1-\frac{x}{n}\right) \leq -\frac{x}{n}$$

$$\left(1-\frac{x}{n}\right)^n \leq e^{-x}$$

For all $x\in(0,+\infty)$ $\quad |f_n(x)| \leq |(\ln x)e^{-x}|$

explain $\int_0^{+\infty} |\ln(x)e^{-x}|\,dx < +\infty$

$$|\ln x\, e^{-x}| \leq |\ln x\, e^{-x}|\mathbb{1}_{(0,1)}(x) + |\ln x\, e^{-x}|\mathbb{1}_{(1,+\infty)}(x)$$

$$\leq |\ln x|\mathbb{1}_{(0,1)}(x) + xe^{-x}\mathbb{1}_{(1,+\infty)}(x)$$

Show that $\int_0^1 |\ln x|\,dx < +\infty$

By M.C.T $\quad \int_0^1 |\ln x|\,dx = \lim_{n\to+\infty}\int_{\frac{1}{n}}^1 |\ln x|\,dx$

But $\int_{\frac{1}{n}}^1 \ln x\,dx = [x\ln x - x]_{\frac{1}{n}}^1$

$$\int_0^1 |\ln x|\,dx < +\infty$$

Date

Show that $\int_1^{+\infty} xe^{-x}\,dx < +\infty$

$\exists A>0 \quad \forall x\geq 1 \quad 0 \leq xe^{-x} < \frac{A}{x^2}$

and $\int_1^{+\infty} \frac{A}{x^2}\,dx < +\infty$

We showed that $g \in \mathcal{L}([0,+\infty))$

So by dominated convergence theorem

$$\lim_{n\to+\infty} \int_0^{+\infty} f_n = \int_0^{+\infty} \lim f_n = \int_0^{+\infty} (\ln x)e^{-x}\,dx$$

Series of positive functions

Let $f_n\ X \longrightarrow [0,+\infty]$ be a sequence measurable functions

Then $\sum_{n=0}^{\infty} \int f_n = \int \sum_{n=0}^{\infty} f_n$

proof: Set $F_n = \sum_{k=0}^{n} f_k$ $\quad F_n$ is measurable, valued in $[0,+\infty]$,

and $F_n \leq F_{n+1}$

By M.C.T. $\quad \int \lim_{n\to+\infty} F_n = \lim_{n\to+\infty} \int F_n$

Example: Write $\int_0^{+\infty} \frac{x}{e^x-1}\,dx$ as the sum of a series.

For $x \geq 0$ $\quad \frac{x}{e^x-1} = \frac{xe^{-x}}{1-e^{-x}} = (xe^{-x})\frac{1}{1-e^{-x}}$

$$= xe^{-x}\sum_{n=0}^{\infty}(e^{-x})^n$$

$$\int_0^{+\infty}\frac{x}{e^x-1}\,dx = \sum_{n=0}^{\infty}\int_0^{+\infty} x(e^{-x})^{n+1}\,dx = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Fubbini's thm

Integration by parts

Integral depending on a parameter, differentiability

Theorem: let $(X, \mathcal{A}, \mu)$ be a measure space
$[a,b]$ closed and bounded in $\mathbb{R}$

$$f : X \times [a,b] \longrightarrow \overline{\mathbb{R}}$$
$$(x,t) \longmapsto f(x,t)$$

assume (i) $\forall t \in [a,b] \quad x \longrightarrow f(x,t) \in L^1(X)$

(ii) $\frac{\partial f}{\partial t}$ is defined for all t in $[a,b]$ for almost all x

(iii) $\exists g \in L(X)$ $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x)$, $\forall t \in [a,b]$ for almost all x in X

Then $F : [a,b] \longrightarrow \mathbb{R}$

$F(t) = \int f(x,t)\,dx$ is differentiable on $[a,b]$

and $F'(t) = \int \frac{\partial f}{\partial t}(x,t)\,dx$

proof: fix t^* in $[a,b]$
and let t_n be a sequence in $[a,b]$ s.t.
$t_n \neq t^*$ and $\lim t_n = t^*$

$$\frac{F(t^*) - F(t_n)}{t^* - t_n} =$$

$$\frac{f(x,t^*) - f(x,t_n)}{t^* - t_n} = \frac{\partial f(x,u)}{\partial t}$$, for some u between t^* and t_n

By (iii) $\left|\frac{f(x,t^*) - f(x,t_n)}{t^* - t_n}\right| \leq g(x)$ a.e,

$$\lim_{n \to +\infty} \frac{f(x,t^*) - f(x,t_n)}{t^* - t_n} = \frac{\partial f}{\partial t}(x,t^*) \quad \text{a.e in } X$$

Apply the D.C.T to find that $\lim_{n\to+\infty}\int \frac{f(x,t^*)-f(x,t_n)}{t^*-t_n}dx = \int \frac{\partial f}{\partial t}(x,t^*)dx$

Example:
Define $F(t)=\int_0^{+\infty}\frac{e^{-tx^2}}{1+x^2}dx$

Show that F is differentiable in $(0,+\infty)$
Fix a, and b such that $0<a<b$

$0<a<t<b$

$$\left|\frac{e^{-tx^2}}{1+x^2}\right| \leq \frac{1}{1+x^2}$$

$$\frac{\partial}{\partial t}\left(\frac{e^{-tx^2}}{1+x^2}\right) = \frac{-x^2e^{-tx^2}}{1+x^2}$$

dominance condition

$$\left|\frac{-x^2e^{-tx^2}}{1+x^2}\right| \leq \frac{x^2}{1+x^2}e^{-ax^2} \leq e^{-ax^2} \quad \text{as } a>0$$

$$\int_0^{+\infty}e^{-ax^2}dx<+\infty \qquad g\in L^1((0,+\infty))$$

by dominated convergence we conclude that F is differentiable on $[a,b]$ and $F'(t)=\int_0^{+\infty}\frac{-x^2e^{-tx^2}}{1+x^2}dx$

As a and b are arbitrary this holds on $(0,+\infty)$

$$F(t)-F'(t)=\int_0^{+\infty}e^{-tx^2}dx$$

set $tx^2=s^2$ $\quad t^{\frac{1}{2}}x=s$ $\quad t^{\frac{1}{2}}dx=ds$

$$F(t)-F'(t)=t^{-\frac{1}{2}}\left(\int_0^{+\infty}e^{-s^2}ds\right)=\frac{\sqrt{\pi}}{2\sqrt{t}}$$

True or false?
Assume $x\longrightarrow f(x,t)$ is in $L(\mathbb{R})$ $\forall\, t\in[a,b]$
assume $\frac{\partial f}{\partial t}(x,t)$ exists. for all t in $[a,b]$

and $x\longrightarrow \frac{\partial f}{\partial t}(x,t)$ is in $L(\mathbb{R})$

Is it true that $\frac{\partial}{\partial t}\int f(x,t)\,dx = \int \frac{\partial f}{\partial x}(x,t)\,dx$?

False:

set $F(t)=\int_0^{+\infty} t^3 e^{-t^2 x}\,dx \quad t\in[0,1]$

$f:[0,1]\to\mathbb{R} \quad f=\mathbb{1}_{\{0\}} \geqslant 0 \quad \int f=0$

$g:[0,1]\to\mathbb{R} \quad g=\mathbb{1}_{[0,1]\cap\mathbb{Q}} \quad \int g=0=m([0,1]\cap\mathbb{Q})$

since $[0,1]\cap\mathbb{Q}$ is countable

Note: that f and g are zero almost everywhere

Classes of functions defined a.e.

