Stochastic Process

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1 Basic notions

1.1 Processes and σ-field

- stochastic process: let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. $X : [0, \infty) \times \Omega \mapsto \mathbb{R}$
- filtration $\{\mathcal{F}_t\}_{t \geq 0}$:
  - definition: $\mathcal{F}_t \subset \mathcal{F}, \forall t$, and $\mathcal{F}_s \subset \mathcal{F}_t, \forall s \leq t$
  - right continuous: define $\mathcal{F}_{t+} = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t > 0$
  - meaning of right continuous: there is no information just after time t which is not already given time t or before
  - null sets N: $\inf \{\mathbb{P}(A) : N \subset A, A \in \mathcal{F}\} = 0$
  - complete: $\mathcal{F}_t$ contains every null set
  - usual conditions: a filtration is right continuous and complete
- $\mathcal{F}_\infty := \sigma\left(\cup_{t \geq 0} \mathcal{F}_t\right) := \bigvee_{t \geq 0} \mathcal{F}_t$
- the arbitrary intersection of σ-fields is a σ-field, but the union of two σ-fields need not to be a σ-field: let $\Omega = \{a, b, c\}$, let $\mathcal{A}_1 = \{\{a\}, \{b, c\}, \emptyset, \Omega\}$, $\mathcal{A}_2 = \{\{b\}, \{a, c\}, \emptyset, \Omega\}$.
- adapted: a stochastic process X is adapted to a filtration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$ measurable for each t.
- minimal augmented filtration generated by X: the smallest filtration that is right continuous and complete and w.r.t. which the process X is adapted
  - let $\{\mathcal{F}_t^{00}\}$ be the smallest filtration w.r.t. which X is adapted
    $$\mathcal{F}_t^{00} = \sigma(X_s : s \leq t)$$
    we say $\{\mathcal{F}_t^{00}\}$ be the filtration generated by X.
  - let $\mathcal{N}$ be the collection of null sets, so that $\mathcal{N} = \{A \subset \Omega : \mathbb{P}^*(A) = 0\}$, let
    $$\mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N}).$$
  - let
    $$\mathcal{F}_t = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}^0.$$
- distinguishable and versions
  - distinguishable: $\mathbb{P}(X_t \neq Y_t, \text{for some } t > 0) = 0$
  - versions (modification): $\mathbb{P}(X_t \neq Y_t) = 0$, for each $t \geq 0$
  - example that two process that are versions of each other but are not indistinguishable: let $\Omega = [0, 1]$, $\mathcal{F}$ be the Borel σ-field on $[0, 1]$, $\mathbb{P}$ be the Lebesgue measure on $[0, 1]$, $X(t, \omega) = 0$ for all t and ω, and $Y(t, \omega) = 1$ if $t = \omega$ and $Y(t, \omega) = 0$ otherwise. Note that $t \to X(t, \omega)$ are continuous for each ω, but the function $t \to Y(t, \omega)$ are not continuous for any ω.

- paths (trajectories): the function $t \to X(t,\omega)$. There will be one path for each ω
- continuous process: if the paths of X are continuous functions, except for a set of ω's in a null set
- function which is right continuous with left limits:

$$\lim_{h>0, h\downarrow 0} f(t+h) = f(t) \quad \text{and} \quad \lim_{h<0, h\uparrow 0} f(t+h) \text{ exists}, \quad \forall t > 0.$$

- cadlag: paths that are right continuous with left limits

1.2 Laws and state space

2 Brownian motion

2.1 Definition and basic properties

- definition of Brownian motion
  - $\mathcal{F}_t$ measurable for each $t \geq 0$
  - $W_0 = 0$, a.s. (standrad Brownian motion)
  - $W_t - W_s \sim \mathcal{N}(0, t-s)$, $\forall s < t$ ($W_t - W_s$ has the same law with W_{t-s})
  - $W_t - W_s$ is independent of $\mathcal{F}_s$ whenever $s. < t$
  - W_t has continuous paths
- d-dimensional Brownian motion: $\left(W_t^{(1)}, W_t^{(2)}, \cdots, W_t^{(d)}\right)$
- Wiener measure: $\mathbb{P}_W(A) = \mathbb{P}(W \in A)$ for all Borel subsets A of $C([0,\infty))$
- $Y_t = aW_{t/a^2}$ is a Brownian motion started at 0
- jointly normal: A sequence of random variables $X_1, X_2, \cdots, X_n$ is said to be jointly normal if there exists a sequence of i.i.d. normal random variables $Z_1, Z_2, \cdots, Z_m$ with mean zero and variance one and constants b_{ij} and a_i such that

