Stochastic Process

Jiamin Jian

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Reference: Bass, Richard F. *Stochastic Processes.* Vol. 33. Cambridge University Press, 2011.

1 Basic notions

1.1 Processes and σ-field

- Stochastic process: let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. $X : [0, \infty) \times \Omega \mapsto \mathbb{R}$
- Filtration $\{\mathcal{F}_t\}_{t \geq 0}$:
  - definition: $\mathcal{F}_t \subset \mathcal{F}, \forall t$, and $\mathcal{F}_s \subset \mathcal{F}_t, \forall s \leq t$
  - right continuous: define $\mathcal{F}_{t+} = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t > 0$
  - meaning of right continuous: there is no information just after time t which is not already given time t or before
  - null sets N: $\inf \{\mathbb{P}(A) : N \subset A, A \in \mathcal{F}\} = 0$
  - complete: $\mathcal{F}_t$ contains every null set
  - usual conditions: a filtration is right continuous and complete
- $\mathcal{F}_\infty := \sigma\left(\cup_{t \geq 0} \mathcal{F}_t\right) := \bigvee_{t \geq 0} \mathcal{F}_t$
- The arbitrary intersection of σ-fields is a σ-field, but the union of two σ-fields need not to be a σ-field: let $\Omega = \{a, b, c\}$, let $\mathcal{A}_1 = \{\{a\}, \{b, c\}, \emptyset, \Omega\}$, $\mathcal{A}_2 = \{\{b\}, \{a, c\}, \emptyset, \Omega\}$.
- Adapted: a stochastic process X is adapted to a filtration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$ measurable for each t.
- Minimal augmented filtration generated by X: the smallest filtration that is right continuous and complete and w.r.t. which the process X is adapted
  - let $\{\mathcal{F}_t^{00}\}$ be the smallest filtration w.r.t. which X is adapted
  $$\mathcal{F}_t^{00} = \sigma(X_s : s \leq t)$$
  we say $\{\mathcal{F}_t^{00}\}$ be the filtration generated by X.
  - let $\mathcal{N}$ be the collection of null sets, so that $\mathcal{N} = \{A \subset \Omega : \mathbb{P}^*(A) = 0\}$, let
  $$\mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N}).$$
  - let
  $$\mathcal{F}_t = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}^0.$$
- Distinguishable and versions
  - distinguishable: $\mathbb{P}(X_t \neq Y_t, \text{for some } t > 0) = 0$
  - versions (modification): $\mathbb{P}(X_t \neq Y_t) = 0$, for each $t \geq 0$
  - example that two process that are versions of each other but are not indistinguishable: let $\Omega = [0, 1]$, $\mathcal{F}$ be the Borel σ-field on $[0, 1]$, $\mathbb{P}$ be the Lebesgue measure on $[0, 1]$, $X(t, \omega) = 0$ for all t and ω, and $Y(t, \omega) = 1$ if $t = \omega$ and $Y(t, \omega) = 0$ otherwise. Note that $t \to X(t, \omega)$ are continuous for each ω, but the function $t \to Y(t, \omega)$ are not continuous for any ω.

- Paths (trajectories): the function $t \to X(t,\omega)$. There will be one path for each ω
- Continuous process: if the paths of X are continuous functions, except for a set of ω's in a null set
- Function which is right continuous with left limits:

$$\lim_{h>0, h\downarrow 0} f(t+h) = f(t) \quad \text{and} \quad \lim_{h<0, h\uparrow 0} f(t+h) \text{ exists}, \quad \forall t > 0.$$

- Cadlag: paths that are right continuous with left limits

1.2 Laws and state space

2 Brownian motion

2.1 Definition and basic properties

- Definition of Brownian motion
  - $\mathcal{F}_t$ measurable for each $t \geq 0$
  - $W_0 = 0$, a.s. (standrad Brownian motion)
  - $W_t - W_s \sim \mathcal{N}(0, t-s)$, $\forall s < t$ ($W_t - W_s$ has the same law with W_{t-s})
  - $W_t - W_s$ is independent of $\mathcal{F}_s$ whenever $s. < t$
  - W_t has continuous paths
- d-dimensional Brownian motion: $\left(W_t^{(1)}, W_t^{(2)}, \cdots, W_t^{(d)}\right)$
- Wiener measure: $\mathbb{P}_W(A) = \mathbb{P}(W \in A)$ for all Borel subsets A of $C([0,\infty))$
- $Y_t = aW_{t/a^2}$ is a Brownian motion started at 0
- Jointly normal: A sequence of random variables $X_1, X_2, \cdots, X_n$ is said to be jointly normal if there exists a sequence of i.i.d. normal random variables $Z_1, Z_2, \cdots, Z_m$ with mean zero and variance one and constants b_{ij} and a_i such that

