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let $f : [0, \frac{\pi}{2}] \longrightarrow R$ be a continuous function

Find $\lim_{n \to +\infty} \int_0^{\frac{\pi}{2}} (\sin x)^n f(x)\, dx$

Almost everywhere in $[0, \frac{\pi}{2}]$

$$(\sin x)^n f(x) \longrightarrow 0 \quad \text{as} \quad n \to +\infty$$

$$|(\sin x)^n f(x)| \leq |f(x)| \quad \forall x \in [0, \tfrac{\pi}{2}]$$

but $f \in L^1([0, \frac{\pi}{2}])$

use D.C.T.

$$\lim_{n \to +\infty} \int_0^{\frac{\pi}{2}} (\sin x)^n f(x)\, dx = 0$$

The same argument is valid, if $f \in L^1([0, \frac{\pi}{2}])$

In particular,

$$\lim_{n \to +\infty} \int_0^{\frac{\pi}{2}} \frac{(\sin x)^n}{\sqrt{\frac{\pi}{2} - x}}\, dx = 0$$

Find

$$\lim_{n \to +\infty} \int_0^{\frac{\pi}{2} - \frac{1}{n}} \frac{(\sin x)^n}{\frac{\pi}{2} - x}\, dx$$

$$\forall x \in [0, \tfrac{\pi}{2}] \quad \left| \frac{(\sin x)^n \mathbb{1}_{[0, \frac{\pi}{2} - \frac{1}{n}]}}{\frac{\pi}{2} - x} \right| \leq \frac{1}{\frac{\pi}{2} - x}$$

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If $\mu(X) < +\infty$ and $f \in L^\infty(X)$

Show that $\lim_{p \to +\infty} \left(\int |f|^p \right)^{\frac{1}{p}} = \|f\|_\infty$

If $X = (0,1)$ $f: X \to \mathbb{R}$, measurable

s.t. $f \notin L^\infty(X)$

$|f|^p \in L^1(X) \quad \forall p > 1$

$f = \log|x|$ or $\log\left(\frac{1}{x}\right)$

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If $f \in L^1(\mathbb{R})$, $f*f \in L^1(\mathbb{R})$

Example: $f(x) = \frac{1}{\sqrt{|x|}} \mathbb{1}_{[-1,1]}$

$$f^2(x) = \frac{1}{|x|} \mathbb{1}_{[-1,1]} \notin L^1([-1,1])$$

By M.C.T. $\int_{-1}^{-\frac{1}{n}} \frac{dx}{|x|} + \int_{\frac{1}{n}}^{1} \frac{dx}{|x|} \longrightarrow \int f^2 = +\infty$

$$f*f \in L^1(\mathbb{R})$$

$$f*f(x) = \int_{\mathbb{R}} f(x-y) f(y)\, dy.$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{|x-y|}} \frac{1}{\sqrt{|y|}} \mathbb{1}_{[-1,1]}(x-y)\, dy$$

lemma: Let $f, g : \mathbb{R}^d \longrightarrow [0, +\infty]$ be measurable

Define $f*g(x) = \int_{\mathbb{R}^d} f(x-y) g(y)\, dy$

Then $x \longrightarrow f*g(x)$ is measurable

$d = 1$

if A and B are open intervals

Then $\mathbb{1}_A * \mathbb{1}_B$ is measurable

if A is open in $\mathbb{R}$, A is a countable disjoint union of open intervals.

Thanks to the M.C.T.

$\mathbb{1}_A * \mathbb{1}_B$ is measurable if A is open and B is an open interval

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Now let A be a Lebesgue measurable subset of $\mathbb{R}$

Show that there is a decreasing sequences open set V_n such that $A \subset V_n$

$$m\left(\bigcap_{n=1}^{\infty} V_n \setminus A\right) = 0$$

to $\frac{1}{n}$, we know that there is an open subset W_n in $\mathbb{R}$ and closed subset F_n in $\mathbb{R}$ such that

$$F_n \subset A \subset W_n$$

and $m(W_n \setminus F_n) < \frac{1}{n}$

in particular $m(W_n \setminus A) < \frac{1}{n}$

Set $V_n = \bigcap_{k=1}^{n} W_k$

V_n is a decreasing sequence of subsets

V_n is open

$m(V_n \setminus A) < \frac{1}{n}$

$V_n \setminus A \subset V_1 \setminus A$ and $m(V_1 \setminus A) < 1$

by the decreasing set property,

$$m\left(\bigcap_{n=1}^{\infty} V_n \setminus A\right) = \lim_{n \to +\infty} m(V_n \setminus A) = 0$$

