GCE August, 2014

Jiamin JIAN

Exercise 1:

(i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is bounded. Given an example, with proof, of such a function f whose improper Riemann integral on $(-\infty, \infty)$ exists and finite, but which is not in $L^1(\mathbb{R})$.

(ii) Suppose $-\infty < a < b < \infty$. Prove that if the proper Riemann integral of a function g on $[a, b]$ exists, then the Lebesgue integral of g on $[a, b]$ exists and equals the value of the proper Riemann integral.

Solution:

(i) Let

$$f(x) = \frac{\sin x}{x} \mathbb{I}_{[0,\infty)}(x),$$

and we want to show the integral of $f(x)$ on $\mathbb{R}$ is exists by the Cauchy convergence theorem for the improper Riemann integral. For any $A_2 > A_1 > 0$, we have

$$\int_{A_1}^{A_2} \frac{\sin x}{x}\, dx = \frac{\cos A_1}{A_1} - \frac{\cos A_2}{A_2} - \int_{A_1}^{A_2} \frac{\cos x}{x^2}\, dx,$$

then we know that

$$\Big| \int_{A_1}^{A_2} \frac{\sin x}{x}\, dx \Big| \leq \frac{1}{A_1} + \frac{1}{A_2} + \int_{A_1}^{A_2} \frac{1}{x^2}\, dx = \frac{2}{A_1}.$$

For any $\epsilon > 0$, choose $A = \frac{2}{\epsilon}$, when $A_2 > A_1 > A$, we have

$$\Big| \int_{A_1}^{A_2} \frac{\sin x}{x}\, dx \Big| \leq \frac{2}{A_1} < \frac{2}{A} < \epsilon,$$

thus we know that $\int_0^\infty \frac{\sin x}{x}\, dx$ exists. Next we show that $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$. Note that

$$\begin{aligned}
\lim_{a\to\infty} \int_0^a \frac{\sin t}{t}\, dt &= \lim_{a\to\infty} \int_0^a \int_0^\infty e^{-tx} \sin x\, dx\, dt \\
&= \int_0^\infty \int_0^\infty e^{-tx} \sin x\, dx\, dt \\
&=: \int_0^\infty I(t)\, dt.
\end{aligned}$$

Since

$$I(t) = \int_0^\infty e^{-tx} \sin x\, dx = 1 - t^2 I(t),$$

we know that $I(t) = \frac{1}{1+t^2}$ and

$$\lim_{a\to\infty} \int_0^a \frac{\sin t}{t}\, dt = \int_0^\infty \frac{1}{1+t^2}\, dt = \frac{\pi}{2}.$$

Next we need to show that $f(x)$ is not in $L^1(\mathbb{R})$. Let $N \in \mathbb{N}$ and $N > 1$, we have

$$\begin{aligned}
\int_0^{2\pi N} \left|\frac{\sin x}{x}\right| dx &= \sum_{n=0}^{N-1} \int_{2n\pi}^{2\pi(n+1)} \left|\frac{\sin x}{x}\right| dx \\
&\geq \sum_{n=0}^{N-1} \frac{1}{2(n+1)\pi} \int_{2n\pi}^{2\pi(n+1)} |\sin x|\, dx \\
&= \sum_{n=0}^{N-1} \frac{1}{2(n+1)\pi} \int_0^{2\pi} |\sin x|\, dx \\
&= \sum_{n=0}^{N-1} \frac{2}{(n+1)\pi}.
\end{aligned}$$

Let $N \to \infty$, we know that $\int_0^\infty |\frac{\sin x}{x}|\, dx$ diverges, so $f(x)$ is not in $L^1(\mathbb{R})$ but improper Riemann integral of $f(x)$ on $(-\infty, \infty)$ exists and $f(x)$ is finite.