$$X_i = \sum_{j=1}^{m} b_{ij} Z_j + a_i, \quad \forall i = 1, 2, \cdots, n$$

  In matrix notation $X = BZ + A$.
- Gaussion process $\{X_t\}_{t\geq 0}$: for each $n \geq 1$ and $t_1 < t_2 < \cdots < t_n$, the collection of random variables $X_{t_1}, X_{t_2}, \cdots, X_{t_n}$ is a jointly normal collection.
- the Brownian motion W is a Gaussian process.
- $\mathrm{Cov}(W_t, W_s) = s \wedge t$
- If W is a process such that all the finite-dimensional distributions are jointly normal, $\mathbb{E}[W_s] = 0$ for all s, $\mathrm{Cov}(W_t, W_s) = s$ whenever $s \leq t$, and the paths of W_t are continuous, then W is a Brownian motion.
- Let W_t be a Brownian motion w.r.t. $\{\mathcal{F}_t^{00}\}$, where $\mathcal{F}_t^{00} = \sigma(W_s : s \leq t)$. Let $\mathcal{N}$ be the collection of null sets, $\mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N})$, and $\mathcal{F}_t = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^0$. Then
  - W is a Brownian motion w.r.t the filtration $\{\mathcal{F}_t\}$.
  - $\mathcal{F}_t = \mathcal{F}_t^0$ for each t.
  - W is a Brownian motion w.r.t. the filtration generated by W, then it is also a Brownian motion w.r.t. the minimal augmented filtration.
- Let $t_0 > 0$ and let X, Y be random variables taking values in $C([0, t_0])$ which have the same finite-dimensional distributions. Then the laws of X and Y are equal.
  - it shows that if W and W' are both Brownian motions, they have all the same properties.
  - But if X and Y have the same finite-dimensional distributions, they may have different properties. The example is $X(t,\omega) = 0, \forall t, \omega$; $Y = 1$ if $t = \omega$ and 0 otherwise.

3 Martingales

3.1 Definition and examples

- definition of a continuous-time martingale
  - (1) $\mathbb{E}[|M_t|] < \infty$ for each t
  - (2) M_t is $\mathcal{F}_t$ measurable for each t
  - (3) $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, a.s., if $s < t$
- submartingale and supermartingale
  - submartingale: (3) $\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s$, a.s., if $s < t$
  - supermartingale: (3) $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$, a.s., if $s < t$
  - if $s < t$, then $\mathbb{E}[M_s] \leq \mathbb{E}[M_t]$ if M is a submartingale, and $\mathbb{E}[M_s] \geq \mathbb{E}[M_t]$ if M is a supermartingale. Thus submartingales tends to increase on average, and supermartingale tends to decrease on average.
- examples of martingales
  - $M_t = W_t$
  - $M_t = W_t^2 - t$
  - $M_t = e^{aW_t - \frac{1}{2}a^2 t}$, $a \in \mathbb{R}$
  - Let X be an integrable $\mathcal{F}$ measurable random variable, and let $M_t = \mathbb{E}[X|\mathcal{F}_t]$

3.2 Doob's inequality

Suppose M_t is a martingale or non-negative submartingale with paths that are right continuous with left limits. Then

- $$\mathbb{P}\left(\sup_{s\leq t}|M_s| \geq \lambda\right) \leq \frac{\mathbb{E}[|M_t|]}{\lambda}$$
- If $1 < p < \infty$, then
$$\mathbb{E}\left[\sup_{s\leq t}|M_s|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_t|^p\right]$$

