Stochastic Process

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Reference: Bass, Richard F. *Stochastic Processes*. Vol. 33. Cambridge University Press, 2011.

1 Basic notions

1.1 Processes and σ-field

- Stochastic process: let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. $X : [0, \infty) \times \Omega \mapsto \mathbb{R}$
- Filtration $\{\mathcal{F}_t\}_{t\geq 0}$:
  - definition: $\mathcal{F}_t \subset \mathcal{F}, \forall t$, and $\mathcal{F}_s \subset \mathcal{F}_t, \forall s \leq t$
  - right continuous: define $\mathcal{F}_{t+} = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon}$, if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t > 0$
  - meaning of right continuous: there is no information just after time t which is not already given time t or before
  - null sets N: $\inf\{\mathbb{P}(A) : N \subset A, A \in \mathcal{F}\} = 0$
  - complete: $\mathcal{F}_t$ contains every null set
  - usual conditions: a filtration is right continuous and complete
- $\mathcal{F}_\infty := \sigma\left(\cup_{t\geq 0}\mathcal{F}_t\right) := \bigvee_{t\geq 0} \mathcal{F}_t$
- The arbitrary intersection of σ-fields is a σ-field, but the union of two σ-fields need not to be a σ-field: let $\Omega = \{a, b, c\}$, let $\mathcal{A}_1 = \{\{a\}, \{b, c\}, \emptyset, \Omega\}$, $\mathcal{A}_2 = \{\{b\}, \{a, c\}, \emptyset, \Omega\}$.
- Adapted: a stochastic process X is adapted to a filtration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$ measurable for each t.
- Minimal augmented filtration generated by X: the smallest filtration that is right continuous and complete and w.r.t. which the process X is adapted
  - let $\{\mathcal{F}_t^{00}\}$ be the smallest filtration w.r.t. which X is adapted
$$\mathcal{F}_t^{00} = \sigma(X_s : s \leq t)$$
    we say $\{\mathcal{F}_t^{00}\}$ be the filtration generated by X.
  - let $\mathcal{N}$ be the collection of null sets, so that $\mathcal{N} = \{A \subset \Omega : \mathbb{P}^*(A) = 0\}$, let
$$\mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N}).$$
  - let
$$\mathcal{F}_t = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon}^0.$$
- Distinguishable and versions
  - distinguishable: $\mathbb{P}(X_t \neq Y_t, \text{for some } t > 0) = 0$
  - versions (modification): $\mathbb{P}(X_t \neq Y_t) = 0$, for each $t \geq 0$
  - example that two process that are versions of each other but are not indistinguishable: let $\Omega = [0, 1]$, $\mathcal{F}$ be the Borel σ-field on $[0, 1]$, $\mathbb{P}$ be the Lebesgue measure on $[0, 1]$, $X(t, \omega) = 0$ for all t and ω, and $Y(t, \omega) = 1$ if $t = \omega$ and $Y(t, \omega) = 0$ otherwise. Note that $t \to X(t, \omega)$ are continuous for each ω, but the function $t \to Y(t, \omega)$ are not continuous for any ω.

- Paths (trajectories): the function $t \to X(t,\omega)$. There will be one path for each ω
- Continuous process: if the paths of X are continuous functions, except for a set of ω's in a null set
- Function which is right continuous with left limits:

$$\lim_{h>0,h\downarrow 0} f(t+h) = f(t) \quad \text{and} \quad \lim_{h<0,h\uparrow 0} f(t+h) \text{ exists}, \quad \forall t > 0.$$

- Cadlag: paths that are right continuous with left limits

1.2 Laws and state space

2 Brownian motion

2.1 Definition and basic properties

- Definition of Brownian motion
  - $\mathcal{F}_t$ measurable for each $t \geq 0$
  - $W_0 = 0$, a.s. (standrad Brownian motion)
  - $W_t - W_s \sim \mathcal{N}(0, t-s)$, $\forall s < t$ ($W_t - W_s$ has the same law with W_{t-s})
  - $W_t - W_s$ is independent of $\mathcal{F}_s$ whenever $s. < t$
  - W_t has continuous paths
- d-dimensional Brownian motion: $\left(W_t^{(1)}, W_t^{(2)}, \cdots, W_t^{(d)}\right)$
- Wiener measure: $\mathbb{P}_W(A) = \mathbb{P}(W \in A)$ for all Borel subsets A of $C([0,\infty))$
- $Y_t = aW_{t/a^2}$ is a Brownian motion started at 0
- Jointly normal: A sequence of random variables $X_1, X_2, \cdots, X_n$ is said to be jointly normal if there exists a sequence of i.i.d. normal random variables $Z_1, Z_2, \cdots, Z_m$ with mean zero and variance one and constants b_{ij} and a_i such that

$$X_i = \sum_{j=1}^{m} b_{ij} Z_j + a_i, \quad \forall i = 1, 2, \cdots, n$$

  In matrix notation $X = BZ + A$.

