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Exercise 1:

Suppose that u is a real-valued function defined on $[0,1]$, that $u \geq 0$ and that $u \in L^1([0,1])$. Define $E_n := \{x \in [0,1] : n-1 \leq u(x) \leq n\}$ for each positive integer n. Show that

$$\sum_{n=1}^{\infty} n|E_n| < +\infty.$$

Solution:

As $u \in L^1([0,1])$ and $u(x) \geq 0$, we have

$$\int_0^1 |u(x)|\, dx = \int_0^1 u(x)\, dx < +\infty.$$

Let $A_n = \{x \in [0,1] : n-1 \leq u(x) < n\}$, $\forall n \in \mathbb{N}$, then we have $E_n \subset A_n \cup A_{n+1}$ and $|E_n| \leq |A_n| + |A_{n+1}|$. Thus

$$\sum_{n=1}^{\infty} n|E_n| \leq \sum_{n=1}^{\infty} n|A_n| + \sum_{n=1}^{\infty} n|A_{n+1}|.$$

We want to show that $\sum_{n=1}^{\infty} n|A_n| < \infty$ and $\sum_{n=1}^{\infty} n|A_{n+1}| < \infty$. Note that

$$A_n = \{x \in [0,1] : n-1 \leq u(x) < n\} = \{x \in [0,1] : n \leq u(x)+1 < n+1\},$$

then we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n|A_n| &= \sum_{n=1}^{\infty} \int_{A_n} n \leq \sum_{n=1}^{\infty} \int_{A_n} (u(x)+1)\, dx \\
&= \int_{\bigcup_{n=1}^{\infty} A_n} (u(x)+1)\, dx \\
&\leq \int_0^1 u(x)\, dx + \int_0^1 1\, dx \\
&< \infty.
\end{aligned}$$

And similarly, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n|A_{n+1}| &= \sum_{n=1}^{\infty} \int_{A_{n+1}} n \leq \sum_{n=1}^{\infty} \int_{A_{n+1}} u(x)\, dx \\
&= \int_{\bigcup_{n=1}^{\infty} A_{n+1}} u(x)\, dx \\
&\leq \int_0^1 u(x)\, dx < \infty.
\end{aligned}$$

Thus we have $\sum_{n=1}^{\infty} n|E_n| \leq \sum_{n=1}^{\infty} n|A_n| + \sum_{n=1}^{\infty} n|A_{n+1}| < +\infty$.

Exercise 2:

Show that a subset E of a metric space X is open if and only if there is a continuous real-valued function f on X such that $E = \{x \in X : f(x) > 0\}$.

Solution:

If there is a continuous real-valued function f on X such that $E = \{x \in X : f(x) > 0\}$, we want to show that E is an open set. Since $(0, +\infty)$ is an open subset of $\mathbb{R}$, $E = \{x \in X : f(x) > 0\} = f^{-1}((0, +\infty))$ is also an open set as f is continuous on X. We can also verify the statement by definition. Suppose $y \in E$, since $E = \{x \in X : f(x) > 0\}$, we have $f(y) > 0$. Since f in continuous on X, we know that there exists a δ such that when $d(x, y) < \delta$, then $|f(x) - f(y)| < f(y)$, which implies $-f(y) < f(x) - f(y) < f(y)$, hence we have $f(x) > 0$. Then we know that there exists a $\delta > 0$, when $x \in B_\delta(y)$, we have $f(x) > 0$. Thus for any $y \in E$, there exists a $\delta > 0$ such that $B_\delta(y) \subset E$. So we know that E is an open set.