$$X_i = \sum_{j=1}^{m} b_{ij} Z_j + a_i, \quad \forall i = 1, 2, \cdots, n$$

  In matrix notation $X = BZ + A$.
- Gaussion process $\{X_t\}_{t\geq 0}$: for each $n \geq 1$ and $t_1 < t_2 < \cdots < t_n$, the collection of random variables $X_{t_1}, X_{t_2}, \cdots, X_{t_n}$ is a jointly normal collection.
- The Brownian motion W is a Gaussian process.
- $\text{Cov}(W_t, W_s) = s \wedge t$
- If W is a process such that all the finite-dimensional distributions are jointly normal, $\mathbb{E}[W_s] = 0$ for all s, $\text{Cov}(W_t, W_s) = s$ whenever $s \leq t$, and the paths of W_t are continuous, then W is a Brownian motion.
- Let W_t be a Brownian motion w.r.t. $\{\mathcal{F}_t^{00}\}$, where $\mathcal{F}_t^{00} = \sigma(W_s : s \leq t)$. Let $\mathcal{N}$ be the collection of null sets, $\mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N})$, and $\mathcal{F}_t = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^0$. Then
  - W is a Brownian motion w.r.t the filtration $\{\mathcal{F}_t\}$.
  - $\mathcal{F}_t = \mathcal{F}_t^0$ for each t.
  - W is a Brownian motion w.r.t. the filtration generated by W, then it is also a Brownian motion w.r.t. the minimal augmented filtration.
- Let $t_0 > 0$ and let X, Y be random variables taking values in $C([0, t_0])$ which have the same finite-dimensional distributions. Then the laws of X and Y are equal.
  - it shows that if W and W' are both Brownian motions, they have all the same properties.
  - But if X and Y have the same finite-dimensional distributions, they may have different properties. The example is $X(t,\omega) = 0, \forall t, \omega$; $Y = 1$ if $t = \omega$ and 0 otherwise.

3 Martingales

3.1 Definition and examples

- Definition of a continuous-time martingale
  - (1) $\mathbb{E}[|M_t|] < \infty$ for each t
  - (2) M_t is $\mathcal{F}_t$ measurable for each t
  - (3) $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, a.s., if $s < t$
- Submartingale and supermartingale
  - submartingale: (3) $\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s$, a.s., if $s < t$
  - supermartingale: (3) $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$, a.s., if $s < t$
  - if $s < t$, then $\mathbb{E}[M_s] \leq \mathbb{E}[M_t]$ if M is a submartingale, and $\mathbb{E}[M_s] \geq \mathbb{E}[M_t]$ if M is a supermartingale. Thus submartingales tends to increase on average, and supermartingale tends to decrease on average.
- Examples of martingales
  - $M_t = W_t$
  - $M_t = W_t^2 - t$
  - $M_t = e^{aW_t - \frac{1}{2}a^2 t}$, $a \in \mathbb{R}$
  - Let X be an integrable $\mathcal{F}$ measurable random variable, and let $M_t = \mathbb{E}[X|\mathcal{F}_t]$

3.2 Doob's inequality

Suppose M_t is a martingale or non-negative submartingale with paths that are right continuous with left limits. Then

- $$\mathbb{P}\left(\sup_{s\leq t}|M_s| \geq \lambda\right) \leq \frac{\mathbb{E}[|M_t|]}{\lambda}$$
- If $1 < p < \infty$, then $$\mathbb{E}\left[\sup_{s\leq t}|M_s|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_t|^p\right]$$

3.3 Stopping time

- Definition: A random variable $T : \Omega \to [0, \infty]$ is a stopping time if $\{\omega \in \Omega : T < t\} \in \mathcal{F}_t$ for all t
- Boundedness: T is a finite stopping time if $T < \infty$ a.s., and T is a bounded stopping time if there exists $K \in [0, \infty)$ such that $T \leq K$ a.s.
- Some properties: suppose $\{F_t\}$ satisfies the usual condition
  - T is a stopping time if and only if $\{\omega : T \leq t\} \in \mathcal{F}_t$ for all t
  - if $T = t$ a.s., then T is a stopping time
  - if S and T are stopping times, then so $S \vee T$ and $S \wedge T$
  - if $T_n, n = 1, 2, \cdots$, are stopping times with $T_1 \leq T_2 \leq \cdots$, then so is $\sup_n T_n$
  - if $T_n, n = 1, 2, \cdots$, are stopping times with $T_1 \geq T_2 \geq \cdots$, then so is $\inf_n T_n$
  - if $s \geq 0$ and S is a stopping time, then so $S + s$
- For a Borel measurable set A, let $T_A = \inf\{t > 0 : X_t \in A\}$. Suppose $\mathcal{F}_t$ satisfies the usual conditions and X_t has continuous paths,
  - if A is open, then T_A is a stopping time

- if A is closed, then T_A is a stopping time

- Approximation of stopping time from the right: if T is a finite stopping time, define

$$T_n(\omega) = \frac{k+1}{2^n} \quad \text{if } \frac{k}{2^n} \le T(\omega) < \frac{k+1}{2^n}.$$

  Note that $\{T_n\}$ are stopping times decreasing to T.