- Gaussion process $\{X_t\}_{t\geq 0}$: for each $n \geq 1$ and $t_1 < t_2 < \cdots < t_n$, the collection of random variables $X_{t_1}, X_{t_2}, \cdots, X_{t_n}$ is a jointly normal collection.
- The Brownian motion W is a Gaussian process.
- $\mathrm{Cov}(W_t, W_s) = s \wedge t$
- If W is a process such that all the finite-dimensional distributions are jointly normal, $\mathbb{E}[W_s] = 0$ for all s, $\mathrm{Cov}(W_t, W_s) = s$ whenever $s \leq t$, and the paths of W_t are continuous, then W is a Brownian motion.
- Let W_t be a Brownian motion w.r.t. $\{\mathcal{F}_t^{00}\}$, where $\mathcal{F}_t^{00} = \sigma(W_s : s \leq t)$. Let $\mathcal{N}$ be the collection of null sets, $\mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N})$, and $\mathcal{F}_t = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^0$. Then
  - W is a Brownian motion w.r.t the filtration $\{\mathcal{F}_t\}$.
  - $\mathcal{F}_t = \mathcal{F}_t^0$ for each t.
  - W is a Brownian motion w.r.t. the filtration generated by W, then it is also a Brownian motion w.r.t. the minimal augmented filtration.
- Let $t_0 > 0$ and let X, Y be random variables taking values in $C([0, t_0])$ which have the same finite-dimensional distributions. Then the laws of X and Y are equal.
  - it shows that if W and W' are both Brownian motions, they have all the same properties.
  - But if X and Y have the same finite-dimensional distributions, they may have different properties. The example is $X(t,\omega) = 0, \forall t, \omega$; $Y = 1$ if $t = \omega$ and 0 otherwise.

3 Martingales

3.1 Definition and examples

- Definition of a continuous-time martingale
  - (1) $\mathbb{E}[|M_t|] < \infty$ for each t
  - (2) M_t is $\mathcal{F}_t$ measurable for each t
  - (3) $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, a.s., if $s < t$
- Submartingale and supermartingale
  - submartingale: (3) $\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s$, a.s., if $s < t$
  - supermartingale: (3) $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$, a.s., if $s < t$
  - if $s < t$, then $\mathbb{E}[M_s] \leq \mathbb{E}[M_t]$ if M is a submartingale, and $\mathbb{E}[M_s] \geq \mathbb{E}[M_t]$ if M is a supermartingale. Thus submartingales tends to increase on average, and supermartingale tends to decrease on average.
- Examples of martingales
  - $M_t = W_t$
  - $M_t = W_t^2 - t$
  - $M_t = e^{aW_t - \frac{1}{2}a^2 t}$, $a \in \mathbb{R}$
  - Let X be an integrable $\mathcal{F}$ measurable random variable, and let $M_t = \mathbb{E}[X|\mathcal{F}_t]$

3.2 Doob's inequality

Suppose M_t is a martingale or non-negative submartingale with paths that are right continuous with left limits. Then

- $$\mathbb{P}\left(\sup_{s\leq t} |M_s| \geq \lambda\right) \leq \frac{\mathbb{E}[|M_t|]}{\lambda}$$
- If $1 < p < \infty$, then
$$\mathbb{E}\left[\sup_{s\leq t} |M_s|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_t|^p\right]$$

3.3 Stopping time

- Definition: A random variable $T : \Omega \to [0, \infty]$ is a stopping time if $\{\omega \in \Omega : T < t\} \in \mathcal{F}_t$ for all t
- Boundedness: T is a finite stopping time if $T < \infty$ a.s., and T is a bounded stopping time if there exists $K \in [0, \infty)$ such that $T \leq K$ a.s.
- Some properties: suppose $\{F_t\}$ satisfies the usual condition
  - T is a stopping time if and only if $\{\omega : T \leq t\} \in \mathcal{F}_t$ for all t
  - if $T = t$ a.s., then T is a stopping time
  - if S and T are stopping times, then so $S \vee T$ and $S \wedge T$
  - if $T_n, n = 1, 2, \cdots$, are stopping times with $T_1 \leq T_2 \leq \cdots$, then so is $\sup_n T_n$
  - if $T_n, n = 1, 2, \cdots$, are stopping times with $T_1 \geq T_2 \geq \cdots$, then so is $\inf_n T_n$
  - if $s \geq 0$ and S is a stopping time, then so $S + s$
- For a Borel measurable set A, let $T_A = \inf\{t > 0 : X_t \in A\}$. Suppose $\mathcal{F}_t$ satisfies the usual conditions and X_t has continuous paths,
  - if A is open, then T_A is a stopping time

- if A is closed, then T_A is a stopping time

- Approximation of stopping time from the right: if T is a finite stopping time, define

$$T_n(\omega) = \frac{k+1}{2^n} \quad \text{if } \frac{k}{2^n} \leq T(\omega) < \frac{k+1}{2^n}.$$

  Note that $\{T_n\}$ are stopping times decreasing to T.

- Define $\mathcal{F}_T = \{A \in \mathcal{F} : \text{ for each } t \geq 0, A \cap \{\omega : T \leq t\} \in \mathcal{F}_t\}$, suppose $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration satisfying the usual conditions
  - $\mathcal{F}_T$ is σ-field
  - if $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$
  - if $\mathcal{F}_{T+} = \cap_{\epsilon>0}\mathcal{F}_{T+\epsilon}$, then $\mathcal{F}_{T+} = \mathcal{F}_T$
  - if X_t has right-continuous paths, then X_T is $\mathcal{F}_T$ measurable

3.4 The optional stopping theorem

Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions. If M_t is a martingale or non-negative submartingale whose paths are right continuous, $\sup_{t\geq 0} \mathbb{E}[|M_t^2|] < \infty$, and T is a finite stopping time, then $\mathbb{E}[M_T] \geq \mathbb{E}[M_0]$.

3.5 Convergence and regularity

Let $\mathcal{D}_n = \{k/2^n : k \geq 0\}, \mathcal{D} = \cup_n \mathcal{D}_n$.