On the other direction, we want to show that if $E \subset X$ is open, there exists a continuous function f on X such that $E = \{x \in X : f(x) > 0\}$. For $E \subset X$, let

$$f(x) = d(x, E^c) = \min\{d(x, y) : y \in E^c\},$$

where E^c is the complementary set of E. Then we have when $x \in E^c$, $f(x) = 0$ and when $x \in E$, $f(x) > 0$, so we have $E = \{x \in x : f(x) > 0\}$. Next we need to show f is continuous on X . Let $x, y \in X$ and p is the any point in E^c, then

$$d(x, p) \leq d(x, y) + d(y, p),$$

and so

$$d(x, E^c) \leq d(x, y) + d(y, p)$$

as $d(x, E^c)$ is the minimum. Then we have $d(y, p) \geq d(x, E^c) - d(x, y)$ for all $p \in E^c$, thus we can get that $d(y, E^c) \geq d(x, E^c) - d(x, y)$, which is equivalent to

$$d(x, E^c) - d(y, E^c) \leq d(x, y).$$

Similarly, we can change the position of x and y and get the inequality

$$d(y, E^c) - d(x, E^c) \leq d(x, y).$$

Thus we have for any $x, y \in X$,

$$|d(x, E^c) - d(y, E^c)| \leq d(x, y).$$

Let $\epsilon > 0$ be given, we can choose $\delta = \epsilon$ such that $|d(x, E^c) - d(y, E^c)| < d(x, y) = \epsilon$ whenever $d(x, y) < \delta$. Thus f is a continuous function on X.

Exercise 3:

Consider the sequence of functions $\{f_n\}$ defined on the non-negative reals: $[0, +\infty)$ where $f_n(x) = 2nxe^{-nx^2}$. Let g be a continuous and bounded function on $[0, +\infty)$ valued in $\mathbb{R}$.

(i) Find with proof

$$\lim_{n\to\infty} \int_0^\infty f_n(t)g(t)\,dt.$$

(ii) Define for x in $[0, +\infty)$,

$$g_n(x) = \int_0^\infty f_n(t)g(x+t)\,dt.$$

Assuming g is zero outside the interval $[0, M]$, where $M > 0$, does the sequence g_n have a limit in $L^1([0, +\infty))$?

(iii) If h is in $L^1([0, +\infty))$, define for x in $[0, +\infty)$,

$$h_n(x) = \int_0^\infty f_n(t)h(x+t)\,dt.$$

Show that h_n is measurable on $[0, +\infty)$ and is in $L^1([0, +\infty))$.

(iv) Find, if it exists, with proof, the limit of h_n in $L^1([0, +\infty))$.

Solution:

(i) Denote $y = nt^2$, we have

$$\int_0^\infty 2nte^{-nt^2} g(t)\,dt = \int_0^\infty e^{-y} g\Big(\sqrt{\frac{y}{n}}\Big)\,dy.$$

Since $g(x)$ is a continuous and bounded function on $[0, +\infty)$, suppose that $|g(x)| \le C$ for any $x \in [0, +\infty)$, then we know that $|e^{-y}g(\sqrt{\frac{y}{n}})| \le Ce^{-y}$ and $Ce^{-y} \in L^1([0, +\infty))$ as $\int_0^\infty |Ce^{-y}|\,dy = C < +\infty$. And for any fixed $y \in [0, +\infty)$, when $n \to \infty$, $g(\sqrt{\frac{y}{n}}) \to g(0)$ and then $e^{-y}g(\sqrt{\frac{y}{n}}) \to e^{-y}g(0)$. By the dominate convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty f_n(t)g(t)\,dt &= \int_0^\infty \lim_{n\to\infty} e^{-y} g\Big(\sqrt{\frac{y}{n}}\Big)\,dy \\
&= \int_0^\infty e^{-y} g(0)\,dy \\
&= g(0).
\end{aligned}$$

(ii) Since $f_n(x) = 2nxe^{-nx^2}$, denote $y = nt^2$, we have

$$g_n(x) = \int_0^\infty f_n(t)g(x+t)\,dt = \int_0^\infty e^{-y} g\Big(x + \sqrt{\frac{y}{n}}\Big)\,dy.$$

