Solvability of McKean-Vlasov FBSDEs by Schauder's Theorem and its application to the MFG system

Jiamin Jian

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Reference: Carmona René, and François Delarue. *Probabilistic theory of mean field games with applications I: Mean Field FBSDEs, Control, and Games*, volume 83 of Probability Theory and Stochastic Modelling. 2018.

1 Solvability of McKean-Vlasov FBSDEs by Schauder's Theorem

1.1 Problem setup

Let $(\Omega, \mathcal{F}_T, \mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a completed filtered probability space satisfying the usual conditions, on which $W = (W_t)_{0\le t\le T}$ is a standard Brownian motion. Our goal is to prove existence (but not necessarily uniqueness) of a solution to a fully coupled McKean-Vlasov forward-backward system

$$\begin{cases} dX_t = B(t, X_t, Y_t, Z_t, \mathcal{L}(X_t, Y_t))\, dt + \Sigma(t, X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dW_t, \\ dY_t = -F(t, X_t, Y_t, Z_t, \mathcal{L}(X_t, Y_t))\, dt + Z_t\, dW_t, \end{cases} \tag{1}$$

with initial condition $X_0 = \xi$ for some $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R})$ and terminal condition $Y_T = G(X_T, \mathcal{L}(X_T))$. The coefficient are assumed to be deterministic. The functions

- $B, F : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R} \times \mathbb{R}) \to \mathbb{R}$,
- $\Sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R} \times \mathbb{R}) \to \mathbb{R}$,
- $G : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$.

We can see that the McKean-Vlasov constraint in the system (1) involves the full-fledged distribution of the process $(X_t, Y_t)_{0\le t\le T}$.

1.2 Assumptions and main result

A 1 (Nondegenerate MKV FBSDE)**.** *There exists a constant $L \ge 1$ such that*

- *(A1) For any $t \in [0, T], x, x', y, y', z, z' \in \mathbb{R}, \nu, \nu' \in \mathcal{P}_2(\mathbb{R} \times \mathbb{R})$ and $\mu \in \mathcal{P}_2(\mathbb{R})$*

$$\begin{aligned} &|(B, F)(t, x', y', z', \nu) - (B, F)(t, x, y, z, \nu)| \le L|(x, y, z) - (x', y', z')|, \\ &|\Sigma(t, x', y', \nu) - \Sigma(t, x, y, \nu)| \le L|(x, y) - (x', y')|, \\ &|G(x', \mu) - G(x, \mu)| \le L|x - x'|. \end{aligned}$$

  Moreover, for any $(t, x, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, the coefficients $B(t, x, y, z, \cdot), F(t, x, y, z, \cdot), \Sigma(t, x, y, \cdot)$ and $G(x, \cdot)$ are continuous in the measure argument with respect to the 2-Wasserstein distance.

- *(A2) The functions Σ and G are bounded by L.*

- *(A3) For any $t \in [0, T], x, y, z \in \mathbb{R}$ and $\nu \in \mathcal{P}_2(\mathbb{R} \times \mathbb{R})$,*

$$\begin{aligned} &|B(t, x, y, z, \nu)| \le L\left[1 + |y| + |z| + M_2(\nu)\right], \\ &|F(t, x, y, z, \nu)| \le L\left[1 + |y| + |z| + M_2(\nu \circ \pi^{-1})\right], \quad \pi(x, y) = y. \end{aligned}$$

- *(A4) The function Σ is positive and continuous.*

Remark 1. *$M_2(\nu)$ denotes the square root of the second moment of ν. We also notice that A(3) in Assumption 1 can be rewritten as*

$$
\begin{aligned}
|B(t,x,y,z,\mathcal{L}(X,Y))| &\le L\left[1+|y|+|z|+\mathbb{E}\left[|X|^2+|Y|^2\right]^{\frac{1}{2}}\right],\\
|F(t,x,y,z,\mathcal{L}(X,Y))| &\le L\left[1+|y|+|z|+\mathbb{E}\left[|Y|^2\right]^{\frac{1}{2}}\right],
\end{aligned}
$$

for any square-integrable random variable X and Y.

Theorem 2. *Under assumption 1, for any random variable $\xi \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R})$, the FBSDE system (1) has a solution $(X,Y,Z)=(X_t,Y_t,Z_t)_{0\le t\le T}$ with $X_0=\xi$ as initial condition.*

1.3 Idea to prove the main result

We expect Y_t and X_t to be connected by a deterministic relationship of the form $Y_t=\varphi(t,X_t)$, where φ be a function from $[0,T]\times\mathbb{R}$ to $\mathbb{R}$. If that is indeed the case, the law of the pair (X_t,Y_t) is entirely determined by the law of X_t since the distribution $\mathcal{L}(X_t,Y_t)$ of (X_t,Y_t) is equal to $(I,\varphi(t,\cdot))(\mathcal{L}(X_t))=(\mathcal{L}(X_t))\circ(I,\varphi(t,\cdot))^{-1}$. For a probability measure μ in $\mathbb{R}$ and for a measurable mapping ψ from $\mathbb{R}$ to $\mathbb{R}$, we shall denote by $\psi\diamond\mu$ the image of μ under the map $(I,\psi):x\to(x,\psi(x))$, that is $\psi\diamond\mu=\mu\circ(I,\psi)^{-1}$. With this notion, it is natural to look for a function $\varphi:[0,T]\times\mathbb{R}\to\mathbb{R}$ and a flow $t\to\mu_t\in\mathcal{P}_2(\mathbb{R})$ such that