- Let $\{M_t : t \in \mathcal{D}\}$ be either a martingale, a submartingale, or a supermartingale w.r.t. $\{\mathcal{F}_t : t \in \mathcal{D}\}$ and suppose $\sup_{t\in\mathcal{D}} \mathbb{E}[|M_t|] < \infty$. Then
  - (1) $\lim_{t\to\infty} M_t$ exists, a.s.
  - (2) With probability one M_t has left and right limits along $\mathcal{D}$.

- Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions, and let M_t be a martingale w.r.t. $\{\mathcal{F}_t\}$. Then M has a version that is also a martingale and that in addition has paths that are right continuous with left limits.

- Increasing paths: a process A_t has increasing paths if the function $t \to A_t(\omega)$ is increasing for almost every ω

- Suppose $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions and suppose A_t is an adapted process with paths that are increasing, are right continuous with left limits, and $A_\infty = \lim_{t\to\infty} A_t$ exists, a.s. Suppose X is non-negative integrable random variable, and M_t is a version of the martingale $\mathbb{E}[X|\mathcal{F}_t]$ which has paths that are right continuous with left limits. Suppose $\mathbb{E}[XA_\infty] < \infty$. Then

$$\mathbb{E}\left[\int_0^\infty X\, dA_s\right] = \mathbb{E}\left[\int_0^\infty M_s\, dA_s\right].$$

  The above equation also can be rewritten as

$$\mathbb{E}\left[\int_0^\infty X\, dA_s\right] = \mathbb{E}\left[\int_0^\infty \mathbb{E}[X|\mathcal{F}_s]\, dA_s\right].$$

  From above, for each t, we also have

$$\mathbb{E}\left[\int_0^t X\, dA_s\right] = \mathbb{E}\left[\int_0^t \mathbb{E}[X|\mathcal{F}_s]\, dA_s\right].$$

3.6 Some applications of martingales

- If W_t is a Brownian motion, then

$$\mathbb{P}\left(\sup_{s\leq t} W_s \geq \lambda\right) \leq e^{-\frac{\lambda^2}{2t}}, \quad ,\lambda > 0,$$

and

$$\mathbb{P}\left(\sup_{s\leq t} |W_s| \geq \lambda\right) \leq 2e^{-\frac{\lambda^2}{2t}}, \quad ,\lambda > 0.$$

- Let W be a Brownian motion, let $a, b > 0$ and let $T = \inf\{t > 0 : W_t \notin [-a, b]\}$. Then

$$\mathbb{P}(W_T = -a) = \frac{b}{a+b}, \quad \mathbb{P}(W_T = b) = \frac{a}{a+b},$$

and

$$\mathbb{E}[T] = ab.$$

- Suppose M_t is a martingale with continuous paths and with $M_0 = 0$ a.s., $T = \inf\{t > 0 : M_t \notin [-a, b]\}$, and $T < \infty$ a.s. Then

$$\mathbb{P}(M_T = -a) = \frac{b}{a+b}, \quad \mathbb{P}(M_T = b) = \frac{a}{a+b}.$$

- Let W be a Brownian motion, let $a, b > 0$ and let $T = \inf\{t > 0 : W_t \notin [-a, b]\}$. Then

$$\mathbb{E}\left[e^{-r^2T/2}\mathbb{1}_{\{W_T=-a\}}\right] = \frac{e^{rb} - e^{-rb}}{e^{r(a+b)} - e^{-r(a+b)}}$$

and

$$\mathbb{E}\left[e^{-r^2T/2}\mathbb{1}_{\{W_T=b\}}\right] = \frac{e^{ra} - e^{-ra}}{e^{r(a+b)} - e^{-r(a+b)}}.$$

4 Markov properties of Brownian motion

4.1 Markov properties

- Markov property: let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions, and let W be a Brownian motion w.r.t. $\{\mathcal{F}_t\}$. If u is a fixed time, then $Y_t = W_{t+u} - W_u$ is a Brownian motion independent of $\mathcal{F}_u$.

- Strong Markov property: let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions, and let W be a Brownian motion w.r.t. $\{\mathcal{F}_t\}$. If T is a stopping time, then $Y_t = W_{t+T} - W_T$ is a Brownian motion independent of $\mathcal{F}_T$.

- General process: let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions, and let X be a process adapted to $\{\mathcal{F}_t\}$. Suppose X has paths that are right continuous with left limits and suppose $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same law with X_{t-s} whenever $s < t$. If T is a finite stopping time, then $Y_t = W_{t+T} - W_T$ is a process that is independent of $\mathcal{F}_T$ and X and Y have the same law.

4.2 Applications

- The reflection of Brownian motion: let W_t be a Brownian motion, $b > 0$, $T = \inf\{t : W_t \geq b\}$, and $x < b$. Then

$$\mathbb{P}\left(\sup_{s\leq t} W_s \geq b, W_t < x\right) = \mathbb{P}\left(W_t > 2b - x\right).$$

- Let W_t be a Brownian motion w.r.t. a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let T be a finite stopping time and $s > 0$. If $a < b$, then

$$\mathbb{P}\left(W_{T+s} \in [a, b] | \mathcal{F}_T\right) \leq \frac{|b-a|}{\sqrt{2\pi s}}.$$

5 The Poisson process

The Poisson process is the prototype of a pure jump process, and it is the building block for Lévy process.