- Define $\mathcal{F}_T = \{A \in \mathcal{F} : \text{ for each } t \ge 0, A \cap \{\omega : T \le t\} \in \mathcal{F}_t\}$, suppose $\{\mathcal{F}_t\}_{t\ge 0}$ is a filtration satisfying the usual conditions
  - $\mathcal{F}_T$ is σ-field
  - if $S \le T$, then $\mathcal{F}_S \subset \mathcal{F}_T$
  - if $\mathcal{F}_{T+} = \cap_{\epsilon>0}\mathcal{F}_{T+\epsilon}$, then $\mathcal{F}_{T+} = \mathcal{F}_T$
  - if X_t has right-continuous paths, then X_T is $\mathcal{F}_T$ measurable

3.4 The optional stopping theorem

Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions. If M_t is a martingale or non-negative submartingale whose paths are right continuous, $\sup_{t\ge 0} \mathbb{E}[|M_t^2|] < \infty$, and T is a finite stopping time, then $\mathbb{E}[M_T] \ge \mathbb{E}[M_0]$.

3.5 Convergence and regularity

Let $\mathcal{D}_n = \{k/2^n : k \ge 0\}, \mathcal{D} = \cup_n \mathcal{D}_n$.

- Let $\{M_t : t \in \mathcal{D}\}$ be either a martingale, a submartingale, or a supermartingale w.r.t. $\{\mathcal{F}_t : t \in \mathcal{D}\}$ and suppose $\sup_{t\in\mathcal{D}} \mathbb{E}[|M_t|] < \infty$. Then
  - (1) $\lim_{t\to\infty} M_t$ exists, a.s.
  - (2) With probability one M_t has left and right limits along $\mathcal{D}$.

- Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions, and let M_t be a martingale w.r.t. $\{\mathcal{F}_t\}$. Then M has a version that is also a martingale and that in addition has paths that are right continuous with left limits.

- Increasing paths: a process A_t has increasing paths if the function $t \to A_t(\omega)$ is increasing for almost every ω

- Suppose $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions and suppose A_t is an adapted process with paths that are increasing, are right continuous with left limits, and $A_\infty = \lim_{t\to\infty} A_t$ exists, a.s. Suppose X is non-negative integrable random variable, and M_t is a version of the martingale $\mathbb{E}[X|\mathcal{F}_t]$ which has paths that are right continuous with left limits. Suppose $\mathbb{E}[XA_\infty] < \infty$. Then

$$\mathbb{E}\left[\int_0^\infty X\, dA_s\right] = \mathbb{E}\left[\int_0^\infty M_s\, dA_s\right].$$

  The above equation also can be rewritten as

$$\mathbb{E}\left[\int_0^\infty X\, dA_s\right] = \mathbb{E}\left[\int_0^\infty \mathbb{E}[X|\mathcal{F}_s]\, dA_s\right].$$

  From above, for each t, we also have

$$\mathbb{E}\left[\int_0^t X\, dA_s\right] = \mathbb{E}\left[\int_0^t \mathbb{E}[X|\mathcal{F}_s]\, dA_s\right].$$

3.6 Some applications of martingales

- If W_t is a Brownian motion, then
$$\mathbb{P}\left(\sup_{s\le t} W_s \ge \lambda\right) \le e^{-\frac{\lambda^2}{2t}}, \quad ,\lambda > 0,$$
and
$$\mathbb{P}\left(\sup_{s\le t} |W_s| \ge \lambda\right) \le 2e^{-\frac{\lambda^2}{2t}}, \quad ,\lambda > 0.$$

- Let W be a Brownian motion, let $a, b > 0$ and let $T = \inf\{t > 0 : W_t \notin [-a, b]\}$. Then
$$\mathbb{P}(W_T = -a) = \frac{b}{a+b}, \quad \mathbb{P}(W_T = b) = \frac{a}{a+b},$$
and
$$\mathbb{E}[T] = ab.$$

- Suppose M_t is a martingale with continuous paths and with $M_0 = 0$ a.s., $T = \inf\{t > 0 : M_t \notin [-a, b]\}$, and $T < \infty$ a.s. Then
$$\mathbb{P}(M_T = -a) = \frac{b}{a+b}, \quad \mathbb{P}(M_T = b) = \frac{a}{a+b}.$$

- Let W be a Brownian motion, let $a, b > 0$ and let $T = \inf\{t > 0 : W_t \notin [-a, b]\}$. Then
$$\mathbb{E}\left[e^{-r^2T/2}\mathbb{1}_{\{W_T=-a\}}\right] = \frac{e^{rb} - e^{-rb}}{e^{r(a+b)} - e^{-r(a+b)}}$$
and
$$\mathbb{E}\left[e^{-r^2T/2}\mathbb{1}_{\{W_T=b\}}\right] = \frac{e^{ra} - e^{-ra}}{e^{r(a+b)} - e^{-r(a+b)}}.$$

4 Markov properties of Brownian motion

4.1 Markov properties

- Markov property: let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions, and let W be a Brownian motion w.r.t. $\{\mathcal{F}_t\}$. If u is a fixed time, then $Y_t = W_{t+u} - W_u$ is a Brownian motion independent of $\mathcal{F}_u$.

