The master equation

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1 The master equation

- Author: Daniel Lacker
- Notes: Mean field games and interacting particle systems
- Link: http://www.columbia.edu/~dl3133/MFGSpring2018.pdf

1.1 Introduction

- Develops an analytic approach to mean field games.
- In some ways the master equation plays the role of an HJB equation.
- For mean field games, the value function should depend on $(t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$.
- The value $V(t, x, m)$ should be read as "the remaining value at time t for an agent with current state x given that the distribution of the continuum of other agents is m".

1.2 Calculus on $\mathcal{P}(\mathbb{R}^d)$

- Definition: a function $U : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is C^1
- Define the intrinsic derivative $D_m U : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$
- Give three examples to show how to get the intrinsic derivatives
- Define the second order derivative of functions on $\mathcal{P}(\mathbb{R}^d)$
- Lemma: shows that $D^s_m U(m, v, v') = D^2_m U(m, v', v)$ and $D^2_m U(m, v, v') = D_m[D_m U(\cdot, v)](m, v')$ if U is C^2
- Proposition: shows how the derivatives on $\mathcal{P}(\mathbb{R}^d)$ interact with empirical measures
- Lemma: shows that for a function $U = U(x, m)$ on $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$, the derivatives in x and in m commute.

1.3 Itô's formula for $F(X_t, \mu_t)$

Derive an Itô's formula for smooth functions of an Itô's process and a conditional measure flow associated to a potentially different Itô's process

$$\begin{aligned} dX_t &= b(X_t, \mu_t)\, dt + \sigma(X_t, \mu_t)\, dW_t + \gamma(X_t, \mu_t)\, dB_t, \quad X_0 = \xi \\ d\bar{X}_t &= \bar{b}(\bar{X}_t, \mu_t)\, dt + \bar{\sigma}(\bar{X}_t, \mu_t)\, dW_t + \bar{\gamma}(\bar{X}_t, \mu_t)\, dB_t, \quad \bar{X}_0 = \bar{\xi} \end{aligned}$$

where W, B are independent Brownian motions and $\mu_t = \mathcal{L}(\bar{X}_t | \mathcal{F}^B_t)$, where $(\mathcal{F}^B_t)_{t \in [0,T]}$ is the filtration generated by B. The Itô's formula reads

$$dF(X_t, \mu_t) = \big(LF(X_t, \mu_t) + \bar{L}F(X_t, \mu_t)\big)\, dt + \text{martingale},$$

where the operators L and $\bar{L}$ is referred to the notes.

Note that when $b = \bar{b}, \sigma = \bar{\sigma}, \gamma = \bar{\gamma}$ and $\xi = \bar{\xi}$, then $X = \bar{X}$, and the Itô's formula describes the dynamics of functions of the pair (X_t, μ_t) coming from a McKean-Vlasov equation.

1.4 Feynman-Kac formula

Derive PDEs for expectations of functionals of the pair (X_t, μ_t). Let $(t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$, and consider processes X and $\bar{X}$ that solves the SDEs

$$
\begin{aligned}
dX_s^{t,x,m} &= b(X_s^{t,x,m}, \mu_s^{t,m})\, ds + \sigma(X_s^{t,x,m}, \mu_s^{t,m})\, dW_s + \gamma(X_s^{t,x,m}, \mu_s^{t,m})\, dB_s \\
d\bar{X}_s^{t,m} &= \bar{b}(\bar{X}_s^{t,m}, \mu_s^{t,m})\, ds + \bar{\sigma}(\bar{X}_s^{t,m}, \mu_s^{t,m})\, dW_s + \bar{\gamma}(\bar{X}_s^{t,m}, \mu_s^{t,m})\, dB_s,
\end{aligned}
$$

on $s \in [t, T]$, with initial conditions $X_t^{t,x,m} = x$ and $\bar{X}_t^{t,m} \sim m$, and with $\mu_s^{t,m} = \mathcal{L}\left(\bar{X}_s^{t,m} | (B_r - B_t)_{r\in[t,s]}\right)$. Define the value function

$$
V(t, x, m) = \mathbb{E}\left[g\left(X_T^{t,x,m}, \mu_T^{t,m}\right) + \int_t^T f\left(X_s^{t,x,m}, \mu_s^{t,m}\right) ds \right],
$$

for some given nice functions g and f. Suppose $U : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is smooth and satisfies

$$
\begin{aligned}
\partial_t U(t, x, m) + LU(t, x, m) + \bar{L}U(t, x, m) + f(x, m) &= 0 \\
U(T, x, m) &= g(x, m).
\end{aligned}
$$

Then $U = V$.