- Definition: Let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions. A Poisson process with parameter $\lambda > 0$ is a stochastic process X satisfying the following properties:
  - (1) $X_0 = 0$ a.s.
  - (2) The paths of X_t are right continuous with left limits
  - (3) If $s < t$, then $X_t - X_s$ is a Poisson random variable with parameter $\lambda(t-s)$
  - (4) If $s < t$, then $X_t - X_s$ is independent of $\mathcal{F}_s$
- $X_{t-} = \lim_{s\to t, s<t} X_s$ be the left-hand limit at time t, and $\Delta X_t = X_t - X_{t-}$ be the size of the jump at time t
- Let X be a Poisson process,
  - with probability one, the paths of X_t are increasing
  - with probability one, the paths of X_t are constant except for jumps of size 1
  - there are only finitely many jumps in each finite time interval
- Let $T_1 = \inf\{t : \Delta X_t = 1\}$, the time of the first jump. Define $T_{i+1} = \inf\{t > T_i : \Delta X_t = 1\}$, so T_i is the time of the i-th jump. The random variables $T_1, T_2 - T_1, \cdots, T_{i+1} - T_i, \cdots$ are independent exponential random variables with parameter λ.
- The construction of Poisson process: let $U_1, U_2, \cdots$ be independent exponential random variable with parameter λ and let $T_j = \sum_{i=1}^{j} U_i$. Define

$$X_t(\omega) = k, \text{ if } T_k(\omega) \le t < T_{k+1}(\omega).$$

- The densities shows that an exponential random variable has a Gamma distribution with parameter λ and 1. Then by the invariant summation of Gamma distribution, T_j is a Gamma random variable with parameters λ and j. Thus

$$\mathbb{P}(X_t < k) = \mathbb{P}(T_k > t) = \int_t^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}\, dx.$$

  Performing the integration by parts repeatedly shows that

$$\mathbb{P}(X_t < k) = \sum_{i=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!},$$

  thus X_t is a Poisson random variable with parameter λt.
- Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions. Suppose $X_0 = 0$ a.s., X has paths that are right continuous with left limits, $X_t - X_s$ is independent of $\mathcal{F}_s$ if $s < t$ and $X_t - X_s$ has the same law with X_{t-s} whenever $s < t$. If the paths of X are piecewise constant, increasing, all the jumps of X are of size 1, and X is not identically 0, then X is a Poisson process.

6 Construction of Brownian motion

There are several ways of constructing Brownian motion, none of them easy. Here gives two constructions. The first is the one that Wiener used, which is based on Fourier series. The second uses martingale techniques, which is due to Lévy.

- Wiener's construction
  - The main step is to construct W_t for $t \in [0, 1]$.

- Take independent copies $Y^{(1)}, Y^{(2)}, \cdots$ each on $[0,1]$, then let

$$W_t = \left(\sum_{i=0}^{[t]-1} Y_1^{(i)} \right) + Y_{t-[t]}^{[t]}.$$

- Fix $t \in [0, \pi]$, the Fourier series for the function $f(s) = s \wedge t$ is

$$s \wedge t = \frac{st}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(ks)\sin(kt)}{k^2}.$$

- Let $Z_0, Z_1, \cdots$ be i.i.d. standard normal random variables and let

$$W_t = \frac{t}{\sqrt{\pi}} Z_0 + \sum_{k=1}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin(kt)}{k} \right) Z_k,$$

  then W_t is a Gaussian process, has mean zero and $\mathrm{Cov}(W_s, W_t) = s \wedge t$. And we also can show that W_t has continuous paths. Thus W as constructed above has the correct finite-dimensional distributions to be a Brownian motion.

- Martingale method
  - Proceed as in the previous method to construct $\{W_t : 0 \le t \le \pi\}$, where W_t is a Gaussian process with $\mathbb{E}[W_t] = 0$ and $\mathrm{Cov}(W_s, W_t) = s \wedge t$, and we need to show that W has a version with continuous paths.
  - First we show that W is a martingale, and so has a version with paths that are right continuous with left limits. We use Doob's inequalities to control the oscillation of W over short time intervals, and then use the Borel–Cantelli lemma to show continuity.
  - Theorem: if $\{W_t : 0 \le t \le 1\}$ is a Gaussian process with $\mathbb{E}[W_t] = 0$ and $\mathrm{Cov}(W_s, W_t) = s \wedge t$ for all $0 \le s, t \le 1$, then there is a version of W that is a Brownian motion on $[0, 1]$.
  - There is nothing special about the trigonometric polynomials in the martingale method. Let $\langle f, g \rangle = \int_0^1 f(r)g(r)\,dr$ be the inner product for the Hilbert space $L^2([0,1])$; we consider only real-valued functions for simplicity. Let $\{\varphi_n\}$ be a complete orthonormal system for $L^2([0,1])$: we have $\langle \varphi_m, \varphi_n \rangle = 0$ if $m \ne n$ and $\langle \varphi_n, \varphi_n \rangle = 1$ for each n, and $f = 0$ a.e. if $\langle f, \varphi_n \rangle = 0$ for all n. Let

$$a_n(t) = \langle \mathbb{1}_{[0,t]}, \varphi_n \rangle = \int_0^t \varphi_n(r)\,dr.$$

    If $Z_0, Z_1, \cdots$ be i.i.d. standard normal random variables and let

$$W_t = \sum_{n=1}^{\infty} a_n(t) Z_k.$$

    Then we have

$$\mathrm{Cov}(W_s, W_t) = \sum_{n=1}^{\infty} a_n(s)a_n(t) = \sum_{n=1}^{\infty} \langle \mathbb{1}_{[0,s]}, \varphi_n \rangle \langle \mathbb{1}_{[0,t]}, \varphi_n \rangle = \langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \rangle = s \wedge t.$$

    Thus W defined above is a mean zero Gaussian process on $[0,1]$ with the same covariances as a Brownian motion.

