GCE, MA503

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Exercise 1:

(i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is bounded. Given an example, with proof, of such a function f whose improper Riemann integral on $(-\infty, \infty)$ exists and finite, but which is not in $L^1(\mathbb{R})$.

(ii) Suppose $-\infty < a < b < \infty$. Prove that if the proper Riemann integral of a function g on $[a, b]$ exists, then the Lebesgue integral of g on $[a, b]$ exists and equals the value of the proper Riemann integral.

Solution:

(i) We set

$$f(x) = \frac{\sin x}{x} \mathbb{I}_{[0,\infty)}(x),$$

and we want to show the integral of $f(x)$ on $\mathbb{R}$ is converges by the Cauchy convergence theorem for the improper Riemann integral. For any $A_2 > A_1 > 0$, we have

$$\int_{A_1}^{A_2} \frac{\sin x}{x}\, dx = \frac{\cos A_1}{A_1} - \frac{\cos A_2}{A_2} - \int_{A_1}^{A_2} \frac{\cos x}{x^2}\, dx,$$

then we know that

$$\Big|\int_{A_1}^{A_2} \frac{\sin x}{x}\, dx\Big| \leq \frac{1}{A_1} + \frac{1}{A_2} + \int_{A_1}^{A_2} \frac{1}{x^2}\, dx = \frac{2}{A_1}.$$

For any $\epsilon > 0$, we set $A = \frac{2}{\epsilon}$, when $A_2 > A_1 > A$, we have

$$\Big|\int_{A_1}^{A_2} \frac{\sin x}{x}\, dx\Big| \leq \frac{2}{A_1} < \frac{2}{A} < \epsilon,$$

thus we know that $\int_0^\infty \frac{\sin x}{x}\,dx$ converges. Next we show that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. We have

$$\begin{aligned}
\lim_{a\to\infty}\int_0^a \frac{\sin t}{t}\,dt &= \lim_{a\to\infty}\int_0^\infty e^{-tx}\sin x\,dx\,dt \\
&= \int_0^\infty\int_0^\infty e^{-tx}\sin x\,dx\,dt \\
&=: \int_0^\infty I(t)\,dt,
\end{aligned}$$

and since

$$I(t) = \int_0^\infty e^{-tx}\sin x\,dx = 1 - t^2 I(t),$$

we know that $I(t) = \frac{1}{1+t^2}$ and

$$\lim_{a\to\infty}\int_0^a \frac{\sin t}{t}\,dt = \int_0^\infty \frac{1}{1+t^2}\,dt = \frac{\pi}{2}.$$

Next we need to show that $f(x)$ is not in $L^1(\mathbb{R})$. Let $N \in \mathbb{N}$ and $N > 1$, we have

$$\begin{aligned}
\int_0^{2\pi N} \left|\frac{\sin x}{x}\right| dx &= \sum_{n=0}^{N-1}\int_{2n\pi}^{2\pi(n+1)} \left|\frac{\sin x}{x}\right| dx \\
&\geq \sum_{n=0}^{N-1}\frac{1}{2(n+1)\pi}\int_{2n\pi}^{2\pi(n+1)} |\sin x|\,dx \\
&= \sum_{n=0}^{N-1}\frac{1}{2(n+1)\pi}\int_0^{2\pi} |\sin x|\,dx \\
&= \sum_{n=0}^{N-1}\frac{2}{(n+1)\pi}.
\end{aligned}$$

Let $N \to \infty$, we know that $\int_0^\infty |\frac{\sin x}{x}|\,dx$ diverges, so $f(x)$ is not in $L^1(\mathbb{R})$ but improper Riemann integral of $f(x)$ on $(-\infty,\infty)$ exists and $f(x)$ is finite.

(ii) Riemann integral is defined for functions g on a closed and bounded interval $[a,b]$ as follows: for any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, the corresponding lower sum $L(g,P)$ and upper sum $U(g,P)$ are defined by

$$L(g,P) = \sum_{i=1}^n \inf_{x\in[x_{i-1},x_i]} g(x)(x_i - x_{i-1})$$

$$U(g,P) = \sum_{i=1}^n \sup_{x\in[x_{i-1},x_i]} g(x)(x_i - x_{i-1})$$

Function g is Riemann integrable if $\sup_P L(g,P) = \inf_P U(g,P)$, and the integral $\int_a^b f(x)\,dx$ then equals to this common value. For every partition P , define the functions

$$\phi_{g,P} = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} g(x), \quad \text{if } x \in (x_{i-1}, x_i)$$

$$\psi_{g,P} = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} g(x), \quad \text{if } x \in (x_{i-1}, x_i)$$

At the nodes x_i, the functions $\phi_{g,P}$ and $\psi_{g,P}$ are equal to 0. Then $\phi_{g,P}$ and $\psi_{g,P}$ are step functions, and by definition, the lower and upper sums are their integrals,

$$L(g,P) = \int \phi_{g,P}, \quad U(g,P) = \int \psi_{g,P},$$

with respect to Lebesgue measure and

$$\phi_{g,P} \le g \le \psi_{g,P}$$

except at the nodes x_i.

It is known from the theory of Riemann integration that if g is Riemann integrable, then there exists a sequence of partitions P_k such that

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} L(g,P) = \lim_{k\to\infty} U(g,P)$$

and P_{k+1} is a refinement of P_k, thus

$$\phi_{g,P_k} \le \phi_{g,P_{k+1}} \le g \le \psi_{g,P_{k+1}} \le \psi_{g,P_k}$$

except at the nodes of the partitions P_k, which is a countable set. Hence

$$\begin{aligned}
\int |\phi_{g,P_{k+m}} - \phi_{g,P_k}| &= \int \phi_{g,P_{k+m}} - \phi_{g,P_k} \\
&= \int \phi_{g,P_{k+m}} - \int \phi_{g,P_k} \\
&= L(g,P_{k+m}) - L(g,P_k) \\
&= |L(g,P_{k+m}) - L(g,P_k)|.
\end{aligned}$$

Since the sequence $\{L(g,P_k)\}$ converges, it is Cauchy sequence in $\mathbb{R}$, and, consequently, $\{\phi_{g,P_k}\}$ is L^1 Cauchy sequence of step maps. Similarly, $\{\psi_{g,P_k}\}$ is L^1 Cauchy sequence of step maps. So we have $\{\phi_{g,P_k}\}$ and $\{\psi_{g,P_k}\}$ converge a.e. on [a, b], and since $\phi_{g,P} \le g \le \psi_{g,P}$ a.e., they converge to f a.e. Thus the limits of the sequences of the integrals of the step maps $\phi_{g,P}$ and $\psi_{g,P}$ equal to the Lebesgue integral of f. Since the integrals of the step maps equal to the lower and upper Riemann sums, whose limit is the Riemann integral, the Riemann integral equals to the Lebesgue integral.

Exercise 2:

Let f_n be a sequence of measurable functions from $[0,1]$ to $\mathbb{R}$. Assume that each function f_n is finite almost everywhere. Show that f_n converges in measure to zero if and only if

$$\lim_{n\to\infty}\int_0^1 \frac{|f_n|}{1+|f_n|}=0$$

Hint: Recall that by definition f_n converges in measure to f if and only if, given any $\epsilon>0$,

$$\lim_{n\to\infty}|\{|f_n-f|>\epsilon\}|=0.$$

Solution:

Firstly suppose that $f_n\to 0$ in measure, for any fixed $\epsilon>0$, we have

$$\begin{aligned}\int_0^1 \frac{|f_n|}{1+|f_n|}\,d\mu &= \int_{\{|f_n|\ge\epsilon\}\cap[0,1]}\frac{|f_n|}{1+|f_n|}\,d\mu+\int_{\{|f_n|<\epsilon\}\cap[0,1]}\frac{|f_n|}{1+|f_n|}\,d\mu\\ &\le \mu(|f_n|\ge\epsilon)+\epsilon\mu(\{|f_n|\le\epsilon\}\cap[0,1])\\ &\le \mu(|f_n|\ge\epsilon)+\epsilon,\end{aligned}$$

thus we know that $\limsup_{n\to\infty}\int_0^1\frac{|f_n|}{1+|f_n|}\,d\mu\le\epsilon$. Let $\epsilon\to 0$, we have $\lim_{n\to\infty}\int_0^1\frac{|f_n|}{1+|f_n|}\,d\mu=0$.

On the other hand, suppose $\lim_{n\to\infty}\int_0^1\frac{|f_n|}{1+|f_n|}\,d\mu=0$, for any $\epsilon>0$, we have

$$\begin{aligned}\mu(|f_n|\ge\epsilon) &= \int_{|f_n|\ge\epsilon}1\,d\mu\\ &= \frac{1+\epsilon}{\epsilon}\int_{|f_n|\ge\epsilon}\frac{\epsilon}{1+\epsilon}\,d\mu\\ &\le \frac{1+\epsilon}{\epsilon}\int_0^1\frac{|f_n|}{1+|f_n|}\,d\mu,\end{aligned}$$

thus when $n\to\infty$, we have

$$\lim_{n\to\infty}\mu(|f_n|\ge\epsilon)\le\lim_{n\to\infty}\frac{1+\epsilon}{\epsilon}\int_0^1\frac{|f_n|}{1+|f_n|}\,d\mu=0.$$

Hence we know that $\lim_{n\to\infty}\mu(|f_n|\ge\epsilon)=0$ and f_n converges in measure to 0.

Exercise 3:

(i) Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n a converging sequence in $L^1(X)$. Show that f_n has a sub-sequence which is convergent almost everywhere.

(ii) Find a sequence g_n in $L^1([0,1])$ such that: g_n converges in $L^1([0,1])$ and for all x in $[0,1]$ the sequence $g_n(x)$ diverges.

(iii) In the measure space $(X, \mathcal{A}, \mu)$, let A_n be a sequence of element of $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$ and let f be in $L^1(X)$. Show that $\lim_{n\to\infty} \int_{A_n} f = 0$.

Solution:

(i) Firstly we show that when f_n converges to f in $L^1(X)$, then f_n converges to f in measure. For $n \geq 1$ and $\epsilon > 0$, let $A = \{|f_n - f| > \epsilon\}$. Note that

$$|f_n - f| \geq 1_A |f_n - f| \geq \epsilon 1_A,$$

integrating across the inequality yields

$$\int_X |f_n - f| \, d\mu \geq \epsilon \mu(A).$$

That is

$$\mu(|f_n - f| \geq \epsilon) \leq \frac{1}{\epsilon} \int_X |f_n - f| \, d\mu.$$

Since the right hand side converges to 0 as $n \to \infty$, we have

$$\lim_{n\to\infty} \mu(|f_n - f| \geq \epsilon) = 0.$$

Therefore we know that f_n converges to f in measure.

Next we show that if f_n converges to f in measure, then there exists a sub-sequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ pointwise almost everywhere. Since f_n converges to f in measure, we can find $n_1 < n_2 < \cdots$ such that

$$\mu(|f - f_{n_k}| > \frac{1}{k}) \leq \frac{1}{2^k}, \quad \forall n \geq n_k.$$

Define $E_k = \{|f - f_{n_k}| > \frac{1}{k}\}$ and $H_m = \bigcup_{k=m}^{\infty} E_k$, then we have

$$\mu(E_k) \leq \frac{1}{2^k}, \quad \mu(H_m) \leq \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}}.$$

Set $Z = \bigcap_{m=1}^{\infty} H_m$, then

$$\mu(Z) \leq \mu(H_m) \leq \frac{1}{2^{m-1}}.$$

So we have $\mu(Z) = 0$. If $x \in Z$, then $x \notin H_m$ for some m, hence $x \notin E_k$ for all $k \geq m$, which implies

$$|f(x) - f_{n_k}| \leq \frac{1}{k}.$$

Thus $f_{n_k} \to f(x)$ for all $x \notin Z$. Since Z has zero measure, we therefore have pointwise convergence of f_{n_k} to f almost everywhere.

Thus we know that when f_n converges to f in $L^1(X)$, then f_n converges to f in measure, and then there exists a sub-sequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ pointwise almost everywhere.

(ii) We suppose that

$$g_n(x) = \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x),$$

whenever $k \geq 0, 2^k \leq n < 2^{k+1}$. For any $n \in \mathbb{N}$, we have

$$\int_0^1 |g_n(x)|\, dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x)\, dx = \frac{1}{2^k} < +\infty,$$

so we know that $g_n \in L^1((0,1))$. And similarly we have

$$\int_0^1 |g_n(x) - 0|\, dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x)\, dx = \frac{1}{2^k} < \frac{2}{n},$$

then when $n \to +\infty$, $\int_0^1 |g_n(x) - 0|\, dx \to 0$, thus we get $g_n \to 0$ in $L^1([0,1])$. But for any $x \in [0,1]$, and for any $N \in \mathbb{N}$, we can find a $n > N$ with $f_n(x) = 1$. Thus f_n can not converges to 0 anywhere for $x \in (0,1)$. And $g_n(x)$ is a sequence of indicator functions of intervals of decreasing length, marching across the unit interval $[0,1]$ over and over again, thus we know that $g_n(x)$ diverges.

(iii) We denote

$$f_n(x) = f(x)\mathbb{I}_{A_n}(x),$$

where $\mathbb{I}_{A_n}(\cdot)$ is a indicator function on A_n. Since A_n is a sequence in $\mathcal{A}$ such that $\mu(A_n) \to 0$ as $n \to +\infty$, then we know that $f_n(x)$ converges to 0 almost everywhere. As

$$|f_n(x)| = |f(x)\mathbb{I}_{A_n}(x)| \leq |f(x)|$$

and $f \in L^1(X)$, we know that f is a dominate function of f_n. By the dominate convergence theorem, we have

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu = \int_X 0\, d\mu = 0,$$

thus we have

$$\lim_{n\to\infty}\int_X f_n(x)\,d\mu = \lim_{n\to\infty}\int_{A_n} f\,d\mu = 0.$$

So, we know that $\int_{A_n} f$ converges to zero.

Exercise 4:
Suppose $f \in L^1(\mathbb{R})$ is such that $f > 0$, almost everywhere. Show that $\int f > 0$.

Solution:
Since $f > 0$, we have

$$\int f\,d\mu > \int_{\{f\geq \frac{1}{n}\}} f\,d\mu \geq \frac{1}{n}\mu(\{f \geq \frac{1}{n}\}).$$

Let's argue by contraction. Suppose that $\mu(\{f \geq \frac{1}{n}\}) = 0$ for any n, since $\{f > 0\} = \bigcup_{n=1}^{\infty}\{f \geq \frac{1}{n}\}$, we have

$$\mu(\{f > 0\}) = \mu\Big(\bigcup_{n=1}^{\infty}\{f \geq \frac{1}{n}\}\Big) \leq \sum_{n=1}^{\infty}\mu\Big(\{f \geq \frac{1}{n}\}\Big) = 0,$$

which is contradictory with the condition $f > 0$ almost everywhere. So there exists $n \in \mathbb{N}$ such that $\mu(\{f \geq \frac{1}{n}\}) > 0$. Thus we know that

$$\int f\,d\mu \geq \frac{1}{n}\mu(\{f \geq \frac{1}{n}\}) > 0.$$

2 GCE January, 2015

Exercise 1:

Construct a subset $A \subset \mathbb{R}$ such that A is closed, contains no intervals, is uncountable, and has Lebesgue measure $\frac{1}{2}$ (i.e. $|A| = \frac{1}{2}$). Also explain why your set A has each of the above properties.

Hint: One possible approach here is to adjust the construction of the Cantor set to achieve a Cantor-like set with measure $\frac{1}{2}$, but you don't need to have seen the Cantor set to answer the question.

Solution:

We follow the construction of Cantor set by deleting the open middle forth from a set of line segment. We start by deleting the open middle $(\frac{3}{8}, \frac{5}{8})$ from the interval $[0, 1]$, leaving two line segments $A_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$. Next we do the same thing by deleting $(\frac{5}{32}, \frac{7}{32})$ and $(\frac{25}{32}, \frac{27}{32})$, then we have

$$A_2 = [0, \frac{5}{32}] \cup [\frac{7}{32}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{25}{32}] \cup [\frac{27}{32}, 1].$$

This process is continued as $n \to \infty$, we can get the Cantor-like set A.

Since we only delete the open interval from $[0, 1]$ each time, then the union of the intervals we deleted is an open set, thus the Cantor-like set A is closed. We denote $A^c = [0, 1] \setminus A$, then we have

$$|A^c| = \sum_{n=1}^{\infty} \frac{2^{n-1}}{4^n} = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2},$$

thus we know that the measure of Cantor-like set is $\frac{1}{2}$ and it is uncountable. Next we need to show the set A contains no intervals. Suppose the interval $(\alpha, \beta) \in A$. For the n-th time we delete the interval whose measure is $\frac{1}{4^n}$, so when $n \to \infty$, it is far smaller than $\beta - \alpha$, then we have to separate the interval (α, β). Thus similarly with the Cantor set, the Cantor-like set contains no intervals.

Exercise 2:

(i) Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n a sequence in $L^1(X)$. Let f be in $L^1(X)$. Assume that $\int f_n$ converges to $\int f$, f_n converges to f almost everywhere, and for each n, $f_n \geq 0$, almost everywhere. Show that f_n converges to f in $L^1(X)$.

Hint: Set $g_n = \min(f_n, f)$. Note that $|f_n - f| = f + f_n - 2g_n$.

(ii) Find a sequence f_n in $L^1(\mathbb{R})$ and f in $L^1(\mathbb{R})$ such that $\int f_n$ converges to $\int f$, f_n converges to f almost everywhere, but f_n does not converge to f in $L^1(\mathbb{R})$.

Solution:

(i) We set $g_n = \min(f_n, f)$, then $|f_n - f| = f + f_n - 2g_n$, thus we can get

$$\int_X |f_n - f| \, d\mu = \int_X (f + f_n - 2g_n) \, d\mu.$$

Since $f \in L^1(X)$ and $f_n \in L^1(X)$, then we know that $g_n \in L^1(X)$, so we have

$$\int_X |f_n - f| \, d\mu = \int_X f \, d\mu + \int_X f_n \, d\mu - 2 \int_X g_n \, d\mu.$$

And by the definition of g_n, we know that g_n converges to f almost everywhere as f_n converges to f almost everywhere. As $f_n \geq 0$ almost everywhere, then $f \geq 0$ a.e. Since $|g_n| \leq |f|$ and $f \in L^1(X)$, by the dominate convergence theorem, we know that

$$\begin{aligned}
\lim_{n\to\infty} \int_X |f_n - f| \, d\mu &= \int_X f \, d\mu + \lim_{n\to\infty} \int_X f_n \, d\mu - 2 \lim_{n\to\infty} \int_X g_n \, d\mu \\
&= 2 \int_X f \, d\mu - 2 \int_X \lim_{n\to\infty} g_n \, d\mu \\
&= 2 \int_X f \, d\mu - 2 \int_X f \, d\mu = 0,
\end{aligned}$$

hence we get f_n converges to f in $L^1(X)$.

(ii) We denote

$$f_n(x) = \begin{cases} \frac{1}{n}, & x \in [-n, 0] \\ -\frac{1}{n}, & x \in (0, n] \end{cases}$$

and $f(x) = 0$, since $|f_n \leq \frac{1}{n}|$, we have f_n converges to f almost everywhere. As

$$\int_{\mathbb{R}} f_n \, d\mu = \int_{-n}^{0} \frac{1}{n} \, d\mu + \int_0^n \Big(-\frac{1}{n} \Big) \, d\mu = 1 - 1 = 0,$$

we know that f_n in $L^1(\mathbb{R})$ and $\int f_n$ converges to $\int f$. But since

$$\int_{\mathbb{R}} |f_n - f| \, d\mu = \int_{-n}^{n} \frac{1}{n} \, d\mu = 2,$$

we can get that f_n does not converge to f in $L^1(\mathbb{R})$.

Exercise 3:

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(i) Let f be in $L^1([0, \infty))$. Show that

$$\lim_{x\to 0^+} \int_0^\infty f(t) e^{-xt} \, dt = \int_0^\infty f(t) \, dt$$

(ii) Let $[a, b]$ be an interval in $\mathbb{R}$. If $\tilde{f}$ is continuous on $[a, b]$ and monotonic, and g' is continuous on $[a, b]$, we can prove that there is a c in $[a, b]$ such that

$$\int_a^b \tilde{f}g = g(a)\int_a^c \tilde{f} + g(b)\int_c^b \tilde{f}.$$

Using this result, show that if g is as specified above and f is in $L^1([a, b])$, there is a c in $[a, b]$ such that

$$\int_a^b fg = g(a)\int_a^c f + g(b)\int_c^b f.$$

(iii) Let f be in $L^\infty([0, \infty))$. Assume that there is a constant L in $\mathbb{R}$ such that $\lim_{x\to\infty} \int_0^x f = L$. Show that

$$\lim_{x\to 0^+} \int_0^\infty f(t)e^{-xt}\, dt = L.$$

Solution:

(i) When $x \geq 0$ and $t \geq 0$, we know that $|f(t)e^{-xt}| \leq |f(t)|$. As $f \in L^1([0, \infty))$ and for any fixed t, $\lim_{x\to 0^+} f(t)e^{-xt} = f(t)$, by the dominate convergence theorem, we have

$$\lim_{x\to 0^+} \int_0^\infty f(t)e^{-xt}\, dt = \int_0^\infty \lim_{x\to 0^+} f(t)e^{-xt}\, dt = \int_0^\infty f(t)\, dt.$$

(ii) Since $\tilde{f}$ is continuous on $[a, b]$, we can introduce $F(x) = \int_a^x \tilde{f}$, and we know that $F'(x) = \tilde{f}(x)$. Then through integral by parts, we have

$$\begin{aligned}
\int_a^b f\tilde{(x)}g(x)\, dx &= \int_a^b g(x)\, dF(x) \\
&= g(b)F(b) - g(a)F(a) - \int_a^b g'(x)F(x)\, dx \\
&= g(b)\int_a^b \tilde{f}(x)\, dx - g(a)\int_a^a \tilde{f}(x)\, dx - \int_a^b g'(x)F(x)\, dx \\
&= g(b)\int_a^b \tilde{f}(x)\, dx - \int_a^b g'(x)F(x)\, dx.
\end{aligned}$$

Since g is differentiable on $[a, b]$ and monotonic, and g' is continuous on $[a, b]$, we know that g' is integrable in $[a, b]$ and $g'(x) \geq 0$ for all $x \in [a, b]$. By the mean value theorem for integral, there exists $c \in [a, b]$, and

$$\int_a^b g'(x)F(x)\, dx = F(c)\int_a^b g'(x)\, dx = F(c)(g(b) - g(a)).$$

Thus for this $c \in [a,b]$, we have

$$\begin{aligned}
\int_a^b f(\tilde{x})g(x)\,dx &= g(b)\int_a^b \tilde{f}(x)\,dx - F(c)(g(b)-g(a)) \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - (g(b)-g(a))\int_a^c \tilde{f}(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - g(b)\int_a^c \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx \\
&= g(b)\int_c^b \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx.
\end{aligned}$$

Since $C_c([a,b])$ is dense in $L^1([a,b])$, then we know that for any $f \in L^1([0,1])$, there exists a function sequence $\{f_n\} \subset C_c([a,b])$ and $\int_a^b |f_n - f| \to 0$ as $n \to +\infty$. Since g is differentiable on $[a,b]$ and monotonic, we know there exists $K > 0$, and $\forall x \in [a,b]$, we have $|g(x)| \le K$. So, we have

$$\lim_{n\to+\infty}\int_a^b |gf - gf_n| \le K \lim_{n\to+\infty}\int_a^b |f - f_n| = 0,$$

then by the conclusion we get from (i) we have

$$\int_a^b fg = \lim_{n\to+\infty}\int_a^b f_n g = \lim_{n\to+\infty}\Big(g(a)\int_a^{c_n} f_n + g(b)\int_{c_n}^b f_n\Big),$$

where c_n is depends on f_n for each n.