- Strong Markov property: let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions, and let W be a Brownian motion w.r.t. $\{\mathcal{F}_t\}$. If T is a stopping time, then $Y_t = W_{t+T} - W_T$ is a Brownian motion independent of $\mathcal{F}_T$.

- General process: let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions, and let X be a process adapted to $\{\mathcal{F}_t\}$. Suppose X has paths that are right continuous with left limits and suppose $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same law with X_{t-s} whenever $s < t$. If T is a finite stopping time, then $Y_t = W_{t+T} - W_T$ is a process that is independent of $\mathcal{F}_T$ and X and Y have the same law.

4.2 Applications

- The reflection of Brownian motion: let W_t be a Brownian motion, $b > 0$, $T = \inf\{t : W_t \ge b\}$, and $x < b$. Then
$$\mathbb{P}\left(\sup_{s\le t} W_s \ge b, W_t < x\right) = \mathbb{P}\left(W_t > 2b - x\right).$$

- Let W_t be a Brownian motion w.r.t. a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let T be a finite stopping time and $s > 0$. If $a < b$, then
$$\mathbb{P}\left(W_{T+s} \in [a, b] | \mathcal{F}_T\right) \le \frac{|b-a|}{\sqrt{2\pi s}}.$$

5 The Poisson process

The Poisson process is the prototype of a pure jump process, and it is the building block for Lévy process.

- Definition: Let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions. A Poisson process with parameter $\lambda > 0$ is a stochastic process X satisfying the following properties:
  - (1) $X_0 = 0$ a.s.
  - (2) The paths of X_t are right continuous with left limits
  - (3) If $s < t$, then $X_t - X_s$ is a Poisson random variable with parameter $\lambda(t-s)$
  - (4) If $s < t$, then $X_t - X_s$ is independent of $\mathcal{F}_s$
- $X_{t-} = \lim_{s\to t, s<t} X_s$ be the left-hand limit at time t, and $\Delta X_t = X_t - X_{t-}$ be the size of the jump at time t
- Let X be a Poisson process,
  - with probability one, the paths of X_t are increasing
  - with probability one, the paths of X_t are constant except for jumps of size 1
  - there are only finitely many jumps in each finite time interval
- Let $T_1 = \inf\{t : \Delta X_t = 1\}$, the time of the first jump. Define $T_{i+1} = \inf\{t > T_i : \Delta X_t = 1\}$, so T_i is the time of the i-th jump. The random variables $T_1, T_2 - T_1, \cdots, T_{i+1} - T_i, \cdots$ are independent exponential random variables with parameter λ.
- The construction of Poisson process: let $U_1, U_2, \cdots$ be independent exponential random variable with parameter λ and let $T_j = \sum_{i=1}^{j} U_i$. Define

$$X_t(\omega) = k, \text{ if } T_k(\omega) \le t < T_{k+1}(\omega).$$

- The densities shows that an exponential random variable has a Gamma distribution with parameter λ and 1. Then by the invariant summation of Gamma distribution, T_j is a Gamma random variable with parameters λ and j. Thus

$$\mathbb{P}(X_t < k) = \mathbb{P}(T_k > t) = \int_t^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}\, dx.$$

  Performing the integration by parts repeatedly shows that

$$\mathbb{P}(X_t < k) = \sum_{i=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!},$$

  thus X_t is a Poisson random variable with parameter λt.
- Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions. Suppose $X_0 = 0$ a.s., X has paths that are right continuous with left limits, $X_t - X_s$ is independent of $\mathcal{F}_s$ if $s < t$ and $X_t - X_s$ has the same law with X_{t-s} whenever $s < t$. If the paths of X are piecewise constant, increasing, all the jumps of X are of size 1, and X is not identically 0, then X is a Poisson process.

6 Construction of Brownian motion

There are several ways of constructing Brownian motion, none of them easy. Here gives two constructions. The first is the one that Wiener used, which is based on Fourier series. The second uses martingale techniques, which is due to Lévy.

- Wiener's construction
  - The main step is to construct W_t for $t \in [0, 1]$.

- Take independent copies $Y^{(1)}, Y^{(2)}, \cdots$ each on $[0,1]$, then let

$$W_t = \left(\sum_{i=0}^{[t]-1} Y_1^{(i)}\right) + Y_{t-[t]}^{[t]}.$$

- Fix $t \in [0,\pi]$, the Fourier series for the function $f(s) = s \wedge t$ is

$$s \wedge t = \frac{st}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin(ks)\sin(kt)}{k^2}.$$

- Let $Z_0, Z_1, \cdots$ be i.i.d. standard normal random variables and let

$$W_t = \frac{t}{\sqrt{\pi}} Z_0 + \sum_{k=1}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin(kt)}{k}\right) Z_k,$$

then W_t is a Gaussian process, has mean zero and $\text{Cov}(W_s, W_t) = s \wedge t$. And we also can show that W_t has continuous paths. Thus W as constructed above has the correct finite-dimensional distributions to be a Brownian motion.