7 Path properties of Brownian motion

The paths of Brownian motion are continuous, but they are not differentiable. We can see that the paths satisfy a Hölder continuity condition with $\alpha < \frac{1}{2}$. A precise description of the oscillatory behavior of Brownian motion is given by the law of iterated logarithm.

- Hölder continuity: a function $f : [0,1] \to \mathbb{R}$ is said to be Hölder continuous of order α if there is a constant M such that
$$|f(t) - f(s)| \leq M|t-s|^{\alpha}, \quad s,t \in [0,1].$$
- Borel-Cantelli lemma: suppose that $\{A_n\}_{n\geq 1}$ is a sequence of events in a probability space. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then
$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = \mathbb{P}\left(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k\right) := \mathbb{P}(A(i.o.)) = 0.$$
It means with probability one only a finite number of the events occur.
- Second Borel-Cantelli lemma: If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, and the events $\{A_n\}$ are independent, then
$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 1.$$
- If $\alpha < \frac{1}{2}$, the paths of Brownian motion are Hölder continuous of order α on $[0,1]$.
- Law of the iterated logarithm(LIL): Let W be a Brownian motion, we have
$$\limsup_{t\to\infty} \frac{|W_t|}{\sqrt{2t\log\log t}} = 1, a.s. \text{ and } \limsup_{t\to 0} \frac{|W_t|}{\sqrt{2t\log\log\frac{1}{t}}} = 1, a.s.$$
- With probability one, the paths of Brownian motion are nowhere differentiable.

8 The continuity of paths

It is often important to know whether a stochastic process has continuous paths. An important condition is the Kolmogorov continuity criterion.

- Dyadic rationals: let $\mathcal{D}_n = \{k/2^n : k \leq 2^n\}$, the set $\mathcal{D} = \cup_n \mathcal{D}_n$ is known as the set of dyadic rationals in $[0,1]$.
- Suppose $\{X_t : t \in \mathcal{D}\}$ is a real-valued process and there exist c_1, ϵ and $p > 0$ such that
$$\mathbb{E}[|X_t - X_s|^p] \leq c_1|t-s|^{1+\epsilon}, \quad s,t \in \mathcal{D},$$
then the following hold
  - there exists c_2 depending only on c_1, p and ϵ such that for $M > 0$,
$$\mathbb{P}\left(\sup_{s,t\in\mathcal{D}, s\neq t} \frac{|X_t - X_s|}{|t-s|^{\frac{\epsilon}{4p}}} \geq M\right) \leq \frac{c_2}{M^p}.$$
  - with probability one, X_t is uniformly continuous on $\mathcal{D}$.
- Suppose X takes values in some metric space $\mathcal{S}$ with metric $d_{\mathcal{S}}$ and there exist c_1, ϵ and $p > 0$ such that
$$\mathbb{E}[d_{\mathcal{S}}(X_t, X_s)^p] \leq c_1|t-s|^{1+\epsilon}, \quad s,t \in \mathcal{D},$$
then the following hold
  - there exists c_2 depending only on c_1, p and ϵ such that for $M > 0$,
$$\mathbb{P}\left(\sup_{s,t\in\mathcal{D}, s\neq t} \frac{d_{\mathcal{S}}(X_t, X_s)}{|t-s|^{\frac{\epsilon}{2p}}} \geq M\right) \leq \frac{c_2}{M^p}.$$
  - with probability one, X_t is uniformly continuous on $\mathcal{D}$.

- Suppose there exist c_1, ϵ, N and $p > 0$ such that if $n \le N$,

$$\mathbb{E}[d_{\mathcal{S}}(X_t, X_s)^p] \le c_1 |t - s|^{1+\epsilon}, \quad s, t \in \mathcal{D}_n,$$

  then there exists c_2 depending on c_1, p and ϵ but not N such that for $M > 0$ and $n \le N$, we have

$$\mathbb{P}\left(\sup_{s,t \in \mathcal{D}_n, s \neq t} \frac{d_{\mathcal{S}}(X_t, X_s)}{|t - s|^{\frac{\epsilon}{2p}}} \ge M \right) \le \frac{c_2}{M^p}.$$

- If $\alpha < \frac{1}{2}$, then the paths of a one-dimensional Brownian motion $\{W_t : 0 \le t \le 1\}$ are Hölder continuous of order α with probability one.

9 Continuous semimartingales

9.1 Definitions

- A process X has increasing paths (or X is an increasing process): the functions $t \to X_t(\omega)$ are increasing with probability one.
- A process X with paths of bounded variation: with probability one, the functions $t \to X_t(\omega)$ are of bounded variation.
- A process X with paths of locally bounded variation: if there exists stopping times $R_n \to \infty$ such that the process $X_{t \wedge R_n}$ has paths of bounded variation for each n.
- Uniform integrable:
- A martingale is a uniformly martingale: if the family of random variables $\{M_t\}$ is uniformly integrable.
- A process X is a local martingale: if there exists stopping times $R_n \to \infty$ such that the process $M_t^n = X_{t \wedge R_n}$ is a uniformly integrable martingale for each n.
- Continuous martingale: a martingale whose paths are continuous.
- Semimartingale: a process X of the form $X_t = M_t + A_t$, where M_t is a local martingale and A_t is a process whose paths are locally of bounded variation. As a consequence of the Doob-Meyer decomposition we will see that submartingales and supermartingales are semimartingales.
- Example: a Brownian motion W_t is a martingale and is a local martingale, but is not a uniformly integrable martingale and is not a square integrable martingale.