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Thus,

$$1_B(x-y)\,1_{V_n}(y) \longrightarrow 1_B(x-y)\,1_A(y)\quad, \text{a.e}$$

$$0 \leq 1_B(x-y)1_{V_n}(y) \leq 1_B(x-y)1_{V_1}(y)$$

Thus, $\int 1_B(x-y)1_{V_n}(y) \longrightarrow \int 1_B(x-y)1_A(y)\,dy$

Thus $1_A * 1_B$ is measurable

For all Lebesgue measurable sets A and all open intervals B

Repeat the same argument to obtain that $1_A * 1_B$ is measurable if A and B are any 2 Lebesgue measurable subsets of $\mathbb{R}$.

Apply linearity,
if $f * g : \mathbb{R} \longrightarrow [0,+\infty]$
are simple functions
$f * g$ is measurable

Finally, to obtain the result for any measurable functions $f, g\ \mathbb{R} \longrightarrow [0,+\infty]$ (simple approximation functions)

use f_n, g_n an increasing sequences of simple functions

converging to f and g and apply the MCT

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Integrability of convolution products.

Proposition: Let $f \in L^1(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$ $p = 1, 2, \text{ or } +\infty$

then $f * g \in L^p(\mathbb{R}^d)$

and $\|f*g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$

Case 1: $p=1$ assume that f and g are valued in $[0, +\infty]$

$$f*g\,(x) = \int f(x-y)g(y)\,dy \geqslant 0$$

Thus

$$\|f*g\|_{L^1} = \int \int f(x-y)g(y)\,dy\,dx$$

Fubbini's Theorem $= \int g(y)\left(\int f(x-y)\,dx\right)dy$

$$= \left(\int g\right)\cdot\left(\int f\right) = \|f\|_{L^1}\|g\|_{L^1}$$

Now if f, g are in $L^1(\mathbb{R}^d)$

$$|f*g(x)| \leq \int |f(x-y)g(y)|\,dy = \int |f|(x-y)|g|(y)\,dy$$

$$= |f|*|g|(x)$$

but we have shown that

$$|f|*|g| \in L^1(\mathbb{R}^d)$$

Thus $f*g \in L^1(\mathbb{R}^d)$

and $\|f*g\|_{L^1} \leq \| |f|*|g| \|_{L^1} = \||f|\|_{L^1} \||g|\|_{L^1}$

$$= \|f\|_{L^1}\|g\|_{L^1}$$

Case 2: $p=2$

First assume that f and g are valued in $[0, +\infty]$

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$$\left(\int f(x-y)\,g(y)\,dy\right)^2 = \left(\int f(x-y)g(y)dy\right)\left(\int f(x-y')g(y')dy'\right)$$

Fubbini's Theorem $= \iint f(x-y)f(x-y')g(y)g(y')\,dy\,dy'$

$$= \iint g(x-y)\,g(x-y')\,f(y)\,f(y')\,dy\,dy'$$

Thus $\int\left((f*g)(x)\right)^2 dx =$

Fubini's) $\iint f(y)f(y')\int\left(g(x-y)\,g(x-y')\,dx\right)dy\,dy'$

By Cauchy schwautz inequality

$$\leq \iint f(y)\,f(y')\left(\int g^2(x-y)\,dx\right)^{\frac{1}{2}}\left(\int g^2(x-y')\,dx\right)^{\frac{1}{2}}dy\,dy'$$

$$= \|g\|_{L^2}^2 \iint f(y)f(y')\,dy\,dy'$$

$$= \|g\|_{L^2}^2\,\|f\|_{L^1}^2$$

Thus $\|f*g\|_{L^2} \leq \|g\|_{L^2}\|f\|_{L^1}$

if $f\in L^1(\mathbb{R}^d)$, $g\in L^2(\mathbb{R}^d)$ and f and g are valued on $[0,+\infty]$