$(X,\mathcal{A},\mu)$ measure space

Let f,g,h be measurable functions from X ~~to~~ $\overline{\mathbb{R}}$

(i) $f=f$ a.e. (reflexivity)

(ii) if $f=g$ a.e. then $g=f$ a.e. (symmetry)

(iii) if $f=g$ a.e. and $g=h$ a.e. then $f=h$ a.e. (transitivity)

proof of (iii) $\exists A\in\mathcal{A}$ s.t. $\mu(A)=0$ for $\forall x\in A^c$, $f(x)=g(x)$

$\exists B\in\mathcal{A}$ s.t. $\mu(B)=0$ for $\forall x\in B^c$, $g(x)=h(x)$

thus $\forall x\in A^c\cap B^c$ $f(x)=h(x)$

$(A^c\cap B^c)^c=A\cup B$

$\mu(A\cup B)\leqslant\mu(A)+\mu(B)=0$

thus $f=h$ a.e.

Proposition:

If $f\in\mathcal{L}(X)$ then $A=\{x : |f(x)|=+\infty\}$ has measure zero

proof: $\int_A|f| = +\infty\cdot\mu(A)$

but $\int_A|f|\leq\int_X|f|<+\infty$

thus $\mu(A)=0$

Def: $L^1(X)$ to be the space of functions f defined a.e. on X valued in $\overline{\mathbb{R}}$ and such that f is measurable and $\int |f| < +\infty$

$L^1(X)$ is a vector space

$\forall s \in \mathbb{R}$ $\forall f \in L^1(X)$, $sf \in L^1(X)$

addition let f, g be in $L^1(X)$

let $A = \{x \in X, |f(x)| = +\infty$ or $|g(x)| = +\infty\}$

$\mu(A) = 0$

Evaluate $f(x) + g(x)$ on $X \setminus A$

Proposition: Define $\|\cdot\|_1$ $L^1(X) \longrightarrow \mathbb{R}^+$ by $\|f\|_1 = \int |f|$

$\|\cdot\|_1$ defines a norm on $L^1(X)$

proof: Let f, g be in $L^1(X)$

s be in $\mathbb{R}$

$\int |sf| = |s| \int |f|$ so $\|sf\|_1 = |s| \|f\|_1$

$|f(x) + g(x)| \leq |f(x)| + |g(x)|$ a.e.

thus $\int |f(x) + g(x)| \leq \int |f(x)| + \int |g(x)|$

or $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$

let f be in $L^1(X)$ such that $\|f\|_1 = 0$ that is $\int |f| = 0$

Let A_n be a set of $= \{x \in X : |f(x)| \geq \frac{1}{n}\}$

$0 = \int |f| \geq \int_{A_n} |f| \geq \int_{A_n} \frac{1}{n} = \frac{1}{n} \mu(A_n)$

thus $\mu(A_n) = 0$

$\bigcup_{n=1}^{\infty} A_n = \{x \in X : |f(x)| > 0\}$ increasing union.

By using increasing union propeity,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to+\infty} \mu(A_n) = 0$$

this shows that $|f| = 0$ a.e.

Thm: $L^1(X)$ is complete

lemma 1: Let (X, d) be a metric space

Let x_n be a Cauchy sequence in X.

If x_n has a convergence subsequence then X_n converges.

lemma 2: Let (X, d) be metric space

Let x_n be a Cauchy sequence in X

There is a subsequence y_n of x_n such that

$$d(y_n, y_{n+1}) \leq 2^{-n} \quad \forall n \in \mathbb{N}$$

proof: Let f_n be Cauchy sequence in $L^1(X)$

pick g_n a subsequence of f_n such that

$$\| g_{n+1} - g_n \|_1 \leq 2^{-n}$$

Define $G: X \to \overline{\mathbb{R}}$ $\quad G(x) = \sum_{n=1}^{\infty} |g_{n+1}(x) - g_n(x)|$

$(X, \mathcal{A}, \mu)$

$L^1(X) = \{$ measurable functions f defined a.e. on X, s.t. $\int |f| < +\infty \}$

$L^1(X)$ is a vector space

$\|f\|_1 = \int |f|$ defines a norm on $L^1(X)$

proof of the theorem.

Let f_n be a Cauchy sequence in $L^1(X)$

There is a subsequence g_n of f_n such that

$$\| g_{n+1} - g_n \| \leq 2^{-n}$$

Date

set $G = \sum_{n=1}^{\infty} |g_{n+1} - g_n|$

$G : X \longrightarrow [0, +\infty]$ is measurable

$\int G = \sum_{n=1}^{\infty} \int |g_{n+1} - g_n|$, by the M.C.T

so $\int G \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < +\infty$,

thus $G \in L^1(X)$

$G(x) < +\infty$ a.e,

for almost all x,

$\sum_{n=1}^{\infty} |g_{n+1}(x) - g(x)| < +\infty$,

that is the series $\sum_{n=1}^{\infty} (g_{n+1}(x) - g(x))$ is absolutely convergent.

$\Downarrow$

Note that $\sum_{n=1}^{p} g_{n+1}(x) - g_n(x) = g_{p+1}(x) - g_1(x)$ then it is also convergent

thus $g_p(x)$ converges in $\mathbb{R}$ as $p \to +\infty$, for almost all x

Denote by $f(x)$ this limit

Now we apply the D.C.T as $g_{p+1}(x) - g_1(x) \longrightarrow f(x) - g_1(x)$ a.e,

$|g_{p+1} - g_1| \leq \left|\sum_{n=1}^{p} g_{n+1}(x) - g_n(x)\right| \leq \sum_{n=1}^{p} |g_{n+1} - g_n| \leq G$

and $G \in L^1(X)$

We conclude that $f - g_1 \in L^1(X)$ and $\|g_{p+1} - g_1 - (f - g_1)\|_1 \longrightarrow 0$

that is

$f \in L^1(X)$

and $\|g_{p+1} - f\|_1 \to 0$

Remark: The exact same argument can show the following

Theorem: Let f_n be convergent sequence in $L^1(X)$

There is a subsequence f_{n_k} of f_n and g in $L^1(X)$ such that

$f_{n_k}(x)$ converges a.e. and $|f_{n_k}(x)| \leq g(x)$ a.e.

The Lebesgue space L^2

$(X, \mathcal{A}, \mu)$ measure space

$L^2(X) = \{$ measurable functions f from X to $\overline{\mathbb{R}}$, defined a.e.) and such that $\int f^2 < +\infty \}$

Note that $f \in L^2(X) \Leftrightarrow f^2 \in L^1(X)$

Prop and definition:

$L^2(X)$ is a vector space

the formula

$\langle f, g \rangle = \int fg$ defines an inner product on $L^2(X)$

Set $\|f\|_2 = \langle f, f \rangle^{\frac{1}{2}} = \left(\int f^2\right)^{\frac{1}{2}}$

proof : Let $f, g \in L^2(X)$

show that $f \cdot g \in L^1(X)$

fg is measurable $\quad |fg| \leqslant \frac{f^2}{2} + \frac{g^2}{2}$

$$(|f| - |g|)^2 = f^2 + g^2 - 2|fg| \geqslant 0$$

As $f, g \in L^2(X)$, $\int f^2 < +\infty$ and $\int g^2 < +\infty$

so $\int |fg| < +\infty$

so $fg \in L^1(X)$

Let $f, g \in L^2(X)$ and $s \in \mathbb{R}$

$sf \in L^2(X)$ is clear

$$(f+g)^2 = f^2 + 2fg + g^2 \in L^1(X)$$

so $(f+g) \in L^2(X)$

Assume that f is in $L^2(X)$ and $\|f\|_2 = 0$

$\int f^2 = 0 \Rightarrow f^2 = 0$ a.e.