Since $\{c_n\} \subset [a,b]$ and $[a,b]$ is compact, there exists a subsequence of $\{c_n\}$, which is denoted as $\{c_{n_k}\}$, converges to c and $c \in [a,b]$. Thus we have

$$\begin{aligned}
\int_a^b fg &= \lim_{k\to+\infty}\Big(g(a)\int_a^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^b f_{n_k}\Big) \\
&= \lim_{k\to+\infty}\Big(g(a)\int_a^{c} f_{n_k} + g(a)\int_c^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^c f_{n_k} + g(b)\int_c^b f_{n_k}\Big) \\
&= g(a)\int_a^c f + g(b)\int_c^b f + \lim_{k\to+\infty}\Big(g(a)\int_c^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^c f_{n_k}\Big) \\
&= g(a)\int_a^c f + g(b)\int_c^b f.
\end{aligned}$$

(iii) For any $K > 0$, we have

$$\lim_{x\to0^+}\int_0^\infty f(t)e^{-xt}\,dt = \lim_{x\to0^+}\Big(\int_0^K f(t)e^{-xt}\,dt + \int_K^\infty f(t)e^{-xt}\,dt\Big)$$

let $K \to \infty$, we can get

$$\lim_{x\to 0^+} \int_0^\infty f(t)e^{-xt}\,dt = \lim_{x\to 0^+} \lim_{K\to\infty} \int_0^K f(t)e^{-xt}\,dt,$$

then we know that

$$\begin{aligned}
\lim_{x\to 0^+} \int_0^\infty f(t)e^{-xt}\,dt &= \lim_{x\to 0^+} \lim_{K\to\infty} \Big(\int_0^K f(t)\,dt + \int_0^K f(t)(e^{-xt}-1)\,dt \Big) \\
&= L + \lim_{x\to 0^+} \lim_{K\to\infty} \int_0^K f(t)(e^{-xt}-1)\,dt \\
&= L + \lim_{K\to\infty} \lim_{x\to 0^+} \int_0^K f(t)(e^{-xt}-1)\,dt
\end{aligned}$$

as $\int_0^K f(t)e^{-xt}\,dt$ is continuous with x and K. As $f(t) \in L^\infty([0,\infty))$, we have

$$\int_0^K |f(t)|\,dt \le K\|f\|_\infty < \infty,$$

then we know that $f(t) \in L^1([0,K])$. And since $|f(t)(e^{-xt}-1)| \le |f(t)|$ when $x \ge 0, t \ge 0$, by the dominate convergence theorem, we have

$$\lim_{x\to 0^+} \int_0^K f(t)(e^{-xt}-1)\,dt = \int_0^K f(t) \lim_{x\to 0^+} (e^{-xt}-1)\,dt = 0,$$

hence we can get

$$\lim_{x\to 0^+} \int_0^\infty f(t)e^{-xt}\,dt = L.$$

3 GCE May, 2015

Exercise 1:

Give an example of $f_n, f \in L^1(\mathbb{R})$ such that $f_n \to f$ uniformly, but $\|f_n\|_1$ does not converge to $\|f\|_1$.

Solution:

Example 1: We suppose that $f_n(x) = \frac{1}{n}\mathbb{I}_{[1,n]}(x)$ and $f(x) = 0$. Since

$$|f_n(x) - 0| = |\frac{1}{n}\mathbb{I}_{[1,n]}(x) - 0| \leq \frac{1}{n},$$

we know that $f_n \to f$ uniformly. As $\|f(x)\|_1 = 0$ and

$$\|f_n\|_1 = \int_{\mathbb{R}} |f_n(x)|\, dx = \int_1^n \frac{1}{n}\, dx = \frac{n-1}{n} \to 1$$

as $n \to \infty$. So we have $\|f_n\|_1$ does not converge to $\|f\|_1$.

Example 2: We set $f(x) = 0$ and

$$f_n(x) = \Big(-\frac{1}{2^{2n}} + \frac{1}{2^n}\Big) \cdot \mathbb{I}_{[0,2^{2n}]}(x).$$

Since $|f_n(x) - f(x)| < \frac{1}{2^n}$, we know that $f_n \to f$ uniformly. And as

$$\|f_n(x)\|_1 = \int_0^{2^{2n}} -\frac{1}{2^{2n}} + \frac{1}{2^n}\, dx = 2^n - 1 \to +\infty,$$

we have $\|f_n\|_1$ does not converge to $\|f\|_1$.

Exercise 2:

Show that for all $\epsilon > 0$ and all $f \in L^1(\mathbb{R}), \exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$ for some f_n with $|f_n| \leq n$ and $f_n = 0$ on $\mathbb{R} \setminus [-n, n]$.

Solution:

We suppose that

$$f_n(x) = f \cdot \mathbb{I}_{\{x:|f(x)|\leq n\}\cap\{x\in[-n,n]\}}(x),$$

so we know that $f_n = 0$ on $\mathbb{R} \setminus [-n, n]$ and $|f_n| \leq n$. Next we need to show that $\exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$. We know that

$$\begin{aligned}
\|f_n - f\|_1 &= \int_{\mathbb{R}} |f_n - f|\, dx \\
&= \int_{\{|f|\geq n\}\cup\{x\in\mathbb{R}\setminus[-n,n]\}} |f|\, dx \\
&\leq \int_{\{|f|\geq n\}} |f|\, dx + \int_{-\infty}^{-n} |f|\, dx + \int_n^{+\infty} |f|\, dx.
\end{aligned}$$

Since

$$\int_{\{|f|\geq n\}} |f|\,dx = \int_{\mathbb{R}} |f|\mathbb{I}_{\{|f|>n\}}(x)\,dx$$

and $|f|\mathbb{I}_{\{|f|>n\}}(x)$ goes to 0 pointwise and $|f|\mathbb{I}_{|f|>n}(x) < |f| \in L^1(\mathbb{R})$, by the dominate convergence theorem, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}} |f|\mathbb{I}_{|f|>n}\,dx = \int_{\mathbb{R}} \lim_{n\to\infty}|f|\mathbb{I}_{|f|>n}(x)\,dx = 0.$$

Similarly, since

$$\int_n^{+\infty} |f|\,dx = \int_{\mathbb{R}} |f|\mathbb{I}_{[n,+\infty)}(x)\,dx,$$

and $|f|\mathbb{I}_{[n,+\infty)}(x) \to 0$ as $n \to \infty$ pointwise and $|f|\mathbb{I}_{[n,+\infty)}(x) \leq |f| \in L^1(\mathbb{R})$, by the dominate convergence theorem, we can get

$$\lim_{n\to\infty}\int_{\mathbb{R}} |f|\mathbb{I}_{[n,+\infty)}(x)\,dx = \int_{\mathbb{R}} \lim_{n\to\infty}|f|\mathbb{I}_{[n,+\infty)}(x)\,dx = 0.$$

Then we also can get

$$\lim_{n\to\infty}\int_{-\infty}^{-n} |f|\,dx = 0.$$

Thus we have

$$\lim_{n\to\infty}\|f_n - f\|_1 \leq \lim_{n\to\infty}\Big(\int_{\{|f|\geq n\}} |f|\,dx + \int_{-\infty}^{-n} |f|\,dx + \int_n^{+\infty} |f|\,dx\Big) = 0,$$

hence we know that $\exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$.

Exercise 3:

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(i) If f is in $L^1(X) \cap L^\infty(X)$, show that $|f|^p \in L^1(X)$ for all p in $(1, \infty)$.

(ii) If f is in $L^1(X) \cap L^\infty(X)$, show that

$$\lim_{p\to\infty}\Big(\int |f|^p\Big)^{\frac{1}{p}} = \|f\|_\infty.$$

(iii) Set $A = \{x \in X : |f(x)| > 0\}$. If f is in $L^\infty(X), \mu(A) < \infty$, and $\mu(A) \neq 1$, find

$$\lim_{p\to 0^+}\Big(\int |f|^p\Big)^{\frac{1}{p}}.$$

(iv) We now assume that the set A defined in (iii) satisfies $\mu(A) = 1$, that f is in $L^\infty(X)$, and $\ln|f|$ is in $L^1(X)$, find

$$\lim_{p\to 0^+}\Big(\int |f|^p\Big)^{\frac{1}{p}}.$$

Solution:

(i) We need to show $|f|^p \in L^1(X)$, so we just need to show that for any $p \in (1, \infty)$, $f \in L^p(X)$. For any $p \in (1, \infty)$, since $f \in L^1(X) \cap L^\infty(X)$, we have

$$\begin{aligned} \|f\|_p &= \Big(\int_X |f|^p \, d\mu \Big)^{\frac{1}{p}} \\ &= \Big(\int_X |f||f|^{p-1} \, d\mu \Big)^{\frac{1}{p}} \\ &\leq (\|f\|_\infty)^{\frac{p-1}{p}} (\|f\|_1)^{\frac{1}{p}} < \infty, \end{aligned}$$

thus we know that $f \in L^p(X)$. So, we know that $|f|^p \in L^1(X)$ for all p in $(1, \infty)$.

(ii) We denote $t \in [0, \|f\|_\infty)$, then the set

$$A = \{x \in X : |f(x)| \geq t\}$$

has positive and bounded measure. Since

$$\begin{aligned} \|f\|_p &= \Big(\int_{(0,1)} |f|^p \, d\mu \Big)^{\frac{1}{p}} \geq \Big(\int_A |f|^p \, d\mu \Big)^{\frac{1}{p}} \\ &\geq \Big(t^p \mu(A) \Big)^{\frac{1}{p}} = t(\mu(A))^{\frac{1}{p}}, \end{aligned}$$

if $\mu(A)$ is finite, then when $p \to +\infty$, we have $(\mu(A))^{\frac{1}{p}} \to 1$ and if $\mu(A) = \infty$, then $(\mu(A)^{\frac{1}{p}}) = \infty$, in both cases we have

$$\liminf_{p \to +\infty} \|f\|_p \geq t.$$

Since t is arbitrary and $t \in [0, \|f\|_\infty)$, we have

$$\liminf_{p \to +\infty} \|f\|_p \geq \|f\|_\infty.$$

On the other hand, as $f(x)$ is in $L^1(X)$, we have

$$\begin{aligned} \|f\|_p &= \Big(\int_X |f|^p \, d\mu \Big)^{\frac{1}{p}} \\ &= \Big(\int_X |f||f|^{p-1} \, d\mu \Big)^{\frac{1}{p}} \\ &\leq (\|f\|_\infty)^{\frac{p-1}{p}} (\|f\|_1)^{\frac{1}{p}}. \end{aligned}$$

Since $\|f\|_1 < +\infty$, then when $p \to +\infty$, we know that

$$\limsup_{p \to +\infty} \|f\|_p \leq \|f\|_\infty.$$

Thus we have

$$\limsup_{p\to+\infty}\|f\|_p \le \|f\|_\infty \le \liminf_{p\to+\infty}\|f\|_p,$$

then we know that $\|f\|_p \to \|f\|_\infty$ as $p\to\infty$.

(iii) When $\mu(A)<1$, we have

$$\begin{aligned}\int_X |f|^p\,d\mu &= \int_A |f|^p\,d\mu\\ &\le \|f\|_\infty^p \mu(A).\end{aligned}$$

Since $f\in L^\infty(X)$ and $\mu(A)<1$, we know that

$$\lim_{p\to 0^+}\left(\int |f|^p\right)^{\frac{1}{p}} \le \lim_{p\to 0^+}\|f\|_\infty (\mu(A))^{\frac{1}{p}} = 0$$

But if we set $f=1$ and $\mu(X)<\infty$, we know that $f\in L^\infty(X)$, if $\mu(A)>1$, we have

$$\lim_{p\to 0^+}\left(\int |f|^p\right)^{\frac{1}{p}} = \lim_{p\to 0^+}(\mu(A))^{\frac{1}{p}} = \infty.$$

Thus the limit is not exist.

(iv) Since we have $A=\{x\in X : |f|>0\}$, then

$$\begin{aligned}\int_X |f|^p\,d\mu &= \int_{\{x\in X:|f|>0\}} |f|^p\,d\mu + \int_{\{x\in X:|f|=0\}} |f|^p\,d\mu\\ &= \int_A |f|^p\,d\mu.\end{aligned}$$

And we denote that $F(p)=\log(\int_A |f|^p\,d\mu)$, then we know that

$$\lim_{p\to 0^+}\left(\int |f|^p\right)^{\frac{1}{p}} = \lim_{p\to 0^+} e^{\frac{F(p)}{p}}.$$

As $F(0)=\log(\mu(A))=0$ and e^x is continuous, then we have

$$\begin{aligned}\lim_{p\to 0^+}\left(\int |f|^p\right)^{\frac{1}{p}} &= \lim_{p\to 0^+}\exp\left\{\frac{F(p)-F(0)}{p-0}\right\}\\ &= \exp\left\{\lim_{p\to 0^+}\frac{F(p)-F(0)}{p-0}\right\}\\ &= e^{F'(0)}.\end{aligned}$$

As $F(p)=\log(\int_A |f|^p\,d\mu)$ and $\ln|f|$ is in $L^1(X)$, we have

$$F'(p) = \frac{\int_A |f|^p\cdot\log|f|\,d\mu}{\int_A |f|^p\,d\mu},$$

thus we have $F^{'}(0) = \frac{\int_A \log|f|\,d\mu}{\mu(A)} = \int_A \log|f|\,d\mu$. Then we know that

$$\begin{aligned}\lim_{p\to 0^+}\left(\int f^p\right)^{\frac{1}{p}} &= e^{F^{'}(0)} \\ &= \exp(\int_A \log|f|\,d\mu).\end{aligned}$$

4 GCE August, 2015

Exercise 1:

Use the Fubini theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = \pi^{\frac{n}{2}}$$

Here $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Hint: For $n = 2$, use polar coordinates.

Solution:

Firstly, we define

$$I(a) = \int_{-a}^{a} e^{-x^2}\, dx,$$

then we have

$$I^2(a) = \int_{-a}^{a} e^{-x^2}\, dx \int_{-a}^{a} e^{-y^2}\, dy.$$

As $(-a, a)$ is an interval with finite measure and $|e^{-x^2}| \leq 1$, by the Fubini theorem, we have

$$I^2(a) = \int_{-a}^{a} \int_{-a}^{a} e^{-(x^2+y^2)}\, dx\, dy.$$

Take the transformation as follows,

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

then we know that

$$\int_0^{2\pi} \int_0^{a} re^{-r^2}\, dr\, d\theta < I^2(a) < \int_0^{2\pi} \int_0^{\sqrt{2}a} re^{-r^2}\, dr\, d\theta,$$

thus we can get

$$(1 - e^{-a^2})\pi < I^2(a) < (1 - e^{-2a^2})\pi.$$

Let $a \to \infty$, we have

$$\lim_{a\to\infty} I^2(a) = \int_{\mathbb{R}^2} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = \pi,$$

then we know that $\lim_{a\to\infty} I(a) = \int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}$. For the n dimensional domain, we have

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2}\, d\mathbf{x} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-(x_1^2+x_2^2+\cdots x_n^2)}\, dx_1 dx_2 \cdots dx_n \\ &= \Big(\int_{\mathbb{R}} e^{-x_1^2}\, dx_1 \Big)^n = \pi^{\frac{n}{2}}. \end{aligned}$$

Exercise 2:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and f be in $L^1(X)$. Let for all positive integers n set $B_n = \{x \in X : n-1 \leq |f(x)| < n\}$.

(i) Show that $\mu(B_n) < \infty$ for all $n \geq 2$.

(ii) Show that $\sum_{n=2}^{\infty} n\mu(B_n) < \infty$.

(iii) Define $C_n = \{x \in X : n-1 \leq |f(x)| \leq n\}$. Is the sum $\sum_{n=2}^{\infty} n\mu(C_n)$ finite?

(iv) Show that

$$\sum_{n=2}^{\infty} \sum_{m=2}^{n} \frac{m^2}{n^2} \mu(B_m) < \infty.$$

(v) Show that for $n \geq 2$

$$\int |f|^2 1_{\{|f|<n\}} = \int |f|^2 1_{\{|f|<1\}} + \sum_{m=2}^{n} \int |f|^2 1_{B_m}$$

and infer that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<n\}} < \infty$$

Solution:

(i) Since $f \in L^1(X)$, we have $\int_X |f| \, d\mu < \infty$. For $n \geq 2$, we know that

$$\int_X |f| \, d\mu \geq \int_{B_n} |f| \, d\mu \geq (n-1) \int_{B_n} 1 \, d\mu = (n-1)\mu(B_n).$$

When $n \geq 2$, we have $n - 1 \geq 1$, then we know that $\mu(B_n) < \infty$. So, for any $n \geq 2$, we have $\mu(B_n) < \infty$.

(ii) Since $B_n = \{x \in X : n-1 \leq |f(x)| < n\} = \{x \in X : n \leq |f(x)| + 1 < n+1\}$, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n\mu(B_n) &= \sum_{n=2}^{\infty} \int_{B_n} n \, d\mu \\
&\leq \sum_{n=2}^{\infty} \int_{B_n} |f(x)| + 1 \, d\mu \\
&= \sum_{n=2}^{\infty} \int_{B_n} |f(x)| \, d\mu + \sum_{n=2}^{\infty} \int_{B_n} 1 \, d\mu \\
&\leq 2 \int_{\bigcup_{n=2}^{\infty} B_n} |f(x)| \, d\mu \\
&\leq 2 \int_X |f(x)| \, d\mu < \infty.
\end{aligned}$$

(iii) We claim that the sum $\sum_{n=2}^{\infty} n\mu(C_n)$ is finite. As $C_n = \{x \in X : n-1 \leq |f(x)| \leq n\} \subset B_n \cup B_{n+1}$, then we have

$$\mu(C_n) \leq \mu(B_n \cup B_{n+1}) \leq \mu(B_n) + \mu(B_{n+1}),$$

therefore, we know that

$$\sum_{n=2}^{\infty} n\mu(C_n) \leq \sum_{n=2}^{\infty} n\mu(B_n) + \sum_{n=2}^{\infty} n\mu(B_{n+1}).$$

Since $\int_{B_{n+1}} |f| \, d\mu \geq n \int_{B_{n+1}} 1 \, d\mu = n\mu(B_{n+1})$, then we have

$$\begin{aligned} \sum_{n=2}^{\infty} n\mu(B_{n+1}) &\leq \sum_{n=2}^{\infty} \int_{B_{n+1}} |f| \, d\mu \\ &= \int_{\bigcup_{n=2}^{\infty} B_{n+1}} |f| \, d\mu \\ &< \int_X |f| \, d\mu < \infty. \end{aligned}$$

As we showed $\sum_{n=2}^{\infty} n\mu(B_n) < \infty$ in (ii), hence we have

$$\sum_{n=2}^{\infty} n\mu(C_n) \leq \sum_{n=2}^{\infty} n\mu(B_n) + \sum_{n=2}^{\infty} n\mu(B_{n+1}) < \infty.$$

(iv) We can rewrite the $\sum_{n=2}^{\infty} \sum_{m=2}^{n} \frac{m^2}{n^2} \mu(B_m)$ and then we have

$$\begin{aligned} \sum_{n=2}^{\infty} \sum_{m=2}^{n} \frac{m^2}{n^2} \mu(B_m) &= \sum_{m=2}^{\infty} \mu(B_m) m^2 \sum_{n=m}^{\infty} \frac{1}{n^2} \\ &= \sum_{m=2}^{\infty} m\mu(B_m) \sum_{n=m}^{\infty} \frac{m}{n^2}. \end{aligned}$$

Next we need to show that $\sum_{n=m}^{\infty} \frac{m}{n^2}$ is bounded. When $m \geq 2$, we have

$$\sum_{n=m}^{\infty} \frac{m}{n^2} < m \int_{m-1}^{\infty} \frac{1}{x^2} \, dx = \frac{m}{m-1} \leq 2,$$

then we know that

$$\sum_{n=2}^{\infty} \sum_{m=2}^{n} \frac{m^2}{n^2} \mu(B_m) < 2 \sum_{m=2}^{\infty} m\mu(B_m) < \infty.$$

(v) Firstly, we show that

$$\int |f|^2 1_{\{|f|<n\}} \, d\mu = \int |f|^2 1_{\{|f|<1\}} \, d\mu + \sum_{m=2}^{n} \int |f|^2 1_{B_m} \, d\mu.$$

By calculation, we have

$$\begin{aligned}
\int |f|^2 1_{\{|f|<n\}}\, d\mu &= \int |f|^2 1_{\{|f|<1\}}\, d\mu + \int |f|^2 1_{\{1\le|f|<n\}}\, d\mu \\
&= \int |f|^2 1_{\{|f|<1\}}\, d\mu + \int |f|^2 \sum_{m=2}^{n} 1_{\{m-1\le|f|<m\}}\, d\mu \\
&= \int |f|^2 1_{\{|f|<1\}}\, d\mu + \sum_{m=2}^{n} \int |f|^2 1_{\{m-1\le|f|<m\}}\, d\mu \\
&= \int |f|^2 1_{\{|f|<1\}}\, d\mu + \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu,
\end{aligned}$$

then we get the equation we wanted. Next we show that $\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<n\}} < \infty$. As

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<n\}}\, d\mu \\
=\ & \sum_{n=1}^{\infty} \frac{1}{n^2} \Big(\int |f|^2 1_{\{|f|<1\}}\, d\mu + \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu \Big) \\
=\ & \sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<1\}}\, d\mu + \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu.
\end{aligned}$$

For the first term in the right hand side of the above equation, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<1\}}\, d\mu < \sum_{n=1}^{\infty} \frac{1}{n^2} \int_X |f|\, d\mu < \infty.$$

And for the second term in the right hand side of the above equation, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu &= \sum_{n=2}^{\infty} \frac{1}{n^2} \sum_{m=2}^{n} \int |f|^2 1_{B_m}\, d\mu \\
&\le \sum_{n=2}^{\infty} \frac{1}{n^2} \sum_{m=2}^{n} \int m^2 1_{B_m}\, d\mu \\
&= \sum_{n=2}^{\infty} \sum_{m=2}^{n} \frac{m^2}{n^2} \mu(B_m) < \infty.
\end{aligned}$$

Thus we can get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \int |f|^2 1_{\{|f|<n\}}\, d\mu < \infty.$$

Exercise 3:

Prove or disprove: suppose that $f, g : \mathbb{R} \to \mathbb{R}$, with f being a measurable function, and g being a continuous function. Then $f \circ g$ is measurable. By definition, $(f \circ g)(x) = f(g(x))$, that is, it is the composition of the two functions.