if f, g are valued in $\overline{\mathbb{R}}$

as previously $|(f*g)(x)| \leq |f| * |g| \ (x)$

so $|f*g(x)|^2 \leq \left(|f|*|g|\,(x)\right)^2$

Thus shows that

$f*g\in L^2(\mathbb{R}^d)$

and $\|f*g\|_{L^2} \leq \||f|*|g|\|_{L^2} \leq \||f|\|_{L^1}\,\||g|\|_{L^2}$

$$= \|f\|_{L^1}\,\|g\|_{L^2}$$

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case $f \in L^1(\mathbb{R}^d)$ $g \in L^\infty(\mathbb{R}^d)$

$$|f * g(x)| \leq \int |g(x-y)||f(y)|\,dy$$

$$\leq \|g\|_\infty \int |f(y)|\,dy = \|g\|_\infty \|f\|_{L^1}$$

So $\|f * g\|_{L^\infty} \leq \|g\|_{L^\infty} \|f\|_{L^1}$

We proved that for $f, g : \mathbb{R}^d \to [0, +\infty]$ measurable

$$f * g = g * f$$

$$(f * g) * h = f * (g * h)$$

If f and g are in $L^1(\mathbb{R}^d)$

$$f * g = (f^+ - f^-) * g$$

$$= f^+ * g - f^- * g$$

$$= f^+ * (g^+ - g^-) - f^- * (g^+ - g^-)$$

$$= f^+ * g^+ - f^+ * g^- - f^- * g^+ + f^- * g^-$$

Lemma: Let f be in $C_c(\mathbb{R}^d)$ and g be in $L^p(\mathbb{R}^d)$ $p = 1, 2$, or $+\infty$

Then $f * g \in L^p(\mathbb{R}^d) \cap C(\mathbb{R}^d)$

In fact, $f * g$ is uniformly continuous in $\mathbb{R}^d$

proof. Introduce $K = \overline{\{x \in \mathbb{R}^d,\ f(x) \neq 0\}}$

Since $f \in C_c(\mathbb{R}^d)$, K is compact.

$$|f| \leq 1_K \sup |f|$$

so $f \in L^1(\mathbb{R}^d)$

and $f * g \in L^p(\mathbb{R}^d)$

As $f \in C_c(\mathbb{R}^d)$, f is uniformly continuous in $\mathbb{R}^d$.

Fix $\varepsilon > 0$, $\exists \alpha > 0$ $\forall u, v \in \mathbb{R}^d$

$$|u - v| < \alpha \Rightarrow |f(u) - f(v)| < \varepsilon$$

let x, z be in $\mathbb{R}^d$ such that $|x - z| < \alpha$

<u>case $p=1$</u>:

$$|f*g(x) - f*g(z)|$$

$$= \left| \int f(x-y) - f(z-y)\, g(y)\, dy \right|$$

$$\leq \sup_{y \in \mathbb{R}^d} |f(x-y) - f(z-y)| \int |g(y)|\, dy < \varepsilon \|g\|_{L^1}$$

<u>case $p=2$</u>:

$$|f*g(x) - f*g(z)|$$

$$\leq \int |f(x-y) - f(z-y)|\, |g(y)|\, dy$$

Cauchy schwartz inequality

$$\leq \left(\int |f(x-y) - f(z-y)|^2 dy \right)^{\frac{1}{2}} \|g\|_{L^2}$$

$$< \varepsilon \left[m(K + B(0, \alpha)) \right]^{\frac{1}{2}} \|g\|_{L^2}$$

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$p = +\infty$

$$\begin{aligned} & |f*g(x) - f*g(z)| \\ \le & \int |f(x-y) - f(z-y)|\,|g(y)|\,dy \\ \le & \|g\|_\infty \int |f(x-y) - f(z-y)|\,dy \\ \le & \varepsilon \|g\|_\infty \left[m\left(K + B(0,\alpha)\right)\right] \end{aligned}$$

Notations.