Next we want to show that g_n converges to g in $L^1([0,+\infty))$. Since

$$\begin{aligned}\int_0^\infty |g_n(x)-g(x)|\,dx &= \int_0^\infty \Big|\int_0^\infty e^{-y} g\Big(x+\sqrt{\frac{y}{n}}\Big)\,dy - g(x)\Big|\,dx \\ &= \int_0^\infty \Big|\int_0^\infty e^{-y} g\Big(x+\sqrt{\frac{y}{n}}\Big)\,dy - \int_0^\infty g(x)e^{-y}\,dy\Big|\,dx \\ &= \int_0^\infty \Big|\int_0^\infty e^{-y}\Big(g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big)\,dy\Big|\,dx \\ &\le \int_0^\infty\int_0^\infty e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dy\,dx,\end{aligned}$$

and by Fubini theorem,

$$\begin{aligned}\int_0^\infty\int_0^\infty e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dy\,dx &= \int_0^\infty\int_0^{M-\sqrt{\frac{y}{n}}} e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy \\ &+ \int_0^\infty\int_{M-\sqrt{\frac{y}{n}}}^M e^{-y}|g(x)|\,dx\,dy,\end{aligned}$$

when $n\to\infty$, we have

$$\int_0^\infty\int_{M-\sqrt{\frac{y}{n}}}^M e^{-y}|g(x)|\,dx\,dy \to 0,$$

thus we know that

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty |g_n(x)-g(x)|\,dx &\le \lim_{n\to\infty}\int_0^\infty\int_0^{M-\sqrt{\frac{y}{n}}} e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy \\ &\le \lim_{n\to\infty}\int_0^\infty\int_0^M e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy.\end{aligned}$$

Since $e^{-y}|g(x+\sqrt{\frac{y}{n}})-g(x)| \le 2Ce^{-y}$ and $2Ce^{-y}\in L^1([0,+\infty))$, by the dominate convergence theorem we have

$$\lim_{n\to\infty}\int_0^\infty\int_0^M e^{-y}\Big|g\Big(x+\sqrt{\frac{y}{n}}\Big)-g(x)\Big|\,dx\,dy = 0,$$

thus

$$\lim_{n\to\infty}\int_0^\infty |g_n(x)-g(x)|\,dx = 0.$$

Hence g_n converges to g in $L^1([0,+\infty))$.

(iii) Since $C_c([0,+\infty))$ is dense in $L^1([0,+\infty))$ and $h(x)\in L^1([0,+\infty))$, we can find a sequence $\{h^k\}_{k=1}^\infty$ such that $h^k\to h$ in $L^1([0,+\infty))$. We want to show h_n is measurable by showing it is the limit of a sequence of measurable functions. By the result we got from (ii), for any $k\in\mathbb{N}$, we have $h_n^k = \int_0^\infty f_n(t)g(t)\,dt$ converges to $h^k(x)$ in $L^1([0,+\infty))$. Firstly we show that $h_n^k(x)$ converges to $h_n(x)$ almost everywhere. For any $x\in[0,+\infty)$, we have

$$
\begin{aligned}
|h_n(x) - h_n^k(x)| &= \Big| \int_0^\infty f_n(t) h(x+t)\, dt - \int_0^\infty f_n(t) h^k(x+t)\, dt \Big| \\
&= \Big| \int_0^\infty f_n(t)(h(x+t) - h^k(x+t))\, dt \Big| \\
&\leq \int_0^\infty f_n(t) |h(x+t) - h^k(x+t)|\, dt,
\end{aligned}
$$

we denote $z = x + t$, then

$$
|h_n(x) - h_n^k(x)| \leq \int_x^\infty f_n(z - x) |h(z) - h^k(z)|\, dz.
$$

Since $f_n(x) = 2nxe^{-nx^2}$, when $x = \frac{1}{\sqrt{2n}}$, the $f_n(x)$ gets the maximum value as $\sqrt{2n}e^{-\frac{1}{2}}$, thus we have