3.3 Stopping time

- definition: A random variable $T : \Omega \to [0, \infty]$ is a stopping time if $\{\omega \in \Omega : T < t\} \in \mathcal{F}_t$ for all t
- boundedness: T is a finite stopping time if $T < \infty$ a.s., and T is a bounded stopping time if there exists $K \in [0, \infty)$ such that $T \leq K$ a.s.
- Some properties: suppose $\{F_t\}$ satisfies the usual condition
  - T is a stopping time if and only if $\{\omega : T \leq t\} \in \mathcal{F}_t$ for all t
  - if $T = t$ a.s., then T is a stopping time
  - if S and T are stopping times, then so $S \vee T$ and $S \wedge T$
  - if $T_n, n = 1, 2, \cdots$, are stopping times with $T_1 \leq T_2 \leq \cdots$, then so is $\sup_n T_n$
  - if $T_n, n = 1, 2, \cdots$, are stopping times with $T_1 \geq T_2 \geq \cdots$, then so is $\inf_n T_n$
  - if $s \geq 0$ and S is a stopping time, then so $S + s$
- For a Borel measurable set A, let $T_A = \inf\{t > 0 : X_t \in A\}$. Suppose $\mathcal{F}_t$ satisfies the usual conditions and X_t has continuous paths,
  - if A is open, then T_A is a stopping time

- if A is closed, then T_A is a stopping time

- approximation of stopping time from the right: if T is a finite stopping time, define

$$T_n(\omega) = \frac{k+1}{2^n} \quad \text{if } \frac{k}{2^n} \leq T(\omega) < \frac{k+1}{2^n}.$$

  Note that $\{T_n\}$ are stopping times decreasing to T.

- define $\mathcal{F}_T = \{A \in \mathcal{F} : \text{ for each } t \geq 0, A \cap \{\omega : T \leq t\} \in \mathcal{F}_t\}$, suppose $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration satisfying the usual conditions
  - $\mathcal{F}_T$ is σ-field
  - if $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$
  - if $\mathcal{F}_{T+} = \cap_{\epsilon>0}\mathcal{F}_{T+\epsilon}$, then $\mathcal{F}_{T+} = \mathcal{F}_T$
  - if X_t has right-continuous paths, then X_T is $\mathcal{F}_T$ measurable

3.4 The optional stopping theorem

Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions. If M_t is a martingale or non-negative submartingale whose paths are right continuous, $\sup_{t\geq 0} \mathbb{E}[|M_t^2|] < \infty$, and T is a finite stopping time, then $\mathbb{E}[M_T] \geq \mathbb{E}[M_0]$.

3.5 Convergence and regularity

Let $\mathcal{D}_n = \{k/2^n : k \geq 0\}, \mathcal{D} = \cup_n \mathcal{D}_n$.

- Let $\{M_t : t \in \mathcal{D}\}$ be either a martingale, a submartingale, or a supermartingale w.r.t. $\{\mathcal{F}_t : t \in \mathcal{D}\}$ and suppose $\sup_{t\in\mathcal{D}} \mathbb{E}[|M_t|] < \infty$. Then
  - (1) $\lim_{t\to\infty} M_t$ exists, a.s.
  - (2) With probability one M_t has left and right limits along $\mathcal{D}$.

- Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions, and let M_t be a martingale w.r.t. $\{\mathcal{F}_t\}$. Then M has a version that is also a martingale and that in addition has paths that are right continuous with left limits.

- increasing paths: a process A_t has increasing paths if the function $t \to A_t(\omega)$ is increasing for almost every ω

- Suppose $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions and suppose A_t is an adapted process with paths that are increasing, are right continuous with left limits, and $A_\infty = \lim_{t\to\infty} A_t$ exists, a.s. Suppose X is non-negative integrable random variable, and M_t is a version of the martingale $\mathbb{E}[X|\mathcal{F}_t]$ which has paths that are right continuous with left limits. Suppose $\mathbb{E}[XA_\infty] < \infty$. Then

$$\mathbb{E}\left[\int_0^\infty X\, dA_s\right] = \mathbb{E}\left[\int_0^\infty M_s\, dA_s\right].$$

  The above equation also can be rewritten as

$$\mathbb{E}\left[\int_0^\infty X\, dA_s\right] = \mathbb{E}\left[\int_0^\infty \mathbb{E}[X|\mathcal{F}_s]\, dA_s\right].$$