$C(\mathbb{R}^d) = \{$ continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$ $\}$

$C_c(\mathbb{R}^d) = \{$ continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$, with compact support $\}$

$C^q(\mathbb{R}^d) = \{$ functions f from $\mathbb{R}^d$ to $\mathbb{R}$ such that $\forall\, j \in \mathbb{N}^d$ with $|j| \le q$, $D_j f$ exists in $\mathbb{R}^d$ and is continuous in $\mathbb{R}^d$ $\}$

where $j = (j_1, j_2, \cdots j_d)$ in $\mathbb{N}^d$

$|j| = j_1 + \cdots + j_d$

$D_j = (\partial_1)^{j_1} \cdots (\partial_d)^{j_d}$

$C_c^q(\mathbb{R}^d) = \{$ functions in $C^q(\mathbb{R}^d)$ with compact support $\}$

$C^\infty(\mathbb{R}^d) = \bigcap_{q=1}^{\infty} C^q(\mathbb{R}^d)$

$C_c^\infty(\mathbb{R}^d) = \{ f$ in $C^\infty(\mathbb{R}^d)$ with compact support $\}$

If V is any open subset of $\mathbb{R}^d$, we define similarly $C(V)$, $C_c(V)$, ...

Show that $C_c^\infty(\mathbb{R}^d)$ is non-empty

If $d=1$ set $f(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } |x|<1 \\ 0 & \text{o.w.} \end{cases}$

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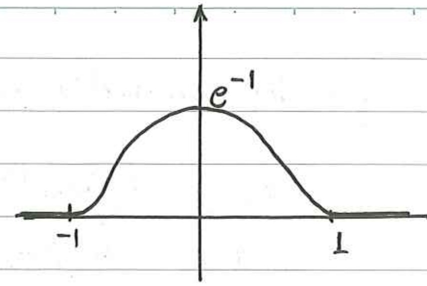
If $|x|<1$ $f^{(q)}(x)=e^{\frac{1}{x^2-1}} R_q(x)$

where $R_q(x)$ is a rational fraction which is continuous in $(-1,1)$



Fix q in $\mathbb{N}$ $\lim_{\substack{x\to 1\\ x<1}} \frac{e^{\frac{1}{x^2-1}}}{(x-1)^2}=0$

if $d\geqslant 2$ set $f(x)=\begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x|<1 \\ 0 & \text{if } |x|\geqslant 1\end{cases}$

Proposition:

Let f be in $C_c^q(\mathbb{R}^d)$ and $g\in L^p(\mathbb{R}^d)$

Then

$$f*g(\mathbb{R}^d)\in C^q(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$$

and for j in $\mathbb{N}^d$ such that $|j|\leq q$

$$D_j(f*g)=(D_jf)*g$$

proof: if $q=1$

$f\in L^1(\mathbb{R}^d)$ $\partial_j f\in C_c^1(\mathbb{R}^d)$, thus $\partial_j f\in L^1(\mathbb{R}^d)$

$$f*g(x)=\int f(x-y)g(y)\,dy$$

if $p=1$ Clearly $y\longrightarrow f(x-y)g(y)$ and $y\longrightarrow \partial_j f(x-y)g(y)$ are in $L^1(\mathbb{R}^d)$

Now $|\partial_j f(x-y)g(y)|\leq \sup_{\mathbb{R}^d}|\partial_j f||g(y)|$

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Thus by dominated convergence

$f * g$ is differentiable in $\mathbb{R}^d$

and $\partial_j (f * g) = (\partial_j f) * g$

Since $\partial_j f \in C_c(\mathbb{R}^d)$, due to the previous proposition

$(\partial_j f) * g$ is continuous in $\mathbb{R}^d$

if $p=2$ Fix $x_0 \in \mathbb{R}^d$, let x be in $B(x_0, \alpha)$

$$|\partial_j f(x-y) g(y)| \leq |g(y)| \, 1_{(-K + B(x_0, \alpha))}(y)$$

same conclusion

case $p = +\infty$ Fix $x_0 \in \mathbb{R}^d$ For x in $B(x_0, \alpha)$

$$|\partial_j f(x-y) g(y)| \leq \|g\|_\infty \, 1_{(-K + B(x_0, \alpha))}$$

Repeat the same argument to obtain the result for any q

Corollary If $f \in C_c^\infty(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$

then $f * g \in C^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$

and $\forall j \in \mathbb{N}^d$, $D_j(f * g) = (D_j f) * g$

Theorem Let V be a (non empty) open subset of $\mathbb{R}^d$

Then $C_c^\infty(V)$ is dense in $L^p(V)$ if $p = 1, 2,$

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proof Fix $f \in L^p(V)$

We proved that $\exists\, g \in C_c(V)$ such that $\|f-g\|_{L^p(V)} < \varepsilon$