$f = 0$ a.e.

$f = 0$ in $L^2(X)$

Date

Theorem: $L^2(X)$ is complete

proof: HWpb

$$G(x) = \sum_{n=1}^{\infty} |g_{n+1} - g_n|$$

$$\int \left(\sum_{n=1}^{\infty} |g_{n+1} - g| \right)^2 < +\infty$$

Def: Let $(X, \mathcal{A}, \mu)$ be a measure space

$L^\infty(X) = \{ f : X \to [0, +\infty]$, measurable functions, such that

f is measurable and

$\exists M \in (0, +\infty)$ $|f(x)| \leq M$ a.e. $\}$

Let $f, g \in L^\infty(X)$, $s \in \mathbb{R}$

$\exists M, N \in [0, +\infty)$ s.t. $|f(x)| \leq M$ and $|g(x)| \leq N$

for $x \in X \setminus A$, for $x \in X \setminus B$

$\mu(A) = \mu(B) = 0$

$|sf(x)| \leq |s|M$, $\forall x \in X \setminus A$

$\mu(A \cup B) = 0$

$\forall x \in X \setminus (A \cup B)$ $|f(x) + g(x)| \leq M + N$

so sf and $f + g \in L^\infty(X)$

for $f \in L^\infty(X)$, define $\|f\|_\infty = \inf\{ M \in \mathbb{R}$, s.t. $|f(x)| \leq M$ a.e. $\}$

lemma: For $f \in L^\infty(X)$, $|f(x)| \leq \|f\|_\infty$ a.e.

$\{ M \in \mathbb{R} : |f(x)| \leq M$, a.e. $\}$ is an interval of $\mathbb{R}$

Set $A_n = \{ x \in X$ s.t. $|f(x)| > \|f\|_\infty + \frac{1}{n} \}$

$\mu(A_n) = 0$

$A_n \in \mathcal{A}$ A_n is an increasing sequence

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \right) = \lim_{n \to +\infty} \mu(A_n) = 0$$

(by the increasing sequence of sets property)

$$\bigcup_{n=1}^{\infty} A_n = \{x \in X : |f(x)| > \|f\|_\infty\}$$

in other words, $|f(x)| \le \|f\|_\infty$ a.e.

thus if $f \in L^\infty(X)$ and $\|f\|_\infty = 0$ as

$|f(x)| \le 0$ a.e.

then $f = 0$ a.e. in $L^\infty(X)$

Proposition: Let V be an open subset of $\mathbb{R}$ and $f \in C(V) \cap L^\infty(V)$

$$\|f\|_\infty = \sup_V |f|$$

proof HWPb

Theorem: $L^\infty(X)$ is complete

proof: Let f_n be a Cauchy sequence in $L^\infty(X)$

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall p, q \in \mathbb{N}$

if $p > q > N \Rightarrow \|f_p - f_q\|_\infty < \varepsilon$

Set $E_{m,n} = \{x \in X : |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\}$

By the definition $\mu(E_{m,n}) = 0$

$$Y = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{m,n}$$

Y is a measurable set and $\mu(Y) = 0$

Consider the sequence f_n on $X \setminus Y$

$\forall x \in X \setminus Y$ $f_n(x)$ is Cauchy sequence in $\mathbb{R}$,

then it converges to some limit $f(x)$ in $\mathbb{R}$

$f : X \setminus Y \to \mathbb{R}$

f is the pointwise limit of the sequence of measurable functions f_n, thus f is measurable

Set $f(x) = 0$, for x in Y

Now for $x \in X \setminus Y$ $p, q > N$

$\|f_p - f_q\|_\infty < \varepsilon$, so $|f_p(x) - f_q(x)| < \varepsilon$

let $p \to +\infty$ $|f(x) - f_q(x)| < \varepsilon$, $\forall x \in X \setminus Y$ and $\forall q > N$

thus, as $\mu(Y) = 0$

$|f(x) - f_q(x)| \leq \varepsilon$ a.e. in X for $q > N$

so $\|f - f_q\|_\infty \leq \varepsilon$ for $q > N$

thus f_n converges to f in $L^\infty(X)$

Definition: We say that the measure space $(X, \mathcal{A}, \mu)$ is finite if $\mu(X) < +\infty$

Example : X is measurable on $\mathbb{R}$ and bounded

Proposition. Let X be a finite measure space

Then $L^\infty(X) \subseteq L^2(X) \subseteq L^1(X)$

the corresponding injections are continuous.

$L : (V_1, N_1) \longrightarrow (V_2, N_2)$

$\exists c > 0$ $N_2(L(x)) \leq c N_1(x)$ $\forall x \in V_1$

proof: Let f be in $L^\infty(X)$

$|f(x)| \leq \|f\|_\infty$ a.e.

$$\int |f(x)|^2 \leq \int \|f\|_\infty^2 = \|f\|_\infty^2 \mu(X)$$

this shows that $f \in L^2(X)$

$$\|f\|_2 \leq (\mu(X))^{\frac{1}{2}} \|f\|_\infty$$

Thus, the injection $L^\infty(X) \longrightarrow L^2(X)$ is continuous.

Let g be in $L^2(X)$

$\int |g|$

as $\mu(X)$ is finite $\mu(X) < +\infty$

any constant is in $L^2(X)$

$$\|g\|_1 = \int |g| = \int 1|g| \leq \|1\|_{L^2(X)} \|g\|_{L^2(X)}$$
$$= \left(\int_X 1\right)^{\frac{1}{2}} \|g\|_{L^2(X)} = \mu(X)^{\frac{1}{2}} \|g\|_{L^2(X)}$$

This shows that $g \in L^1(X)$ and the injection from $L^2(X) \longrightarrow L^1(X)$ is continuous.

If X is a bounded open set in $\mathbb{R}$, then $L^\infty(X) \subsetneqq L^2(X) \subsetneqq L^1(X)$

Density results

Definition: Let (X, d) be a metric space. We say that the subset of D of X is dense in X if $\overline{D} = X$

Example: $\mathbb{Q}$ is dense in $\mathbb{R}$

Prop: Let (X, d) be a metric space and D be a subset of X, the following statements are equivalent:

(i) D is dense in X

(ii) $\forall x \in X$, $\exists$ sequence x_n valued in D and convergent to x

(iii) $\forall x \in X$, $\forall \varepsilon > 0$, $\exists y \in D$, $d(x, y) < \varepsilon$

(iv) For any open subset V of X, $V \cap D \neq \phi$

Let $(X, \mathcal{A}, \mu)$ be a measure space

Define $S(X) = \{$simple functions on $X\}$

$$\sum_{n=1}^{p} C_n \mathbb{1}_{A_n}$$

Prop: $S(X) \cap L^p(X)$ is dense in $L^p(X)$, if $p = 1, 2$ or $+\infty$

proof: case $p = 1$

Let f be in $L^1(X)$

Set $f = f^+ - f^-$

We know that there is a sequence of simple functions s.t.

$$f_n \leqslant f_{n+1} \leqslant f^+$$

and $f_n(x) \longrightarrow f^+(x)$ pointwise

$$|f_n| \leqslant f^+ \text{ and } f^+ \in L^1(X) \qquad f_n \in L^1(X) \cap S(X)$$

by the D.C.T

$$\|f_n - f^+\|_1 \longrightarrow 0$$

Similarly, $\exists$ sequence g_n in $L^1(X) \cap S(X)$

s.t. $\|g_n - f^-\|_1 \longrightarrow 0$

conclusion

$$\|f - (f_n - g_n)\|_1 \longrightarrow 0$$

$$f_n - g_n \in S(X) \cap L^1(X)$$

Case $p=2$ identical

case $p = +\infty$

Let $f \in L^\infty(X)$, $\qquad f = f^+ - f^-$

$\exists$ sequence $f_n \in S(X)$

s.t.