Solution:

No, the statement is not true and we can find a counter example as follows. Suppose that C is the Cantor set and we define a mapping ϕ: for any $x \in C$, let $0.c_1c_2c_3\cdots$ be its ternary expansion, where $c_n = 0$ or $c_n = 2$, $n = 1, 2, \cdots$ and let

$$\phi(x) = 0.\frac{c_1}{2}\frac{c_2}{2}\frac{c_3}{2}\cdots,$$

where the expansion on the right is now interpreted as a binary expansion in terms of digits 0 and 1. It is clear that the image of C, under ϕ, is a subset of $[0, 1]$. And next we extend the domain to the entire unit interval $[0, 1]$. If $x \in [0, 1]\backslash C$, then x is a member of one of the open intervals (a, b) removed from $[0, 1]$ in the construction of C, and therefore $\phi(a) = \phi(b)$. And we define $\phi(x) = \phi(a) = \phi(b)$. Since $\phi(\cdot)$ is increasing on $[0, 1]$, and since the range of $\phi(\cdot)$ is the entire interval $[0, 1]$, $\phi(\cdot)$ has no jump discontinuities. Since a monotonic function can have no discontinuities other than jump discontinuities, we know that $\phi(\cdot)$ is continuous. Then we define

$$\varphi(x) = x + \phi(x),\ x \in [0, 1]$$

with range $[0, 2]$. Since $\phi(\cdot)$ is increasing on $[0, 1]$ and continuous there, φ is strictly increasing and topological there (continuous and one-to-one with a continuous inverse on the range φ). Since each open interval removed from $[0, 1]$ in the construction of the Cantor set C is mapped by φ onto an interval of $[0, 2]$ of the equal length, $\mu(\varphi(I \setminus C)) = \mu(I \setminus C) = 1$. Since C is a set of measure zero, φ is an example of a topological mapping that maps a set of measure zero onto a set of positive measure.

Now let D is a non-measurable subset of $\varphi(C)$ and let $E = \varphi^{-1}(D)$. Then the characteristic function $f = 1_E(x)$ of the set E is measurable and $g = \varphi^{-1}$ is continuous, but the composite function $f(g(x))$ is non-measurable characteristic function of the non-measurable set D.

Claim: suppose that $f, g : \mathbb{R} \to \mathbb{R}$, with f being a measurable function, and g being a continuous function. Then $g \circ f$ is measurable.

Proof: Since $f : (\mathbb{R}, \mathcal{B}_\mathbb{R}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ is Lebesgue-measurable and as $g : \mathbb{R} \to \mathbb{R}$ is continuous, it is Borel-measurable. Take any $B \in \mathcal{B}_\mathbb{R}$, we want to show that $(g \circ f)^{-1}(B) \in \mathcal{B}_\mathbb{R}$. By measurability of g, since $B \in \mathcal{B}_\mathbb{R}$, we have $B' = g^{-1}(B) \in \mathcal{B}_\mathbb{R}$. By the measurability of f, this implies that $f^{-1}(B^{'}) \in \mathcal{B}_{\mathbb{R}}$. This shows that $g \circ f$ is measurable for the σ-algebras $\mathcal{B}_{\mathbb{R}}$.

5 GCE January, 2016

Exercise 1:

Let f_n be a sequence of continuous functions from $[0,1]$ to $\mathbb{R}$ which is uniformly convergent. Let x_n be in $[0,1]$ such that $f_n(x_n) \geq f_n(x)$, for all x in $[0,1]$.

(i) Is the sequence x_n convergent?

(ii) Show that the sequence $f_n(x_n)$ is convergent.

Solution:

(i) No, the sequence x_n may not convergent. We assume that $f_n(x) = 0$ for all $x \in [0,1]$. And for any $k \in \mathbb{N}$ we set the sequence x_n is

$$x_n = \begin{cases} 0, & n = 2k \\ 1, & n = 2k-1, \end{cases}$$

Then we know that $x_n \in [0,1]$ and $f_n(x_n) = 0 = f_n(x)$ for any $x \in [0,1]$, but the sequence x_n is not convergent.

(ii) We suppose f_n is uniformly converges to f on $[0,1]$. Since f_n is continuous, then f is also a continuous function. For any $y \in [0,1]$, there exist a x, such that $f(y) \leq f(x)$. And since f_n is uniformly converges to f on $[0,1]$, for any $\epsilon > 0$, there exists a $N_1 \in \mathbb{N}$, when $n > N_1$, for any $y \in [0,1]$, we have

$$|f_n(y) - f(y)| < \epsilon,$$

which is equivalent to $f(y) - \epsilon < f_n(y) < f(y) + \epsilon$. We use the x_n to substitute the y, then we have $f_n(x_n) \leq f(x_n) + \epsilon \leq f(x) + \epsilon$.

On the other hand, for the above x, we have $f_n(x_n) \geq f_n(x)$. As f_n is uniformly converges to f on $[0,1]$, for the above $\epsilon > 0$, there exists a $N_2 \in \mathbb{N}$, when $n > N_2$, for the above x, we have $f_n(x) > f(x) - \epsilon$. And then we have $f_n(x_n) > f(x) - \epsilon$. Thus for the above ϵ and x, there exists a N^*, which is the biggest one we related, then when $n > N^*$, we have

$$f(x) - \epsilon < f_n(x_n) < f(x) + \epsilon.$$

So, we know that the sequence $f_n(x_n)$ is convergent.

Exercise 2:

Let $\mathbb{I}$ be the set of all irrational number ($\mathbb{I} \subset \mathbb{R}$).

(i) Using that $\mathbb{Q} = \mathbb{R} \setminus \mathbb{I}$ (the set of all rationals) is countable, show that given $\epsilon > 0$, there is a closed subset $F \subset \mathbb{I}$ such that $|\mathbb{I} \setminus F| < \epsilon$.

(ii) Is F compact? Please explain why or why not.

Solution:

(i) We rearrange the rational number and denote it as $\{a_n\}_{n=1}^{\infty}$. It is a countable set. For $\epsilon > 0$, and for each $a_n \in \mathbb{Q}$, we can find an open set

$$a_n \in (a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}),$$

then we know that $\cup_{n=1}^{\infty}(a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}})$ is an open coverage of $\mathbb{Q}$, and

$$\Big| \cup_{n=1}^{\infty} (a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}})\Big| \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

We denote $S = \cup_{n=1}^{\infty}(a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}})$, then $\mathbb{R} \setminus S \subset \mathbb{R} \setminus \mathbb{Q} = \mathbb{I}$. We set $F = \mathbb{R} \setminus S$, as S is an open set, then F is closed. And we have

$$|\mathbb{I} \setminus F| = |\mathbb{I}| - |\mathbb{R} \setminus S| = |\mathbb{I}| - |\mathbb{R}| + |S| < \epsilon.$$

(ii) No, $F$is not a compact set. Suppose F is compact, then F is closed and bounded, thus F has finite measure. Since we have $(\mathbb{I} \setminus F) \cup F$, then there exists a $M > 0$ such that

$$|\mathbb{I}| = |(\mathbb{I} \setminus F) \cup F| \leq |\mathbb{I} \setminus F| + |F| < \epsilon + M,$$

which is contradictory with $|\mathbb{I}| = \infty$. Thus F is not compact.

Exercise 3:

Find with proof:

$$\lim_{n\to\infty} \int_0^1 \frac{1 + nx^3}{(1 + x^2)^n}\, dx$$

Solution:

For $x \in (0, 1)$, we denote $f_n(x) = \frac{1+nx^3}{(1+x^2)^n}$. Firstly, for $x \in (0, 1)$, since $(1 + x^2)^n \geq 1 + nx^2$, then we have

$$f_n(x) \leq \frac{1 + nx^3}{1 + nx^2} \leq 1 \in L^1((0, 1)).$$

And for $x \in (0, 1)$, since $(1 + x^2)^n \geq \frac{1}{2}n(n - 1)x^4$, we have

$$f_n(x) = \frac{1 + nx^3}{(1 + x^2)^n} \leq \frac{2 + 2nx^3}{n(n - 1)x^4} = \frac{\frac{2}{x^4}}{n(n - 1)} + \frac{\frac{1}{x}}{n - 1},$$

so for any fixed $x \in (0,1)$, we have $\lim_{n\to\infty} f_n(x) = 0$, thus we know that $f_n(x)$ converges to 0 pointwise. By the dominate convergence theorem, we have

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^3}{(1+x^2)^n}\,dx = \int_0^1 \lim_{n\to\infty} \frac{1+nx^3}{(1+x^2)^n}\,dx = 0.$$

Exercise 4:

Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) = 1$. Let f be in $L^1(X)$ such that $f \geq 0$ almost everywhere.

(i) show that

$$\lim_{p\to 0^+} \int f^p = \mu(\{x \in X : f(x) > 0\})$$

(ii) If $\mu(\{x \in X : f(x) > 0\}) < 1$, find

$$\lim_{p\to 0^+} \Big(\int f^p\Big)^{\frac{1}{p}}.$$

Solution:

(i) Since

$$\begin{aligned}\int_X f^p\,d\mu &= \int_{\{x\in X: f>0\}} f^p\,d\mu + \int_{\{x\in X: f=0\}} f^p\,d\mu \\ &= \int_{\{x\in X: f>0\}} f^p\,d\mu,\end{aligned}$$

as f be in $L^1(X)$ and $f \geq 0$ almost everywhere, by the Fatou's lemma,

$$\mu(\{x \in X : f(x) > 0\}) = \int \mathbb{I}_{\{x\in X: f>0\}}(x)\,d\mu \leq \liminf_{p\to 0^+} \int_{\{x\in X: f>0\}} f^p\,d\mu.$$

On the other hand, we know that

$$\begin{aligned}\int_{\{x\in X: f>0\}} f^p\,d\mu &= \int_{\{x\in X: 0<f<n\}} f^p\,d\mu + \int_{\{x\in X: f\geq n\}} f^p\,d\mu \\ &\leq \int_{\{x\in X: f\geq n\}} f^p\,d\mu + n^p\mu(\{x \in X : f(x) > 0\}).\end{aligned}$$

For $0 < p < 1$, when $x \in \{x \in X : f(x) > n\}$, we have $f^p < f$, thus we have

$$\begin{aligned}\limsup_{p\to 0^+} \int_{\{x\in X: f>0\}} f^p\,d\mu &\leq \mu(\{x \in X : f(x) > 0\}) + \limsup_{p\to 0^+} \int_{\{x\in X: f\geq n\}} f^p\,d\mu \\ &\leq \mu(\{x \in X : f(x) > 0\}) + \int_{\{x\in X: f\geq n\}} f\,d\mu \\ &\leq \mu(\{x \in X : f(x) > 0\}) + \int_X f\,\mathbb{I}_{\{x\in X: f\geq n\}}(x)\,d\mu\end{aligned}$$

Since $f \cdot \mathbb{I}_{\{x\in X:f\geq n\}}(x) \leq f \in L^1(X)$ and $\lim_{n\to\infty} f\,\mathbb{I}_{\{x\in X:f\geq n\}}(x) = 0$, by the dominate convergence theorem, we have

$$\limsup_{p\to 0^+} \int_{\{x\in X:f>0\}} f^p \, d\mu \leq \mu(\{x \in X : f(x) > 0\}),$$

thus we know that

$$\lim_{p\to 0^+} \int f^p = \mu(\{x \in X : f(x) > 0\}).$$

(ii) **Method 1:**

As $\mu(X) = 1$ and $f \in L^1(X)$, we know that $f \in L^\infty(X)$. We denote $S = \{x \in X : f > 0\}$, then

$$\begin{aligned} \int_X f^p \, d\mu &= \int_{\{x\in X:f>0\}} f^p \, d\mu + \int_{\{x\in X:f=0\}} f^p \, d\mu \\ &= \int_S f^p \, d\mu \\ &\leq \int_S \|f\|_\infty^p \, d\mu \\ &= \|f\|_\infty^p \mu(S), \end{aligned}$$

thus we have

$$\lim_{p\to 0^+} \left(\int f^p\right)^{\frac{1}{p}} \leq \lim_{p\to 0^+} \|f\|_\infty (\mu(S))^{\frac{1}{p}} = 0$$

as $\mu(S) < 1$.

Method 2:

We denote $S = \{x \in X : f > 0\}$, then

$$\begin{aligned} \int_X f^p \, d\mu &= \int_{\{x\in X:f>0\}} f^p \, d\mu + \int_{\{x\in X:f=0\}} f^p \, d\mu \\ &= \int_S f^p \, d\mu. \end{aligned}$$

And we denote that $F(p) = \log(\int_S f^p \, d\mu)$, then we know that

$$\lim_{p\to 0^+} \left(\int f^p\right)^{\frac{1}{p}} = \lim_{p\to 0^+} e^{\frac{F(p)}{p}}.$$

As $F(0) = \log(\mu(S))$, then we have

$$\begin{aligned} \lim_{p\to 0^+} \left(\int f^p\right)^{\frac{1}{p}} &= \lim_{p\to 0^+} \exp\left\{\frac{F(p) - \log(\mu(S)) + \log(\mu(S))}{p}\right\} \\ &= \lim_{p\to 0^+} (\mu(S))^{\frac{1}{p}} \exp\left\{\frac{F(p) - \log(\mu(S))}{p - 0}\right\}. \end{aligned}$$

As $F(p) = \log(\int_S f^p \, d\mu)$, we have

$$F^{'}(p) = \frac{\int_S f^p \cdot \log f \, d\mu}{\int_S f^p \, d\mu},$$

thus we have $F^{'}(0) = \frac{\int_S \log f \, d\mu}{\mu(S)}$. Then we know that

$$\begin{aligned}
\lim_{p\to 0^+} \left(\int f^p \right)^{\frac{1}{p}} &= \lim_{p\to 0^+} (\mu(S))^{\frac{1}{p}} \exp\left\{ \lim_{p\to 0^+} \frac{F(p) - F(0)}{p - 0} \right\} \\
&= \lim_{p\to 0^+} (\mu(S))^{\frac{1}{p}} e^{F^{'}(0)} \\
&= 0
\end{aligned}$$

as $\mu(S) < 1$.

6 GCE May, 2016

Exercise 1:

A real-valued function f is increasing on a closed interval $[a, b] \subset \mathbb{R}$ if and only if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$.

(i) Using the definition of measurable, show that f is measurable on $[a, b]$.

(ii) Show that f is continuous, except perhaps a countable number of points.

Solution:

(i) For any $c \in \mathbb{R}$, we denote $S = f^{-1}([c, +\infty])$, by the definition of S, we know that $S = \{x \in [a, b] | f(x) \geq c\}$. For any $x \in S$, if $y > x$ and $y \in [a, b]$, as f is increasing, we have $f(y) \geq f(x) \geq c$. So, we have $y \in S$. It is equivalent to that if $x \in S$, for any $y \in [a, b]$ and $y \geq x$, we have $y \in S$. This means S can only be , $[a, b]$, $(a, b]$, $[\inf S, b]$ and $(\inf S, b]$, all of the sets are measurable, thus we know that f is measurable.

(ii) Let $f(x^-)$ and $f(x^+)$ denote the left and the right hand limits of f respectively. Let A be the set of points where f is not continuous. Then for any $x \in A \subset [a, b]$, we can find a rational number $f^*(x) \in \mathbb{Q}$, such that $f(x^-) < f(x^*) < f(x^+)$. Since f is increasing function, then for $x_1, x_2 \in A$ and $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$, also we have $f(x_1^+) \leq f(x_2^-)$. Thus we have $f(x_1^*) < f(x_1^+) \leq f(x_2^-) < f(x_2^*)$, then we know that $f(x_1^*) < f(x_2^*)$. Then there exists a injection between A and a subsets of rational number $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, then we know that A is also countable. Thus f is continuous except perhaps a countable number of points.

Exercise 2:

If f is Lebesgue integrable on $\mathbb{R}$, define

$$F(x) = \int_0^x f \, d\mu$$

where $\mu(E)$ is the Lebesgue measurable set $E \subset \mathbb{R}$. Show that

(i) F is continuous.

(ii) If $\mu(E) = 0$, then $\mu(F(E)) = 0$.

Solution:

(i) Suppose $\{x_n\}$ is a sequence and $x_n \to x_0$ as n goes to infinity. Then we need to show that $F(x_n)$ converges to $F(x_0)$, i.e.

$$\lim_{n \to +\infty} \int_0^{x_n} f \, d\mu = \int_0^{x_0} f \, d\mu.$$

Since we have

$$\lim_{n\to+\infty}\int_0^{x_n} f\,d\mu = \lim_{n\to+\infty}\int_0^{\infty} f\,\mathbb{I}_{[0,x_n]}(x)\,d\mu$$

and

$$|f\,\mathbb{I}_{[0,x_n]}(x)| \le |f| \in L^1(\mathbb{R}),$$

by the dominate convergence theorem, we have

$$\lim_{n\to+\infty}\int_0^{\infty} f\,\mathbb{I}_{[0,x_n]}(x)\,d\mu = \int_0^{\infty}\lim_{n\to+\infty} f\,\mathbb{I}_{[0,x_n]}(x)\,d\mu.$$

Next we need to show that

$$\lim_{n\to+\infty}\mathbb{I}_{[0,x_n]}(x) = \mathbb{I}_{[0,x_0]}(x).$$

If $x_n \to x_0$, then for any $0 < t < x_0$, there exists a $N_1 \in \mathbb{N}$, such that $t < x_n$ for any $n > N_1$, and hence we have $\mathbb{I}_{[0,x_n]}(t) = 1$ for all $n > N_1$. Similarly, for $t > x_0$, there exists a $N_2 \in \mathbb{N}$ such that $\mathbb{I}_{[0,x_n]}(t) = 0$ for all $n > N_2$. Since $\{x_0\}$ is a singleton, which has zero measure,, thus we have

$$\lim_{n\to+\infty}\mathbb{I}_{[0,x_n]}(x) = \mathbb{I}_{[0,x_0]}(x)\; a.e.$$

Then we have

$$\lim_{n\to+\infty}\int_0^{\infty} f\,\mathbb{I}_{[0,x_n]}(x)\,d\mu = \int_0^{\infty} f\,\mathbb{I}_{[0,x_0]}(x)\,d\mu = \int_0^{x_0} f\,d\mu,$$

from which we know F is continuous.

(ii) We need to show that the continuous image of a zero measure set is also a zero measure set. For $E \in \mathbb{R}$ and $\mu(E) = 0$, we can find a disjoint sequence E_n such that $E \subset \cup_{n=1}^{\infty} E_n$ and for any $\epsilon > 0$ we have $\mu(\cup_{n=1}^{\infty} E_n) < \epsilon$. And then we have $F(E) \subset F(\cup_{n=1}^{\infty} E_n)$. Then we know that

$$\mu(F(E)) \le \mu(F(\cup_{n=1}^{\infty} E_n)).$$

Since F is continuous, if f is lipchitz continuous or f is absolutely continuous, then there exists a constant $K > 0$ and we have $\mu(F(\cup_{n=1}^{\infty} E_n)) \le K\mu(\cup_{n=1}^{\infty} E_n) < K\epsilon$. So, we know that $\mu(F(E)) = 0$.