(ii) Riemann integral is defined for functions g on a closed and bounded interval $[a, b]$ as follows: for any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, the corresponding lower sum $L(g, P)$ and upper sum $U(g, P)$ are defined by

$$L(g, P) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} g(x)(x_i - x_{i-1})$$

$$U(g, P) = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} g(x)(x_i - x_{i-1})$$

Function g is Riemann integrable if $\sup_P L(g, P) = \inf_P U(g, P)$, and the integral $\int_a^b f(x)\, dx$ then equals to this common value. For every partition P , define the functions

$$\phi_{g,P} = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} g(x), \quad \text{if } x \in (x_{i-1}, x_i)$$

$$\psi_{g,P} = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} g(x), \quad \text{if } x \in (x_{i-1}, x_i)$$

At the nodes x_i, the functions $\phi_{g,P}$ and $\psi_{g,P}$ are equal to 0. Then $\phi_{g,P}$ and $\psi_{g,P}$ are step functions, and by definition, the lower and upper sums are their integrals,

$$L(g, P) = \int \phi_{g,P}, \quad U(g, P) = \int \psi_{g,P},$$

with respect to Lebesgue measure and

$$\phi_{g,P} \leq g \leq \psi_{g,P}$$

except at the nodes x_i.

It is known from the theory of Riemann integration that if g is Riemann integrable, then there exists a sequence of partitions P_k such that

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} L(g,P) = \lim_{k\to\infty} U(g,P)$$

and P_{k+1} is a refinement of P_k, thus

$$\phi_{g,P_k} \le \phi_{g,P_{k+1}} \le g \le \psi_{g,P_{k+1}} \le \psi_{g,P_k}$$

except at the nodes of the partitions P_k, which is a countable set. Hence

$$\begin{aligned} \int |\phi_{g,P_{k+m}} - \phi_{g,P_k}| &= \int \phi_{g,P_{k+m}} - \phi_{g,P_k} \\ &= \int \phi_{g,P_{k+m}} - \int \phi_{g,P_k} \\ &= L(g,P_{k+m}) - L(g,P_k) \\ &= |L(g,P_{k+m}) - L(g,P_k)|. \end{aligned}$$

Since the sequence $\{L(g,P_k)\}$ converges, it is Cauchy sequence in $\mathbb{R}$, and, consequently, $\{\phi_{g,P_k}\}$ is L^1 Cauchy sequence of step maps. Similarly, $\{\psi_{g,P_k}\}$ is L^1 Cauchy sequence of step maps. So we have $\{\phi_{g,P_k}\}$ and $\{\psi_{g,P_k}\}$ converge a.e. on [a, b], and since $\phi_{g,P} \le g \le \psi_{g,P}$ a.e., they converge to f a.e. Thus the limits of the sequences of the integrals of the step maps $\phi_{g,P}$ and $\psi_{g,P}$ equal to the Lebesgue integral of f. Since the integrals of the step maps equal to the lower and upper Riemann sums, whose limit is the Riemann integral, the Riemann integral equals to the Lebesgue integral.

Exercise 2:

Let f_n be a sequence of measurable functions from $[0,1]$ to $\mathbb{R}$. Assume that each function f_n is finite almost everywhere. Show that f_n converges in measure to zero if and only if

$$\lim_{n\to\infty}\int_0^1 \frac{|f_n|}{1+|f_n|} = 0$$

Hint: Recall that by definition f_n converges in measure to f if and only if, given any $\epsilon > 0$,

$$\lim_{n\to\infty} |\{|f_n - f| > \epsilon\}| = 0.$$

Solution:

Firstly suppose that $f_n \to 0$ in measure, for any fixed $\epsilon > 0$, we have

$$\begin{aligned}
\int_0^1 \frac{|f_n|}{1+|f_n|}\,d\mu &= \int_{\{|f_n|\geq\epsilon\}\cap[0,1]} \frac{|f_n|}{1+|f_n|}\,d\mu + \int_{\{|f_n|<\epsilon\}\cap[0,1]} \frac{|f_n|}{1+|f_n|}\,d\mu \\
&\leq \mu(|f_n|\geq\epsilon) + \epsilon\mu(\{|f_n|\leq\epsilon\}\cap[0,1]) \\
&\leq \mu(|f_n|\geq\epsilon) + \epsilon,
\end{aligned}$$

thus we know that $\limsup_{n\to\infty}\int_0^1 \frac{|f_n|}{1+|f_n|}\,d\mu \leq \epsilon$. Let $\epsilon \to 0$, we have $\lim_{n\to\infty}\int_0^1 \frac{|f_n|}{1+|f_n|}\,d\mu = 0$.

