MA - 503 Midterm - Fall 2016
Write your name on each page that you are turning in.

Exercise 1

We define l^1 to be the space of real sequences u_n, $n \geq 1$, such that $\sum_{n=1}^{\infty} |u_n| < \infty$. Recall that l^1 is a normed space for the norm $\|u\|_1 = \sum_{n=1}^{\infty} |u_n|$. Prove that l^1 is complete.

Hint: Let u^k be a Cauchy sequence in l^1. First show that for a fixed n, $k \to u_n^k$ is Cauchy.

Exercise 2

Let V be a normed space and S a non-empty subset of V. Assume $\exists \alpha > 0$, $\forall x \in S$, $B(x, \alpha) \subset S$. Find, with proof, S.

Exercise 3

Denote by $l(I)$ the length of any interval I. Define m^* and m^{**} from $\mathcal{P}(\mathbb{R})$ to $[0, \infty]$ by setting

$$m^*(A) = \inf\{\sum_{k=1}^{\infty} l(I_k) : A \subset \bigcup_{k=1}^{\infty} I_k, \quad I_k \text{ is an open and bounded interval}\},$$

$$m^{**}(A) = \inf\{\sum_{k=1}^{\infty} l(J_k) : A \subset \bigcup_{k=1}^{\infty} J_k, \quad J_k \text{ is any interval}\}.$$

Prove that $m^* = m^{**}$.

Exercise 4

Let m^* be defined as in exercise 2. Let λ be a real number in $(0, \infty)$.

(i). For A in $\mathcal{P}(\mathbb{R})$ show that $m^*(\lambda A) = \lambda m^*(A)$.

(ii). Recall that E, a subset of $\mathbb{R}$, is measurable if, by definition

$$\forall A \in \mathcal{P}(\mathbb{R}), \quad m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Show that E is measurable if and only if λE is measurable.

Exercise 5

For each of the following statements, prove or disprove.

(i). Let (X, ρ) be a metric space, V an open subset of X, $f : X \to \mathbb{R}$ a function such

that the restriction of f to V and the restriction of f to V^c are continuous. Then f is continuous on X.
(ii). Let (X, ρ) be a metric space, V an open subset of X, such that V^c is also open in X. Let $f : X \to \mathbb{R}$ a function such that the restriction of f to V and the restriction of f to V^c are continuous. Then f is continuous on X.
(iii). Let $(Y, \mathcal{A}, \mu)$ be a measure space, A be in $\mathcal{A}$, $f : Y \to \mathbb{R}$ a function such that the restriction of f to A and the restriction of f to A^c are measurable. Then f is measurable on Y.