Exercise 3:

Let f be in $L^1(\mathbb{R})$ such that $f \ge 0$ almost everywhere and $\int_{\mathbb{R}} f = 1$. Set $f_n(x) = nf(nx)$. Let g be in $L^\infty(\mathbb{R})$.

(i) Let x_0 be in $\mathbb{R}$. Assume that g is continuous at x_0. show that

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x_0-y)g(y)\,dy = g(x_0).$$

(ii) If g is uniformly continuous, is this limit uniformly in x_0?
(iii) If h is in $L^1(\mathbb{R})$ show that the function in x

$$\int_{\mathbb{R}} f_n(x-y)h(y)\,dy$$

converges to h in $L^1(\mathbb{R})$.

Solution:

(i) We denote $z = x_0 - y$, so we have

$$\int_{\mathbb{R}} f_n(x_0-y)g(y)\,dy = \int_{\mathbb{R}} f_n(z)g(x_0-z)\,dz = \int_{\mathbb{R}} nf(nz)g(x_0-z)\,dz,$$

and then we denote $u = nz$,

$$\int_{\mathbb{R}} nf(nz)g(x_0-z)\,dz = \int_{\mathbb{R}} f(u)g(x_0-\frac{u}{n})\,du.$$

Since $f\in L^1(\mathbb{R})$ and $g(x)\in L^\infty(\mathbb{R})$, there exists a $M>0$ such that

$$|f(u)g(x_0-\frac{u}{n})| \le Mf(u)\in L^1(\mathbb{R}),$$

by the dominate convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x_0-y)g(y)\,dy &= \lim_{n\to\infty}\int_{\mathbb{R}} f(u)g(x_0-\frac{u}{n})\,du \\
&= \int_{\mathbb{R}}\lim_{n\to\infty} f(u)g(x_0-\frac{u}{n})\,du \\
&= \int_{\mathbb{R}} f(u)g(x_0)\,du \\
&= g(x_0)
\end{aligned}$$

as g is continuous at x_0.

(ii) We need to show that $\int_{\mathbb{R}} f_n(x-y)g(y)\,dy$ is uniformly converges to $g(x)$ when g is uniformly continuous on $\mathbb{R}$. By the definition of $f_n(x)$, we have

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} nf(nx)\,dx = \int_{\mathbb{R}} f(nx)\,d(nx) = 1.$$

For any $x \in \mathbb{R}$,

$$\begin{aligned}\Big|\int_{\mathbb{R}} f_n(x-y)g(y)\,dy - g(x)\Big| &= \Big|\int_{\mathbb{R}} f_n(z)g(x-z)\,dz - g(x)\Big| \\ &= \Big|\int_{\mathbb{R}} f_n(z)g(x-z)\,dz - \int_{\mathbb{R}} f_n(z)g(x)\,dz\Big| \\ &\leq \int_{\mathbb{R}} f_n(z)|g(x-z)-g(x)|\,dz \\ &= \int_{\mathbb{R}} nf(nz)|g(x-z)-g(x)|\,dz,\end{aligned}$$

we denote $u = nz$, then we have

$$\Big|\int_{\mathbb{R}} f_n(x-y)g(y)\,dy - g(x)\Big| \leq \int_{\mathbb{R}} f(u)\Big|g\Big(x-\frac{u}{n}\Big)-g(x)\Big|\,du.$$

As $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, there exists a $M > 0$ such that

$$\Big|f(u)\Big(g(x-\frac{u}{n})-g(x)\Big)\Big| \leq 2Mf(u) \in L^1(\mathbb{R}),$$

by the dominate convergence theorem, we have

$$\lim_{n\to\infty}\Big|\int_{\mathbb{R}} f_n(x-y)g(y)\,dy - g(x)\Big| \leq \int_{\mathbb{R}} \lim_{n\to\infty} f(u)\Big|g\Big(x-\frac{u}{n}\Big)-g(x)\Big|\,du.$$

Since g is uniformly continuous on $\mathbb{R}$, for any $x \in \mathbb{R}$, and for any $\epsilon > 0$, there exists a $N \in \mathbb{N}$, which is independent of x, such that when $n > N$, we have $g(x-\frac{u}{n})-g(x) < \epsilon$. So, for the above ϵ and N, when $n > N$ we have

$$\int_{\mathbb{R}} f(u)\Big|g\Big(x-\frac{u}{n}\Big)-g(x)\Big|\,du \leq \int_{\mathbb{R}} f(u)\epsilon\,du = \epsilon$$

thus we know that $\int_{\mathbb{R}} f_n(x-y)g(y)\,dy$ is uniformly converges to $g(x)$.

(iii) As $h \in L^1(\mathbb{R})$ and $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, for any $\epsilon > 0$, there exists a function $g \in C_c(\mathbb{R})$, such that

$$\|g-h\|_1 < \epsilon.$$

We denote $\int_{\mathbb{R}} f_n(x-y)h(y)\,dy = h_n(x)$ and $\int_{\mathbb{R}} f_n(x-y)g(y)\,dy = g(x)$, then we have

$$\|h(x)-h_n(x)\|_1 \leq \|h(x)-g(x)\| + \|g(x)-g_n(x)\| + \|g_n(x)-h_n(x)\|.$$

For the above ϵ, as $\|g-h\|_1 < \epsilon$ and by the result we get from (ii), $g_n(x)$ is uniformly converges to $g(x)$, we have $\|g_n(x)-g(x)\| < \epsilon$, then we have

$$\lim_{n\to\infty}\|h(x)-h_n(x)\|_1 = \lim_{n\to\infty}\|g_n(x)-h_n(x)\|.$$

Next we need to verify the term $\|g_n(x) - h_n(x)\|$, since

$$\begin{aligned}
\|g_n(x) - h_n(x)\| &= \Big\| \int_{\mathbb{R}} f_n(x-y)h(y)\,dy - \int_{\mathbb{R}} f_n(x-y)g(y)\,dy \Big\| \\
&= \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} f_n(x-y)(h(y)-g(y))\,dy \Big|\,dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} f_n(x-y)|h(y)-g(y)|\,dy\,dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) \Big| h\Big(x - \frac{u}{n}\Big) - g\Big(x - \frac{u}{n}\Big) \Big|\,du\,dx,
\end{aligned}$$

by Fubini's theorem, we have

$$\|g_n(x) - h_n(x)\| \leq \int_{\mathbb{R}} f(u) \int_{\mathbb{R}} \Big| h\Big(x - \frac{u}{n}\Big) - g\Big(x - \frac{u}{n}\Big) \Big|\,dx\,du.$$

Since $f \in L^1(\mathbb{R})$, $h \in L^1(\mathbb{R})$ and $g \in C_c(\mathbb{R})$, there exists a $M > 0$ such that

$$f(u) \Big| h\Big(x - \frac{u}{n}\Big) - g\Big(x - \frac{u}{n}\Big) \Big| \leq 2Mf(u) \in L^1(\mathbb{R}),$$

by the dominate convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \|g_n(x) - h_n(x)\| &\leq \int_{\mathbb{R}} f(u) \lim_{n\to\infty} \int_{\mathbb{R}} \Big| h\Big(x - \frac{u}{n}\Big) - g\Big(x - \frac{u}{n}\Big) \Big|\,dx\,du \\
&= \int_{\mathbb{R}} f(u) \lim_{n\to\infty} \Big\| h\Big(x - \frac{u}{n}\Big) - g\Big(x - \frac{u}{n}\Big) \Big\|\,du = 0.
\end{aligned}$$

Thus we know that

$$\lim_{n\to\infty} \|h(x) - h_n(x)\|_1 = 0,$$

which means $\int_{\mathbb{R}} f_n(x-y)h(y)\,dy$ converges to h in $L^1(\mathbb{R})$.

7 GCE August, 2016

Exercise 1:

Suppose that u is a real-valued function defined on $[0,1]$, that $u \geq 0$ and that $u \in L^1([0,1])$. Define $E_n := \{x \in [0,1] : n-1 \leq u(x) \leq n\}$ for each positive integer n. Show that

$$\sum_{n=1}^{\infty} n|E_n| < +\infty.$$

Solution:

As $u \in L^1([0,1])$ and $u(x) \geq 0$, we have

$$\int_0^1 |u(x)|\, dx = \int_0^1 u(x)\, dx < +\infty.$$

And since

$$\begin{aligned} \int_0^1 u(x)\, dx &= \sum_{n=1}^{\infty} \int_{E_n} u(x)\, dx \\ &\geq \sum_{n=1}^{\infty} (n-1)|E_n| \\ &= \sum_{n=1}^{\infty} n|E_n| + \sum_{n=1}^{\infty} |E_n|, \end{aligned}$$

and $\sum_{n=1}^{\infty} |E_n| < +\infty$, then we have

$$\sum_{n=1}^{\infty} n|E_n| < +\infty.$$

Exercise 2:

Show that a subset E of a metric space X is open if and only if there is a continuous real-valued function f on X such that $E = \{x \in x : f(x) > 0\}$.

Solution:

If there is a continuous real-valued function f on X such that $E = \{x \in x : f(x) > 0\}$, we want to show that E is an open set. Since $(0, +\infty)$ is an open set, $E = \{x \in x : f(x) > 0\} = f^{-1}((0, +\infty))$ is also an open set as f is continuous on X. We can also verify the statement by definition. Suppose $y \in E$, since $E = \{x \in x : f(x) > 0\}$, we have $f(y) > 0$. Since f in continuous on X, we know that there exists a δ such that when $d(x,y) < \delta$, then $|f(x) - f(y)| < f(y)$, which implies $-f(y) < f(x) - f(y) < f(y)$, hence we have $f(x) > 0$. Then we know that there exists a $\delta > 0$, when $x \in B_\delta(y)$, we have $f(x) > 0$. Thus for any $y \in E$, there exists a δ, and $B_\delta(y) \subset E$. So we know that E is an open set.

On the other direction, we want to show that if $E \subset X$ is open, there exists a continuous function f on X such that $E = \{x \in x : f(x) > 0\}$. For $E \in X$, we denote

$$f(x) = d(x, E^c) = \min_{y \in E^c} d(x, y).$$

Then we have when $x \in E^c$, $f(x) = 0$ and when $x \in E$, $f(x) > 0$, so we have $E = \{x \in x : f(x) > 0\}$. Next we need to show f is continuous on X . Let $x, y \in X$ and p is the any point in E^c, then

$$d(x,p) \leq d(x,y) + d(y,p),$$

and so

$$d(x, E^c) \leq d(x,y) + d(y,p)$$

as $d(x, A)$ is the minimum. Then we have $d(y,p) \geq d(x,E^c) - d(x,y)$ for all $p \in E^c$, thus we can get that $d(y,E^c) \geq d(x,E^c) - d(x,y)$, which is equivalent to

$$d(x,E^c) - d(y,E^c) \leq d(x,y).$$

Similarly, we can change the position of x and y then get

$$d(y,E^c) - d(x,E^c) \leq d(x,y),$$

so we have for any $x, y \in X$,

$$|d(x,E^c) - d(y,E^c)| \leq d(x,y).$$

Then for any $\epsilon > 0$, there exists a $\delta = \epsilon$, such that when $d(x,y) < \delta$, we have $|d(x,E^c) - d(y,E^c)| < d(x,y) = \epsilon$. So, we have showed that f is a continuous function on X.

Exercise 3:

Consider the sequence of functions $\{f_n\}$ defined on the non-negative reals: $[0, +\infty)$ where $f_n(x) = 2nxe^{-nx^2}$. Let g be a continuous and bounded function on $[0, +\infty)$ valued in $\mathbb{R}$.

(i) Find with proof

$$\lim_{n \to \infty} \int_0^\infty f_n(t) g(t)\, dt.$$

(ii) Define for x in $[0, +\infty)$,

$$g_n(x) = \int_0^\infty f_n(t) g(x+t)\, dt.$$

Assuming g is zero outside the interval $[0, M]$, where $M > 0$, does the sequence g_n have a limit in $L^1([0, +\infty))$?

(iii) If h is in $L^1([0, +\infty))$, define for x in $[0, +\infty)$,

$$h_n(x) = \int_0^\infty f_n(t) h(x+t)\, dt.$$

Show that h_n is measurable on $[0, +\infty)$ and is in $L^1([0, +\infty))$.

(iv) Find, if it exists, with proof, the limit of h_n in $L^1([0, +\infty))$.

Solution:

(i) We denote $y = nt^2$, then we have

$$\int_0^\infty 2nte^{-nt^2} g(t)\, dt = \int_0^\infty e^{-y} g\Big(\sqrt{\frac{y}{n}}\Big)\, dy.$$

Since $g(x)$ is a continuous and bounded function on $[0, +\infty)$, we suppose that $|g(x)| \le C$ for any $x \in [0, +\infty)$. Then we know that $|e^{-y} g(\sqrt{\frac{y}{n}})| \le Ce^{-y}$ and $Ce^{-y} \in L^1([0, +\infty))$ as $\int_0^\infty |Ce^{-y}|\, dy = C < +\infty$. And for any fixed $y \in [0, +\infty)$, when $n \to \infty$, $g(\sqrt{\frac{y}{n}}) \to g(0)$ and then $e^{-y} g(\sqrt{\frac{y}{n}}) \to e^{-y} g(0)$. By the dominate convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty f_n(t) g(t)\, dt &= \int_0^\infty \lim_{n\to\infty} e^{-y} g\Big(\sqrt{\frac{y}{n}}\Big)\, dy \\
&= \int_0^\infty e^{-y} g(0)\, dy \\
&= g(0).
\end{aligned}$$

(ii) Since $f_n(x) = 2nxe^{-nx^2}$, we denote $y = nt^2$, then we have

$$g_n(x) = \int_0^\infty f_n(t) g(x+t)\, dt = \int_0^\infty e^{-y} g\Big(x + \sqrt{\frac{y}{n}}\Big)\, dy.$$

Next we want to show that g_n converges to g in $L^1([0, +\infty))$. Since

$$\begin{aligned}
\int_0^\infty |g_n(x) - g(x)|\, dx &= \int_0^\infty \Big| \int_0^\infty e^{-y} g\Big(x + \sqrt{\frac{y}{n}}\Big)\, dy - g(x) \Big|\, dx \\
&= \int_0^\infty \Big| \int_0^\infty e^{-y} g\Big(x + \sqrt{\frac{y}{n}}\Big)\, dy - \int_0^\infty g(x) e^{-y}\, dy \Big|\, dx \\
&= \int_0^\infty \Big| \int_0^\infty e^{-y} \Big(g\Big(x + \sqrt{\frac{y}{n}}\Big) - g(x) \Big)\, dy \Big|\, dx \\
&\le \int_0^\infty \int_0^\infty e^{-y} \Big| g\Big(x + \sqrt{\frac{y}{n}}\Big) - g(x) \Big|\, dy\, dx,
\end{aligned}$$

and by Fubini theorem,

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dy\,dx &= \int_0^\infty \int_0^{M-\sqrt{\frac{y}{n}}} e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy\\ &+ \int_0^\infty \int_{M-\sqrt{\frac{y}{n}}}^{M} e^{-y}|g(x)|\,dx\,dy,\end{aligned}$$

when $n\to\infty$, we have

$$\int_0^\infty \int_{M-\sqrt{\frac{y}{n}}}^{M} e^{-y}|g(x)|\,dx\,dy \to 0,$$

thus we know that

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty |g_n(x)-g(x)|\,dx &\le \lim_{n\to\infty}\int_0^\infty \int_0^{M-\sqrt{\frac{y}{n}}} e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy\\ &\le \lim_{n\to\infty}\int_0^\infty \int_0^{M} e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy.\end{aligned}$$

Since $e^{-y}|g(x+\sqrt{\frac{y}{n}})-g(x)|\le 2Ce^{-y}$ and $2Ce^{-y}\in L^1([0,+\infty))$, then by the dominate convergence theorem we have

$$\lim_{n\to\infty}\int_0^\infty \int_0^{M} e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy = 0,$$

thus we know that

$$\lim_{n\to\infty}\int_0^\infty |g_n(x)-g(x)|\,dx = 0.$$

So, we have showed that g_n converges to g in $L^1([0,+\infty))$.

(iii) Since $C_c([0,+\infty))$ is dense in $L^1([0,+\infty))$ and $h(x)\in L^1([0,+\infty))$, we can find a sequence $\{h^k\}_{k=1}^\infty$ such that $h^k\to h$ in $L^1([0,+\infty))$. We want to show h_n is measurable by showing it is the limit of a sequence of measurable functions. By the result we got from (ii), for any $k\in\mathbb{N}$, we have $h_n^k=\int_0^\infty f_n(t)g(t)\,dt$ converges to $h^k(x)$ in $L^1([0,+\infty))$. Firstly we show that $h_n^k(x)$ converges to $h_n(x)$ almost everywhere. For any $x\in[0,+\infty)$, we have

$$\begin{aligned}|h_n(x)-h_n^k(x)| &= \Big|\int_0^\infty f_n(t)h(x+t)\,dt-\int_0^\infty f_n(t)h^k(x+t)\,dt\Big|\\ &= \Big|\int_0^\infty f_n(t)(h(x+t)-h^k(x+t))\,dt\Big|\\ &\le \int_0^\infty f_n(t)|h(x+t)-h^k(x+t)|\,dt,\end{aligned}$$

we denote $z = x + t$, then

$$|h_n(x) - h_n^k(x)| \le \int_x^\infty f_n(z-x)|h(z) - h^k(z)|\, dz.$$

Since $f_n(x) = 2nxe^{-nx^2}$, when $x = \frac{1}{\sqrt{2n}}$, the $f_n(x)$ gets the maximum value as $\sqrt{2n}e^{-\frac{1}{2}}$, thus we have

$$\begin{aligned}
|h_n(x) - h_n^k(x)| &\le \int_x^\infty f_n(z-x)|h(z) - h^k(z)|\, dz \\
&\le \|f_n\|_\infty \int_x^\infty |h(z) - h^k(z)|\, dz \\
&\le \|f_n\|_\infty \int_0^\infty |h(z) - h^k(z)|\, dz \\
&= \|f_n\|_\infty \|h - h^k\|_1 \to 0
\end{aligned}$$

as $k \to +\infty$. Then we show that h_n^k is continuous. This means we want to show that for $x \in [0, +\infty)$, let $x_j \to x$, then $h_n^k(x_j) \to h_n^k(x)$. By the definition of $h_n^k(x_j)$, we have

$$h_n^k(x_j) = \int_0^\infty f_n(t) h^k(x_j + t)\, dt = \int_0^\infty e^{-y} h^k\Big(x_j + \sqrt{\frac{y}{n}}\Big)\, dy.$$

And since $h^k \in C_c([0, +\infty))$, $|e^{-y}h^k(x_j + \sqrt{\frac{y}{n}}| \le \|h^k\|_\infty e^{-y} \in L^1([0, +\infty))$, by the dominate convergence theorem, we have

$$\lim_{j\to\infty} h_n^k(x_j) = \int_0^\infty \lim_{j\to\infty} e^{-y} h^k\Big(x_j + \sqrt{\frac{y}{n}}\Big)\, dy = \int_0^\infty e^{-y} h^k\Big(x + \sqrt{\frac{y}{n}}\Big)\, dy = h_n^k(x),$$

thus we know that h_n^k is uniformly continuous. From above, we have $h_n^k \to h_n$ almost everywhere and h_n^k is uniformly continuous, then we have h_n is the limit of a sequence of measurable functions. So, we get that h_n is measurable on $[0, +\infty)$.

Next we show that h_n is in $L^1([0, +\infty))$. Since

$$\begin{aligned}
\|h_n\|_1 &= \int_0^\infty |h_n(x)|\, dx \\
&= \int_0^\infty \Big| \int_0^\infty f_n(t) h(x+t)\, dt \Big|\, dx \\
&\le \int_0^\infty \int_0^\infty |f_n(t) h(x+t)|\, dt\, dx,
\end{aligned}$$

by Fubini theorem, we have

$$\begin{aligned}
\|h_n\|_1 &\le \int_0^\infty \int_0^\infty |f_n(t)h(x+t)|\,dx\,dt \\
&= \int_0^\infty f_n(t)\Big(\int_0^\infty |h(x+t)|\,dx\Big)\,dt \\
&= \int_0^\infty f_n(t)\Big(\int_t^\infty |h(z)|\,dz\Big)\,dt \\
&\le \int_0^\infty f_n(t)\Big(\int_0^\infty |h(z)|\,dz\Big)\,dt \\
&= \|h\|_1 \int_0^\infty f_n(t)\,dt \\
&= \|h\|_1 < +\infty.
\end{aligned}$$

Thus we know that h_n is in $L^1([0,+\infty))$.

(iv) We want to show that h_n converges to h in $L^1([0,+\infty))$. Let $\epsilon > 0$, since $C_c([0,+\infty))$ is dense in $L^1([0,+\infty))$, then there exists a $g \in C_c([0,+\infty))$ such that $\|h-g\|_1 < \epsilon$. So we have

$$\begin{aligned}
\|h_n - h\|_1 &= \|h_n - g_n + g_n - g + g - f\|_1 \\
&\le \|h_n - g_n\|_1 + \|g_n - g\|_1 + \|g - f\|_1,
\end{aligned}$$

where the definition of g_n is as question (ii). By the result we get form (ii), for the ϵ above, we have $\|g_n - g\| < \epsilon$, then we know that

$$\|h_n - h\|_1 < \|h_n - g_n\|_1 + 2\epsilon.$$

Next we need to deal with $\|h_n - g_n\|_1$. Since

$$\begin{aligned}
\|h_n - g_n\|_1 &= \int_0^\infty |h_n(x) - g_n(x)|\,dx \\
&\le \int_0^\infty \int_0^\infty f_n(t)|h(x+t) - g(x+t)|\,dt\,dx,
\end{aligned}$$

we denote $z = x + t$ and by Fubini theorem we have

$$\begin{aligned}
\|h_n - g_n\|_1 &\leq \int_0^\infty \int_0^\infty f_n(t)|h(x+t) - g(x+t)|\, dt\, dx \\
&= \int_0^\infty f_n(t) \int_t^\infty |h(z) - g(z)|\, dz\, dt \\
&\leq \int_0^\infty f_n(t) \int_0^\infty |h(z) - g(z)|\, dz\, dt \\
&= \int_0^\infty f_n(t)\|h - g\|_1 dt \\
&= \|h - g\|_1 \int_0^\infty f_n(t) dt \\
&= \|h - g\|_1 < \epsilon.
\end{aligned}$$

Thus we know that

$$\|h_n - h\|_1 < \|h_n - g_n\|_1 + 2\epsilon < 3\epsilon$$

for any $\epsilon > 0$. So, we have showed that h_n converges to h in $L^1([0, +\infty))$.