$$0 \leqslant f_n \leqslant f_{n+1} \leqslant f^+$$

But $f^+(X) \leqslant \|f\|_\infty$ a.e.

so $0 \leqslant f_n(x) \leqslant \|f\|_\infty$ a.e.

$$f_n \in L^\infty(X)$$

$$A_{n,k} = \left\{ x \in X : \frac{k-1}{2^n} < f^+(x) \leqslant \frac{k}{2^n} \right\} \qquad 1 \leqslant k \leqslant n \cdot 2^n$$

$$f_n^+ = \sum_{k=1}^{n 2^n} \frac{k-1}{2^n} \mathbb{1}_{A_{n,k}}$$

For all n large enough $n > \|f\|_\infty$

$$0 \leq f^+(x) - f_n^+(x) \leq \frac{1}{2^n} \quad \text{a.e.}$$

So, $\|f^+ - f_n^+\|_\infty \longrightarrow 0$

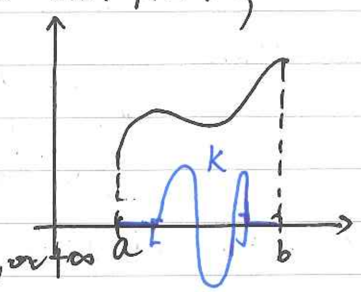
Similar construction for f^-

<u>Density of continuous function</u>.

V is open subset of $\mathbb{R}$

$C(V) = \{$ continuous functions from V to $\mathbb{R}\}$

$C_c(V) = \{ f$ in $C(V)$ s.t. $\exists K$ compact s.t. $K \subseteq V$, and $f(x) = 0 \ \forall x \in V \setminus K \}$



In general, $C(V) \not\subseteq L^1(V)$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$

$f \in C(V)$ but $f \notin L^p(\mathbb{R})$ $p = 1, 2,$ or $+\infty$

However, $C_c(V) \subset L^p(V)$ $p = 1, 2,$ or $+\infty$

Let $f \in C_c(V)$

let K be compact, s.t. $\forall x \in V \setminus K$ $f(x) = 0$

$f(x)$ is bounded on K, by M

$$|f| \leq M \mathbb{1}_K$$

so $f \in L^\infty(V)$

$$\int |f| \leq M\, m(K) < +\infty \quad \Rightarrow \quad f \in L^1(X)$$

$$\int f^2 \leq M^2 m(K) < +\infty \quad \Rightarrow \quad f \in L^2(X)$$

Theorem $C_c(V)$ is dense in $L^p(V)$ $p = 1, 2$

if $p = +\infty$ $C_c(V)$ is NOT dense in $L^\infty(V)$

$C(V)$ is closed in $L^\infty(V)$

Remark: if (X, d) is a metric space

$A \subseteq X$, A is closed and dense

$A = X$

proof:

Assume $p=1$ $m(V) < +\infty$

Let $A \in \mathcal{A}$ s.t. $A \subseteq V$

$m(A) < +\infty$ $1_A \in L^1(V)$

Here, we use the following property of the Lebesgue measure

Fix $\varepsilon > 0$

$\exists$ compact set K, $\exists$ open set W such that

$K \subseteq A \subseteq W \subseteq V$

and $m(W \setminus K) < \varepsilon$

Hwpb, $\exists$ continuous function $f: V \to \mathbb{R}$ s.t.

$f(x) = 1$ if $x \in K$

0 if $x \in V \setminus W$

$0 \leq f(x) \leq 1$ $\forall x \in W$

$$\int_V |1_A - f| = \int_W |1_A - f| = \int_{W \setminus K} |1_A - f| \leq 2m(W \setminus K) = 2\varepsilon$$

Let s be any function in $S(V) \cap L^1(V)$

$$s = \sum_{n=1}^{p} c_n \mathbb{1}_{A_n}, \quad A_n \in \mathcal{A}, \; c_n \in \mathbb{R}$$

use linear combinations

$\exists g \in C_c(V)$, $\|s - g\|_1 < \varepsilon$

let $f \in L^1(V)$:

we proved that $S(V) \cap L^1(V)$ is dense in $L^1(V)$

$\exists$ simple function $s \in S(V) \cap L^1(V)$
s.t. $\|s - f\|_1 < \varepsilon$
$\exists g \in C_c(V)$, $\|g - s\|_1 < \varepsilon$

so $\|f - g\|_1 < 2\varepsilon$

if $m(V) = +\infty$
$V_n = V \cap (-n, n)$
$m(V_n) < +\infty$
let $f \in L^1(V)$
$\|f \mathbb{1}_{V_n} - f\|_1 \longrightarrow 0$

because $f\mathbb{1}_{V_n}(x) \to f(x)$ a.e.
$|f \mathbb{1}_{V_n}| \leq |f|$ and $f \in L^1(X)$

By the D.C.T $\|f\mathbb{1}_{V_n} - f\|_1 \to 0$

Fix $\varepsilon > 0$ $\exists n \in \mathbb{N}$ s.t. $\|f\mathbb{1}_{V_n} - f\|_1 < \varepsilon$
apply the previous step to $f\mathbb{1}_{V_n}$
$\exists g \in C_c(V)$ s.t. $\|g - f\mathbb{1}_{V_n}\|_1 < \varepsilon$

conclusion $\|f - g\|_1 < 2\varepsilon$

if $p = 2$ 'similar proof'
if $p = +\infty$, $C(V) \cap L^\infty(V)$ is closed in $L^\infty(V)$
$\forall f \in C(V) \cap L^\infty(V)$
$\|f\|_\infty = \sup_V |f|$

let f_n ~~be~~ a Cauchy-sequence in $C(V) \cap L^\infty(V)$
f_n converges uniformly to f a continuous function in V

thus, $f \in C(V) \cap L^\infty(V)$

converge theorems

(1) D.C.T. $f_n \to f$ a.e. $|f_n| \le g$, $g \in L^1(X)$

then $\|f_n - f\|_1 \longrightarrow 0$

(2) "converse" if $f_n \longrightarrow f$ in $L^1(X)$

then $\exists$ subsequence f_{n_k} s.t. $f_{n_k}(x) \longrightarrow f(x)$ a.e.

and $\exists\, g \in L^1(X)$ s.t. $|f_{n_k}| \le g$

⚠ But it could be that for all x in X, $f_n(x)$ diverges

Egorov's

Let $(X, \mathcal{A}, \mu)$ be a finite measure space ($\mu(X) < +\infty$)

Let f_n be a sequence of measurable functions which converges pointwise a.e. to f

Assume f is valued in $\mathbb{R}$, then $\forall \varepsilon > 0$, $\exists A \in \mathcal{A}$

s.t. $f_n \longrightarrow f$ uniformly in $X \setminus A$

$\sup_{x \in X\setminus A} |f_n(x) - f(x)| \longrightarrow 0$ and $\mu(A) < \varepsilon$

prove that $C_c(V)$ is the dense in $L^1(V)$

Observe that

$V = \{x \in \mathbb{R},\ d(x, V^c) > 0\}$

Set $K_n = \{x \in \mathbb{R},\ d(x, V^c) \ge \frac{1}{n}\} \cap [-n, n]$

K_n is compact. $K_n \subseteq K_{n+1}$

$\bigcup_{n=1}^{\infty} K_n = V$

Let A be a Lebesgue measurable subset of $\mathbb{R}$, s.t. $A \subseteq V$.