Set $h(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x|<1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$

Set $\int h > 0$

Define $h_n(x) = \dfrac{n^d h(nx)}{\int h}$

set $z = nx$

$$\int h_n(x) = \frac{\int h(z)\,dz}{\int h} = 1$$

$h_n(x) = 0$ if $|x| \geq \frac{1}{n}$

$g \in C_c(\mathbb{R}^d)$ Set K be a compact set such that

$\forall x \in V \setminus K \quad g(x) = 0$

$d(K, V^c) > 0$.

If n is such that $\frac{1}{n} < d(K, V^c)$

then $h_n * g = \int_{\mathbb{R}^d} h_n(x-y)\, g(y)\, dy$

is a function which is also compactly supported on V.

its support is in $K + B(0, \frac{1}{n})$

According to the previous proposition,

$h_n * g \in C^\infty(\mathbb{R}^d)$

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We first show that

$$h_n * g \longrightarrow g \text{ uniformly in } \mathbb{R}^d$$

Since g is continuous on the compact set K
it is uniformly continuous on K.
g is uniformly continuous on $\mathbb{R}^d$

$\exists \alpha > 0 \quad \forall u, v \in \mathbb{R}^d$
$|u-v| < \alpha \Rightarrow |g(u) - g(v)| < \varepsilon$

$$|h_n * g(x) - g(x)| = \left| \int g(x-y) h_n(y) dy - \int h_n(y) g(x) dy \right|$$

since $\int h_n = 1$

$$\leq \underbrace{\int_{|y|<\alpha} |g(x-y) - g(x)| |h_n(y)| dy}_{I_1} + \underbrace{\int_{|y|>\alpha} |g(x-y) - g(x)| |h_n(y)| dy}_{I_2}$$

$$I_1 \leq \int_{|y|<\alpha} \varepsilon h_n(y) dy < \varepsilon$$

$$I_2 \leq 2 \sup |g| \int_{|y| \geq \alpha} \frac{n^d h(ny)}{(\int h)} dy$$

$z = ny \qquad \leq 2 \sup |g| \int_{|z| \geq n\alpha} \frac{h(z)}{(\int h)} dz$

$$= 0 \quad \text{if } n > \frac{1}{\alpha}$$

$$h_n * g \longrightarrow g \text{ uniformly in } V$$

Dominance

$$|h_n * g(x)|^p \leq \underbrace{\left(2 \max_V |g|\right)^p \mathbb{1}_{K+B(0,\frac{1}{m})}}_{\in L^1(V)}$$

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By D.C.T. $h_n * g \longrightarrow g$ in $L^p(V)$ $p=1,2,$

so $\exists\, N \in \mathbb{N}$

$$\| h_N * g - g \|_{L^p(V)} < \varepsilon$$

By triangle inequality $\| f - h_N * g \|_{L^p(V)} < 2\varepsilon$

Application

Let f be in $L^1([a,b])$

Show that $\lim\limits_{n \to +\infty} \int_a^b \sin(nx) f(x)\,dx = 0$

Fix $\varepsilon > 0$. Let g be in $C_c^\infty([a,b])$ s.t. $\int_a^b |f-g| < \varepsilon$

$$\left| \int_a^b \sin(nx)\, g(x)\,dx \right| = \left| -\int_a^b \frac{-\cos(nx)}{n} g'(x)\,dx \right|$$

$$\leq \frac{1}{n} \int_a^b |g'| \longrightarrow 0 \text{ as } n \longrightarrow +\infty$$

Thus $\exists N$ s.t. $\forall n > N$

$$\left| \int_a^b \sin(nx)\, g(x)\,dx \right| < \varepsilon$$

for $n > N$, $\left| \int_a^b \sin(nx) f(x)\,dx \right| \leq \left| \int_a^b \sin(nx)\big(f(x) - g(x) \quad\big)\,dx \right|$

$$+ \int_a^b \sin(nx)\, g(x)\,dx \leq \int_a^b |f-g| + \varepsilon = 2\varepsilon$$

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Application to Fourier Series in $L^2([-\pi,\pi])$

In this section, functions will be valued in $\mathbb{C}$

$f \in L^2([-\pi,\pi])$ if, by definition

Ref and Imf are in $L^2([-\pi,\pi])$

Let f and g be in $L^2([-\pi,\pi])$.