9.2 Square integrable martingales

- Definition: a martingale is square integrable martingale if there exists a $\mathcal{F}_\infty$ measurable random variable M_∞ such that $\mathbb{E}[M_\infty^2] < \infty$ and $M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]$ for all t.
- Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions and M a right continuous process. The following are equivalent:
  - (1) M is a square integrable martingale.
  - (2) M is a martingale with $\sup_{t \ge 0} \mathbb{E}[M_t^2] < \infty$.
  - (3) M is a martingale with $\mathbb{E}[\sup_{t \ge 0} M_t^2] < \infty$.
- If M is a square integrable martingale and $S < T$ are finite stopping times, then $\mathbb{E}[M_T | \mathcal{F}_S] = M_S$. This conclusion is also valid if M is a uniformly integrable martingale.
- Suppose M is a square integrable martingale and T is a stopping time, then $X_t = M_{t \wedge T}$ is a martingale with respect to $\{\mathcal{F}_{t \wedge T}\}$.
- Suppose $\{\mathcal{F}_t\}$ is a filtration satisfying the usual conditions and M is a process that is adapted to $\{\mathcal{F}_t\}$ such that M_t is integrable for each t. If $\mathbb{E}[M_T] = 0$ for every bounded stopping time T, then M_t is a martingale.

- If $\mathbb{E}[M_t] = 0$ for all $t \geq 0$, is M_t a martingale? The answer is no. The counter example is as follows: let $M_t = B_t \mathbb{1}_{0 \leq t < 1} + (B_t^2 - t)\mathbb{1}_{t \geq 1}$.

- Suppose M_t is a square integrable martingale. Then
$$\mathbb{E}[(M_T - M_S)^2 | \mathcal{F}_S] = \mathbb{E}[M_T^2 - M_S^2 | \mathcal{F}_S].$$

- Suppose $M_0 = 0$, M_t is a continuous local martingale, and the paths of M_t are locally of bounded variation. Then M is identically 0 a.s., that is $\mathbb{P}(M_t = 0, \forall t \geq 0) = 1$.

9.3 Quadratic variation

- Definition: a continuous square integrable martingale M_t has quadratic variation $\langle M \rangle_t$ if $M_t^2 - \langle M \rangle_t$ is a martingale, where $\langle M \rangle_t$ is a continuous adapted increasing process with $\langle M \rangle_0 = 0$.

- Example: W is a Brownian motion, t_0 is fixed and $M_t = W_{t \wedge t_0}$, the quadratic variation of M is just $\langle M \rangle_t = t \wedge t_0$. Brownian motion itself does not fit perfectly into the framework of stochastic integration because it is not a square integrable martingale, although it is a martingale.

- Class D: a process X is of process D if $\{Z_T : T \text{ is a finite stopping time}\}$ is a uniformly integrable family of random variables.

- Existence and uniqueness: let M_t be a continuous square integrable martingale, there exists a continuous adapted increasing process $\langle M \rangle_t$ with $\langle M \rangle_0 = 0$ and with increasing paths such that $M_t^2 - \langle M \rangle_t$ is a martingale. If A_t is a continuous adapted increasing process such that $M_t^2 - A_t$ is a martingale, then $\mathbb{P}(A_t \neq \langle M \rangle_t \text{ for some } t) = 0$.

- By the definition of $\langle M \rangle_t$, we have
$$\mathbb{E}\left[(M_T - M_S)^2 - (\langle M \rangle_T - \langle M \rangle_S) | \mathcal{F}_S\right] = \mathbb{E}\left[M_T^2 - M_S^2 - (\langle M \rangle_T - \langle M \rangle_S) | \mathcal{F}_S\right] = 0.$$

- Covariation: if M and N are two square integrable martingales, define $\langle M, N \rangle_t$ by
$$\langle M, N \rangle_t = \frac{1}{2}\left[\langle M + N \rangle_t - \langle M \rangle_t - \langle N \rangle_t\right] = \frac{1}{4}[\langle M + N \rangle_t - \langle M - N \rangle_t].$$

- Another definition: let M be a square integrable martingale and let $t_0 > 0$, then $\langle M \rangle_{t_0}$ is the limit in probability of
$$\sum_{k=0}^{[2^n t_0]} \left(M_{\frac{k+1}{2^n}} - M_{\frac{k}{2^n}}\right)^2,$$
where $[2^n t_0]$ is the largest integer less than or equal to $2^n t_0$.

9.4 The Doob-Meyer decomposition

- Suppose A^1 and A^2 are two increasing adapted continuous processes starting at zero with $A^i_\infty = \lim_{t \to \infty} A^i_t < \infty$, a.s. for $i = 1, 2$, and suppose there exists a positive real K such that for all t,
$$\mathbb{E}[A^i_\infty - A^i_t | \mathcal{F}_t] \leq K, \quad a.s. \quad i = 1, 2.$$
Let $B_t = A^1_t - A^2_t$. Suppose there exists a non-negative random variable V with $\mathbb{E}[V^2] < \infty$ such that for all t,
$$|\mathbb{E}[B_\infty - B_t | \mathcal{F}_t]| \leq \mathbb{E}[V | \mathcal{F}_t], a.s.,$$
then
$$\mathbb{E}\left[\sup_{t \geq 0} B_t^2\right] \leq 8\mathbb{E}\left[V^2\right] + 8\sqrt{2}K\left(\mathbb{E}[V^2]\right)^{\frac{1}{2}}.$$

- Doob-Meyer decomposition: suppose Z_t is a continuous adapted supermartingale of class D, then there exists an increasing adapted continuous process A_t with paths locally of bounded variation starts at 0 and a continuous local martingale M_t such that
$$Z_t = M_t - A_t.$$
If M' and A' are two other such process with $Z_t = M'_t - A'_t$, then $M_t = M'_t$ and $A_t = A'_t$ for all t, a.s.