1.5 Verification theorem

Let $(t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$, and consider the controlled processes X and $\bar{X}$ that solves the SDEs

$$
\begin{aligned}
dX_s^{t,x,m} &= b(X_s^{t,x,m}, \mu_s^{t,m}, \alpha_s)\, ds + \sigma(X_s^{t,x,m}, \mu_s^{t,m}, \alpha_s)\, dW_s + \gamma(X_s^{t,x,m}, \mu_s^{t,m})\, dB_s \\
d\bar{X}_s^{t,m} &= \bar{b}(\bar{X}_s^{t,m}, \mu_s^{t,m})\, ds + \bar{\sigma}(\bar{X}_s^{t,m}, \mu_s^{t,m})\, dW_s + \bar{\gamma}(\bar{X}_s^{t,m}, \mu_s^{t,m})\, dB_s,
\end{aligned}
$$

on $s \in [t, T]$, with initial conditions $X_t^{t,x,m} = x$ and $\bar{X}_t^{t,m} \sim m$, and with $\mu_s^{t,m} = \mathcal{L}\left(\bar{X}_s^{t,m} | (B_r - B_t)_{r\in[t,s]}\right)$. Define the value function

$$
V(t, x, m) = \sup_{\alpha} \mathbb{E}\left[g\left(X_T^{t,x,m}, \mu_T^{t,m}\right) + \int_t^T f\left(X_s^{t,x,m}, \mu_s^{t,m}, \alpha_s\right) ds \right],
$$

for some given nice functions g and f. Define the Hamiltonian

$$
H(x, m, y, z) = \sup_{a\in A} \left[y \cdot b(x, m, a) + \frac{1}{2} \mathrm{Tr}\left[z\left(\sigma\sigma^\top + \gamma\gamma^\top\right)(x, m, a)\right] + f(x, m, a) \right].
$$

Suppose $U : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is smooth and satisfies

$$
\begin{aligned}
\partial_t U(t, x, m) + H(x, m, D_x U(t, x, m), D_x^2 U(t, x, m)) + \bar{L}U(t, x, m) &= 0 \\
U(T, x, m) &= g(x, m).
\end{aligned}
$$

Suppose also that there exists a measurable function $\alpha : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to A$ such that $\alpha(t, x, m)$ attains the supremum in $H(x, m, D_x U(t, x, m), D_x^2 U(t, x, m))$ for each (t, x, m) and also the SDE

$$
dX_t = b(X_t, \mu_t, \alpha(t, X_t, \mu_t))dt + \sigma(X_t, \mu_t, \alpha(t, X_t, \mu_t))dW_t + \gamma(X_t, \mu_t)dB_t
$$

is well-posed. Then $U = V$, and $\alpha(t, X_t, \mu_t)$ is an optimal control.

1.6 The master equation for mean field game

Let $b = \bar{b}, \sigma = \bar{\sigma}, \gamma = \bar{\gamma}$ and $\xi = \bar{\xi}$, then $X = \bar{X}$, the dynamics of X and $\bar{X}$ are the same.

In order to define the value function for mean field fame, we must take care that the equilibrium is unique. We assume that for each $(t, m) \in [0, T] \times \mathcal{P}(\mathbb{R}^d)$, there is a unique mean field equilibrium $(\mu_s^{t,m})_{s\in[t,T]}$ starting from (t, m).

Define the value function as

$$
V(t, x, m) = \sup_{\alpha} \mathbb{E}\left[g\left(X_T^{t,x,m}, \mu_T^{t,m}\right) + \int_t^T f\left(X_s^{t,x,m}, \mu_s^{t,m}, \alpha_s\right) ds \right],
$$

where

$$dX_s = b(X_s, \mu_s^{t,m}, \alpha(s))ds + \sigma(X_s, \mu_s^{t,m}, \alpha(s))dW_s + \gamma(X_s, \mu_s^{t,m})dB_s, \quad s \in [t, T]$$

with $X_t = x$. This quantity $V(t, x, m)$ is the remaining in-equilibrium value, after time t, to a player starting from state x at time t, given that the distribution of other agents starts at m at time t.

Define the Hamiltonian

$$H(x, m, y, z) = \sup_{a \in A} \left[y \cdot b(x, m, a) + \frac{1}{2}\mathrm{Tr}\left[z \left(\sigma\sigma^\top + \gamma\gamma^\top \right)(x, m, a) \right] + f(x, m, a) \right].$$

Let $\alpha(x, m, y, z)$ denote a maximizer. Define

$$\begin{aligned} \hat{b}(x, m, y, z) &= b(x, m, \alpha(x, m, y, z)), \\ \hat{\sigma}(x, m, y, z) &= \sigma(x, m, \alpha(x, m, y, z)). \end{aligned}$$