Set $A_n = A \cap K_n$

$A_n \subseteq A_{n+1}$

$\bigcup_{n=1}^{\infty} A_n = A$

Definition:

Let (X, d) be a metric space

$A \subset X$ any subset of X

$d(x, A) = \inf\{d(x, a), a \in A\}$

Equivalently

(i) $a \in A$ $d(x, A) \le d(x, a)$

(ii) there is a sequence a_n in A s.t. $\lim_{n \to +\infty} d(x, a_n) = d(x, A)$

Assume that $m(A) < +\infty$

Fix $\varepsilon > 0$,

$\exists N \in \mathbb{N}$ s.t. $m(A_n) > m(A) - \varepsilon$

We know that there is a closed set F and open set W such that

$$F \subset A_N \subset W$$

and $m(W \setminus F) < \varepsilon$

F is compact

We assume $W \subset \{x \in \mathbb{R} : d(x, V^c) > \frac{1}{2N}\} \cap (-2N, 2N)$

$\exists$ a continuous function such that

$$\begin{cases} f(x) = 1, & \forall x \in F \\ 0 \leq f \leq 1 \\ f(x) = 0 & \forall x \in W^c \end{cases}$$

$f \in C_c(V)$

$$\|f - 1_{A_N}\|_1 = \int_V |f - 1_{A_N}| = \int_{V \setminus F} |f - 1_{A_N}|$$

$$= \int_{W \setminus F} |f - 1_{A_N}| \leq 2m(W \setminus F) < 2\varepsilon$$

$\|1_{A_n} - 1_A\|_1 \longrightarrow 0$ because of the D.C.T

Clearly for $x \in \mathbb{R}$ $1_{A_n}(x) \longrightarrow 1_A(x)$

$A_n = A \cap K_n$ $\quad |1_{A_n} - 1_A| \leq 1_A$

$A_n \subseteq A_{n+1} \subseteq \cdots \subseteq A$ thus $\exists p$ s.t. $\|1_{A_p} - 1_A\| < \varepsilon$

set $R = \max\{p, N\}$

There is an f in $C_c(V)$ such that $\|1_A - f\|_1 < 2\varepsilon$

By the linearity,

for every $g \in S(V) \cap L^1(V)$

$\exists f \in C_c(V)$ s.t. $\|f - g\|_1 < \varepsilon$

Let $h \in L^1(V)$, we have proved that
$\exists\, g \in S(V) \cap L^1(V)$ such that $\|h-g\|_1 < \varepsilon$

thus $\|h-f\|_1 < 2\varepsilon$

proof of Egorov's theorem

Step 1: Show that $\forall \alpha > 0\ \forall \beta > 0\ \exists A \in \mathcal{A},\ \exists N \in \mathbb{N}$
$|f_n - f| < \alpha$ on A $\forall n \geq N$
and $\mu(X \setminus A) < \beta$

Introduce

$$E_n = \{x \in X : |f(x) - f_k(x)| < \alpha,\ \forall k \geq n\}$$

measurable functions $f_k \to f$

$$E_n \in \mathcal{A}, \text{ because } E_n = \bigcap_{k=n}^{\infty} \underbrace{\{x \in X : |f(x) - f_k(x)| < \alpha\}}_{\in \mathcal{A}}$$

countable intersection $E_n \in \mathcal{A}$

Increasing set property $E_n \subseteq E_{n+1}$

$$\bigcup_{n=1}^{\infty} E_n = X \setminus B$$

where $\mu(B) = 0$

By the increasing set property,

$$\lim_{n \to +\infty} \mu(E_n) = \mu(X)$$

thus $\exists N$ s.t. $\mu(\bigcup_{n=1}^{N} E_n) > \mu(X) - \beta$

set $A = \bigcup_{n=1}^{N} E_n$

Fix $\varepsilon > 0$, set $\alpha = \frac{1}{n}$, and $\beta = \frac{\varepsilon}{2^n}$

using step 1:
$\exists A_n \in \mathcal{A},\ \exists N(n)$ s.t. $|f_p - f| < \frac{1}{n}$ on A_n $\forall p \geq N(n)$
and $m(X \setminus A) < \frac{\varepsilon}{2^n}$,

define $A = \bigcup_{n=1}^{\infty} A_n$

$\mu(X \setminus A) < \varepsilon$

and $f_n \to f$ uniformly on A

Product spaces

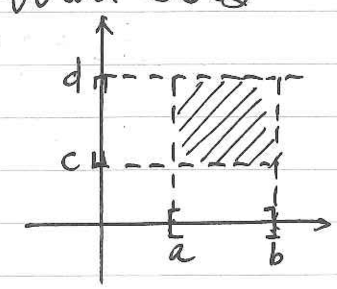
Lebesgue measure in $\mathbb{R}^d$

Definition: let X, Y be 2 measure spaces

Denote by $\mathcal{A}$ and $\mathcal{B}$ their σ-algebra resp.

The product σ-algebra on $X \times Y$ is the smallest σ-algebra containing all the subsets $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$

Definition: The measure space $(X, \mathcal{A}, \mu)$ is said to be σ-finite if $\exists A_n \in \mathcal{A}$ s.t. $\mu(A_n) < +\infty$ and $X = \bigcup_{n=1}^{\infty} A_n$



Remark: $\mathbb{R}$ with the Lebesgue measure is σ-finite

set $A_n = [-n, n]$

- Any measure subspace of σ-finite measure space is also σ-finite

The product measure theorem

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be 2 σ-finite measure space, There exist a unique measure $\mathcal{A} \otimes \mathcal{B} \longrightarrow [0, +\infty]$ denoted by $\mu \otimes \nu$ such that $\mu \otimes \nu\,(A \times B) = \mu(A) \cdot \nu(B)$ $\forall A \in \mathcal{A}, B \in \mathcal{B}$

Proposition: Every open set in $\mathbb{R}^d$ is Lebesgue measurable (and every closed set too)

proof: Define the cube $Q_n(a)$, where $n \in \mathbb{N}$, $a = (a_1, a_2, \cdots a_d)$ in $\mathbb{Z}^d$

$Q_n(a) = \{x \in \mathbb{R}^d : 10^{-n} a_j < x_j \leq 10^{-n}(a_j + 1)\}$

Denote $C_n = \bigcup_{a \in \mathbb{Z}^d} Q_n(a)$, C_n is countable

$C = \bigcup_{n=1}^{\infty} C_n$ is countable

Let V be an open subset $V = \bigcup Q_n(a)$
s.t. $Q_n(a) \subset V$

it is clear that $\bigcup_{Q_n(a) \subseteq V} Q_n(a) \subseteq V$

conversely, let $x \in V$: show that $\exists Q_n(a)$ s.t. $\{x\} \subset Q_n(a) \subset V$

$x = (x_1, \cdots x_d)$ as V is open and $x \in V$

$\exists \varepsilon > 0$ s.t. if $-\varepsilon < x_1 - y_1 < \varepsilon$

$\vdots$

$-\varepsilon < x_d - y_d < \varepsilon$,

then $y = (y_1, \cdots y_d)$ is in V

$\exists p_1, \cdots p_d, q_1, \cdots q_d \in \mathbb{Q}$ s.t.