On the other hand, suppose $\lim_{n\to\infty}\int_0^1 \frac{|f_n|}{1+|f_n|}\,d\mu = 0$, for any $\epsilon > 0$, we have

$$\begin{aligned}
\mu(|f_n|\geq\epsilon) &= \int_{|f_n|\geq\epsilon} 1\,d\mu \\
&= \frac{1+\epsilon}{\epsilon}\int_{|f_n|\geq\epsilon} \frac{\epsilon}{1+\epsilon}\,d\mu \\
&\leq \frac{1+\epsilon}{\epsilon}\int_0^1 \frac{|f_n|}{1+|f_n|}\,d\mu,
\end{aligned}$$

thus when $n\to\infty$,

$$\lim_{n\to\infty}\mu(|f_n|\geq\epsilon) \leq \lim_{n\to\infty}\frac{1+\epsilon}{\epsilon}\int_0^1 \frac{|f_n|}{1+|f_n|}\,d\mu = 0.$$

Hence $\lim_{n\to\infty}\mu(|f_n|\geq\epsilon) = 0$ and f_n converges in measure to 0.

Exercise 3:

(i) Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n a converging sequence in $L^1(X)$. Show that f_n has a sub-sequence which is convergent almost everywhere.

(ii) Find a sequence g_n in $L^1([0,1])$ such that: g_n converges in $L^1([0,1])$ and for all x in $[0,1]$ the sequence $g_n(x)$ diverges.

(iii) In the measure space $(X, \mathcal{A}, \mu)$, let A_n be a sequence of element of $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$ and let f be in $L^1(X)$. Show that $\lim_{n\to\infty} \int_{A_n} f = 0$.

Solution:

(i) Firstly we show that when f_n converges to f in $L^1(X)$, then f_n converges to f in measure. For $n \geq 1$ and $\epsilon > 0$, let $A = \{|f_n - f| > \epsilon\}$. Note that

$$|f_n - f| \geq 1_A |f_n - f| \geq \epsilon 1_A,$$

integrating across the inequality yields

$$\int_X |f_n - f| \, d\mu \geq \epsilon\mu(A).$$

That is

$$\mu(|f_n - f| \geq \epsilon) \leq \frac{1}{\epsilon} \int_X |f_n - f| \, d\mu.$$

Since the right hand side converges to 0 as $n \to \infty$, we have

$$\lim_{n\to\infty} \mu(|f_n - f| \geq \epsilon) = 0.$$

Therefore we know that f_n converges to f in measure.

Next we show that if f_n converges to f in measure, then there exists a sub-sequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ pointwise almost everywhere. Since f_n converges to f in measure, we can find $n_1 < n_2 < \cdots$ such that

$$\mu(|f - f_{n_k}| > \frac{1}{k}) \leq \frac{1}{2^k}, \quad \forall n \geq n_k.$$

Define $E_k = \{|f - f_{n_k}| > \frac{1}{k}\}$ and $H_m = \bigcup_{k=m}^{\infty} E_k$, then we have

$$\mu(E_k) \leq \frac{1}{2^k}, \quad \mu(H_m) \leq \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}}.$$

Set $Z = \bigcap_{m=1}^{\infty} H_m$, then

$$\mu(Z) \leq \mu(H_m) \leq \frac{1}{2^{m-1}}.$$

So we have $\mu(Z) = 0$. If $x \in Z$, then $x \notin H_m$ for some m, hence $x \notin E_k$ for all $k \geq m$, which implies

$$|f(x) - f_{n_k}| \leq \frac{1}{k}.$$

Thus $f_{n_k} \to f(x)$ for all $x \notin Z$. Since Z has zero measure, we therefore have pointwise convergence of f_{n_k} to f almost everywhere.