The master equation then takes the form

$$\begin{aligned} 0 =& \partial_t U(t, x, m) + H(x, m, D_x U(t, x, m), D_x^2 U(t, x, m)) \\ &+ \int_{\mathbb{R}^d} D_m U(t, x, m, v) \cdot \hat{b}(v, m, D_x U(t, v, m), D_x^2 U(t, v, m)) m(dv) \\ &+ \int_{\mathbb{R}^d} \frac{1}{2}\mathrm{Tr}\left[D_v D_m U(t, x, m, v)(\hat{\sigma}\hat{\sigma}^\top + \gamma\gamma^\top)(v, m, D_x U(t, v, m), D_x^2 U(t, v, m)) \right] m(dv) \\ &+ \int\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}\mathrm{Tr}\left[D_m^2 U(t, x, m, v, \tilde{v})\gamma(v, m)\gamma^\top(\tilde{v}, m) \right] m(dv) m(d\tilde{v}) \\ &+ \int_{\mathbb{R}^d} \mathrm{Tr}\left[D_x D_m U(t, x, m, v)\gamma(x, m)\gamma^\top(v, m) \right] m(dv), \end{aligned}$$

with the terminal condition $U(T, x, m) = g(x, m)$.

Suppose $U = U(t, x, m)$ is a smooth solution of the above master equation, then $U = V$. Moreover, the equilibrium control is given by $\alpha(t, x, m) = \alpha(x, m, D_x U(t, x, m), D_x^2 U(t, x, m))$, assuming that this function is nice enough for the Mckean-Vlasov SDE

$$\begin{aligned} dX_t = b(X_t, \mu_t, \alpha(t, X_t, \mu_t))dt + \sigma(X_t, \mu_t, \alpha(t, X_t, \mu_t))dW_t + \gamma(X_t, \mu_t)dB_t \\ \mu_t = \mathcal{L}(X_t | \mathcal{F}_t^B), \end{aligned}$$

to be well-posed. Finally, the unique solution $\mu = (\mu_t)_{t \in [0,T]}$ of this McKean-Vlasov equation is the unique mean field equilibrium.

1.7 Simplifications of the master equation

- Drift control and constant volatility: assume that the MFG is as follows:

  $$\begin{aligned} \alpha^* \in \arg\max_\alpha \mathbb{E}\left[g(X_T, \mu_T) + \int_t^T f_1(X_t, \mu_t) - f_2(X_t, \alpha_t) dt \right] \\ dX_t = \alpha_t dt + \sigma dW_t + \gamma dB_t,\ \mu_t = \mathcal{L}(X_t | \mathcal{F}_t^B), \quad \forall t \in [0, T]. \end{aligned}$$

  In this example we have that $A = \mathbb{R}^d$, and the Hamiltonian is

  $$H(x, y) = \sup_{a \in \mathbb{R}^d} (a \cdot y - f_2(x, a)).$$

  Hence the optimizer in the Hamiltonian is $\alpha(x, y) = D_y H(x, y)$. Next, we can specialize to the case where $\sigma = 1$ and $\gamma = 0$.

- A linear quadratic model: consider the MFG problem is given by

  $$\begin{aligned} \alpha^* \in \arg\max_\alpha \mathbb{E}\left[-\frac{\lambda}{2}|\bar{X}_T - X_T| - \int_t^T \frac{1}{2}|\alpha_t|^2 dt \right] \\ dX_t = \alpha_t dt + dW_t,\ \bar{X}_t = \mathbb{E}[X_t]. \end{aligned}$$

In this example, we have that the Hamiltonian is $H(x,y) = \sup_a(a \cdot y - \frac{1}{2}|a|^2) = \frac{1}{2}|y|^2$ with the optimizer $\alpha(x,y) = y$. We can write down the master equation.

In order to try to get a solution, we start making a first ansatz, which consists of assuming that the solution depends only on the mean of the distribution and not on the whole probability distribution, i.e. there exists a function $F : [0,T] \times (\mathbb{R}^d)^2$ such that $U(t,x,m) = F(t,x,\bar{m})$. The final ansatz is to separate the spatial and temporal variables, i.e. assume that

$$F(t,x,y) = \frac{1}{2}f(t)|y-x|^2 + g(t).$$

Then we can solve the master equation by solving an ODE system.

1.8 Bibliographic notes

- It is not clear yet if there is a satisfactory viscosity theory for the master equation, with a key challenge posed by the the infinite-dimensional state space $\mathcal{P}(\mathbb{R}^d)$.
- The strong well-posedness results and explain how the master equation, when it admits a smooth function, can be used to prove the convergence of the n-player Nash equilibria to the mean field equilibrium. This kind of analysis is extremely fruitful when it works, but it notably requires uniqueness for the MFE and a quite regular solution of the master equation.