• Martingale method

- Proceed as in the previous method to construct $\{W_t : 0 \leq t \leq \pi\}$, where W_t is a Gaussian process with $\mathbb{E}[W_t] = 0$ and $\text{Cov}(W_s, W_t) = s \wedge t$, and we need to show that W has a version with continuous paths.
- First we show that W is a martingale, and so has a version with paths that are right continuous with left limits. We use Doob's inequalities to control the oscillation of W over short time intervals, and then use the Borel–Cantelli lemma to show continuity.
- Theorem: if $\{W_t : 0 \leq t \leq 1\}$ is a Gaussian process with $\mathbb{E}[W_t] = 0$ and $\text{Cov}(W_s, W_t) = s \wedge t$ for all $0 \leq s, t \leq 1$, then there is a version of W that is a Brownian motion on $[0,1]$.
- There is nothing special about the trigonometric polynomials in the martingale method. Let $\langle f, g \rangle = \int_0^1 f(r)g(r)\, dr$ be the inner product for the Hilbert space $L^2([0,1])$; we consider only real-valued functions for simplicity. Let $\{\varphi_n\}$ be a complete orthonormal system for $L^2([0,1])$: we have $\langle \varphi_m, \varphi_n \rangle = 0$ if $m \neq n$ and $\langle \varphi_n, \varphi_n \rangle = 1$ for each n, and $f = 0$ a.e. if $\langle f, \varphi_n \rangle = 0$ for all n. Let

$$a_n(t) = \langle \mathbb{1}_{[0,t]}, \varphi_n \rangle = \int_0^t \varphi_n(r)\, dr.$$

If $Z_0, Z_1, \cdots$ be i.i.d. standard normal random variables and let

$$W_t = \sum_{n=1}^{\infty} a_n(t) Z_k.$$

Then we have

$$\text{Cov}(W_s, W_t) = \sum_{n=1}^{\infty} a_n(s) a_n(t) = \sum_{n=1}^{\infty} \langle \mathbb{1}_{[0,s]}, \varphi_n \rangle \langle \mathbb{1}_{[0,t]}, \varphi_n \rangle = \langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \rangle = s \wedge t.$$

Thus W defined above is a mean zero Gaussian process on $[0,1]$ with the same covariances as a Brownian motion.

7 Path properties of Brownian motion

The paths of Brownian motion are continuous, but they are not differentiable. We can see that the paths satisfy a Hölder continuity condition with $\alpha < \frac{1}{2}$. A precise description of the oscillatory behavior of Brownian motion is given by the law of iterated logarithm.

- Hölder continuity: a function $f : [0,1] \to \mathbb{R}$ is said to be Hölder continuous of order α if there is a constant M such that
$$|f(t) - f(s)| \leq M|t-s|^{\alpha}, \quad s,t \in [0,1].$$
- Borel-Cantelli lemma: suppose that $\{A_n\}_{n\geq 1}$ is a sequence of events in a probability space. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then
$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = \mathbb{P}\left(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k\right) := \mathbb{P}(A(i.o.)) = 0.$$
It means with probability one only a finite number of the events occur.
- Second Borel-Cantelli lemma: If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, and the events $\{A_n\}$ are independent, then
$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 1.$$
- If $\alpha < \frac{1}{2}$, the paths of Brownian motion are Hölder continuous of order α on $[0,1]$.
- Law of the iterated logarithm(LIL): Let W be a Brownian motion, we have
$$\limsup_{t\to\infty} \frac{|W_t|}{\sqrt{2t\log\log t}} = 1, a.s. \text{ and } \limsup_{t\to 0} \frac{|W_t|}{\sqrt{2t\log\log\frac{1}{t}}} = 1, a.s.$$
- With probability one, the paths of Brownian motion are nowhere differentiable.

8 The continuity of paths

It is often important to know whether a stochastic process has continuous paths. An important condition is the Kolmogorov continuity criterion.

- Dyadic rationals: let $\mathcal{D}_n = \{k/2^n : k \leq 2^n\}$, the set $\mathcal{D} = \cup_n \mathcal{D}_n$ is known as the set of dyadic rationals in $[0,1]$.
- Suppose $\{X_t : t \in \mathcal{D}\}$ is a real-valued process and there exist c_1, ϵ and $p > 0$ such that
$$\mathbb{E}[|X_t - X_s|^p] \leq c_1|t-s|^{1+\epsilon}, \quad s,t \in \mathcal{D},$$
then the following hold
  - there exists c_2 depending only on c_1, p and ϵ such that for $M > 0$,
$$\mathbb{P}\left(\sup_{s,t\in\mathcal{D}, s\neq t} \frac{|X_t - X_s|}{|t-s|^{\frac{\epsilon}{4p}}} \geq M\right) \leq \frac{c_2}{M^p}.$$
  - with probability one, X_t is uniformly continuous on $\mathcal{D}$.
- Suppose X takes values in some metric space $\mathcal{S}$ with metric $d_{\mathcal{S}}$ and there exist c_1, ϵ and $p > 0$ such that
$$\mathbb{E}[d_{\mathcal{S}}(X_t, X_s)^p] \leq c_1|t-s|^{1+\epsilon}, \quad s,t \in \mathcal{D},$$
then the following hold
  - there exists c_2 depending only on c_1, p and ϵ such that for $M > 0$,
$$\mathbb{P}\left(\sup_{s,t\in\mathcal{D}, s\neq t} \frac{d_{\mathcal{S}}(X_t, X_s)}{|t-s|^{\frac{\epsilon}{2p}}} \geq M\right) \leq \frac{c_2}{M^p}.$$
  - with probability one, X_t is uniformly continuous on $\mathcal{D}$.