Thus we know that when f_n converges to f in $L^1(X)$, then f_n converges to f in measure, and then there exists a sub-sequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ pointwise almost everywhere.

(ii) For each $n \in \mathbb{N}$, let

$$g_n(x) = \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x),$$

whenever $k \in \mathbb{N}, 2^k \leq n < 2^{k+1}$. For any $n \in \mathbb{N}$, we have

$$\int_0^1 |g_n(x)| \, dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x) \, dx = \frac{1}{2^k} < +\infty,$$

so we know that $g_n \in L^1((0,1))$. And similarly we have

$$\int_0^1 |g_n(x) - 0| \, dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x) \, dx = \frac{1}{2^k} < \frac{2}{n},$$

then when $n \to +\infty$, $\int_0^1 |g_n(x) - 0| \, dx \to 0$, thus we get $g_n \to 0$ in $L^1([0,1])$. But for any $x \in [0,1]$, and for any $N \in \mathbb{N}$, we can find a $n > N$ with $f_n(x) = 1$. Thus f_n can not converges to 0 anywhere for $x \in (0,1)$. Then $g_n(x)$ is a sequence of indicator functions of intervals of decreasing length, marching across the unit interval $[0,1]$ over and over again, thus we know that $g_n(x)$ diverges.

(iii) We denote

$$f_n(x) = f(x)\mathbb{I}_{A_n}(x),$$

where $\mathbb{I}_{A_n}(\cdot)$ is a indicator function on A_n. Since A_n is a sequence in $\mathcal{A}$ such that $\mu(A_n) \to 0$ as $n \to +\infty$, then we know that $f_n(x)$ converges to 0 almost everywhere. As

$$|f_n(x)| = |f(x)\mathbb{I}_{A_n}(x)| \leq |f(x)|$$

and $f \in L^1(X)$, we know that f is a dominate function of f_n. By the dominate convergence theorem, we have

$$\lim_{n\to\infty} \int_X f_n(x) \, d\mu = \int_X 0 \, d\mu = 0,$$

thus

$$\lim_{n\to\infty} \int_X f_n(x) \, d\mu = \lim_{n\to\infty} \int_{A_n} f \, d\mu = 0.$$

So, we know that $\int_{A_n} f$ converges to zero.

<u>Exercise 4:</u>

Suppose $f \in L^1(\mathbb{R})$ is such that $f > 0$, almost everywhere. Show that $\int f > 0$.

Solution:

Since $f > 0$, we have

$$\int f\,d\mu > \int_{\{x\in\mathbb{R}:f\geq\frac{1}{n}\}} f\,d\mu \geq \frac{1}{n}\mu\Big(\Big\{x \in \mathbb{R} : f \geq \frac{1}{n}\Big\}\Big).$$

Let's argue by contraction. Suppose that $\mu(\{x \in \mathbb{R} : f \geq \frac{1}{n}\}) = 0$ for all n, since $\{x \in \mathbb{R} : f > 0\} = \bigcup_{n=1}^{\infty}\{x \in \mathbb{R} : f \geq \frac{1}{n}\}$, we have

$$\mu(\{x \in \mathbb{R} : f > 0\}) = \mu\Big(\bigcup_{n=1}^{\infty}\Big\{x \in \mathbb{R} : f \geq \frac{1}{n}\Big\}\Big) \leq \sum_{n=1}^{\infty}\mu\Big(\Big\{x \in \mathbb{R} : f \geq \frac{1}{n}\Big\}\Big) = 0,$$

which is contradictory with the condition $f > 0$ almost everywhere. So there exists $n \in \mathbb{N}$ such that $\mu(\{x \in \mathbb{R} : f \geq \frac{1}{n}\}) > 0$. Thus we know that

$$\int f\,d\mu \geq \frac{1}{n}\mu\Big(\Big\{x \in \mathbb{R} : f \geq \frac{1}{n}\Big\}\Big) > 0.$$