Exercise 4:

Show that a set $E \subset \mathbb{R}$ is Lebesgue measurable if and only if $E = H \cup Z$ where H is a countable union of closed sets and Z has measure zero. You may use the following property: for any Lebesgue measurable subset A of $\mathbb{R}$ and any $\epsilon > 0$, there is a closed subset F of $\mathbb{R}$ such that $F \subset A$ and the measure of $A \setminus F$ is less than ϵ.

Solution:

If $E \subset \mathbb{R}$ is Lebesgue measurable, then we know that $\forall \epsilon > 0$, there is a closed subset H of $\mathbb{R}$ such that $H \subset E$ and the measure of $E \setminus H$ is less than ϵ. We denote $Z = E \setminus H$, then we have $m(Z) = 0$ and $Z \cup H = (E \setminus H) \cup H = E$.

Since H is a countable union of closed sets, then H is a $\mathcal{F}_\sigma$ set and it is measurable. And as Z is a zero measure set, it is also Lebesgue measurable. Thus we know that $E = H \cup Z$ is Lebesgue measurable.

Exercise 5:

Give an example of a sequence f_n in $L^1((0,1))$ such that $f_n \to 0$ in $L^1((0,1))$ but f_n does not converge to zero almost everywhere.

Solution:

We suppose that

$$f_n(x) = \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x),$$

whenever $k \geq 0, 2^k \leq n < 2^{k+1}$. For any $n \in \mathbb{N}$, we have

$$\int_0^1 |f_n(x)|\,dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x)\,dx = \frac{1}{2^k} < +\infty,$$

so we know that $f_n \in L^1((0,1))$. And similarly we have

$$\int_0^1 |f_n(x) - 0|\,dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x)\,dx = \frac{1}{2^k} < \frac{2}{n},$$

then when $n \to +\infty$, $\int_0^1 |f_n(x) - 0|\,dx \to 0$, thus we get $f_n \to 0$ in $L^1((0,1))$. But for any $x \in (0,1)$, and for any $N \in \mathbb{N}$, we can find a $n > N$ with $f_n(x) = 1$. Thus f_n can not converges to 0 anywhere for $x \in (0,1)$.

8 GCE January, 2017

Exercise 1:

Consider the sequence of functions f_n defined on the non-negative reals by $f_n(x) = 2nxP(x)e^{-nx^2}$, where P is a polynomial function.

(i) Is f_n pointwise convergent on $[0, +\infty)$? Is f_n uniformly convergent on $[0, +\infty)$? Explain your answers to both questions.

(ii) Let g_n be a sequence of continuous functions defined on $[0, +\infty)$ and valued in $\mathbb{R}$. Assume that each g_n is in $L^1([0, +\infty))$ and that sequence g_n is uniformly convergent to zero. Prove or disprove: $\lim_{n\to\infty} \int_0^\infty g_n = 0$.

(iii) Determine (with proof) $\lim_{n\to\infty} \int_0^\infty f_n$.

Solution:

(i) For any $x \in [0, \infty)$, as $P(x)$ is a polynomial function, by the L'Hospital's Rule, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 2nxP(x)e^{-nx^2} = \lim_{n\to\infty} \frac{2nxP(x)}{e^{nx^2}} = 0,$$

then for any $\epsilon > 0$, there exist a $N \in \mathbb{N}$, such that $n > N$ we have

$$|2nxP(x)e^{-nx^2} - 0| < \epsilon,$$

thus we know that f_n converges to $f(x) = 0$ pointwise on $[0, \infty)$. But f_n is not uniformly convergent to $f(x) = 0$. We suppose $P(x) = 1$, then we have $f_n(x) = 2nxe^{-nx^2}$. When $x = \frac{1}{\sqrt{n}}$,

$$f_n(x) = 2n\frac{1}{\sqrt{n}}e^{-n\frac{1}{n}} = 2\sqrt{n}e^{-1},$$

so we have

$$\sup_{x\in[0,\infty)} |f_n(x) - 0| \geq 2\sqrt{n}e^{-1} \to \infty$$

when n goes to $+\infty$. Thus we know that f_n is not uniformly converges on $[0, +\infty)$.

(ii) The statement is not true. We suppose

$$g_n(x) = \begin{cases} \frac{4}{n^2}x, & x \in [0, \frac{n}{2}) \\ \frac{4}{n} - \frac{4}{n^2}x, & x \in [\frac{n}{2}, n] \\ 0, & x \in (n, +\infty), \end{cases}$$

then we know that for any $n \in \mathbb{N}$,

$$\int_{[0,\infty)} g_n(x)\, dx = \int_0^{\frac{n}{2}} \frac{4}{n^2}x\, dx + \int_{\frac{n}{2}}^n \frac{4}{n} - \frac{4}{n^2}x\, dx = 1,$$

so we know that $g_n(x) \in L^1([0,\infty))$. When $x \in [0, \frac{n}{2})$, $g_n(x) = \frac{4}{n^2}x \leq \frac{2}{n}$ and when $x \in [\frac{n}{2}, n]$, $g_n(x) = \frac{4}{n} - \frac{4}{n^2}x \leq \frac{2}{n}$, so we know that g_n uniformly converges to 0. But since for any $n \in \mathbb{N}$, $\int_0^\infty g_n(x)\,dx = 1$, then we have

$$\lim_{n\to+\infty} \int_{[0,\infty)} g_n(x)\,dx = \lim_{n\to\infty} 1 = 1.$$

Thus $\lim_{n\to\infty} \int_0^\infty g_n = 0$ can not hold.

(iii) We denote $y = nx^2$, then we have

$$\int_0^\infty 2nxP(x)e^{-nx^2}\,dx = \int_0^\infty e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy.$$

Since $P(x)$ is a polynomial function, for any fixed y, when $n\to\infty$, $P(\sqrt{\frac{y}{n}}) \to P(0)$ and then $e^{-y}P(\sqrt{\frac{y}{n}}) \to e^{-y}P(0)$. Since $P(x)$ is a polynomial function, there exist a $M > 0$, such that when $y \in [M,\infty)$, $e^{-y}P(\sqrt{\frac{y}{n}}) < \frac{1}{y^2}$, then we have

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty f_n &= \lim_{n\to\infty}\int_0^M f_n + \lim_{n\to\infty}\int_M^\infty f_n \\ &= \lim_{n\to\infty}\int_0^M e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy + \lim_{n\to\infty}\int_M^\infty e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy.\end{aligned}$$

Since $P(x)$ is a polynomial function, then $P(\sqrt{\frac{y}{n}})$ is continuous on $y \in [0, M]$, then we have when $y \in [0, M]$,

$$\Big|e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\Big| \leq e^{-y}\|P\|_\infty.$$

Since $e^{-y}\|P\|_\infty \in L^1([0,M])$ and $\frac{1}{y^2} \in L^1([M,+\infty))$, by the dominate convergence theorem, we have

$$\lim_{n\to\infty}\int_0^M e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy = \int_0^M e^{-y}P(0)\,dy = P(0)(1-e^{-M}),$$

and

$$\lim_{n\to\infty}\int_M^\infty e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy = \int_M^\infty e^{-y}P(0)\,dy = P(0)e^{-M}.$$

Thus we know that

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty f_n &= \lim_{n\to\infty}\int_0^M e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy + \lim_{n\to\infty}\int_M^\infty e^{-y}P\Big(\sqrt{\frac{y}{n}}\Big)\,dy \\ &= P(0)(1-e^{-M}) + P(0)e^{-M} \\ &= P(0).\end{aligned}$$

Exercise 2(all answers require proofs:)
Let f_n be the sequence in $L^2(\mathbb{R})$ defined by $f_n = \mathbb{I}_{[n,n+1]}$.
(i) Let g be in $L^2(\mathbb{R})$. Does $\int f_n g$ have a limit as n tends to infinity?
(ii) Does the sequence f_n converge in $L^2(\mathbb{R})$?

Solution:
(i) Firstly we show that $f_n = \mathbb{I}_{[n,n+1]}(x)$ converges to $f(x) = 0$ pointwise on $\mathbb{R}$. Since

$$|f_n - f| = |\mathbb{I}_{[n,n+1]}(x) - 0| = \mathbb{I}_{[n,n+1]}(x),$$

for any fixed $x \in \mathbb{R}$, $\forall \epsilon > 0$,we can find a $N = [x] + 1$, such that $n > N$, we have

$$|f_n - f| = \mathbb{I}_{[n,n+1]}(x) = 0 < \epsilon.$$

Thus we know that f_n converges to $f(x) = 0$ pointwisely on $\mathbb{R}$. Since

$$\Big| \int_{\mathbb{R}} f_n g \, dx \Big| \le \int_{\mathbb{R}} |f_n g| \, dx = \int_n^{n+1} |g(x)| \, dx,$$

by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\Big| \int_{\mathbb{R}} f_n g \, dx \Big| &\le \int_n^{n+1} |g(x)| \, dx \\
&\le \Big(\int_n^{n+1} |g|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_n^{n+1} 1^2 \, dx \Big)^{\frac{1}{2}} \\
&= \Big(\int_n^{n+1} |g|^2 \, dx \Big)^{\frac{1}{2}} \\
&= \Big(\int_{\mathbb{R}} |g|^2 \mathbb{I}_{[n,n+1]}(x) \, dx \Big)^{\frac{1}{2}}.
\end{aligned}$$

Since $|g|^2 \mathbb{I}_{[n,n+1]}(x) \le |g(x)|^2$ and since $g \in L^2(\mathbb{R})$, we have $\int_{\mathbb{R}} |g(x)|^2 \, dx < +\infty$, then we know that $|g(x)|^2 \in L^1(\mathbb{R})$, by the dominate convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \Big| \int_{\mathbb{R}} f_n g \, dx \Big|^2 &\le \lim_{n\to\infty} \Big(\int_{\mathbb{R}} |g|^2 \mathbb{I}_{[n,n+1]}(x) \, dx \Big) \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} \big(|g|^2 \mathbb{I}_{[n,n+1]}(x) \big) \, dx.
\end{aligned}$$

Since $f_n = \mathbb{I}_{[n,n+1]}(x)$ converges to $f(x) = 0$ pointwise on $\mathbb{R}$, we can show that $|g|^2 \mathbb{I}_{[n,n+1]}(x)$ also converges to $f(x) = 0$ pointwise on $\mathbb{R}$, then we have

$$\lim_{n\to\infty} \Big| \int_{\mathbb{R}} f_n g \, dx \Big|^2 \le \int_{\mathbb{R}} \lim_{n\to\infty} \big(|g|^2 \mathbb{I}_{[n,n+1]}(x) \big) \, dx = 0.$$

Thus we know that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)\,dx = 0$.

(ii) Since $f_n = \mathbb{I}_{[n,n+1]}(x)$ converges to $f(x) = 0$ pointwise on $\mathbb{R}$, but

$$\int_{\mathbb{R}} |f_n(x) - 0|^2\,dx = \int_{\mathbb{R}} f_n^2\,dx = \int_n^{n+1} 1\,dx = 1,$$

we know that f_n does not converges to $f(x) = 0$ in $L^2(\mathbb{R})$, then we can get that the sequence f_n does not converge in $L^2(\mathbb{R})$.

Exercise 3:

Let X be a matrix space. For any subset A of X, we denote by $\bar{A}$ the closure of A and $\mathring{A}$ the union of all open subsets contained in A. We set $\partial A = \bar{A} \setminus \mathring{A}$.

(i) Show that A is closed if and only if $\partial A \subset A$.

(ii) Show that A is open if and only if $\partial A \cap A = \emptyset$.

(iii) Is the identity $\partial(\partial B) = \partial B$ valid for all subsets B of X?

(iv) Show that if A is closed then $\partial(\partial A) = \partial A$.

Solution:

(i) When A is closed, we have $A = \bar{A}$, since $\mathring{A}$ the union of all open subsets contained in A, then $\mathring{A} \subset A$. Thus we have $\partial A = \bar{A} \setminus \mathring{A} = A \setminus \mathring{A} \subset A$ as $\mathring{A} \subset A$.

When $\partial A \subset A$, we have $\partial A \cup A \subset A \cup A = A$, then we know that $\bar{A} \subset A$. Since $A \subset \bar{A}$, we can get $\bar{A} = A$, thus A is closed.

(ii) When A is open, since $\mathring{A}$ the union of all open subsets contained in A, then we have $A \subset \mathring{A}$. And we know that $\mathring{A} \subset A$, then we can get $A = \mathring{A}$. As $\partial A = \bar{A} \setminus \mathring{A}$, we have $\partial A = \bar{A} \setminus A$, then it is obviously that $\partial A \cap A = \emptyset$.

When $\partial A \cap A = \emptyset$, we suppose A is not an open set, then there exists a element $x \in A$ such that no open set containing x is a subset of A. Since $\mathring{A}$ the union of all open subsets contained in A, we have $x \notin \mathring{A}$. And as $x \in A$, we know that $x \in \bar{A}$, then we have $x \in \bar{A} \setminus \mathring{A} = \partial A$. Then we can get $x \in \partial A \cap A$, it is contradict with the condition we have. So, the statement that A is not an open set is wrong. Thus we have A is an open set.

(iii) No, the statement is not true. We suppose $B = \mathbb{Q} \cap [0,1]$, which represents the rational number in the interval $[0,1]$. Then we have $\partial B = [0,1]$ and $\partial(\partial B) = \{0,1\}$, which is not equal to ∂B.

(iv) Since $\bar{A}$ is closed and $\mathring{A}$ is open, we have $\partial A = \bar{A} \setminus \mathring{A}$ is closed, then we can get $\overline{\partial A} = \partial A$. By the definition of ∂A, we have $\partial(\partial A) = \overline{\partial A} \setminus \mathring{\partial A} = \partial A \setminus \mathring{\partial A} \subset \partial A$. Next we need to show that $\partial A \subset \partial(\partial A) = \partial A \setminus \mathring{\partial A}$, then we just need to prove that $\mathring{\partial A} = \emptyset$ when A is closed.

When A is closed, since $\partial A = \bar{A} \setminus \mathring{A} = A \setminus \mathring{A}$. As $A \setminus \mathring{A} \subset A$, then we have $\mathring{\partial A} \subset \mathring{A}$. And since the union of subsets in $(A \setminus \mathring{A})$ is the subset of $A \setminus \mathring{A}$, we have $\mathring{\partial A} \subset A \setminus \mathring{A}$. Then we know that $\mathring{\partial A} \subset \mathring{A}$ and $\mathring{\partial A} \subset A \setminus \mathring{A}$. Thus we can get $\mathring{\partial A} \subset \mathring{A} \cap (A \setminus \mathring{A}) = \emptyset$. So, we have showed that $\mathring{\partial A} = \emptyset$. In conclusion, we have $\partial(\partial A) = \partial A$ when A is closed.

Exercise 4:

Let X be a measure space, f_n a sequence in $L^1(X)$ and f an element of $L^1(X)$ such that f_n converges to f almost everywhere and $\lim_{n\to\infty} \int |f_n| = \int |f|$. Show that $\lim_{n\to\infty} \int |f_n - f| = 0$.

Solution:

Since $|f_n - f| \leq |f_n| + |f|$ holds on X, we know that $|f_n| + |f| - |f_n - f|$ is a non-negative function. By the Fatou's lemma, we have

$$\int \lim_{n\to\infty} (|f_n| + |f| - |f_n - f|) \leq \liminf_{n\to\infty} \int (|f_n| + |f| - |f_n - f|).$$

Since f_n converges to f almost everywhere, then we know that $|f_n|$ converges to $|f|$ almost everywhere. Thus we have

$$\lim_{n\to\infty} (|f_n| + |f| - |f_n - f|) = 2|f|.$$

Then we can get that

$$\begin{aligned}
\int 2|f| &\leq \liminf_{n\to\infty} \int (|f_n| + |f| - |f_n - f|) \\
&\leq \liminf_{n\to\infty} \int (|f_n| + |f|) - \limsup_{n\to\infty} \int (|f_n - f|) \\
&= \int 2|f| - \limsup_{n\to\infty} \int (|f_n - f|),
\end{aligned}$$

as $f \in L^1(X)$, then $\int |f| < +\infty$, we have

$$\limsup_{n\to\infty} \int (|f_n - f|) \leq 0.$$

On the other hand, we have

$$0 \leq \liminf_{n\to\infty} \int (|f_n - f|)$$

as $|f_n - f| \geq 0$. Thus we know that

$$\limsup_{n\to\infty} \int (|f_n - f|) \leq 0 \leq \liminf_{n\to\infty} \int (|f_n - f|),$$

which is equivalent to

$$\limsup_{n\to\infty} \int (|f_n - f|) = \liminf_{n\to\infty} \int (|f_n - f|) = 0.$$

So we have

$$\lim_{n\to\infty} \int |f_n - f| = 0.$$

9 GCE May, 2017

Exercise 1:

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let A_n be a sequence in $\mathcal{A}$ such that $\mu(A_n)$ converges to zero.

(i) Prove or disprove: if $f : X \to [0, +\infty)$ is a measurable function and $\mu(X) < +\infty$, then $\int_{A_n} f$ converges to zero.

(ii) Let g be in $L^1(X)$. Show that $\int_{A_n} g$ converges to zero.

Solution:

(i) The statement is not true. We suppose $X = (0, 1]$ and $f(x) = \frac{1}{x^2}$, then we know that $\mu(X) < +\infty$ and $f(x)$ is measurable on X. We set $A_n = [\frac{1}{n^2}, \frac{1}{n}], n \in \mathbb{N}$. Thus we have for all $n \in \mathbb{N}$, $A_n \subset X$. And

$$\mu(A_n) = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} \to 0$$

as n goes to infinity. But for the $\int_{A_n} f$, we have

$$\int_{A_n} f \, d\mu = \int_{\frac{1}{n^2}}^{\frac{1}{n}} \frac{1}{x^2} \, dx = n^2 - n \to +\infty$$

as $n \to +\infty$. So, we know that $\int_{A_n} f$ does not converges to zero.

(ii) We denote

$$g_n(x) = g(x)\mathbb{I}_{A_n}(x),$$

where $\mathbb{I}_{A_n}(\cdot)$ is a indicator function on A_n. Since A_n is a sequence in $\mathcal{A}$ such that $\mu(A_n) \to 0$ as $n \to +\infty$, then we know that $g_n(x)$ converges to 0 almost everywhere. As

$$|g_n(x)| = |g(x)\mathbb{I}_{A_n}(x)| \leq |g(x)|$$

and $g \in L^1(X)$, we know that g is a dominate function of g_n. By the dominate convergence theorem, we have

$$\lim_{n\to\infty} \int_X g_n(x) \, d\mu = \int_X 0 \, d\mu = 0,$$

thus we have

$$\lim_{n\to\infty} \int_X g_n(x) \, d\mu = \lim_{n\to\infty} \int_{A_n} g \, d\mu = 0.$$

So, we know that $\int_{A_n} g$ converges to zero.

Exercise 2:

Let (x, d) be a bounded metric space. For any non empty subset S of X and x in X we define:

$$d(x, S) = \inf\{d(x, s) : s \in S\}.$$

If A and B are two non empty subsets of X we define:

$$d_H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\}.$$

(i) Prove or disprove: If $d_H(A, B) = 0$, are A and B necessarily equal?

(ii) Let $\mathcal{C}$ be the set of all non empty closed subsets of X. Show that d_H defines a metric on $\mathcal{C}$.

Solution:

(i) The statement is not true. By the definition of $d_H(A, B)$, since $d_H(A, B) = 0$, we have

$$\max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\} = 0,$$

then we have $\sup_{x\in A} d(x, B) = \sup_{x\in B} d(x, A) = 0$, so we know that $\forall x \in A, d(x, B) = 0$ and $\forall x \in B, d(x, A) = 0$. For any $x \in A$, since $d(x, B) = \inf\{d(x, y) : y \in B\} = 0$, we can find a sequence $\{y_n\}$, and for any $x \in A$ this sequence converges to x. So we have $B \subset \bar{A}$, where $\bar{A}$ is the closure of A. Similarly, we have $A \subset \bar{B}$.

We suppose $A = [0, 1)$ and $B = [0, 1]$, thus $A \neq B$. Since $A \subset B$, $\forall x \in A$, $\exists y \in B$ such that $x = y$ and $d(x, y) = 0$, we have $\sup_{x\in A} d(x, B) = 0$. On the other hand, when $x \in B$ and $x \in [0, 1)$, since $A = [0, 1)$, we know that foe any $x \in [0, 1)$, there exists a $y \in A$ such that $x = y$ and then $d(x, y) = 0$. And when $x \in B$ and $x = \{1\}$, since $y \in A = [0, 1)$, we have $d(x, A) = \inf\{d(x, y) : y \in A\} = 0$. Thus it is also holds that $\sup_{x\in B} d(x, A) = 0$. Then we know that $d_H(A, B) = 0$ but $A \neq B$. So, A and B is not necessarily equal.

(ii) Since $\mathcal{C}$ is the set of all non empty closed subsets of X, for $A \in \mathcal{C}$ and $B \in \mathcal{C}$, A, B are both closed sets. Next we need to verify the definition of the metric.