  From above, for each t, we also have

$$\mathbb{E}\left[\int_0^t X\, dA_s\right] = \mathbb{E}\left[\int_0^t \mathbb{E}[X|\mathcal{F}_s]\, dA_s\right].$$

3.6 Some applications of martingales

- If W_t is a Brownian motion, then
$$\mathbb{P}\left(\sup_{s\le t} W_s \ge \lambda\right) \le e^{-\frac{\lambda^2}{2t}}, \quad ,\lambda > 0,$$
and
$$\mathbb{P}\left(\sup_{s\le t} |W_s| \ge \lambda\right) \le 2e^{-\frac{\lambda^2}{2t}}, \quad ,\lambda > 0.$$

- Let W be a Brownian motion, let $a, b > 0$ and let $T = \inf\{t > 0 : W_t \notin [-a, b]\}$. Then
$$\mathbb{P}(W_T = -a) = \frac{b}{a+b}, \quad \mathbb{P}(W_T = b) = \frac{a}{a+b},$$
and
$$\mathbb{E}[T] = ab.$$

- Suppose M_t is a martingale with continuous paths and with $M_0 = 0$ a.s., $T = \inf\{t > 0 : M_t \notin [-a, b]\}$, and $T < \infty$ a.s. Then
$$\mathbb{P}(M_T = -a) = \frac{b}{a+b}, \quad \mathbb{P}(M_T = b) = \frac{a}{a+b}.$$

- Let W be a Brownian motion, let $a, b > 0$ and let $T = \inf\{t > 0 : W_t \notin [-a, b]\}$. Then
$$\mathbb{E}\left[e^{-r^2T/2}\mathbb{1}_{\{W_T=-a\}}\right] = \frac{e^{rb} - e^{-rb}}{e^{r(a+b)} - e^{-r(a+b)}}$$
and
$$\mathbb{E}\left[e^{-r^2T/2}\mathbb{1}_{\{W_T=b\}}\right] = \frac{e^{ra} - e^{-ra}}{e^{r(a+b)} - e^{-r(a+b)}}.$$

4 Markov properties of Brownian motion

4.1 Markov properties

- Markov property: let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions, and let W be a Brownian motion w.r.t. $\{\mathcal{F}_t\}$. If u is a fixed time, then $Y_t = W_{t+u} - W_u$ is a Brownian motion independent of $\mathcal{F}_u$.

- Strong Markov property: let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions, and let W be a Brownian motion w.r.t. $\{\mathcal{F}_t\}$. If T is a stopping time, then $Y_t = W_{t+T} - W_T$ is a Brownian motion independent of $\mathcal{F}_T$.

- General process: let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions, and let X be a process adapted to $\{\mathcal{F}_t\}$. Suppose X has paths that are right continuous with left limits and suppose $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same law with X_{t-s} whenever $s < t$. If T is a finite stopping time, then $Y_t = W_{t+T} - W_T$ is a process that is independent of $\mathcal{F}_T$ and X and Y have the same law.

4.2 Applications

- The reflection of Brownian motion: let W_t be a Brownian motion, $b > 0$, $T = \inf\{t : W_t \ge b\}$, and $x < b$. Then
$$\mathbb{P}\left(\sup_{s\le t} W_s \ge b, W_t < x\right) = \mathbb{P}\left(W_t > 2b - x\right).$$

- Let W_t be a Brownian motion w.r.t. a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let T be a finite stopping time and $s > 0$. If $a < b$, then
$$\mathbb{P}\left(W_{T+s} \in [a, b] | \mathcal{F}_T\right) \le \frac{|b-a|}{\sqrt{2\pi s}}.$$

5 The Poisson process

The Poisson process is the prototype of a pure jump process, and it is the building block for Lévy process.