Define $\langle f, g\rangle = \int_{-\pi}^{\pi} f\bar{g}$

set $\|f\|_2 = \langle f, f\rangle^{\frac{1}{2}}$

lemma: We have the Cauchy schwartz inequality for $f, g \in L^2([-\pi,\pi])$

$$|\langle f, g\rangle| \leq \|f\|_2 \|g\|_2$$

proof: for any λ in $\mathbb{C}$,

$$0 \leq \langle f+\lambda g, f+\lambda g\rangle = \int_{-\pi}^{\pi}|f|^2 + \lambda\int_{-\pi}^{\pi} g\bar{f} + \bar{\lambda}\int_{-\pi}^{\pi} f\bar{g} + |\lambda|^2\int_{-\pi}^{\pi}|g|^2$$

$$= \|f\|_2^2 + 2\,\mathrm{Re}\left(\lambda\cdot\int f\bar{g}\right) + |\lambda|^2\|g\|_2^2$$

Set $\int f\bar{g} = \rho e^{i\theta}$ where $\rho \geq 0$ and $\theta \in [0, 2\pi)$

set $\lambda = a e^{-i\theta}$, where $a \in \mathbb{R}$

$$0 \leq \|f\|_2^2 + 2a\cdot\rho + a^2\|g\|_2^2 \quad \text{for all } a \text{ in } \mathbb{R}$$

thus $(2\rho)^2 - 4\|f\|_2^2\|g\|_2^2 \leq 0$

Thus $\rho^2 \leq \|f\|_2^2 \|g\|_2^2$

and $\rho \leq \|f\|_2 \|g\|_2$, where $\rho = \left|\int_{-\pi}^{\pi} f\bar{g}\right|$

Corollary $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$

because $\|f+g\|_2^2 = \langle f+g, f+g\rangle \leq$ (Apply Cauchy schwartz inequality

Remark: If f and g are orthogonal, that is, if $\int_{-\pi}^{\pi} f\bar{g} = 0$

then $\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$

Proposition: Set $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ $n \in \mathbb{Z}$

The e_n's from an orthonormal system in $L^2(-\pi, \pi)$

In other words,

$$\langle e_n, e_m\rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

proof: $\langle e_n, e_m\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}\, dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x} dx$$

is 1 if $n = m$

if $n \neq m$

$$= \frac{1}{2\pi}\left[\frac{e^{i(n-m)x}}{i(n-m)}\right]_{-\pi}^{\pi} = 0$$

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Theorem. The vector space spanned by the e_n, $n \in \mathbb{Z}$ is dense in $L^2(-\pi, \pi)$

prof. Step 1: let f be in $C_c^\infty(-\pi, \pi)$

Set $c_n = \langle f, e_n \rangle$ The n^{th} Fourier coefficients of f

Show that $f(x) = \sum_{n=-\infty}^{+\infty} c_n e_n(x)$

$$\sum_{n=-N}^{N} \frac{c_n e^{inx}}{\sqrt{2\pi}} - f(x) \quad \text{pointwise}$$

convolution

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} e^{in(x-t)} f(t)\, dt - f(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (f(x-t) - f(x)) \sum_{n=-N}^{N} e^{int}\, dt$$

But $$\sum_{n=-N}^{N} e^{int} = \sum_{n=-N}^{N} (e^{it})^n = (e^{it})^{-N} \sum_{n=0}^{2N} (e^{it})^n$$

$$= (e^{it})^{-N} \frac{1 - (e^{it})^{2N+1}}{1 - e^{it}} \quad \text{if } t \neq 0.$$

$$= \frac{e^{-iNt} - e^{i(N+1)t}}{1 - e^{it}}$$

$$= \frac{e^{-i(N+\frac{1}{2})t} - e^{i(N+\frac{1}{2})t}}{e^{-it/2} - e^{it/2}} = \frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}$$