(a) $d_H(A, B) \geq 0$: since (X, d) is a metric space, then $d(x, B) \geq 0$ and $d(x, A) \geq 0$, thus we have $d_H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\} \geq 0$.

(b) $d_H(A, B) = 0 \iff A = B$: if A = B, then we have $d(x, B) = 0$ for any $x \in A$ and $d(x, A) = 0$ for any $x \in B$, thus we know that $d_H(A, B) = 0$. If $d_H(A, B) = 0$, by the result we get from (i), we know that $A \subset \bar{B}$ and $B \subset \bar{A}$. Since A and B are both closed sets, then we have $A \subset B$ and $B \subset A$, thus we can get $A = B$.

(c) $d_H(A,B) = d_H(B,A)$: since $d_H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$ and $d_H(B,A) = \max\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\}$, thus we have $d_H(A,B) = d_H(B,A)$.

(d) For $A, B, C \in \mathcal{C}$, $d_H(A,B) \leq d_H(A,C) + d_H(C,B)$: since $d_H(A,C)+d_H(C,B) \geq \sup_{x\in A} d(x,C)+\sup_{x\in C} d(x,B)$, then we know that $d_H(A,C)+d_H(C,B) \geq \sup_{x\in A} d(x,B)$. Similarly, we have $d_H(A,C) + d_H(C,B) \geq \sup_{x\in B} d(x,A)$, thus we can get $d_H(A,C) + d_H(C,B) \geq \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\} = d_H(A,B)$.

Exercise 3:

Let $(X, \mathcal{A}, \mu)$ be a measure space and $\{f_k\}$ a sequence in $L^p(X)$ where $1 \leq p \leq +\infty$. Suppose that $\{f_k\}$ converges in $L^p(X)$ to f. Show that f_k converges in measure to f on X.

Hint: According to the definition f convergence in measure, you need to show that for any positive ϵ, $\mu(\{x \in X : |f_k(x) - f(x)| \geq \epsilon\})$ converges to zero as k tends to infinity.

Solution:

When $p = +\infty$, since the sequence $\{f_k\}$ converges to f in $L^\infty(X)$, then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, when $n > N$, we have $\|f_n - f\|_\infty < \epsilon$. It means that $|f_n - f|$ is less than ϵ almost everywhere. Thus we have $\mu(|f_n - f| > \epsilon) = 0$ when $n \to \infty$. So we get that $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\})$ converges to zero as n tends to infinity.

When $1 \leq p < \infty$, for any $\epsilon > 0$, we have

$$\begin{aligned} \|f_n - f\|_p^p &= \int_X |f_n - f|^p \, d\mu \\ &\geq \int_{\{x\in X: |f_n - f|^p \geq \epsilon^p\}} |f_n - f|^p \, d\mu \\ &\geq \epsilon^p \mu(\{x \in X : |f_n - f|^p \geq \epsilon^p\}) \\ &= \epsilon^p \mu(\{x \in X : |f_n - f| \geq \epsilon\}), \end{aligned}$$

so we know that

$$\mu(\{x \in X : |f_n - f| \geq \epsilon\}) \leq \frac{1}{\epsilon^p} \|f_n - f\|_p^p.$$

Since $\{f_n\}$ converges in $L^p(X)$ to f, we have $\|f_n - f\|_p^p \to 0$ as $n \to \infty$. So, for any $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\})$ converges to zero as n tends to infinity.

Exercise 4:

Suppose $g_n, g \in L^1(\mathbb{R})$, g_n converges to g almost everywhere, and $\int g_n$ converges to $\int g$. Define $f_n(x) := g_n(x + n)$.

(i) Prove or disprove: there exists an f in $L^1(\mathbb{R})$ such that f_n converges to f almost everywhere.

(ii) Prove or disprove: if there is an f as in (i), then $\int f_n$ converges to $\int f$.

Solution:

(i) The statement is not true. We suppose $g_n(x) = (x + \frac{1}{n})\mathbb{I}_{[0,1]}(x)$ and $g(x) = x\mathbb{I}_{[0,1]}(x)$, then we have

$$|g_n(x) - g(x)| = |(x + \frac{1}{n})\mathbb{I}_{[0,1]}(x) - x\mathbb{I}_{[0,1]}(x)| = \frac{1}{n} \to 0$$

when n tends to infinity. So, g_n converges to g almost everywhere. Since

$$\int_{\mathbb{R}} g_n(x)\,dx = \int_0^1 (x + \frac{1}{n})\,dx = \frac{1}{2} + \frac{1}{n} \to \frac{1}{2}$$

as $n \to +\infty$ and

$$\int_{\mathbb{R}} g(x)\,dx = \int_0^1 x\,dx = \frac{1}{2},$$

we know that $\int g_n$ converges to $\int g$. As $f_n(x) := g_n(x+n)$, then $f_n(x) = (x + n + \frac{1}{n})\mathbb{I}_{[0,1]}(x)$, it is diverges as $f_n(x) > n$ for any $x \in [0,1]$.

(ii) The statement is not true. We set $g_n(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ for all $n \in \mathbb{N}$ and $g(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ too. Since $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx = 1$, then we have $g_n(x) \in L^1(\mathbb{R})$ and $g(x) \in L^1(\mathbb{R})$. As $g_n(x) = g(x)$, we know that g_n converges to g almost everywhere. By the definition of $f_n(x)$, we know that $f_n(x) = g_n(x+n) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x+n)^2}{2}}$ and when $f(x) = 0$, for any fix $x \in \mathbb{R}$ we have,

$$|f_n(x) - f(x)| = |\frac{1}{\sqrt{2\pi}}e^{-\frac{(x+n)^2}{2}} - 0| \to 0$$

as $n \to +\infty$. So, we know that f is in $L^1(\mathbb{R})$ and f_n converges to f almost everywhere. But for any $n \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x+n)^2}{2}}\,dx = 1,$$

and we know that

$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} 0\,dx = 0,$$

thus $\int f_n$ does not converges to $\int f$.

10 GCE August, 2017

Exercise 1:

Let h_n be a sequence of non-negative, borel measurable functions on the interval $(0,1)$ such that $h_n \to 0$ in $L^1((0,1))$.

(i) Show $\sqrt{h_n} \to 0$ in $L^1((0,1))$.

(ii) Given an example to show that h_n^2 need not converge to zero in $L^1((0,1))$.

(iii) If g_n is in $L^1(\mathbb{R})$ such that $|g_n|^{\frac{1}{2}}$ is in $L^1(\mathbb{R})$, and g_n converges to zero in $L^1(\mathbb{R})$ as n tends to infinity, does $|g_n|^{\frac{1}{2}}$ converges to zero in $L^1(\mathbb{R})$?

Solution:

(i) We want to show that $\int_0^1 |\sqrt{h_n} - 0|\, d\mu \to 0$ as $n \to \infty$. Since $h_n \to 0$ in $L^1((0,1))$ and by the holder inequality, we have

$$\begin{aligned}
\int_0^1 |\sqrt{h_n} - 0|\, d\mu &\le \Big(\int_0^1 |(\sqrt{h_n})^2|\, d\mu\Big)^{\frac{1}{2}} \Big(\int_0^1 1^2\, d\mu\Big)^{\frac{1}{2}} \\
&= \Big(\int_0^1 h_n\, d\mu\Big)^{\frac{1}{2}} \Big(\int_0^1 1\, d\mu\Big)^{\frac{1}{2}} \\
&= \Big(\int_0^1 |h_n - 0|\, d\mu\Big)^{\frac{1}{2}}.
\end{aligned}$$

So when n goes to infinity, we have $\int_0^1 |\sqrt{h_n} - 0|\, d\mu \to 0$. Thus we know that $\sqrt{h_n} \to 0$ in $L^1((0,1))$.

(ii) We suppose for $n \in \mathbb{N}$,

$$h_n(x) = n^{\frac{3}{2}} x \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x).$$

Then we have

$$\int_0^1 n^{\frac{3}{2}} x \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x)\, dx = n^{\frac{3}{2}} \int_{\frac{1}{n^2}}^{\frac{1}{n}} x\, dx = \frac{1}{2}(\frac{1}{\sqrt{n}} - \frac{1}{n^{\frac{5}{2}}}),$$

when $n \to +\infty$, we get $\|h_n\|_1 \to 0$, so we know that $h_n \to 0$ in $L^1((0,1))$. But for the $h_n^2(x)$, we have

$$\int_0^1 n^3 x^2 \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x)\, dx = n^3 \int_{\frac{1}{n^2}}^{\frac{1}{n}} x^2\, dx = \frac{1}{3} n^3 (\frac{1}{n^3} - \frac{1}{n^6}) = \frac{1}{3} - \frac{1}{3n^3}.$$

When n tends to infinity, $\int_0^1 n^3 x^2 \mathbb{I}_{[\frac{1}{n^2}, \frac{1}{n})}(x)\, dx \to \frac{1}{3}$, which is not goes to 0. So, we know that $h_n^2(x)$ don't converge to zero in $L^1((0,1))$.

(iii) No, $|g_n|^{\frac{1}{2}}$ need not converge to zero in $L^1(\mathbb{R})$. We can give a counter example. Suppose $g_n(x) = \frac{1}{x^2}\mathbb{I}_{[n,n^2]}(x)$, then we have

$$\int_{\mathbb{R}} |g_n(x)|\, dx = \int_n^{n^2} \frac{1}{x^2}\, dx = \frac{1}{n} - \frac{1}{n^2}.$$

When n goes to infinity, we have $\|g_n(x)\|_1 \to 0$, so $g_n(x)$ is in $L^1(\mathbb{R})$ and g_n converges to zero in $L^1(\mathbb{R})$. For the $|g_n|^{\frac{1}{2}} = \frac{1}{x}\mathbb{I}_{[n,n^2]}(x)$, for any $n \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} |g_n(x)|^{\frac{1}{2}}\, dx = \int_n^{n^2} \frac{1}{x}\, dx = \ln n.$$

When n goes to infinity, we have $\int_{\mathbb{R}} |g_n(x)|^{\frac{1}{2}}\, dx \to +\infty$, so $|g_n|^{\frac{1}{2}}$ is in $L^1(\mathbb{R})$ but g_n don't converges to zero in $L^1(\mathbb{R})$.

Exercise 2:

Let f be in $L^\infty((0,1))$. Show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

Solution:

Since $f \in L^\infty((0,1))$ and $\mu((0,1)) = 1 < \infty$, then we know that for any $p \geq 1$, $f \in L^p((0,1))$. We denote $t \in [0, \|f\|_\infty)$, then the set

$$A = \{x \in (0,1) : |f(x)| \geq t\}$$

has positive and bounded measure. Since

$$\begin{aligned} \|f\|_p &= \Big(\int_{(0,1)} |f|^p\, d\mu\Big)^{\frac{1}{p}} \geq \Big(\int_A |f|^p\, d\mu\Big)^{\frac{1}{p}} \\ &\geq \Big(t^p \mu(A)\Big)^{\frac{1}{p}} = t(\mu(A))^{\frac{1}{p}}, \end{aligned}$$

and $\mu(A)$ is finite, then when $p \to +\infty$, we have $(\mu(A))^{\frac{1}{p}} \to 1$ and

$$\liminf_{p\to+\infty} \|f\|_p \geq t.$$

Since t is arbitrary and $t \in [0, \|f\|_\infty)$, we have

$$\liminf_{p\to+\infty} \|f\|_p \geq \|f\|_\infty.$$

On the other hand, as $|f(x)| \leq \|f\|_\infty$ for almost every $x \in (0,1)$, then for $1 \leq q < p$, since $f(x)$ is in $L^p((0,1))$ and $f(x)$ is in $L^q((0,1))$, we have

$$\begin{aligned}
\|f\|_p &= \Big(\int_{(0,1)} |f|^p \, d\mu\Big)^{\frac{1}{p}} \\
&= \Big(\int_{(0,1)} |f|^q |f|^{p-q} \, d\mu\Big)^{\frac{1}{p}} \\
&\leq (\|f\|_\infty)^{\frac{p-q}{p}} (\|f\|_q)^{\frac{q}{p}}.
\end{aligned}$$

Since $\|f\|_q < +\infty$, then when $p \to +\infty$, we know that

$$\limsup_{p \to +\infty} \|f\|_p \leq \|f\|_\infty.$$

Thus we have

$$\limsup_{p \to +\infty} \|f\|_p \leq \|f\|_\infty \leq \liminf_{p \to +\infty} \|f\|_p,$$

then we know that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

Exercise 3:

Let a_n be a sequence in $[0,1]$ such that the set $S = \{a_n : n = 1, 2, \ldots\}$ is dense in $[0,1]$. Set

$$f(x) = \sum_{n=1}^{\infty} \frac{|x - a_n|^{-\frac{1}{2}}}{n^2}.$$

(i) Show that f is in $L^1([0,1])$.

(ii) Is f in $L^2([0,1])$?

(iii) Is there a continuous function

$$g : [0,1] \setminus S \to \mathbb{R}$$

such that $f = g$ almost everywhere?

Solution:

(i) We check $f \in L^1([0,1])$ by definition, since

$$\begin{aligned}
\int_0^1 \sum_{n=1}^{\infty} \frac{|x - a_n|^{-\frac{1}{2}}}{n^2} \, dx &= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 |x - a_n|^{-\frac{1}{2}} \, dx \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \Big[\int_0^{a_n} (a_n - x)^{-\frac{1}{2}} \, dx + \int_{a_n}^1 (x - a_n)^{-\frac{1}{2}} \, dx \Big] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \Big[2(a_n)^{\frac{1}{2}} + 2(1 - a_n)^{\frac{1}{2}} \Big]
\end{aligned}$$

and $a_n \in [0,1]$, then we know that

$$\int_0^1 \sum_{n=1}^{\infty} \frac{|x-a_n|^{-\frac{1}{2}}}{n^2}\,dx \le 4\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$$

as $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Thus we know that $f \in L^1([0,1])$.

(ii) No, we can show that $f \notin L^2([0,1])$. For $x \in [0,1]$, we have

$$\begin{aligned}
\|f\|_2 &= \int_0^1 \Big(\sum_{n=1}^{\infty} \frac{|x-a_n|^{-\frac{1}{2}}}{n^2}\Big)^2\,dx \\
&\ge \int_0^1 \sum_{n=1}^{\infty} \Big(\frac{|x-a_n|^{-\frac{1}{2}}}{n^2}\Big)^2\,dx \\
&= \sum_{n=1}^{\infty} \frac{1}{n^4}\int_0^1 |x-a_n|^{-1}\,dx.
\end{aligned}$$

To show $f \notin L^2([0,1])$, we just need to prove that $\int_0^1 |x-a_n|^{-1}\,dx = +\infty$. We denote $y = x - a_n$, then we have

$$\int_0^1 |x-a_n|^{-1}\,dx = \int_{-a_n}^{1-a_n} |y|^{-1}\,dy.$$

Since there exists $k > 0$ such that $\frac{1}{k} < a_n$, then we have $-\frac{1}{k} < 0 < 1 - a_n$ and

$$\int_0^1 |x-a_n|^{-1}\,dx \ge \int_{-a_n}^{-\frac{1}{k}} |y|^{-1}\,dy = \int_{\frac{1}{k}}^{a_n} y^{-1}\,dy = \ln a_n + \ln k.$$

When $k \to +\infty$, we have $\ln k + \ln a_n \to \infty$. So, we know that $\int_0^1 |x-a_n|^{-1}\,dx = +\infty$. Thus $\|f\|_2 = +\infty$, then we have $f \notin L^2([0,1])$.

(iii) To show that there is a continuous function $g : [0,1] \setminus S \to \mathbb{R}$ such that $f = g$ almost everywhere, we just need to prove that f is continuous in $[0,1] \setminus S$. So for $x \in [0,1] \setminus S$, we want to show that: $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall y \in [0,1] \setminus S$ such that $|x-y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Firstly, we deal with $f(x) - f(y)$, and then we can get

$$\begin{aligned}
|f(x) - f(y)| &= \Big|\sum_{n=1}^{\infty} \frac{|x-a_n|^{-\frac{1}{2}}}{n^2} - \sum_{n=1}^{\infty} \frac{|y-a_n|^{-\frac{1}{2}}}{n^2}\Big| \\
&= \Big|\sum_{n=1}^{\infty} \frac{1}{n^2}(|x-a_n|^{-\frac{1}{2}} - |y-a_n|^{-\frac{1}{2}})\Big| \\
&\le \sum_{n=1}^{\infty} \frac{1}{n^2}\Big||x-a_n|^{-\frac{1}{2}} - |y-a_n|^{-\frac{1}{2}}\Big|.
\end{aligned}$$

Since $g(x) = |x - a_n|^{-\frac{1}{2}}$ is continuous on $(0, 1]$, then $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall y \in (0, 1]$ such that $|x - y| < \delta$, we have

$$\left| |x - a_n|^{-\frac{1}{2}} - |y - a_n|^{-\frac{1}{2}} \right| < \frac{6}{\pi^2}\epsilon.$$

Since S is countable and dense in $[0, 1]$, then for the above ϵ and δ, $\forall y \in [0, 1] \setminus S$ such that $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \left| |x - a_n|^{-\frac{1}{2}} - |y - a_n|^{-\frac{1}{2}} \right| < \frac{\pi^2}{6} \times \frac{6}{\pi^2}\epsilon = \epsilon.$$

Thus we know that $f(x)$ is continuous in $[0, 1] \setminus S$, which is equivalent to $f(x)$ is continuous almost everywhere in $[0, 1]$. So, there exists a continuous function $g : [0, 1] \setminus S \to \mathbb{R}$ such that $f = g$ almost everywhere.

Exercise 4:

Let $\mathcal{R}$ be the set of all rectangles $(a_1, b_1) \times (a_2, b_2)$ in $\mathbb{R}^2$ such that a_1, b_1, a_2, b_2 are rational numbers.

(i) Let V be an open set in $\mathbb{R}^2$. Show that

$$V = \bigcup_{R \in \mathcal{R}, R \subset V} R.$$

(ii) Recall that the Borel sets of $\mathbb{R}^2$ are the sets in the smallest sigma algebra of $\mathbb{R}^2$ containing all open sets. Show that the smallest sigma algebra of $\mathbb{R}^2$ containing $\mathcal{R}$ is equal to the set set of Borel sets of $\mathbb{R}^2$.

Solution:

(i) Since $\bigcup_{R \in \mathcal{R}, R \subset V} R \subset V$, to prove $V = \bigcup_{R \in \mathcal{R}, R \subset V} R$, we just need to show that $V \subset \bigcup_{R \in \mathcal{R}, R \subset V} R$. Suppose that $\vec{x} = (x_1, x_2) \in V$, since V is an open set, then there exists an open ball such that $B(\vec{x}, r) \subset V$, where r is a positive constant and it is called the radius of the ball. So we can find a rectangle $R = (a_1, b_1) \times (a_2, b_2)$, whose center is exactly $\vec{x}$. We denote $d((a_1, b_1), (a_2, b_2))$ is the distance between (a_1, b_1) and (a_2, b_2). Suppose $d((a_1, b_1), (a_2, b_2)) < r$, then when know that $\vec{x} \in R$, $R \subset V$ and $R \in \mathcal{R}$. For any $x \in V$ we can do same thing, so we have $V \subset \bigcup_{R \in \mathcal{R}, R \subset V} R$. Thus we know that $V = \bigcup_{R \in \mathcal{R}, R \subset V} R$.

(ii) We denote $\sigma(\mathcal{R})$ is the sigma algebra on $\mathbb{R}^2$ generated by sets in $\mathcal{R}$. And we denote $\mathcal{B}(\mathbb{R}^2)$ as the Borel sets of $\mathbb{R}^2$. Since R is open rectangle in $\mathbb{R}^2$ and $\mathcal{R} = \{(a_1, b_1) \times (a_2, b_2) | a_i, b_i \in \mathbb{Q}, i = 1, 2\}$, so $\mathcal{R}$ is the open set in $\mathbb{R}^2$. Then we know that $\sigma(\mathbb{R}) \subset$

$\mathcal{B}(\mathbb{R}^2)$. On the other hand, V is open set and by the result we get in (i), we have $V = \bigcup_{R\in\mathcal{R}, R\subset V} R$. Since the number of set R is countable, then we have $V \in \sigma(\mathcal{R})$. Thus the open sets in $\mathbb{R}^2$ is subset of $\sigma(\mathcal{R})$. Since $\mathcal{B}(\mathbb{R}^2)$ is generated by the open sets in $\mathbb{R}^2$, then we have $\mathcal{B}(\mathbb{R}^2) \subset \sigma(\mathcal{R})$. So we can get $\mathcal{B}(\mathbb{R}^2) = \sigma(\mathcal{R})$. Then we know that the smallest sigma algebra of $\mathbb{R}^2$ containing $\mathcal{R}$ is equal to the set set of Borel sets of $\mathbb{R}^2$.

11 GCE May, 2018

Exercise 1:

Let (X, ρ) be a metric space and K_n a sequence of compact subsets of X such that $K_{n+1} \subset K_n$. Set

$$d_n = \sup\{\rho(x, y) : x \in K_n, y \in K_n\}$$

Assuming that d_n converges to zero show that $\bigcap_{n=1}^{\infty} K_n$ is a singleton.

Solution:

Since $\lim_{n\to+\infty} d_n = 0$, it means the diameter of the intersection of the K_n is zero. So, $\bigcap_{n=1}^{\infty} K_n$ is either empty or consists of a single point. For any $n \in \mathbb{N}$, we pick an element $a_n \in K_n$. So we can get a point sequence $\{a_n\}$, and we have $\{a_n : n \in \mathbb{N}\} \in K_1$. Since K_1 is compact, then we know there exists a sub-sequence of a_n, which is denoted as a_{n_k}, converges to a point a. For any $n \in \mathbb{N}$, since each K_n is compact, and a is the limit of a sequence in K_n, we have $a \in K_n$. Thus $a \in \bigcap_{n=1}^{\infty} K_n$. So we know that $\bigcap_{n=1}^{\infty} K_n$ is a singleton.