$$
\begin{aligned}
|h_n(x) - h_n^k(x)| &\leq \int_x^\infty f_n(z - x) |h(z) - h^k(z)|\, dz \\
&\leq \|f_n\|_\infty \int_x^\infty |h(z) - h^k(z)|\, dz \\
&\leq \|f_n\|_\infty \int_0^\infty |h(z) - h^k(z)|\, dz \\
&= \|f_n\|_\infty \|h - h^k\|_1 \to 0
\end{aligned}
$$

as $k \to +\infty$. Then we show that h_n^k is continuous. This means we want to show that for $x \in [0, +\infty)$, let $x_j \to x$, then $h_n^k(x_j) \to h_n^k(x)$. By the definition of $h_n^k(x_j)$, we have

$$
h_n^k(x_j) = \int_0^\infty f_n(t) h^k(x_j + t)\, dt = \int_0^\infty e^{-y} h^k\Big(x_j + \sqrt{\frac{y}{n}}\Big)\, dy.
$$

And since $h^k \in C_c([0, +\infty))$, $|e^{-y}h^k(x_j + \sqrt{\frac{y}{n}}| \leq \|h^k\|_\infty e^{-y} \in L^1([0, +\infty))$, by the dominate convergence theorem, we have

$$
\lim_{j \to \infty} h_n^k(x_j) = \int_0^\infty \lim_{j \to \infty} e^{-y} h^k\Big(x_j + \sqrt{\frac{y}{n}}\Big)\, dy = \int_0^\infty e^{-y} h^k\Big(x + \sqrt{\frac{y}{n}}\Big)\, dy = h_n^k(x),
$$

thus we know that h_n^k is uniformly continuous. From above, we have $h_n^k \to h_n$ almost everywhere and h_n^k is uniformly continuous, then we have h_n is the limit of a sequence of measurable functions. So, we get that h_n is measurable on $[0, +\infty)$.

Next we show that h_n is in $L^1([0, +\infty))$. Since

$$
\begin{aligned}
\|h_n\|_1 &= \int_0^\infty |h_n(x)|\, dx \\
&= \int_0^\infty \Big| \int_0^\infty f_n(t) h(x+t)\, dt \Big|\, dx \\
&\leq \int_0^\infty \int_0^\infty |f_n(t) h(x+t)|\, dt\, dx,
\end{aligned}
$$

by Fubini theorem, we have

$$\begin{aligned}
\|h_n\|_1 &\leq \int_0^\infty \int_0^\infty |f_n(t)h(x+t)|\,dx\,dt \\
&= \int_0^\infty f_n(t)\Big(\int_0^\infty |h(x+t)|\,dx\Big)\,dt \\
&= \int_0^\infty f_n(t)\Big(\int_t^\infty |h(z)|\,dz\Big)\,dt \\
&\leq \int_0^\infty f_n(t)\Big(\int_0^\infty |h(z)|\,dz\Big)\,dt \\
&= \|h\|_1 \int_0^\infty f_n(t)\,dt \\
&= \|h\|_1 < +\infty.
\end{aligned}$$

Thus we know that h_n is in $L^1([0,+\infty))$.

(iv) We want to show that h_n converges to h in $L^1([0,+\infty))$. Let $\epsilon > 0$ be given, since $C_c([0,+\infty))$ is dense in $L^1([0,+\infty))$, there exists a $g \in C_c([0,+\infty))$ such that $\|h-g\|_1 < \epsilon$. By triangle inequality of the L^1 norm,

$$\begin{aligned}
\|h_n - h\|_1 &= \|h_n - g_n + g_n - g + g - f\|_1 \\
&\leq \|h_n - g_n\|_1 + \|g_n - g\|_1 + \|g - f\|_1,
\end{aligned}$$

where the definition of g_n is as question (ii). By the result we get form (ii), for the above ϵ, there exists a $N \in \mathbb{N}$ such that $\|g_n - g\| < \epsilon$ whenever $n \geq N$, then we know that