$x_1 - \varepsilon < p_1 < y_1 < q_1 < x_1 + \varepsilon$

$\vdots$

$x_d - \varepsilon < p_d < y_d < q_d < x_d + \varepsilon$

...

truncate the decimal expansion of $p_1, p_2, \cdots p_d, q_1, \cdots q_d$

Proposition: let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces $f: X \to \overline{\mathbb{R}}$ $g: Y \to \overline{\mathbb{R}}$ be measurable functions. Then $fg: X \times Y \to \overline{\mathbb{R}}$ is measurable and $f+g$: if it is defined, too.

Knapp Basic Real Analysis

proof: for products.

Step 1: assume that $f = c 1_A$ $g = d 1_B$,

where $A \in \mathcal{A}$, $B \in \mathcal{B}$

$$f(x) g(y) = c 1_A(x) \cdot d 1_B(y) = cd 1_{A \times B}(x, y)$$

$A \times B \in \mathcal{A} \otimes \mathcal{B}$

Step 2: by linearity this result holds for all simple functions

Step 3: If f and g are measurable functions, we know that there are two sequences of simple functions f_n, g_n valued in $\mathbb{R}$ such that $f_n \to f$ and $g_n \to g$ pointwise

then $f_n g_n \to fg$ pointwise

since each $f_n g_n$ is simple, measurable in $X \times Y$

it follows that fg is measurable on $X \times Y$

Fubbini's theorem

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be 2 σ-finite measure spaces

① Let $f : X \times Y \to [0, +\infty)$ be measurable

Then $x \longrightarrow \int_Y f(x,y)\, d\nu(y)$ is measurable on X

$y \longrightarrow \int_X f(x,y)\, d\mu(x)$ is measurable on Y

$$\int_{X\times Y} f\, d(\mu \otimes \nu) = \int_X \left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x)$$

$$= \int_Y \left(\int_X f(x,y)\, d\mu(x)\right) d\nu(y)$$

② Let $g : X \times Y \longrightarrow \overline{\mathbb{R}}$ be measurable

If: $\int_X \left(\int_Y |g(x,y)|\, d\nu(y)\right) d\mu(x) < +\infty$

or $\int_Y \left(\int_X |g(x,y)|\, d\mu(x)\right) d\nu(y) < +\infty$

then $g \in L^1(X \times Y)$ and

$$\iint_{X\times Y} g = \int_X \left(\int_Y g(x,y)\, d\nu(y)\right) d\mu(x)$$

$$= \int_Y \left(\int_X g(x,y)\, d\mu(x)\right) d\nu(y)$$

comments on proof: (see Knapp)

Define $P_1 : X \times Y \longrightarrow X \quad (x,y) \longrightarrow x$ two projections

$P_2 : X \times Y \longrightarrow Y \quad (x,y) \longrightarrow y$

if S is measurable in $X \times Y$

then $P_1(S) \in \mathcal{A}$, and $P_2(S) \in \mathcal{B}$

If $f : X \times Y \to \overline{\mathbb{R}}$ is measurable

then $\forall x \in X$, $y \to f(x,y)$ is measurable

$\forall\ y \in Y$, $x \dashrightarrow f(x,y)$ is measurable

Example: compute

$$I = \int_0^{+\infty} \frac{e^{-ax} - e^{-bx}}{x}\, dx$$

$a > 0$, $b > 0$

$$\frac{e^{-ax} - e^{-bx}}{x} = \int_a^b e^{-tx}\, dt$$

$$\int_0^{+\infty} \int_a^b e^{-tx}\, dt\, dx \qquad a > 0,\ b > 0.$$

$(t,x) \longrightarrow e^{-tx} \mathbb{1}_{[0,+\infty)}(x)\, \mathbb{1}_{[a,b]}(t)$ is measurable on $\mathbb{R}^2$

by Fubini's theorem as $e^{-tx} \geq 0$

$$I = \int_a^b \int_0^{+\infty} e^{-tx}\, dx\, dt$$

$$= \int_a^b \left(-\frac{1}{t} e^{-tx} \Big|_0^{+\infty} \right) dt$$

$$= \int_a^b \frac{dt}{t} = \ln t \Big|_a^b = \ln \frac{b}{a}$$

$f: [0,1] \times [0,1] \longrightarrow \mathbb{R}$

$$\begin{cases} f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2} \\ f(0,0) = 0 \end{cases}$$

Careless computation

$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy \right) dx \qquad \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2} \right) = f(x,y)$$

$$I = \int_0^1 \frac{y}{x^2+y^2} \Big|_{y=0}^{y=1} dx = \int_0^1 \frac{1}{x^2+1}\, dx = \frac{\pi}{4}$$

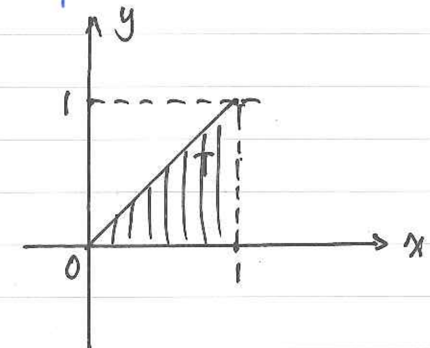
$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2} \right) = f(x,y)$$

$$I = \int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx \right) dy$$

$$= \int_0^1 \frac{-x}{x^2+y^2} \Big|_{x=0}^{x=1} dy = \int_0^1 \frac{-1}{y^2+1}\, dy = -\frac{\pi}{4}$$

$f \notin L^1([0,1]\times[0,1])$ explanation

$$\int_0^1\int_0^1 |f(x,y)|\,dx\,dy = +\infty$$



$$T: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$$

$$\iint_T |f| = \int_0^1\int_0^x |f(x,y)|\,dy\,dx$$

$$= \int_0^1\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx$$

$$= \int_0^1 \left[\frac{y}{x^2+y^2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{x}{2x^2}\,dx = \int_0^1 \frac{1}{2x}\,dx = +\infty$$

$$\iint_{[0,1]\times[0,1]} |f(x,y)|\, \mathbb{1}_{\{(x,y)\in D:\, 0\le y\le x\}}$$

Apply Fubbini's theorem

$$= \int_0^1\left(\int_0^1 |f(x,y)|\, \mathbb{1}_{\{(x,y)\in D;\, 0\le y\le x\}}\,dy\right)dx$$

$$\iint_T |f| = +\infty \le \iint_D |f| \qquad f \notin L^1(D)$$ $\Rightarrow$ can NOT apply Fubbini's theorem

Alternative way of defining the Lebesgue measure on $\mathbb{R}^n$

Define an exterior measure m^* $P(\mathbb{R}^n) \to [0,+\infty]$

$$m^*(S) = \inf\left\{\sum_{k=1}^{\infty} \mathrm{vol}(R_k) : S \subset \bigcup_{k=1}^{\infty} R_k\right\}$$

where $R_k = I_{1,k} \times \cdots I_{n,k}$ and $I_{j,k}$ are open and bounded intervals

From there, define measurable subsets of $\mathbb{R}^n$, and show that m^* is a measure when restricted to those sets

Regularity property

Let E be a Lebesgue measurable subset of $\mathbb{R}^n$, then

$$m(E) = \inf\{m(V) : V \text{ is open and } E \subset V\}$$

$$m(E) = \sup\{m(K) : K \text{ is compact},\ K \subset E\}$$

Translations and dilations

Let E be a measurable subset of $\mathbb{R}^n$, Fix $x \in \mathbb{R}^n$, and $s > 0$

Then
$$E + x = \{y + x,\ y \in E\}$$
$$sE = \{sy,\ y \in E\}$$
are Lebesgue measurable

$$m(E + x) = m(E)$$
$$m(sE) = s^n m(E)$$

Theorem:

Let f be such that • $f \in L^1(\mathbb{R}^n)$ or • f is measurable and $f \geq 0$

Then $$\int_{\mathbb{R}^n} f(y)\, dm(y) = \int_{\mathbb{R}^n} s^n f(x + sz)\, dm(z)$$

$x \in \mathbb{R}^n$, $s > 0$ are fixed.

sketch of proof:

- already proved if $f = 1_E$, where E is measurable in $\mathbb{R}^n$
- by linearity, it must be true for simple functions
- apply approximations by simple functions

More generally

With the same assumptions on f, and on x,

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and invertible mapping,

then $$\int_{\mathbb{R}^n} f(y)\, dm(y) = \int_{\mathbb{R}^n} |\det A|\, f(x + Az)\, dm(z)$$

idea of proof: A can be written as a product of dilations
$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & s & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

and transvections. $$\begin{pmatrix} 1 & 0 & t & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

More general change of variables theorem

where A is a C^1 diffeomorphism between open sets in $\mathbb{R}^n$

A is in C^1 A is differentiable

A^{-1} is also in C^1

Convolution and application to approximation by smooth functions

Let $f, g : \mathbb{R}^n \longrightarrow [0, +\infty]$ be measurable

Define $f * g : \mathbb{R}^n \longrightarrow [0, +\infty]$ the convolution product of f and g by setting $(f*g)(x) = \int f(x-y)g(y)dy$

approximate f by $f \mathbb{1}_{\{|f| \leq n^2\}} \mathbb{1}_{\{|x| \leq n\}}$

which can be approximated by F_n in $C_c(\mathbb{R}^n)$

$(x, y) \longrightarrow F_n(x, y)$ for $\mathbb{R}^{2n}$ to $[0, +\infty]$ is continuous, thus measurable

$(x, y) \longrightarrow F_n(x-y)g(y)$ is measurable

Fubbini's theorem $x \longrightarrow \int F_n(x-y)g(y)dy$ is measurable

Prop: Let $f, g, h : \mathbb{R}^n \longrightarrow [0, +\infty]$ be measurable functions

(i) $f * g = g * f$

(ii) $f * (g * h) = (f * g) * h$

(iii) $(f+g) * h = (f * h) + (g * h)$

(iv) $(\alpha f) * g = \alpha (f * g)$, where $a \in [0, +\infty)$

proof: (iii) and (iv) are clear

(i) change of variables $z = x - y$

$$(f*g)(x) = \int f(z) g(x-z) dz = \int g(x-z) f(z) dz = (g * f)(x)$$

(iii) $f * (g * h)(x)$

$$= \int f(x-y)(g*h)(y)dy$$

$$= \int f(x-y)\left(\int g(y-z)h(z)\, dz\right)dy$$

Apply Fubbini's theorem

$$= \iint f(x-y)g(y-z)h(z)\,dy\,dz$$

$$= \int h(z)\left(\int f(x-y)g(y-z)\,dy\right)dz$$

set $u=y-z$

$$= \int h(z)\left(\int f(x-u-z)g(u)\,du\right)dz$$

$$= \int h(z)(f*g)(x-z)\,dz = (f*g)*h$$

Prop: Let f be in $L^1(\mathbb{R}^n)$ and g in $L^p(\mathbb{R}^n)$ where $p=1, 2,$ or $+\infty$

then $f*g$ is in $L^p(\mathbb{R}^n)$ and

$$\|f*g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}$$

that is, the bilinear product $(f,g)\longrightarrow f*g$ is continuous from $L^1(\mathbb{R}^n)\times L^p(\mathbb{R}^n)\longrightarrow L^p(\mathbb{R}^n)$

proof: if $p=1$

$$\iint |f(x-y)g(y)|\,dy\,dx$$

$$= \iint |f(x-y)g(y)|\,dx\,dy = \int |g(y)|\left(\int |f(x-y)|\,dx\right)dy$$

$$= \int\left(\int |f(z)|\,dz\right)|g(y)|\,dy = \|f\|_{L^1}\cdot\|g\|_{L^1}$$

$$(f*g)(x) = \int f(x-y)g(y)\,dy$$

$$\int |(f*g)(x)|\,dx \leq \iint |f(x-y)g(y)|\,dy\,dx = \|f\|_{L^1}\|g\|_{L^1}$$

so $\|f*g\|_{L^1} \leq \|f\|_{L^1}\cdot\|g\|_{L^1}$

in particular,

$|(f*g)(x)| < +\infty$, a.e. in x

if $p=2$ $\quad f\in L^1,\ g\in L^2$

$$\int\left(\int |g(x-y)||f(y)|dy\right)^2 dx$$

$$=\int\left(\int |g(x-y)||f(y)|dy\right)\left(\int|g(x-y')||f(y')|\,dy'\right)dx$$

$$=\iint\left(\int |g(x-y)||g(x-y')|dx\right)|f(y)|\cdot|f(y')|\,dy\,dy'$$

Apply Cauchy Schwartz inequality

$$\leq \iint\left(\int |g(x-y)|^2dx\right)^{\frac12}\left(\int|g(x-y')|^2dx\right)^{\frac12}|f(y)||f(y')|\,dy\,dy'$$

$$=\|g\|_{L^2}^2\iint|f(y)||f(y')|\,dy\,dy'$$

$$=\|g\|_{L^2}^2\cdot\|f\|_{L^1}^2$$

$$\left|\int f(y)g(x-y)dy\right|<+\infty \quad \text{a.e.}$$

since it is $\leq \int|g(x-y)||f(y)|dy$

and $\|f*g\|_{L^2}^2\leq\|f\|_{L^1}^2\|g\|_{L^2}^2$

Case $p=+\infty$ $\quad f\in L^1\quad g\in L^\infty$

$$(f*g)(x)=\int f(x-y)\,g(y)\,dy \qquad |g(y)|\leq\|g\|_\infty \text{ a.e.}$$

$$|(f*g)(x)|\leq\int|f(x-y)g(y)|\,dy$$

$$\leq\|g\|_\infty\int|f(x-y)|\,dy=\|g\|_\infty\|f\|_{L^1}$$

Example $f(x) = \frac{1}{\sqrt{x}} \mathbb{1}_{(0,1)}(x)$

$f \geq 0 \quad f \in L^1(\mathbb{R})$

$(f \cdot f)(x) = \frac{1}{x} \mathbb{1}_{(0,1)}(x) \notin L^1(\mathbb{R})$

However $(f * f) \in L^1(\mathbb{R})$

$$(f*f)(x) = \int_{\mathbb{R}} f(x-y) f(y) dy$$

$$= \int_0^1 \frac{1}{\sqrt{x-y}} \mathbb{1}_{(0,1)}(x-y) \frac{1}{\sqrt{y}} dy$$

$$(f*f)(0) = \int_0^1 \frac{1}{\sqrt{-y}} \mathbb{1}_{(0,1)}(-y) \frac{1}{\sqrt{y}} dy = 0$$

$$g(x) = \frac{1}{\sqrt{|x|}} \mathbb{1}_{(-1,1)}(x)$$

$g(0) = +\infty \qquad g \geq 0, \ g \in L^1(\mathbb{R})$

$(g*g) \in L^1(\mathbb{R})$

$$(g*g)(x) = \int_{-1}^1 \frac{1}{\sqrt{|x-y|}} \mathbb{1}_{(-1,1)}(x-y) \frac{1}{\sqrt{|y|}} dy$$

$$(g*g)(0) = \int_{-1}^1 \frac{1}{\sqrt{|-y|}} \mathbb{1}_{(-1,1)}(-y) \frac{1}{\sqrt{|y|}} dy$$

$$= \int_{-1}^1 \frac{1}{|y|} dy = +\infty$$

$|(g*g)(x)| < +\infty$ a.e. in x

Approximation of L^p functions $p = 1, 2$ by smooth functions

$C(\mathbb{R}^n)$: the space of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$