Exercise 2:

(i) Let $[a, b]$ be an interval in $\mathbb{R}$. If $\tilde{f}$ is continuous on $[a, b]$, g is differentiable on $[a, b]$ and monotonic, and g' is continuous on $[a, b]$, show that there is a c in $[a, b]$, such that

$$\int_a^b \tilde{f}g = g(a)\int_a^c \tilde{f} + g(b)\int_c^b \tilde{f}.$$

Hint: Introduce $F(x) = \int_a^x \tilde{f}$ and integral by parts.

(ii) Show that if g is as specified above and f is in $L^1([a, b])$, there is a c in $[a, b]$ such that

$$\int_a^b fg = g(a)\int_a^c f + g(b)\int_c^b f.$$

Solution:

(i) Since $\tilde{f}$ is continuous on $[a, b]$, we can introduce $F(x) = \int_a^x \tilde{f}$, so we know that

$F'(x) = \tilde{f}(x)$. Then through integral by parts, we have

$$\begin{aligned}
\int_a^b f(\tilde{x})g(x)\,dx &= \int_a^b g(x)\,dF(x) \\
&= g(b)F(b) - g(a)F(a) - \int_a^b g'(x)F(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - g(a)\int_a^a \tilde{f}(x)\,dx - \int_a^b g'(x)F(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - \int_a^b g'(x)F(x)\,dx.
\end{aligned}$$

Since g is differentiable on $[a, b]$ and monotonic, and g' is continuous on $[a, b]$, we know that g' is integrable in $[a, b]$ and $g'(x) \geq 0$ for all $x \in [a, b]$. By the mean value theorem for integral, there exists $c \in [a, b]$, and

$$\int_a^b g'(x)F(x)\,dx = F(c)\int_a^b g'(x)\,dx = F(c)(g(b) - g(a)).$$

Thus for this $c \in [a, b]$, we have

$$\begin{aligned}
\int_a^b f(\tilde{x})g(x)\,dx &= g(b)\int_a^b \tilde{f}(x)\,dx - F(c)(g(b) - g(a)) \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - (g(b) - g(a))\int_a^c \tilde{f}(x)\,dx \\
&= g(b)\int_a^b \tilde{f}(x)\,dx - g(b)\int_a^c \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx \\
&= g(b)\int_c^b \tilde{f}(x)\,dx + g(a)\int_a^c \tilde{f}(x)\,dx.
\end{aligned}$$

(ii) Since $C_c([a, b])$ is dense in $L^1([a, b])$, then we know that for any $f \in L^1([0, 1])$, there exists a function sequence $\{f_n\} \subset C_c([a, b])$ and $\int_a^b |f_n - f| \to 0$ as $n \to +\infty$. Since g is differentiable on $[a, b]$ and monotonic, we know there exists $K > 0$, and $\forall x \in [a, b]$, we have $|g(x)| \leq K$. So, we have

$$\lim_{n\to+\infty} \int_a^b |gf - gf_n| \leq K \lim_{n\to+\infty} \int_a^b |f - f_n| = 0,$$

then by the conclusion we get from (i) we have

$$\int_a^b fg = \lim_{n\to+\infty} \int_a^b f_n g = \lim_{n\to+\infty} \Big(g(a)\int_a^{c_n} f_n + g(b)\int_{c_n}^b f_n \Big),$$

where c_n is depends on f_n for each n.

Since $\{c_n\} \subset [a,b]$ and $[a,b]$ is compact, there exists a subsequence of $\{c_n\}$, which is denoted as $\{c_{n_k}\}$, converges to c and $c \in [a,b]$. Thus we have

$$\begin{aligned}
\int_a^b fg &= \lim_{k\to+\infty}\Big(g(a)\int_a^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^b f_{n_k}\Big) \\
&= \lim_{k\to+\infty}\Big(g(a)\int_a^{c} f_{n_k} + g(a)\int_c^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^c f_{n_k} + g(b)\int_c^b f_{n_k}\Big) \\
&= g(a)\int_a^c f + g(b)\int_c^b f + \lim_{k\to+\infty}\Big(g(a)\int_c^{c_{n_k}} f_{n_k} + g(b)\int_{c_{n_k}}^c f_{n_k}\Big) \\
&= g(a)\int_a^c f + g(b)\int_c^b f.
\end{aligned}$$

Exercise 3:

Let $\{f_n\}$ be a sequence of functions $f_n : [0,1] \to \mathbb{R}$.

(i) Define equicontinuity for this sequence.

(ii) Show that if each f_n is differentiable on $[0,1]$ and $|f_n'(x)| \le 1$ for all x in $[0,1]$ and $n \in \mathbb{N}$, the sequence is equicontinuous.

(iii) Suppose the sequence is uniformly bounded and that (ii) holds. Show that f_n has a subsequence which converges uniformly to a continuous function.

(iv) Show through an example that the limit may not be differentiable.

Solution:

(i) The definition of equicontinuity of sequence $\{f_n\}$ at point x is as follows: $\forall\epsilon > 0, \exists\delta > 0$, such that $|x-y| < \delta$ and $\forall n \in \mathbb{N}$, we have $|f_n(x) - f_n(y)| < \epsilon$. And the definition of uniformly equicontinuity of sequence $\{f_n\}$ is as follows: $\forall x \in [0,1], \forall\epsilon > 0, \exists\delta > 0$, such that $|x-y| < \delta$ and $\forall n \in \mathbb{N}$, we have $|f_n(x) - f_n(y)| < \epsilon$.

(ii) Since f_n is differentiable on $[0,1]$, by the mean value theorem, we know that $\forall x, y \in [0,1]$, there exists a $c \in [x,y]$ and we have

$$|f_n(y) - f_n(x)| = |f_n'(c)||y-x|.$$

Since $|f_n'(x)| \le 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}$, then we have

$$|f_n(y) - f_n(x)| \le |y-x|.$$

We set $\delta = \epsilon$, then for all $\epsilon > 0$, there exists $\delta = \epsilon$, such that when $|y-x| < \delta$, $\forall n \in \mathbb{N}$, we have $|f_n(y) - f_n(x)| < \epsilon$. So we know the sequence $\{f_n\}$ is equicontinuous.

(iii) By the Arzelà-Ascoli theorem, we can get f_n has a subsequence which converges uniformly to a continuous function directly. Next we can show the proof of Arzelà-Ascoli theorem.

We enumerate $\{x_i\}_{i\in\mathbb{N}}$ as the rational number in $[0,1]$. Since the sequence $\{f_n\}$ is uniformly bounded, then the set of points $\{f_n(x_1)\}$ is bounded, by the Bolzano-Weierstrass theorem, there is a subsequence $\{f_{n1}(x_1)\}$ converges. Repeating the same argument for the sequence points $\{f_{n1}(x_2)\}$, there is a subsequence $\{f_{n2}\}$ of $\{f_{n1}\}$ such that $\{f_{n2}(x_2)\}$ converges. By induction this process can be continued forever, and so there is a chain of subsequences

$$\{f_n\} \supset \{f_{n1}\} \supset \{f_{n2}\} \supset \cdots$$

Such that for each $k \in \mathbb{N}$, the subsequence$\{f_{nk}\}$ converges at point x_k. We choose the diagonal subsequence $\{f_{kk}\}$. Except for the first n functions, $\{f_{kk}\}$ is a subsequence of the nth row $\{f_{nk}\}$. Therefore, the sequence $\{f_{kk}\}$ converges simultaneously on all x_n.

Next we need to show that $\{f_{kk}\}$ is converges uniformly on $[a,b]$. We just need to prove the uniform Cauchy criterion holds. Given any $\epsilon > 0$ and rational $x_k \in [0,1]$, there is an integer $N(\epsilon, x_k)$ such that when $n, m > N$, we have

$$|f_{nn}(x_k) - f_{mm}(x_k)| < \frac{\epsilon}{3}.$$

Since $\bigcap(x_k - \frac{1}{n}, x_k + \frac{1}{n})$ covers the compact interval [0, 1], then by the Heine-Borel theorem there is a finite subcover, we denote the finite subcover as $U_1, \ldots, U_J$. There exists an integer K such that each open interval U_j, $1 \leq j \leq J$, contains a rational number x_k with $1 \leq k \leq K$. Finally, for any $x \in [0,1]$, there are j and k so that x and x_k belong to the same interval U_j. For this k, we have

$$\begin{aligned} |f_{nn}(x) - f_{mm}(x)| &\leq |f_{nn}(x) - f_{nn}(x_k)| + |f_{nn}(x_k) - f_{mm}(x_k)| + |f_{mm}(x_k) - f_{mm}(x)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

for all $N = \max\{N(\epsilon, x_1), \ldots, N(\epsilon, x_K)\}$ as f_n is equicontinuous. So, for the subsequence $\{f_{kk}\}$, the uniform Cauchy criterion holds. Thus we know that $\{f_{kk}\}$ converges to a continuous function.

(iv) We denote $f_n(x) = \sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}}$, $x \in [0,1]$. Since for all $n \in \mathbb{N}$ and $x \in [0,1]$,

$$|f_n'(x)| = \left|\frac{x - \frac{1}{2}}{\sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}}}\right| < 1$$

and $f_n(x) = \sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}} < 2$, by the conclusion we get from (ii) and (iii), we know that the sequence $\{f_n\}$ is equicontinuous and it has a subsequence which converges uniformly to a continuous function. When $n \to +\infty$, we have $f_n(x) \to f(x) = |x - \frac{1}{2}|$, which is not differentiable. So, we know that the limit of this type sequence may not be differentiable.

Exercise 4:

Let f be a lebesgue measurable function such that

$$\int_0^1 f(x)e^{Kx}\, dx = 0$$

for all $K = 1, 2, 3, \ldots$. Show that necessarily $f(x) = 0$ for almost every $0 \leq x \leq 1$.

12 GCE August, 2018

Exercise 1:

Let X and Y be two metric spaces and f a mapping from X to Y.

(i) Show that f is continuous if and only if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.

(ii) Prove or disprove: assume that f is injective. Then f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.

(iii) Prove or disprove: assume that X is compact. Then f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.

Solution:

(i) Firstly, we show that if f is continuous, then for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$. Since $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is closed as f is continuous, where $f^{-1}(\overline{f(A)})$ is the inverse image of $\overline{f(A)}$. Since $A \subset f^{-1}(f(A))$, then we have $A \subset f^{-1}(\overline{f(A)})$. Since the closure of A is contained in any closed set containing A, so we have $\overline{A} \subset f^{-1}(\overline{f(A)})$. Thus we know that for any $x \in \overline{A}$, we have $f(x) \in \overline{f(A)}$, then we get $f(\overline{A}) \subset \overline{f(A)}$.

Secondly, we show that if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$, we have f is continuous. To verify that f is continuous, we just need to show that for any closed set $C \subset Y$, the inverse image of the C under the function f is also a closed set. We denote $D = f^{-1}(C)$, then we want to show D is closed in X. Since $f(\overline{D}) \subset \overline{f(D)} = \overline{f(f^{-1}(C))} = \overline{C} = C$, we know that $f(\overline{D}) \subset C$. Thus we have $\overline{D} \subset f^{-1}(C) = D$, then we know that D is a closed set in X. So, f is continuous.

(ii) The statement is not true. We can give a counter example as following. We suppose $X = \mathbb{R}^+, Y = \mathbb{R}^+$ and $\forall x \in X, f(x) = \frac{1}{x}$. Then $f(x)$ is continuous in X. We set $A = [1, +\infty)$, and we have $A \subset X$. So, $\overline{A} = [1, +\infty) = A$, and we know that $f(\overline{A}) = (0, 1]$. Since $f(A) = (0, 1]$, we have $\overline{f(A)} = [0, 1]$. Thus $f(\overline{A}) \subsetneqq \overline{f(A)}$, and we can not say $f(\overline{A}) = \overline{f(A)}$.

(iii) From the question (i), we know that if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$, we have f is continuous. Then, if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$, we have f is continuous.

Next we should verify if f is continuous, then for every subset A of X, $f(\overline{A}) = \overline{f(A)}$. By the result we get from question (i), we know that if f is continuous, then for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$. We just need to verify $\overline{f(A)} \subset f(\overline{A})$. Since $A \subset \overline{A}$, then $f(A) \subset f(\overline{A})$ and $\overline{f(A)} \subset \overline{f(\overline{A})}$. As $A \subset X$ and X is compact, then $\overline{A}$ is compact. As f is continuous, we have $\overline{f(\overline{A})} = f(\overline{A})$. So we can get $\overline{f(A)} \subset f(\overline{A})$. In summary, when f is continuous, we have $f(\overline{A}) \subset \overline{f(A)}$ and $\overline{f(A)} \subset f(\overline{A})$. Thus if f is continuous, for every subset A of X, we have $f(\overline{A}) = \overline{f(A)}$.

To sum up, we showed that f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.

Exercise 2:

Let $K \subset \mathbb{R}$ have finite measure and let $f \in L^\infty(\mathbb{R})$. Show that the function F defined by

$$F(x) := \int_K f(x+t)\,dt$$

is uniformly continuous on $\mathbb{R}$.

Solution:

We want to show that $\forall \epsilon > 0$, there exists a $\delta > 0$, such that when $|x-y| < \delta$, we have $|F(x) - F(y)| < \epsilon$. We verify the result by definition. Since

$$|F(x) - F(y)| = \Big| \int_K f(x+t)\,dt - \int_K f(y+t)\,dt \Big|,$$

we change the variable and denote $K_1 = \{k + x | k \in K\}$ and $K_2 = \{k + y | k \in K\}$, then we have

$$|F(x) - F(y)| = \Big| \int_{K_1} f(t)\,dt - \int_{K_2} f(t)\,dt \Big|.$$

We denote $\operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| = C$. Since $f \in L^\infty(\mathbb{R})$, then $\forall \epsilon > 0$, there exist a positive number M such that

$$\int_{K_1 \bigcap [-M,M]^c} |f(t)|\,dt < \epsilon.$$

Otherwise, $\exists \epsilon > 0$, and $\forall M > 0$, we have $\int_{K_1 \bigcap [-M,M]^c} |f(t)|\,dt \geq \epsilon$. We set $M \to +\infty$, then $\int_{K_1 \bigcap [-M,M]^c} f(t)\,dt < C\mu\{K_1 \bigcap [-M,M]^c\} \to 0$. It is contradiction. So, for all $\epsilon > 0$, there exist a M, such that

$$\begin{aligned}
|F(x) - F(y)| &= \Big| \int_{K_1} f(t)\,dt - \int_{K_2} f(t)\,dt \Big| \\
&= \Big| \int_{K_1 \bigcap [-M,M]} f(t)\,dt + \int_{K_1 \bigcap [-M,M]^c} f(t)\,dt \\
&\quad - \int_{K_2 \bigcap [-M,M]} f(t)\,dt - \int_{K_2 \bigcap [-M,M]^c} f(t)\,dt \Big| \\
&\leq \Big| \int_{K_1 \bigcap [-M,M]} f(t)\,dt - \int_{K_2 \bigcap [-M,M]} f(t)\,dt \Big| + 2\epsilon.
\end{aligned}$$

We denote $S = (K_1 \bigcap [-M,M]) \Delta (K_2 \bigcap [-M,M])$, then we have

$$|F(x) - F(y)| \leq \int_S |f(t)|\,dt + 2\epsilon \leq C\mu\{S\} + 2\epsilon.$$

As $K_1 \bigcap [-M, M]$ and $K_2 \bigcap [-M, M]$ are finite, and $K_1 = \{k+x | k \in K\}$, $K_2 = \{k+y | k \in K\}$, we can cover the set S by several open sets whose measure is $|y - x|$, then we have

$$|F(x) - F(y)| \leq Cm|y - x| + 2\epsilon,$$

where C is a positive number. We set $\delta = \frac{\epsilon}{Cm}$, then we have

$$|F(x) - F(y)| \leq 3\epsilon,$$

so, $F(x)$ is uniformly continuous on $\mathbb{R}$.

Exercise 3:

Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ such that $f_n \to 0$ a.e.

(i) Show that if $\{f_{2n}\}$ is increasing and $\{f_{2n+1}\}$ is decreasing, then

$$\int f_n \to 0.$$

(ii) Prove or disprove: if $\{f_{kn}\}$ is decreasing for every prime number k, then

$$\int f_n \to 0.$$

(Note on notation: e.g., if $k = 2$, then $\{f_{kn}\} = \{f_{2n}\}$. Note also that 1 is not prime).

Solution:

(i) Firstly, we consider the sequence $\{f_{2n} - f_2\}$. Since $\{f_{2n}\}$ is increasing, $f_{2n} \to 0$ and $\{f_n\} \in L^1(\mathbb{R})$ for all n, then $\{f_{2n} - f_2\}$ is increasing and $f_{2n} - f_2 \to -f_2$ a.e., then by the monotone convergence theorem, we have

$$\lim_{n \to +\infty} \int (f_{2n} - f_2) = \int \lim_{n \to +\infty} (f_{2n} - f_2) = \int -f_2,$$

then we have

$$\lim_{n \to +\infty} \int f_{2n} = 0.$$

Similarly, as $\{f_{2n+1}\}$ is decreasing, we know that $\{f_1 - f_{2n-1}\}$ is a increasing sequence and $f_1 - f_{2n-1} \to f_1$ a.e., by the monotone convergence theorem, we have

$$\lim_{n \to +\infty} \int (f_1 - f_{2n-1}) = \int \lim_{n \to +\infty} (f_1 - f_{2n-1}) = \int f_1,$$

then we have

$$\lim_{n \to +\infty} \int f_{2n-1} = 0.$$

Then we show that for any subsequence of $\{\int f_n\}$, which denoted as $\{\int f_{n_k}\}$, we can find a subsequence of $\{\int f_{n_k}\}$, which is denoted as $\{\int f_{n_{k_l}}\}$, and we have

$$\lim_{n\to+\infty}\int f_{n_{k_l}} = 0.$$

For the subsequence $\{\int f_{n_k}\}$, we take the even number in the indicator set n_k if it is infinite, or we can take the odd number in the indicator set n_k if it is infinite, then we can get the subsequence of $\{\int f_{n_k}\}$, which is denoted as $\{\int f_{n_{k_l}}\}$. Since we have showed that $\lim_{n\to+\infty}\int f_{2n} = 0$ and $\lim_{n\to+\infty}\int f_{2n-1} = 0$, then we know that $\lim_{n\to+\infty}\int f_{n_{k_l}} = 0$. So, we know that

$$\int f_n \to 0.$$

(ii) The statement is not true. We can find a counter example as follows. We define

$$f_p(x) = p\,\mathbb{I}_{[0,\frac{1}{p}]}(x),$$

where p is a prime number and

$$f_m(x) = 2\,\mathbb{I}_{[0,\frac{1}{m}]}(x),$$

where m is a not prime number. Then we know that $\{f_{np}\}$ is decreasing for every prime number p. But we can find a subsequence of $\{f_n\}$, which is denoted as $\{f_p\}$, p is the prime number, and $\lim_{n\to+\infty}\int f_p \neq 0$ as

$$\lim_{p\to+\infty}\int f_p = \lim_{p\to+\infty}\int p\,\mathbb{I}_{[0,\frac{1}{p}]}(x)\,dx = 1.$$

13 GCE January, 2019

Exercise 1:

Let $E := [0,1] - S_{\mathbb{Q}} = [0,1] \bigcap (S_{\mathbb{Q}})^c$ where $S_{\mathbb{Q}} := \{x \in [0,1] | x = \frac{\sqrt{p}}{q} \text{ for some } p, q \in \mathbb{Z}^+\}$. Prove or disprove: There exists a closed, uncountable subset $F \subset E$.

Solution:

This proposition is true. Since $S_{\mathbb{Q}}$ is a countable set, there exists a bijection between $S_{\mathbb{Q}}$ and the positive rational number in the interval $[0,1]$, so we can enumerate the set $S_{\mathbb{Q}}$ as $\{a_n | n \in \mathbb{N}\}$. That is to say we have $S_{\mathbb{Q}} = \{a_n | n \in \mathbb{N}\}$. And then we consider the union

$$\bigcup_{n=1}^{+\infty} (a_n - \frac{\epsilon}{2^n}, a_n + \frac{\epsilon}{2^n}),$$

it is an open set, we denote it as A, then $A = \bigcup_{n=1}^{+\infty} (a_n - \frac{\epsilon}{2^n}, a_n + \frac{\epsilon}{2^n})$. And when $\epsilon \to 0$, we know that $A \subset [0,1]$ and $S_{\mathbb{Q}} \subset A$.

Since A is an open set, then $[0,1] \bigcap (A)^c$ is a closed set. We denote $F = [0,1] \bigcap (A)^c$, since the measure of set A is

$$m(A) = 2 \sum_{n=1}^{+\infty} \frac{\epsilon}{2^n} = 2\epsilon,$$

then we have $m(F) = 1 - 2\epsilon > 0$, so, the set F is uncountable. Since $F \subset E$ and it is both closed and uncountable, then the proposition is true.

For any countable set S, $S \subset [0,1]$, let $E = [0,1] - S$, we can find a closed, uncountable subset $F \subset E$, and we have the supremum of the measure of F is 1.

Exercise 2:

For x in $[-1,1]$ set $P_n(x) = c_n(1-x^2)^n$ where c_n is such that $\int_{-1}^{1} P_n = 1$.

(i) Show that there is a positive constant C such that $c_n \leq C\sqrt{n}$.

(ii) Let f be a real valued continuous function on $[0,1]$ such that $f(0) = f(1) = 0$. Set for x in $[0,1]$

$$f_n(x) = \int_0^1 P_n(x-t) f(t)\, dt$$

Show that f_n is uniformly convergence to f.

(iii) Let g be in $L^1((0,1))$. Defining $g_n(x) = \int_0^1 P_n(x-t) g(t)\, dt$, is g_n uniformly convergence to g in $(0,1)$? Does g_n converge to g in $L^1((0,1))$?