- Suppose there exist c_1, ϵ, N and $p > 0$ such that if $n \le N$,

$$\mathbb{E}[d_{\mathcal{S}}(X_t, X_s)^p] \le c_1 |t-s|^{1+\epsilon}, \quad s, t \in \mathcal{D}_n,$$

then there exists c_2 depending on c_1, p and ϵ but not N such that for $M > 0$ and $n \le N$, we have

$$\mathbb{P}\left(\sup_{s,t \in \mathcal{D}_n, s \ne t} \frac{d_{\mathcal{S}}(X_t, X_s)}{|t-s|^{\frac{\epsilon}{2p}}} \ge M \right) \le \frac{c_2}{M^p}.$$

- If $\alpha < \frac{1}{2}$, then the paths of a one-dimensional Brownian motion $\{W_t : 0 \le t \le 1\}$ are Hölder continuous of order α with probability one.

9 Continuous semimartingales

9.1 Definitions

- A process X has increasing paths (or X is an increasing process): the functions $t \to X_t(\omega)$ are increasing with probability one.
- A process X with paths of bounded variation: with probability one, the functions $t \to X_t(\omega)$ are of bounded variation.
- A process X with paths of locally bounded variation: if there exists stopping times $R_n \to \infty$ such that the process $X_{t \wedge R_n}$ has paths of bounded variation for each n.
- Uniform integrable:
- A martingale is a uniformly martingale: if the family of random variables $\{M_t\}$ is uniformly integrable.
- A process X is a local martingale: if there exists stopping times $R_n \to \infty$ such that the process $M_t^n = X_{t \wedge R_n}$ is a uniformly integrable martingale for each n.
- Continuous martingale: a martingale whose paths are continuous.
- Semimartingale: a process X of the form $X_t = M_t + A_t$, where M_t is a local martingale and A_t is a process whose paths are locally of bounded variation. As a consequence of the Doob-Meyer decomposition we will see that submartingales and supermartingales are semimartingales.
- Example: a Brownian motion W_t is a martingale and is a local martingale, but is not a uniformly integrable martingale and is not a square integrable martingale.

9.2 Square integrable martingales

- Definition: a martingale is square integrable martingale if there exists a $\mathcal{F}_\infty$ measurable random variable M_∞ such that $\mathbb{E}[M_\infty^2] < \infty$ and $M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]$ for all t.
- Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions and M a right continuous process. The following are equivalent:
  - (1) M is a square integrable martingale.
  - (2) M is a martingale with $\sup_{t \ge 0} \mathbb{E}[M_t^2] < \infty$.
  - (3) M is a martingale with $\mathbb{E}[\sup_{t \ge 0} M_t^2] < \infty$.
- If M is a square integrable martingale and $S < T$ are finite stopping times, then $\mathbb{E}[M_T | \mathcal{F}_S] = M_S$. This conclusion is also valid if M is a uniformly integrable martingale.
- Suppose M is a square integrable martingale and T is a stopping time, then $X_t = M_{t \wedge T}$ is a martingale with respect to $\{\mathcal{F}_{t \wedge T}\}$.
- Suppose $\{\mathcal{F}_t\}$ is a filtration satisfying the usual conditions and M is a process that is adapted to $\{\mathcal{F}_t\}$ such that M_t is integrable for each t. If $\mathbb{E}[M_T] = 0$ for every bounded stopping time T, then M_t is a martingale.

- If $\mathbb{E}[M_t] = 0$ for all $t \geq 0$, is M_t a martingale? The answer is no. The counter example is as follows: let $M_t = B_t \mathbb{1}_{0 \leq t < 1} + (B_t^2 - t)\mathbb{1}_{t \geq 1}$.
- Suppose M_t is a square integrable martingale. Then

$$\mathbb{E}[(M_T - M_S)^2 | \mathcal{F}_S] = \mathbb{E}[M_T^2 - M_S^2 | \mathcal{F}_S].$$

- Suppose $M_0 = 0$, M_t is a continuous local martingale, and the paths of M_t are locally of bounded variation. Then M is identically 0 a.s., that is $\mathbb{P}(M_t = 0, \forall t \geq 0) = 1$.