- Definition: Let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions. A Poisson process with parameter $\lambda > 0$ is a stochastic process X satisfying the following properties:
  - (1) $X_0 = 0$ a.s.
  - (2) The paths of X_t are right continuous with left limits
  - (3) If $s < t$, then $X_t - X_s$ is a Poisson random variable with parameter $\lambda(t-s)$
  - (4) If $s < t$, then $X_t - X_s$ is independent of $\mathcal{F}_s$
- $X_{t-} = \lim_{s \to t, s<t} X_s$ be the left-hand limit at time t, and $\Delta X_t = X_t - X_{t-}$ be the size of the jump at time t
- Let X be a Poisson process,
  - with probability one, the paths of X_t are increasing
  - with probability one, the paths of X_t are constant except for jumps of size 1
  - there are only finitely many jumps in each finite time interval
- Let $T_1 = \inf\{t : \Delta X_t = 1\}$, the time of the first jump. Define $T_{i+1} = \inf\{t > T_i : \Delta X_t = 1\}$, so T_i is the time of the i-th jump. The random variables $T_1, T_2 - T_1, \cdots, T_{i+1} - T_i, \cdots$ are independent exponential random variables with parameter λ.
- The construction of Poisson process: let $U_1, U_2, \cdots$ be independent exponential random variable with parameter λ and let $T_j = \sum_{i=1}^{j} U_i$. Define

$$X_t(\omega) = k, \text{ if } T_k(\omega) \le t < T_{k+1}(\omega).$$

- The densities shows that an exponential random variable has a Gamma distribution with parameter λ and 1. Then by the invariant summation of Gamma distribution, T_j is a Gamma random variable with parameters λ and j. Thus

$$\mathbb{P}(X_t < k) = \mathbb{P}(T_k > t) = \int_t^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}\, dx.$$

  Performing the integration by parts repeatedly shows that

$$\mathbb{P}(X_t < k) = \sum_{i=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!},$$

  thus X_t is a Poisson random variable with parameter λt.
- Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions. Suppose $X_0 = 0$ a.s., X has paths that are right continuous with left limits, $X_t - X_s$ is independent of $\mathcal{F}_s$ if $s < t$ and $X_t - X_s$ has the same law with X_{t-s} whenever $s < t$. If the paths of X are piecewise constant, increasing, all the jumps of X are of size 1, and X is not identically 0, then X is a Poisson process.

6 Construction of Brownian motion

There are several ways of constructing Brownian motion, none of them easy. Here gives two constructions. The first is the one that Wiener used, which is based on Fourier series. The second uses martingale techniques, which is due to Lévy.

- Wiener's construction
  - The main step is to construct W_t for $t \in [0, 1]$.

- Take independent copies $Y^{(1)}, Y^{(2)}, \cdots$ each on $[0,1]$, then let

$$W_t = \left(\sum_{i=0}^{[t]-1} Y_1^{(i)} \right) + Y_{t-[t]}^{[t]}.$$

- Fix $t \in [0, \pi]$, the Fourier series for the function $f(s) = s \wedge t$ is

$$s \wedge t = \frac{st}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(ks)\sin(kt)}{k^2}.$$

- Let $Z_0, Z_1, \cdots$ be i.i.d. standard normal random variables and let

$$W_t = \frac{t}{\sqrt{\pi}} Z_0 + \sum_{k=1}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin(kt)}{k} \right) Z_k,$$

  then W_t is a Gaussian process, has mean zero and $\mathrm{Cov}(W_s, W_t) = s \wedge t$. And we also can show that W_t has continuous paths. Thus W as constructed above has the correct finite-dimensional distributions to be a Brownian motion.

- Martingale method
  - Proceed as in the previous method to construct $\{W_t : 0 \le t \le \pi\}$, where W_t is a Gaussian process with $\mathbb{E}[W_t] = 0$ and $\mathrm{Cov}(W_s, W_t) = s \wedge t$, and we need to show that W has a version with continuous paths.
  - First we show that W is a martingale, and so has a version with paths that are right continuous with left limits. We use Doob's inequalities to control the oscillation of W over short time intervals, and then use the Borel–Cantelli lemma to show continuity.
  - Theorem: if $\{W_t : 0 \le t \le 1\}$ is a Gaussian process with $\mathbb{E}[W_t] = 0$ and $\mathrm{Cov}(W_s, W_t) = s \wedge t$ for all $0 \le s, t \le 1$, then there is a version of W that is a Brownian motion on $[0,1]$.
  - There is nothing special about the trigonometric polynomials in the martingale method.