$$\|h_n - h\|_1 < \|h_n - g_n\|_1 + 2\epsilon, \quad n \geq N.$$

Next we need to deal with $\|h_n - g_n\|_1$. Since

$$\begin{aligned}
\|h_n - g_n\|_1 &= \int_0^\infty |h_n(x) - g_n(x)|\,dx \\
&\leq \int_0^\infty \int_0^\infty f_n(t)|h(x+t) - g(x+t)|\,dt\,dx,
\end{aligned}$$

we denote $z = x + t$ and by Fubini theorem we have

$$\begin{aligned}
\|h_n - g_n\|_1 &\leq \int_0^\infty \int_0^\infty f_n(t)|h(x+t) - g(x+t)|\,dt\,dx \\
&= \int_0^\infty f_n(t) \int_t^\infty |h(z) - g(z)|\,dz\,dt \\
&\leq \int_0^\infty f_n(t) \int_0^\infty |h(z) - g(z)|\,dz\,dt \\
&= \int_0^\infty f_n(t)\|h - g\|_1 dt \\
&= \|h - g\|_1 \int_0^\infty f_n(t) dt \\
&= \|h - g\|_1 < \epsilon.
\end{aligned}$$

Thus we know that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|h_n - h\|_1 < \|h_n - g_n\|_1 + 2\epsilon < 3\epsilon, \quad \forall n \geq N.$$

Hence we have h_n converges to h in $L^1([0, +\infty))$.

<u>Exercise 4:</u>

Show that a set $E \subset \mathbb{R}$ is Lebesgue measurable if and only if $E = H \cup Z$ where H is a countable union of closed sets and Z has measure zero. You may use the following property: for any Lebesgue measurable subset A of $\mathbb{R}$ and any $\epsilon > 0$, there is a closed subset F of $\mathbb{R}$ such that $F \subset A$ and the measure of $A \setminus F$ is less than ϵ.

Solution:

If $E \subset \mathbb{R}$ is Lebesgue measurable, we know that $\forall \epsilon > 0$, there is a closed subset F of $\mathbb{R}$ such that $F \subset E$ and the measure of $E \setminus F$ is less than ϵ. Thus for each $n \in \mathbb{N}$, there exists a closed subset H_n of $\mathbb{R}$ such that $F \subset H_n$ and the measure of $E \setminus H_n$ is less than $1/n$. Let $H = \bigcup_{n=1}^{\infty} H_n$, H is a countable union of closed sets and we have $H \subset E$. And let $Z = E \setminus H$, then

$$m(Z) = m(E) - m(H) = m(E) - m\Big(\bigcup_{n=1}^{\infty} H_n\Big) \leq m(E) - m(H_n) < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Thus we have $m(Z) = 0$ and $E = H \cup Z$.

Since H is a countable union of closed sets, then H is a $\mathcal{F}_\sigma$ set and it is Lebesgue measurable. And as Z is a zero measure set, it is also Lebesgue measurable. The collection of Lebesgue measurable set is a σ-algebra, thus we know that $E = H \cup Z$ is Lebesgue measurable.

<u>Exercise 5:</u>

Give an example of a sequence f_n in $L^1((0,1))$ such that $f_n \to 0$ in $L^1((0,1))$ but f_n does not converge to zero almost everywhere.

Solution:

For each $n \in \mathbb{N}$, let

$$f_n(x) = \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x),$$

whenever $k \geq 0, 2^k \leq n < 2^{k+1}$, $k \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

$$\int_0^1 |f_n(x)|\, dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x)\, dx = \frac{1}{2^k} < \frac{2}{n},$$

thus $f_n \in L^1((0,1))$. And similarly we have

$$\int_0^1 |f_n(x) - 0|\, dx = \int_0^1 \mathbb{I}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x)\, dx = \frac{1}{2^k} < \frac{2}{n},$$

then when $n \to +\infty$, $\int_0^1 |f_n(x) - 0|\, dx \to 0$, thus we have $f_n \to 0$ in $L^1((0,1))$. But for all $x \in (0,1)$, and for all $N \in \mathbb{N}$, we can find a $n > N$ such that $f_n(x) = 1$. Therefore f_n can not converges to 0 anywhere for $x \in (0,1)$.