Solution:

(i) Method 1:

Since $\int_{-1}^{1} c_n(1-x^2)^n\,dx = 1$, then we have

$$c_n = \frac{1}{2\int_0^1 (1-x^2)^n\,dx}.$$

Next we need to find a lower bound of the integral term $\int_0^1 (1-x^2)^n\,dx$. Since for $n > 1$,

$$\begin{aligned}\int_0^1 (1-x^2)^n\,dx &\geq \int_0^{\frac{1}{\sqrt{n}}} (1-x^2)^n\,dx \\ &\geq \frac{1}{\sqrt{n}}(1-\frac{1}{n})^n,\end{aligned}$$

then we have $c_n \leq \frac{\sqrt{n}}{2(1-\frac{1}{n})^n}$. We just need to find a lower bound of $(1-\frac{1}{n})^n$. Since $(1-\frac{1}{n})^n = 1 - C_n^1\frac{1}{n} + C_n^2\frac{1}{n^2} + \cdots + (-\frac{1}{n})^n > \frac{1}{3} - \frac{2}{6n^2} > \frac{1}{4}$ as $n > 1$, then we set $C = 2$, we have $c_n \leq C\sqrt{n}$ for $n > 1$. For $n = 1$, we get $c_1 = \frac{3}{4} < 2$, then when $C = 2$, we have $c_n \leq C\sqrt{n}$ holds.

Method 2:

We change the element and define $x = \sin y$, then we have $\int_0^{\frac{\pi}{2}} c_n \cos^{2n+1} y\,dy = \frac{1}{2}$. Since

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1} y\,dy = 2n\int_0^{\frac{\pi}{2}} \cos^{2n-1} y\,dy - 2n\int_0^{\frac{\pi}{2}} \cos^{2n+1} y\,dy,$$

we denote $I_{2n+1} = \int_0^{\frac{\pi}{2}} \cos^{2n+1} y\,dy$, then we have $(2n+1)I_{2n+1} = 2nI_{2n-1}$. Since $I_1 = \int_0^{\frac{\pi}{2}} \cos y\,dy = 1$, we have $\int_0^{\frac{\pi}{2}} \cos^{2n+1} y\,dy = \frac{(2n)!!}{(2n+1)!!}$. And since

$$\begin{aligned}\frac{(2n)!!}{(2n+1)!!} &= \frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3} \\ &\geq \frac{\sqrt{2n+1}\sqrt{2n-1}\sqrt{2n-1}\sqrt{2n-3}\cdots\sqrt{3}\sqrt{1}}{(2n+1)(2n-1)\cdots 3} \\ &= \frac{1}{\sqrt{2n+1}},\end{aligned}$$

then we have $c_n \leq \frac{\sqrt{2n+1}}{2}$. We set $C = 1$, then we have $c_n \leq C\sqrt{n}$.

(ii) Firstly we extend $f(x)$ to a function from $\mathbb{R}$ to $\mathbb{R}$ by zero. Then we have

$$f_n(x) = \int_0^1 P_n(x-t)f(t)\,dt = \int_{\mathbb{R}} P_n(x-t)f(t)\,dt,$$

then we change the element as $x - t = y$, we have

$$f_n(x) = \int_{\mathbb{R}} P_n(y)f(x-y)\,dy.$$

Then we know that

$$\begin{aligned}
|f_n(x) - f(x)| &= \Big| \int_{\mathbb{R}} P_n(y) f(x-y)\, dy - \int_{-1}^{1} P_n(y) f(x)\, dy \Big| \\
&= \Big| \int_{-1}^{1} P_n(y)(f(x-y) - f(x))\, dy + \int_{([-1,1])^c} P_n(y) f(x-y)\, dy \Big| \\
&\leq \int_{-1}^{1} P_n(y) |(f(x-y) - f(x))|\, dy + \int_{([-1,1])^c} |P_n(y) f(x-y)|\, dy.
\end{aligned}$$

Since when $x \in [0,1]$ and $y \in ([-1,1])^c$, we have $x - y > 1$ or $x - y < 0$, then we have $f(x-y) = 0$, so we have

$$|f_n(x) - f(x)| \leq \int_{-1}^{1} P_n(y) |(f(x-y) - f(x))|\, dy.$$

And by the definition of continuous, we have $\forall \epsilon > 0$, there $\exists \delta$, when $|x - y - x| < \delta$, we have $|f(x-y) - f(x)| < \epsilon$. We denote $S = [-1,1] \bigcap [-\delta, \delta]$, since $f(x)$ is continuous in $\mathbb{R}$, we denote $\sup_{x \in [0,1]} f(x) = M$, then we have $M < +\infty$ and

$$\begin{aligned}
|f_n(x) - f(x)| &\leq \int_{-\delta}^{\delta} P_n(y) |(f(x-y) - f(x))|\, dy + \int_{S} P_n(y) |(f(x-y) - f(x))|\, dy \\
&\leq \epsilon \int_{-\delta}^{\delta} P_n(y)\, dy + 2M \int_S P_n(y)\, dy \\
&\leq \epsilon + 2M \int_S c_n (1 - y^2)^n\, dy \\
&\leq \epsilon + 4MC\sqrt{n} \int_{\delta}^{1} (1 - y^2)^n\, dy \\
&\leq \epsilon + 4MC\sqrt{n}(1 - \delta)(1 - \delta^2)^n.
\end{aligned}$$

Since $\lim_{n \to +\infty} 4MC\sqrt{n}(1-\delta)(1-\delta^2)^n = 0$, then we can say that there exists a $N \in \mathbb{N}$, when $n > N$, we have $4MC\sqrt{n}(1-\delta)(1-\delta^2)^n < \epsilon$. Overall, we know that $\forall x \in [0,1], \forall \epsilon > 0$, there exists a $N \in \mathbb{N}$, when $n > N$, we have $|f_n(x) - f(x)| < 2\epsilon$, so that f_n is uniformly converges to f.

(iii) Firstly, the $g_n(x)$ is not uniformly convergent to g in $(0,1)$, we can give an counter example as following. We define

$$g(x) = \begin{cases} 1, & x = \frac{1}{2} \\ 0, & x \in (0, \frac{1}{2}) \bigcup (\frac{1}{2}, 1), \end{cases}$$

obviously $g(x)$ is not continuous in $(0,1)$, but we have $g_n(x) = \int_0^1 P_n(x-t) g(t)\, dt = 0, \forall x \in (0,1)$. Then $g_n(x)$ is continuous in $[0,1]$. Since $g(x)$ is not continuous in $(0,1)$, we can say that $g_n(x)$ is not uniformly convergent to $g(x)$ in $(0,1)$.

Secondly, we can show that $g_n(x)$ convergent to $g(x)$ in $L^1((0,1))$. Since the continuous functions with compact support are dense in L^1 space, then for all $\epsilon > 0$, there exist a continuous function $f(x) \in C_c([0,1])$, such that $\|f-g\|_1 < \epsilon$. We define the $f_n(x)$ as the section (ii), then we have

$$\|g-g_n\|_1 \leq \|g-f\|_1 + \|f-f_n\|_1 + \|f_n-g_n\|_1.$$

Since f_n is uniformly converges to f, for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$, when $n > N$, we have $\|f-f_n\|_1 < \epsilon$. And for the same ϵ, by the property that continuous function is dense in L^1 space, we have $\|f-g\|_1 < \epsilon$. Next we verify that $\|f_n-g_n\|_1 < \epsilon$. Since

$$\begin{aligned}
\|f_n-g_n\|_1 &= \int_0^1 \Big| \int_0^1 P_n(x-t)g(t) - \int_0^1 P_n(x-t)f(t)\,dt \Big| dx \\
&= \int_0^1 \Big| \int_0^1 P_n(x-t)(g(t)-f(t))\,dt \Big| dx \\
&\leq \int_0^1 \int_0^1 P_n(x-t)|g(t)-f(t)|\,dtdx,
\end{aligned}$$

and $P_n(x-t)$ is continuous for $t \in [0,1]$, then we can find the upper bound for $P_n(x-t)$, we denote it as C, then we have

$$\begin{aligned}
\|f_n-g_n\|_1 &\leq \int_0^1 \int_0^1 P_n(x-t)|g(t)-f(t)|\,dtdx \\
&\leq C \int_0^1 \int_0^1 |g(t)-f(t)|\,dtdx \\
&= C \int_0^1 |g(t)-f(t)|\,dt \\
&= C\|g-f\|_1.
\end{aligned}$$

Since $\|g-f\|_1 < \epsilon$, we have $\|g-g_n\|_1 < (2+\frac{1}{C})\epsilon$ for all $\epsilon > 0$. So, we know that $g_n(x)$ convergent to $g(x)$ in $L^1((0,1))$.

Exercise 3:

Give an example of $f_n, f : \mathbb{R} \mapsto [0,\infty)$ such that $f_n \in L^1(\mathbb{R})$ for every $n \in \mathbb{N}$, $f \in L^2(\mathbb{R})$, $f_n \leq f$ for every $n \in \mathbb{N}$, $f_n \to 0$ a.e., and $\int f_n \nrightarrow 0$.

Solution:

We define the $f(x) = \frac{1}{x}\mathbb{I}_{[1,+\infty)}$ and $f_n(x) = \frac{1}{x}\mathbb{I}_{[n,n^2]}$. For a fixed n, we have

$$\int_{\mathbb{R}} |f_n(x)|\,dx = \int_n^{n^2} \frac{1}{x}\,dx = \ln n,$$

so we have $f_n \in L^1(\mathbb{R})$ for every $n \in \mathbb{N}$. And since

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = \int_1^{+\infty} \frac{1}{x^2} \, dx = 1,$$

so we know that $f \in L^2(\mathbb{R})$. Since for all n, f_n is just a part of f and $f > 0$, then we have $f_n \leq f$ for every $n \in \mathbb{N}$. When $n \to +\infty$, we have $f_n(x) \leq \frac{1}{n}$, so that $f_n \to 0$ almost everywhere. And we calculate the integral of f_n, we have

$$\int_{\mathbb{R}} f_n(x) \, dx = \int_n^{n^2} \frac{1}{x} \, dx = \ln n,$$

when $n \to +\infty$, $\ln n \to +\infty$, so we can get $\int f_n \nrightarrow 0$.

14 GCE May, 2019

Exercise1:

Let V be a normed vector space and S a subset of V. Let S^c be the complement of S. Let x be in S and y be in S^c. The line segment $[x, y]$ is by definition the set

$$\{(1-t)x + ty : t \in [0,1]\}.$$

Show that the intersection of $[x, y]$ and ∂S is non empty, where ∂S is the boundary of S (by definition the boundary of S is the set of points that are in the closure of S and that are not in the interior of S).

Solution:

We want to prove that the intersection of $[x, y]$ and ∂S is non empty, then we need to find a $t^* \in [0,1]$, $\forall \delta > 0$, $B((1-t^*)x+t^*y, \delta) \bigcap S \neq \varnothing$ and $B((1-t^*)x+t^*y, \delta) \bigcap S^c \neq \varnothing$, where $B((1-t^*)x + t^*y, \delta) = \{(1-t)x + ty : |t - t^*| < \delta, t \in [0,1]\}$. Then we need to find that t^*. We define

$$Z = \{t : (1-t)x + ty \in S, t \in [0,1]\},$$

and we denote $t^* = \sup Z$. And we denote $B((1-t^*)x + t^*y, \delta) = B_{t^*,\delta}$.

Firstly, we show that $B_{t^*,\delta} \bigcap S \neq \varnothing$. Since $t^* = \sup Z$, by the definition of t^* then we have $\forall \delta > 0, \exists \epsilon = \frac{\delta}{2}, \big(1 - (t^* - \epsilon)x\big) + (t^* - \epsilon)y \in S$. And since $|t^* - \epsilon - t^*| = \epsilon < \delta$, then $\big(1 - (t^* - \epsilon)x\big) + (t^* - \epsilon)y \in B_{t^*,\delta}$, such that $B_{t^*,\delta} \bigcap S \neq \varnothing$.

Secondly, we need verify that $B_{t^*,\delta} \bigcap S^c \neq \varnothing$. Suppose $B_{t^*,\delta} \bigcap S^c = \varnothing$, then we have that $B_{t^*,\delta} \subset S$. Since $t^* = \sup Z$, by the definition of t^* then we have $\forall \delta > 0, \exists \epsilon = \frac{\delta}{2}, \big(1 - (t^* + \epsilon)x\big) + (t^* + \epsilon)y \notin S$. And since $|t^* - \epsilon - t^*| = \epsilon < \delta$, then $\big(1 - (t^* + \epsilon)x\big) + (t^* + \epsilon)y \in B_{t^*,\delta}$. It is contradict with $B_{t^*,\delta} \subset S$, then we know that $B_{t^*,\delta} \bigcap S^c \neq \varnothing$.

Overall, we find $t^* \in [0,1]$, $(1-t^*)x+t^*y \in [x, y]$, $\forall \delta > 0$, we have $B_{t^*,\delta} \bigcap S \neq \varnothing$ and $B_{t^*,\delta} \bigcap S^c \neq \varnothing$, such that $(1-t^*)x + t^*y \in \partial S$. So, we conclude that the intersection of $[x, y]$ and ∂S is non empty.

Exercise2:

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let g be a measurable function defined on X. Set

$$p_g(t) = \mu(x \in X : |g(x)| > t).$$

(i) If f is in $L^1(X)$ show that there is a constant $C > 0$ such that $p_f(t) \leq \frac{C}{t}$.

(ii) Find a measurable function h defined almost everywhere on $\mathbb{R}$ such that $\exists C > 0$, $p_h(t) \leq \frac{C}{t}$ and h is not in $L^1(\mathbb{R})$.

Solution:

(i) Since $f \in L^1(X)$, then $\exists C > 0$, $\int_X |f| \, d\mu \leq C < +\infty$. We can decompose the integral as following:

$$\begin{aligned}
\int_X |f| \, d\mu &= \int_X |f|\mathbb{I}_{\{|f|>t\}} \, d\mu + |f|\mathbb{I}_{\{|f|\leq t\}} \, d\mu \\
&= \int_X |f|\mathbb{I}_{\{|f|>t\}} \, d\mu + \int_X |f|\mathbb{I}_{\{|f|\leq t\}} \, d\mu \\
&\geq \int_X |f|\mathbb{I}_{\{|f|>t\}} \, d\mu \\
&\geq t \int_X \mathbb{I}_{\{|f|>t\}} \, d\mu \\
&= tp_f(t).
\end{aligned}$$

Then we have $tp_f(t) \leq C$, such that $p_f(t) \leq \frac{C}{t}$.

(ii) We suppose that

$$h(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{|x|}, & x \neq 0, \end{cases}$$

then $h(x) \notin L^1(\mathbb{R})$ since $\frac{1}{x} \notin L^1([0, +\infty))$. Since

$$p_t(t) = \int_{\mathbb{R}} \mathbb{I}_{\{|h|>t\}} \, d\mu = \int_{\mathbb{R}} \mathbb{I}_{\{|x|<\frac{1}{t}\}} \, d\mu = \int_{\{|x|<\frac{1}{t}\}} d\mu,$$

hence we can set $C = 2$ and $p_h(t) \leq \frac{C}{t}$ and h is not in $L^1(\mathbb{R})$.

Exercise3:

Let $\{f_n\} : [0,1] \mapsto [0, \infty)$ be a sequence of functions, each of which is non-decreasing on the interval $[0,1]$. Suppose the sequence is uniformly bounded in $L^2([0,1])$. Show that there exists a sub sequence that converges in $L^1([0,1])$.

Solution:

Since f_n is non-decreasing, then for $x \in [0,1]$, we have $\int_x^1 f_n(y) \, dy \geq (1-x)f_n(x)$. On the other hand, since the sequence is uniformly bounded in $L^2([0,1])$, we have $\forall n \in \mathbb{N}$,

$\exists C > 0$, and $\|f_n\|_2 \le C$. And then we have

$$\begin{aligned} \int_x^1 f_n(y)\,dy &= \int_0^1 f_n(y)\mathbb{I}_{[x,1]}(y)\,dy \\ &\le \Big(\int_0^1 f_n^2(y)\,dy\Big)^{\frac{1}{2}}\Big(\int_0^1 \mathbb{I}_{[x,1]}^2(y)\,dy\Big)^{\frac{1}{2}} \\ &\le C(1-x)^{\frac{1}{2}}. \end{aligned}$$

Such that we have $(1-x)f_n(x) \le C(1-x)^{\frac{1}{2}}$, then $f_n(x) \le C(1-x)^{-\frac{1}{2}}$. Until now we find a type of function $f(x) = C(1-x)^{-\frac{1}{2}}$ that can control the sequence f_n, where C is from the bound of f_n in the $L^2([0,1])$.

Exercise4:

Consider the sequence of functions $f_n : [0,1] \mapsto \mathbb{R}$ where $f_1(x) = \sqrt{x}, f_2(x) = \sqrt{x+\sqrt{x}}, f_3(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$, and in general $f_n(x) = \sqrt{x+\sqrt{x+\sqrt{\cdots+\sqrt{x}}}}$ with n roots.

(i) Show that this sequence converges pointwise on $[0,1]$ and find the limit function f such that $f_n \to f$.

(ii) Does this sequence converge uniformly on $[0,1]$? Prove or disprove uniform convergence.

Solution:

(i) Firstly, we show that the sequence $f_n(x)$ is non-decreasing for the fixed x. We use the mathematical induction. For the fixed $x \in [0,1]$, when $k = 1$, since $f_k(x) = \sqrt{x}$ and $f_{k+1}(x) = \sqrt{x+\sqrt{x}}$, then $f_k(x) \le f_{k+1}(x)$. We suppose when $k = n-1$, the formula $f_k(x) \le f_{k+1}(x)$ holds, which is equivalent to $f_{n-1}(x) \le f_n(x)$. We want to verify $f_n(x) \le f_{n+1}(x)$. Since $f_n(x) = \sqrt{x + f_{n-1}(x)}$ and $f_{n+1}(x) = \sqrt{x+f_n(x)}$, when $f_{n-1}(x) \le f_n(x)$, we have $\sqrt{x+f_{n-1}(x)} \le \sqrt{x+f_n(x)}$, such that $f_n(x) \le f_{n+1}(x)$. So when $k = n$, the formula $f_k(x) \le f_{k+1}(x)$ can also hold. Thus we know that the sequence $f_n(x)$ is non-decreasing for the fixed x.

Then, we show that the sequence $f_n(x)$ is uniformly bounded. We also use the mathematical induction. When $k = 1$, $f_k(x) = \sqrt{x} < \sqrt{3}$. We suppose that when $k = n-1$, we have $f_k(x) < \sqrt{3}$. When $k = n$, $f_n(x) = \sqrt{f_{n-1}(x) + x} < \sqrt{\sqrt{3}+1} < \sqrt{3}$. Such that we get a uniform bound of sequence f_n.

Overall, since the sequence $f_n(x)$ is non-decreasing for the fixed x, and the sequence $f_n(x)$ has uniformly bound $\sqrt{3}$, then this sequence converges pointwise on $[0,1]$. We suppose the sequence $f_n(x)$ converges pointwise on $[0,1]$ to $f(x)$. Since $f_{n+1}(x) =$

$\sqrt{x + f_n(x)}$, when $n \to \infty$, we have $f(x) = \sqrt{x + f(x)}$. So we can get $f^2(x) - f(x) - x = 0$, such that $f(x) = \frac{1+\sqrt{1+4x}}{2}$ as $f(x) \geq 0$. When $x = 0$, we have $f_n(x) = 0, \forall n$. Then we have

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1+\sqrt{1+4x}}{2}, & x \in (0, 1]. \end{cases}$$

(ii) Since for all $n \in \mathbb{N}$, $f_n(x)$ is continuous, if the sequence $f_n(x)$ converge uniformly on $[0, 1]$ to $f(x)$, then $f(x)$ should be continuous. Since the $f(x)$ we get in (i) is not a continuous function, then this sequence $f_n(x)$ is not converge uniformly on $[0, 1]$.

Exercise5:

S is a normed space, and we define $B_1 = \{x \in S : \|x\| \leq 1\}$. Prove or disprove: B_1 is compact.

Solution:

The B_1 is not compact, we can find a counter example. We consider $S = l^2$ and $B_1 = \{x \in l^2 : \|x\| = 1\}$.

Firstly, we can show that B_1 is bounded and closed. By the definition of B_1, we know that B_1 is bounded by 1. $\forall x, y \in B_1$, since $\|x\| \leq \|x-y\| + \|y\|$ and $\|y\| \leq \|x-y\| + \|x\|$, we have $|\|x\| - \|y\|| \leq \|x - y\|$, such that the norm is continuous from l^2 to $\mathbb{R}$. Since the image set $\{1\}$ is closed, then we know the inverse image of $\{1\}$ is also closed, which is actually B_1. So, B_1 is bounded and closed.

Next, we verify that $\exists \epsilon > 0$, B_1 cannot be covered by finitely many balls with radius ϵ. We define e_i as follow:

$$e_{i,m} = \begin{cases} 1, & m = i \\ 0, & m \neq i \end{cases},$$

such that $e_i \in l^2$. Clearly, we have $\forall i, j$, if $i \neq j$, we have $\|e_i - e_j\| = \sqrt{2}$. Suppose B_1 can be covered by the finite balls with radius $\frac{\sqrt{2}}{2}$. Since $\{e_i\}_{i=1}^{+\infty}$ is infinity, hence at least one of such ball contains at least e_j and e_k with $j \neq k$. Let x be the center of this ball, then we have $\|e_j - e_k\| \leq \|e_j - x\| + \|e_k - x\| < \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$. It is contradict with $\forall k, j$, if $k \neq j$, we have $\|e_i - e_j\| = \sqrt{2}$. Hence $\exists \epsilon > 0$, B_1 cannot be covered by finitely many balls with radius ϵ. Then we know that B_1 is not compact.