$$
\begin{cases}
dX_t = B(t,X_t,Y_t,Z_t,\varphi(t,\cdot)\diamond\mu_t)\,dt+\Sigma(t,X_t,Y_t,\varphi(t,\cdot)\diamond\mu_t)\,dW_t,\\
dY_t = -F(t,X_t,Y_t,Z_t,\varphi(t,\cdot)\diamond\mu_t)\,dt+Z_t\,dW_t,
\end{cases}
\tag{2}
$$

under the constraints that $Y_t=\varphi(t,X_t)$ and $\mu_t=\mathcal{L}(X_t)$ for $t\in[0,T]$, and with the initial condition $X_0=\xi$ and terminal condition $Y_T=G(X_T,\mathcal{L}(X_T))$.

The strategy we use below consists in recasting the stochastic system (2) into a fixed point problem over the arguments $(\varphi,(\mu_t)_{0\le t\le T})$:

- Step 1: Using $\varphi(t,\cdot)\diamond\mu_t$ as an input and solve (2) as a standard FBSDE.
- Step 2: A fixed point argument such that $Y_t=\varphi(t,X_t)$ and $\mu_t=\mathcal{L}(X_t)$.

The fixed point problem is solved by means of Schauder's fixed point theorem. This provides existence of a fixed point from compactness arguments. However, it is important to keep in mind that it does not say anything about uniqueness. We here recall the statement of Schauder's theorem for the sake of completeness.

Theorem 3. *Let $(V,\|\cdot\|)$ be a normed linear vector space and E be a nonempty closed convex subset of V. Then, any continuous mapping from E to itself which has a relatively compact range has a fixed point.*

1.4 Preliminary Step: Structure of the Solution for a Given Input

Lemma 4. *Fix $T>0$ and on top of assumption 1, assume that, instead (A2), B and F have the following growth property:*

$$
|(B,F)(t,x,y,z,\nu)|\le L\left[1+|y|+|z|\right],
$$

for all $(t,x,y,z,\nu)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathcal{P}(\mathbb{R}\times\mathbb{R})$.

Then, give a deterministic continuous function $\boldsymbol{\nu}:[0,T]\ni t\mapsto\nu_t\in\mathcal{P}(\mathbb{R}\times\mathbb{R})$, a probability measure $\mu\in\mathcal{P}(\mathbb{R})$, and an initial condition $(t,\xi)\in[0,T]\times L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})$, the forward-backward system

$$
\begin{cases}
dX_s = B(s,X_s,Y_s,Z_s,\nu_s\,ds+\Sigma(s,X_s,Y_s,\nu_s)\,dW_s,\\
dY_s = -F(s,X_s,Y_s,Z_s,\nu_s)\,ds+Z_s\,dW_s, \quad s\in[t,T],
\end{cases}
\tag{3}
$$

with $X_t=\xi$ as initial condition and $Y_T=G(X_T,\mu)$ as terminal condition, has a unique solution, denoted by $(X_s^{t,\xi},Y_s^{t,\xi},Z_s^{t,\xi})_{t\le s\le T}$. Moreover, the decoupling field $u:[0,T]\times\mathbb{R}\ni(t,x)\mapsto u(t,x)=Y_t^{t,x}\in\mathbb{R}$ obtained by choosing $\xi=x$ is bounded by a constant γ depending only upon T and L, and 1/2-Hölder continuous in time and Lipschitz continuous in space in the sense that

$$
|u(t,x)-u(t',x')|\le\Gamma\left(|t-t'|^{1/2}+|x-x'|\right),
$$

for some constant Γ only depending upon T and L. In particular, both γ and Γ are independent of ν and μ. Finally, it holds that $Y_s^{t,\xi} = u(s, X_s^{t,\xi})$ for any $t \leq s \leq T$ and $|Z_s^{t,\xi}| \leq \Gamma L$, $ds \otimes \mathbb{P}$ almost everywhere.

For the time being, we use this existence result in the following way. We start with a bounded continuous function φ from $[0,T] \times \mathbb{R}$ into $\mathbb{R}$, and a flow of probability measures $\boldsymbol{\mu} = (\mu_t)_{0 \leq t \leq T}$ in $C([0,T]; \mathcal{P}_2(\mathbb{R}))$, which we want to think of as the flow of marginal laws $(\mathcal{L}(X_t))_{0 \leq t \leq T}$ of the solution. We apply the above existence result for (3) to $\mu = \mu_T$ and $\nu_t = \varphi(t, \cdot) \diamond \mu_t$ for $t \in [0,T]$ and solve

$$\begin{cases} dX_t = B(t, X_t, Y_t, Z_t, \varphi(t, \cdot) \diamond \mu_t)\, dt + \Sigma(t, X_t, Y_t, \varphi(t, \cdot) \diamond \mu_t)\, dW_t, \\ dY_t = -F(t, X_t, Y_t, Z_t, \varphi(t, \cdot) \diamond \mu_t)\, dt + Z_t\, dW_t, \end{cases}$$

with terminal condition $Y_T = G(X_T, \mu_T)$ and initial condition $X_0 = \xi$. The following estimate will be instrumental in the proof of the main result.