$C_c(\mathbb{R}^n)$:

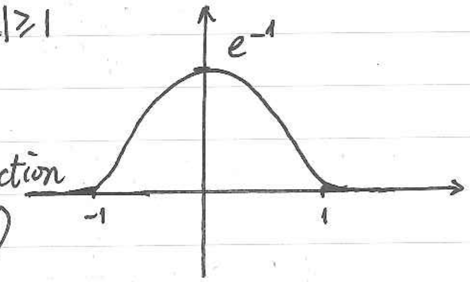
$C^q(\mathbb{R}^n) := \{$functions f from $\mathbb{R}^n$ to $\mathbb{R}$ that posess q continuous derivatives$\}$

$C_c^q(\mathbb{R}^n) = \{$ functions f in $C^q(\mathbb{R}^n)$ such that $\exists$ compact subsets K_f s.t. $f \equiv 0$ in $\mathbb{R}^n \setminus K_f \}$.

$$C^\infty(\mathbb{R}^n) = \bigcap_{q=1}^{\infty} C^q(\mathbb{R}^n)$$

$$C_c^\infty(\mathbb{R}^n) = \bigcap_{q=1}^{\infty} C_c^q(\mathbb{R}^n)$$

example $d=1$ dimension
$$f(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & \text{if } -1 < x < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$



if $-1 < x < 1$ $\quad f^{(n)} = R_n(x)\, e^{\frac{1}{x^2-1}}$

$R_n(x)$ is a rational fraction which is continuous on $(-1, 1)$

for every integer $p > 0$

$$\lim_{x \to 1} \frac{1}{(x-1)^p} e^{\frac{1}{x^2-1}} = 0$$

$$\lim_{x \to -1} \frac{1}{(x+1)^p} e^{\frac{1}{x^2-1}} = 0$$

if $d \geq 2$ set
$$f(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Lemma: If $f \in C_c(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$ then $f * g \in L^p(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and is uniformly continuous on $\mathbb{R}^d$

proof: for $p = 1$
$$|f*g(u) - f*g(v)|$$
$$= \left| \int (f(u-y) - f(v-y))\, g(y)\, dy \right|$$

$\leq \int |f(u-y) - f(v-y)| \, |g(y)| \, dy$

Fix $\varepsilon > 0$, f is uniformly continuous in $\mathbb{R}^d$, $\exists \alpha > 0$ s.t.

if $r, s \in \mathbb{R}^d$ and $|r - s| < \alpha$

then $|f(r) - f(s)| < \varepsilon$

if $|u - v| < \alpha$ $\quad |(f*g)(u) - (f*g)(v)| \leq \varepsilon \int |g(y)| dy = \varepsilon \|g\|_{L^1}$

($(f*g) \in L^1(\mathbb{R}^d)$ was proved earlier)

Lemma: If $f \in C_c^q(\mathbb{R}^d)$ $g \in L^p(\mathbb{R}^d)$

then $(f*g) \in L^p(\mathbb{R}^d) \cap C^q(\mathbb{R}^d)$

and $D_j(f*g) = (D_j f) * g$

where $D_j = (\partial x_1)^{\alpha_1} \cdots (\partial x_d)^{\alpha_d}$

and $\alpha_1 + \cdots \alpha_d \leq q$

proof: $p=1$

∂x_1

$$\frac{(f*g)(x_1+h, x_2, \cdots x_d) - (f*g)(x_1, x_2, \cdots x_d)}{h}$$

$$= \int \underbrace{\frac{f(x_1+h-y_1, x_2-y_2, \cdots x_d-y_d) - f(x_1-y_1, x_2-y_2, \cdots x_d-y_d)}{h}}_{\text{Denote as } T} g(y) \, dy$$

$f \in C_c^q(\mathbb{R}^d)$

$\frac{\partial f}{\partial x_1} \in C_c(\mathbb{R}^d)$, thus $\exists M > 0$, $\forall z \in \mathbb{R}^d$, $\left|\frac{\partial f}{\partial x_1}(z)\right| \leq M$

By mean value theorem

$|T| \leq M$

thus $|Tg(y)| \leq M \underbrace{|g(y)|}_{\in L^1(\mathbb{R}^d)}$ we can apply the D.C.T.

to show that

$$\int T g(y)dy \longrightarrow \int \frac{\partial f}{\partial x_1}(x-y)g(y)dy$$

all the other derivatives can be accounted for likewise

Theorem: $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ if $p=1,2$

In fact if V is open in $\mathbb{R}^d$, $C_c^\infty(V)$ is dense in $L^p(\mathbb{R}^d)$, $p=1,2$

proof: if $V=\mathbb{R}^d$, $p=1$

We already know that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$

Fix $f \in L^1(\mathbb{R}^d)$ and $\varepsilon > 0$

$\exists\ g \in C_c(\mathbb{R}^d)$ s.t. $\|f-g\|_1 < \varepsilon$

set $h(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$

set $C = \int_{\mathbb{R}^d} h$; $C > 0$

then $h_n(x) = \left(n^d h(nx)\right) \cdot C^{-1}$

properties

- $\int_{\mathbb{R}^d} h_n = 0$
- $h_n \geq 0$
- $h_n(x) = 0$ if $|x| \geq \frac{1}{n}$
- $h \in C^\infty(\mathbb{R}^d)$

$(h_n * g) = \int_{\mathbb{R}^d} h_n(x-y)g(y)\,dy$

We know that $h_n * g \in L^1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$

show that $h_n * g(x) \longrightarrow g(x)$ uniformly in $\mathbb{R}^d$.

We note that g is uniformly continuous in $\mathbb{R}^d$

$\exists\, \alpha > 0$, $\forall\, u, v \in \mathbb{R}^d$ $|u-v| < \alpha \Rightarrow |g(u)-g(v)| < \varepsilon$

$$|h_n * g(x) - g(x)| = \left|\int g(x-y)h_n(y)dy - \int g(x)h_n(y)dy\right|$$
$$= \left|\int (g(x-y) - g(x))h_n(y)dy\right|$$
$$\leq \int |g(x-y) - g(x)|\, h_n(y)\, dy$$
$$= \int_{|y|<\alpha} |g(x-y) - g(x)|\, h_n(y)\, dy + \int_{|y|\geq\alpha} |g(x-y) - g(x)|\, h_n(y)\, dy$$
$$\leq \varepsilon + 2\|g\|_\infty \int_{|y|\geq\alpha} h_n(y)dy$$

but $n \to +\infty$ $\int_{|y|\geq\alpha} h_n(y)dy = \int_{|y|\geq\alpha} C^{-1} n^d h(ny)dy$

set $z = ny$ $C^{-1}\int_{|z|\geq n\alpha} h(z)dz$

$$= C^{-1}\int h(z)\,\mathbb{1}_{|z|\geq n\alpha}(z)\,dz$$
$$\to 0 \quad \text{as } n \to +\infty$$

So it is less than ε for some n large enough

$(h_n * g)(x) \longrightarrow g(x)$ uniformly in $\mathbb{R}^d$

since g is in $C_c(\mathbb{R}^d)$

D.C.T can be applied

$$\|h_n * g - g\|_{L^1} \longrightarrow 0$$