9.3 Quadratic variation

- Definition: a continuous square integrable martingale M_t has quadratic variation $\langle M \rangle_t$ if $M_t^2 - \langle M \rangle_t$ is a martingale, where $\langle M \rangle_t$ is a continuous adapted increasing process with $\langle M \rangle_0 = 0$.
- Example: W is a Brownian motion, t_0 is fixed and $M_t = W_{t \wedge t_0}$, the quadratic variation of M is just $\langle M \rangle_t = t \wedge t_0$. Brownian motion itself does not fit perfectly into the framework of stochastic integration because it is not a square integrable martingale, although it is a martingale.
- Class D: a process X is of process D if $\{Z_T : T \text{ is a finite stopping time}\}$ is a uniformly integrable family of random variables.
- Existence and uniqueness: let M_t be a continuous square integrable martingale, there exists a continuous adapted increasing process $\langle M \rangle_t$ with $\langle M \rangle_0 = 0$ and with increasing paths such that $M_t^2 - \langle M \rangle_t$ is a martingale. If A_t is a continuous adapted increasing process such that $M_t^2 - A_t$ is a martingale, then $\mathbb{P}(A_t \neq \langle M \rangle_t \text{ for some } t) = 0$.
- By the definition of $\langle M \rangle_t$, we have

$$\mathbb{E}\left[(M_T - M_S)^2 - (\langle M \rangle_T - \langle M \rangle_S) | \mathcal{F}_S\right] = \mathbb{E}\left[M_T^2 - M_S^2 - (\langle M \rangle_T - \langle M \rangle_S) | \mathcal{F}_S\right] = 0.$$

- Covariation: if M and N are two square integrable martingales, define $\langle M, N \rangle_t$ by

$$\langle M, N \rangle_t = \frac{1}{2}\left[\langle M + N \rangle_t - \langle M \rangle_t - \langle N \rangle_t\right] = \frac{1}{4}[\langle M + N \rangle_t - \langle M - N \rangle_t].$$

- Another definition: let M be a square integrable martingale and let $t_0 > 0$, then $\langle M \rangle_{t_0}$ is the limit in probability of

$$\sum_{k=0}^{[2^n t_0]} \left(M_{\frac{k+1}{2^n}} - M_{\frac{k}{2^n}}\right)^2,$$

where $[2^n t_0]$ is the largest integer less than or equal to $2^n t_0$.

9.4 The Doob-Meyer decomposition

- Suppose A^1 and A^2 are two increasing adapted continuous processes starting at zero with $A_\infty^i = \lim_{t \to \infty} A_t^i < \infty$, a.s. for $i = 1, 2$, and suppose there exists a positive real K such that for all t,

$$\mathbb{E}[A_\infty^i - A_t^i | \mathcal{F}_t] \leq K, \quad a.s. \quad i = 1, 2.$$

Let $B_t = A_t^1 - A_t^2$. Suppose there exists a non-negative random variable V with $\mathbb{E}[V^2] < \infty$ such that for all t,

$$|\mathbb{E}[B_\infty - B_t | \mathcal{F}_t]| \leq \mathbb{E}[V | \mathcal{F}_t], a.s.,$$

then

$$\mathbb{E}\left[\sup_{t \geq 0} B_t^2\right] \leq 8\mathbb{E}\left[V^2\right] + 8\sqrt{2}K\left(\mathbb{E}[V^2]\right)^{\frac{1}{2}}.$$

- Doob-Meyer decomposition: suppose Z_t is a continuous adapted supermartingale of class D, then there exists an increasing adapted continuous process A_t with paths locally of bounded variation starts at 0 and a continuous local martingale M_t such that

$$Z_t = M_t - A_t.$$

If M' and A' are two other such process with $Z_t = M_t' - A_t'$, then $M_t = M_t'$ and $A_t = A_t'$ for all t, a.s.

10 Stochastic integral

10.1 Construction

- Objective: let M_t be a continuous square integrable martingale with respect to a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, and suppose H_t is an adapted process. Under appropriate additional assumptions on H, we want to define
$$N_t = \int_0^t H_s \, dM_s.$$
- A predictable σ-field $\mathcal{P}$ on $[0,\infty) \times \Omega$: $\mathcal{P} = \sigma(X : X$ is left continuous, bounded, and adapted to $\{\mathcal{F}_t\})$.
- Two conditions on the integrand H_t:
  - $H : [0,\infty) \times \Omega \to \mathbb{R}$ is measurable w.r.t. $\mathcal{P}$ (H is predictable).
  - H is integrability:
$$\mathbb{E}\left[\int_0^\infty H_s^2 \, d\langle M \rangle_s\right] < \infty.$$
- Three steps to define $\int_0^t H_s \, dM_S$:
  - When $H_s(\omega) = K(\omega)\mathbb{1}_{(a,b]}(s)$, where K is bounded and $\mathcal{F}_a$ measurable.
  - When H_s is the sum of processes of the form in step 1.
  - When H is predictable and satisfies integrability condition.
- The predictable σ-field $\mathcal{P}$ is generated by the collection $\mathcal{C}$ of precesses of the form
$$X_t = \sum_{i=1}^n K_i(\omega)\mathbb{1}_{(a_i,b_i]}(t),$$
where for each i, K_i is a bounded $\mathcal{F}_{a_i}$ measurable random variable.
- Suppose H is as in step 1, then $N_t = K(M_{t\wedge b} - M_{t\wedge a})$ is a continuous martingale,
$$\mathbb{E}[N_\infty^2] = \mathbb{E}\left[\int_0^\infty K^2\mathbb{1}_{(a,b]}(s) \, d\langle M \rangle_s\right] = \mathbb{E}\left[K^2(\langle M \rangle_b - \langle M \rangle_a)\right],$$
and
$$\langle N \rangle_t = \int_0^t K^2\mathbb{1}_{(a,b]}(s) \, d\langle M \rangle_s.$$
- Suppose
$$H_s(\omega) = \sum_{j=1}^J K_j\mathbb{1}_{(a_j,b_j]}(s),$$
where each K_j is $\mathcal{F}_{a_j}$ measurable and bounded. Define
$$N_t = \sum_{j=1}^J K_j(M_{t\wedge b_j} - M_{t\wedge a_j}).$$
Then N_t is a continuous martingale,
$$\mathbb{E}[N_\infty^2] = \mathbb{E}\left[\int_0^\infty H_s^2 \, d\langle M \rangle_s\right],$$
and
$$\langle N \rangle_t = \int_0^t H_s^2 \, d\langle M \rangle_s.$$