The integral becomes

$$\int_{-\pi}^{\pi} \frac{f(x-t) - f(x)}{\sin(\frac{t}{2})} \sin\left(N+\frac{1}{2}\right)t\, dt$$

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Explain that $t \longrightarrow \dfrac{f(x-t)-f(x)}{\sin(\frac{t}{2})}$ is in $L^1([-\pi,\pi])$

As $f \in C_c^\infty([-\pi,\pi])$

By Mean value theorem

$$|f(x-t)-f(x)| \leq \max_{[-\pi,\pi]} |f'|\,|t|$$

thus $t \longrightarrow \dfrac{f(x-t)-f(x)}{\sin(\frac{t}{2})}$ is in $L^\infty([-\pi,\pi])$

We have shown that $\sum_{n=-\infty}^{+\infty} c_n \dfrac{e^{nx}}{\sqrt{2\pi}}$ is pointwise convergent to $f(x)$

Show that the convergence is uniform

$$|c_n| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx\right|$$

$$= \left|-\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{-in}\,dx\right|$$

$$= \left|\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f''(x)\frac{e^{-inx}}{(-in)^2}\,dx\right|$$

$$\leq \frac{1}{n^2}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}|f''|$$

Thus, $\sum_{n=-\infty}^{\infty} \dfrac{c_n e^{inx}}{\sqrt{2\pi}}$ is uniformly convergent to $f(x)$ in $[-\pi,\pi]$

as $m([-\pi,\pi])$ is finite,

this implies that $\sum_{n=-\infty}^{+\infty} c_n e_n$ converges to f in $L^2([-\pi,\pi])$

Now. Let g be in $L^2([-\pi,\pi])$ and $\varepsilon > 0$

$\exists\ f \in C_c^\infty((-\pi,\pi))$ such that

$$\|f-g\|_{L^2} < \varepsilon$$

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There is a N in $\mathbb{N}$ and $2N+1$ coefficients in $\mathbb{C}$

$c_{-N}, \cdots, c_0, \cdots c_N$

such that

$$\left\|\sum_{n=-N}^{N} c_n e_n - f\right\|_2 < \varepsilon \quad \text{(By Triangle inequality)}$$

Thus $\left\| g - \sum_{n=-N}^{N} c_n e_n \right\|_2 < 2\varepsilon$

Corollary Let f be in $L^2(-\pi, \pi)$

Assume that $\langle f, e_n \rangle = 0 \quad \forall n \in \mathbb{Z}$

then $f = 0$ in $L^2(-\pi, \pi)$

proof: Set $\varepsilon > 0$

There is a trigometric polynomial $p_N = \sum_{n=-N}^{N} c_n e_n$

s.t $\|f - p_N\|_2 < \varepsilon$

But $\langle f, p_N \rangle = 0$

Thus $\|f - p\|_2^2 = \|f\|_2^2 + \|p_N\|_2^2$

so $\|f\|_2^2 < \varepsilon^2$ or $\|f\|_2 < \varepsilon$

As $\varepsilon > 0$ is arbitrary, $\|f\|_2 = 0$

Theorem: Plancherel's theorem

- Let f be in $L^2((-\pi, \pi))$, let $c_n = \langle f, e_n \rangle$

Then $\|f\|_2^2 = \sum_{n=-\infty}^{+\infty} |c_n|^2$

and $\lim_{N \to +\infty} \left\| f - \sum_{n=-N}^{N} c_n e_n \right\|_2 = 0$

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- More precisely, let f and g be in $L^2((-\pi,\pi))$

  Let $c_n = \langle f, e_n \rangle$

  $d_n = \langle g, e_n \rangle$

  Then $\langle f, g \rangle = \int_{-\pi}^{\pi} f\bar{g} = \sum_{n=-\infty}^{+\infty} c_n \bar{d}_n$