10 Stochastic integral

10.1 Construction

- Objective: let M_t be a continuous square integrable martingale with respect to a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, and suppose H_t is an adapted process. Under appropriate additional assumptions on H, we want to define
$$N_t = \int_0^t H_s \, dM_s.$$
- A predictable σ-field $\mathcal{P}$ on $[0,\infty)\times\Omega$: $\mathcal{P} = \sigma(X : X \text{ is left continuous, bounded, and adapted to } \{\mathcal{F}_t\})$.
- Two conditions on the integrand H_t:
  - $H : [0,\infty)\times\Omega \to \mathbb{R}$ is measurable w.r.t. $\mathcal{P}$ (H is predictable).
  - H is integrability:
$$\mathbb{E}\left[\int_0^\infty H_s^2 \, d\langle M\rangle_s\right] < \infty.$$
- Three steps to define $\int_0^t H_s \, dM_S$:
  - When $H_s(\omega) = K(\omega)\mathbb{1}_{(a,b]}(s)$, where K is bounded and $\mathcal{F}_a$ measurable.
  - When H_s is the sum of processes of the form in step 1.
  - When H is predictable and satisfies integrability condition.
- The predictable σ-field $\mathcal{P}$ is generated by the collection $\mathcal{C}$ of precesses of the form
$$X_t = \sum_{i=1}^n K_i(\omega)\mathbb{1}_{(a_i,b_i]}(t),$$
where for each i, K_i is a bounded $\mathcal{F}_{a_i}$ measurable random variable.
- Suppose H is as in step 1, then $N_t = K(M_{t\wedge b} - M_{t\wedge a})$ is a continuous martingale,
$$\mathbb{E}[N_\infty^2] = \mathbb{E}\left[\int_0^\infty K^2 \mathbb{1}_{(a,b]}(s)\, d\langle M\rangle_s\right] = \mathbb{E}\left[K^2(\langle M\rangle_b - \langle M\rangle_a)\right],$$
and
$$\langle N\rangle_t = \int_0^t K^2\mathbb{1}_{(a,b]}(s)\, d\langle M\rangle_s.$$
- Suppose
$$H_s(\omega) = \sum_{j=1}^J K_j \mathbb{1}_{(a_j,b_j]}(s),$$
where each K_j is $\mathcal{F}_{a_j}$ measurable and bounded. Define
$$N_t = \sum_{j=1}^J K_j (M_{t\wedge b_j} - M_{t\wedge a_j}).$$
Then N_t is a continuous martingale,
$$\mathbb{E}[N_\infty^2] = \mathbb{E}\left[\int_0^\infty H_s^2 \, d\langle M\rangle_s\right],$$
and
$$\langle N\rangle_t = \int_0^t H_s^2 \, d\langle M\rangle_s.$$

- Suppose the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions and M_t is a square integrable martingale with continuous paths. Suppose H is of the form

$$H_s(\omega) = \sum_{j=1}^{J} K_j \mathbb{1}_{(a_j,b_j]}(s),$$

where each K_j is bounded and $\mathcal{F}_{a_j}$ measurable. In this case define

$$\int_0^t H_s \, dM_s = \sum_{j=1}^{J} K_j (M_{t\wedge b_j} - M_{t\wedge a_j}).$$

If H is predictable and $\mathbb{E}\left[\int_0^\infty H_s^2 \, d\langle M\rangle_s\right] < \infty$, choose H^n of the form given in above with $\mathbb{E}\left[\int_0^\infty (H_s^n - H_s)^2 \, d\langle M\rangle_s\right] \to 0$, and define

$$N_t = \int_0^t H_s \, dM_s$$

to be the limit respect to the norm of $\int_0^t H_s^n \, dM_s$. Then N_t is a continuous martingale,

$$\mathbb{E}[N_\infty^2] = \mathbb{E}\left[\int_0^\infty H_s^2 \, d\langle M\rangle_s\right],$$

and

$$\langle N\rangle_t = \int_0^t H_s^2 \, d\langle M\rangle_s.$$

Moreover the definition of N_t is independent of the particular choice of the H^n.

10.2 Extensions

- If $\int_0^\infty H_s^2 \, d\langle M\rangle_s < \infty$, a.s., but without the expectation being finite, let

$$T_N = \inf\left\{t : \int_0^\infty H_s^2 \, d\langle M\rangle_s > N\right\}.$$

$M'_t = M_{t\wedge T_N}$ is a square integrable martingale with $\langle M'\rangle_t = \langle M\rangle_{t\wedge T_N} \leq N$. Define $\int_0^t H_s \, dM_s$ to be the quantity $\int_0^t H_s \, dM_{s\wedge T_N}$ if $t \leq T_N$.

- If M_t is a continuous local martingale, let $S_n = \inf\{t : |M_t| \geq n\}$. Then $M_{t\wedge S_n}$ will be uniformly integrable martingale, and it is square integrable. For $t \leq S_n$, we set

$$\int_0^t H_s \, dM_s = \int_0^t H_s \, dM_{s\wedge S_n}$$

and $\langle M\rangle_t = \langle M\rangle_{t\wedge S_n}$.