- Suppose the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions and M_t is a square integrable martingale with continuous paths. Suppose H is of the form

$$H_s(\omega) = \sum_{j=1}^{J} K_j \mathbb{1}_{(a_j,b_j]}(s),$$

where each K_j is bounded and $\mathcal{F}_{a_j}$ measurable. In this case define

$$\int_0^t H_s\, dM_s = \sum_{j=1}^{J} K_j (M_{t\wedge b_j} - M_{t\wedge a_j}).$$

If H is predictable and $\mathbb{E}\left[\int_0^\infty H_s^2\, d\langle M\rangle_s\right] < \infty$, choose H^n of the form given in above with $\mathbb{E}\left[\int_0^\infty (H_s^n - H_s)^2\, d\langle M\rangle_s\right] \to 0$, and define

$$N_t = \int_0^t H_s\, dM_s$$

to be the limit respect to the norm of $\int_0^t H_s^n\, dM_s$. Then N_t is a continuous martingale,

$$\mathbb{E}[N_\infty^2] = \mathbb{E}\left[\int_0^\infty H_s^2\, d\langle M\rangle_s\right],$$

and

$$\langle N\rangle_t = \int_0^t H_s^2\, d\langle M\rangle_s.$$

Moreover the definition of N_t is independent of the particular choice of the H^n.

10.2 Extensions

- If $\int_0^\infty H_s^2\, d\langle M\rangle_s < \infty$, a.s., but without the expectation being finite, let

$$T_N = \inf\left\{t : \int_0^\infty H_s^2\, d\langle M\rangle_s > N\right\}.$$

$M'_t = M_{t\wedge T_N}$ is a square integrable martingale with $\langle M'\rangle_t = \langle M\rangle_{t\wedge T_N} \le N$. Define $\int_0^t H_s\, dM_s$ to be the quantity $\int_0^t H_s\, dM_{s\wedge T_N}$ if $t \le T_N$.

- If M_t is a continuous local martingale, let $S_n = \inf\{t : |M_t| \ge n\}$. Then $M_{t\wedge S_n}$ will be uniformly integrable martingale, and it is square integrable. For $t \le S_n$, we set

$$\int_0^t H_s\, dM_s = \int_0^t H_s\, dM_{s\wedge S_n}$$

and $\langle M\rangle_t = \langle M\rangle_{t\wedge S_n}$.

- Suppose that $X_t = M_t + A_t$ is a semimartingale with continuous paths, so that M is a local martingale and A is a process with paths locally of bounded variation. If $\int_0^\infty H_s^2\, d\langle M\rangle_s + \int_0^t |H_s|\, |dA_s| < \infty$, we define

$$\int_0^t H_s\, dX_s = \int_0^t H_s\, dM_s + \int_0^t H_s\, dA_s,$$

where the first integral on the right is a stochastic integral and the second is a Lebesgue-Stieltjes integral.

- For a semimartingale, we define $\langle X\rangle_t = \langle M\rangle_t$. Given two semimartingales X and Y, we define $\langle X, Y\rangle_t$ by

$$\langle X, Y\rangle_t = \frac{1}{2}\left[\langle X+Y\rangle_t - \langle X\rangle_t - \langle Y\rangle_t\right].$$

11 Itô's formula

- Let X_t be a semimartingale with continuous paths and suppose $f \in C^2$. Then for almost every ω,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s, \quad \forall t \geq 0.$$

- Suppose that $X_t^1, \cdots, X_t^d$ are continuous semimartingales, $X_t = (X_t^1, \cdots, X_t^d)$, and f is a C^2 function on $\mathbb{R}^d$. Then with probability one,

$$f(X_t) = f(X_0) + \int_0^t \sum_{i=0}^{d} \frac{\partial f}{\partial x_i}(X_s)\, dX_s^i + \frac{1}{2}\int_0^t \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\, d\langle X^i, X^j\rangle_s, \quad \forall t \geq 0.$$

- If X and Y are two semimartingales with continuous paths, then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t.$$