Let a_n, $n \in \mathbb{Z}$ be a sequence in $\mathbb{C}$

such that $\sum_{n=-\infty}^{+\infty} |a_n|^2 < +\infty$

then $\sum_{n=-\infty}^{+\infty} a_n e_n$ determines an element in $L^2((-\pi,\pi))$

proof: For f in $L^2((-\pi,\pi))$

$$\langle \sum_{n=-N}^{N} c_n e_n, f - \sum_{n=-N}^{N} c_n e_n \rangle$$

$$= \sum_{n=-N}^{N} c_n \underbrace{\langle e_n, f \rangle}_{\bar{c}_n} - \sum_{n=-N}^{N} |c_n|^2 = 0$$

$$\|f\|_2^2 = \|f - \sum_{n=-N}^{N} c_n e_n + \sum_{n=-N}^{N} c_n e_n\|_2^2$$

$$= \|f - \sum_{n=-N}^{N} c_n e_n\|_2^2 + \sum_{n=-N}^{N} |c_n|^2$$

We have obtained $\sum_{n=-N}^{N} |c_n|^2 \le \|f\|_2^2$

Thus $\sum_{n=-\infty}^{+\infty} |c_n|^2 \le \|f\|_2^2$

Set $F_N = \sum_{n=-N}^{N} c_n e_n$

Show that F_N is Cauchy in $L^2((-\pi,\pi))$

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If $p>q>1$

$$\|F_p - F_q\|_2^2 = \Big\| \sum_{q+1\le|n|\le p} c_n e_n \Big\|_2^2$$

$$= \sum_{q+1\le|n|\le p} |c_n|^2 < \varepsilon \quad \text{, for any fixed } \varepsilon \text{ if } p>q>M.$$

Since $\sum_{n=-\infty}^{+\infty} |c_n|^2$ converges

Thus F_N is Cauchy in $L^2((-\pi,\pi))$

F_N converges to some F in $L^2((-\pi,\pi))$

For fixed n in $\mathbb{Z}$, $\langle F, e_n\rangle = c_n = \langle f, e_n\rangle$

Thus $\langle F-f, e_n\rangle = 0$ for any $n\in\mathbb{Z}$

Thus $F = f$ in $L^2((-\pi,\pi))$

$$\|F_N\|_2^2 = \sum_{n=-N}^{N} |c_n|^2$$

$$\downarrow$$

$$\|F\|_2^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Now let $f, g \in L^2((-\pi,\pi))$

$c_n = \langle f, e_n\rangle$

$d_n = \langle g, e_n\rangle$

$$\sum_{n=-\infty}^{+\infty} |c_n d_n| \le \left(\sum_{n=-\infty}^{+\infty} |c_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=-\infty}^{+\infty} |d_n|^2\right)^{\frac{1}{2}}$$

Thus $\sum_{n=-\infty}^{+\infty} c_n d_n$ is absolutely convergent.

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$$\sum_{n=-N}^{N} c_n \bar{d}_n = \langle \sum_{n=-N}^{N} c_n e_n , \sum_{n=-N}^{N} d_n \rangle$$

$$= \langle f - f + \sum_{n=-N}^{N} c_n e_n , g - g + \sum_{n=-N}^{N} d_n e_n \rangle$$

$$= \langle f - \sum_{|n| \geq N+1} c_n e_n , g - \sum_{|n| \geq N+1} d_n e_n \rangle$$

$$\left| \sum_{n=-N}^{N} c_n \bar{d}_n - \langle f, g \rangle \right|$$

$$\leq \left| \langle f, \sum_{|n| \geq N+1} d_n e_n \rangle \right| + \left| \langle \sum_{|n| \geq N+1} c_n e_n , g \rangle \right|$$

$$+ \left| \langle \sum_{|n| \geq N+1} c_n e_n , \sum_{|n| \geq N+1} d_n e_n \rangle \right|$$

$$\leq \| f \|_2 \left(\sum_{|n| \geq N+1} |d_n|^2 \right)^{\frac{1}{2}} + \| g \|_2 \left(\sum_{|n| \geq N+1} |c_n|^2 \right)^{\frac{1}{2}}$$

$$+ \left(\sum_{|n| \geq N+1} |c_n|^2 \right)^{\frac{1}{2}} \left(\sum_{|n| \geq N+1} |d_n|^2 \right)^{\frac{1}{2}}$$

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