- Suppose that $X_t = M_t + A_t$ is a semimartingale with continuous paths, so that M is a local martingale and A is a process with paths locally of bounded variation. If $\int_0^\infty H_s^2 \, d\langle M\rangle_s + \int_0^t |H_s| \, |dA_s| < \infty$, we define

$$\int_0^t H_s \, dX_s = \int_0^t H_s \, dM_s + \int_0^t H_s \, dA_s,$$

where the first integral on the right is a stochastic integral and the second is a Lebesgue-Stieltjes integral.

- For a semimartingale, we define $\langle X\rangle_t = \langle M\rangle_t$. Given two semimartingales X and Y, we define $\langle X, Y\rangle_t$ by

$$\langle X, Y\rangle_t = \frac{1}{2}\left[\langle X+Y\rangle_t - \langle X\rangle_t - \langle Y\rangle_t\right].$$

11 Itô's formula

- Let X_t be a semimartingale with continuous paths and suppose $f \in C^2$. Then for almost every ω,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s, \quad \forall t \geq 0.$$

- Suppose that $X_t^1, \cdots, X_t^d$ are continuous semimartingales, $X_t = (X_t^1, \cdots, X_t^d)$, and f is a C^2 function on $\mathbb{R}^d$. Then with probability one,

$$f(X_t) = f(X_0) + \int_0^t \sum_{i=0}^d \frac{\partial f}{\partial x_i}(X_s)\,dX_s^i + \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\,d\langle X^i, X^j\rangle_s, \quad \forall t \geq 0.$$

- If X and Y are two semimartingales with continuous paths, then

$$X_tY_t = X_0Y_0 + \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \langle X, Y\rangle_t.$$

12 Some applications of Itô's formula

12.1 Lévy's theorem

- Let M_t be a continuous local martingale with respect to a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions such that $M_0 = 0$ and $\langle M\rangle_t = t$. Then M_t is a Brownian motion with respect to $\{\mathcal{F}_t\}$.

12.2 Time changes of martingales

- Suppose M_t is a continuous local martingale, $M_0 = 0$, $\langle M\rangle_t$ is strictly increasing, and $\lim_{t\to\infty}\langle M\rangle_t = \infty$, a.s. Let

$$\tau(t) = \inf\{u : \langle M\rangle_u \geq t\}.$$

Then $W_t = M_{\tau(t)}$ is a Brownian motion with respect to $\mathcal{F}'_t = \mathcal{F}_{\tau(t)}$.

12.3 Martingale representation

- It says that every martingale adapted to the filtration of a Brownian motion can be expressed as a stochastic integral with respect to the Brownian motion.

- Let $\mathcal{F}_t$ be the minimal augmented filtration generated by a one-dimensional Brownian motion W_t, let $t_0 > 0$, and let Y be $\mathcal{F}_{t_0}$ measurable with $\mathbb{E}[Y^2] < \infty$. There exists a predictable process H_s with $\mathbb{E}[\int_0^{t_0} H_s^2\,ds] < \infty$ such that

$$Y = \mathbb{E}[Y] + \int_0^{t_0} H_s\,dW_s, \; a.s.$$

- Suppose M_t is a right-continuous square integrable martingale with respect to the minimal augmented filtration $\{\mathcal{F}_t\}$ generated by a one-dimensional Brownian motion and suppose $M_0 = 0$. Let $t_0 > 0$. Then there exists a predictable process H_s with $\mathbb{E}[\int_0^{t_0} H_s^2\,ds] < \infty$ such that

$$M_t = \int_0^t H_s\,dW_s, \quad \forall t \leq t_0.$$

- If M_t is a square integrable martingale with respect to the minimal augmented filtration of a one-dimensional Brownian motion W, then M_t has a version with continuous paths.

12.4 The Burkholder-Davis-Gundy inequalities

- Define $M_t^* = \sup_{s \le t} |M_s|$. Let M_t be a continuous local martingale with $M_0 = 0$, a.s., and suppose $2 \le p \le \infty$. There exists a constant c_1 depending on p such that for any finite stopping time T,

$$\mathbb{E}\left[(M_T^*)^p\right] \le c_1 \mathbb{E}\left[\langle M\rangle_T^{p/2}\right].$$

- Let M_t be a continuous local martingale with $M_0 = 0$, a.s., and suppose $2 \le p \le \infty$. There exists a constant c_2 depending on p such that for any finite stopping time T,

$$\mathbb{E}\left[\langle M\rangle_T^{p/2}\right] \le c_2 \mathbb{E}\left[(M_T^*)^p\right].$$

- In fact, the two inequalities are true as long as $p > 0$.

12.5 Stratonovich integrals

- Definition: if X and Y are two continuous semimartingales, the Stratonovich integral, denoted by $\int_0^t X_s \circ dY_s$, is defined by

$$\int_0^t X_s \circ dY_s = \int_0^t X_s \, dY_s + \frac{1}{2}\langle X, Y\rangle_t.$$

- Suppose $f \in C^3$ and X is a continuous semimartingale. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s.$$

- Product rule for Stratonovich integrals: if X and Y are two continuous semimartingales, then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \circ dY_s + \int_0^t Y_s \circ dX_s.$$

- Suppose H and X are continuous semimartingales and $t_0 > 0$. Then $\int H_s \circ dX_s$ is the limit in probability as $n \to \infty$ of

$$\sum_{k=0}^{2^n-1} \frac{H_{kt_0/2^n} + H_{(k+1)t_0/2^n}}{2} \left(X_{(k+1)t_0/2^n} - X_{kt_0/2^n}\right).$$

13 The Girsanov theorem

- Suppose Y_t is a continuous local martingale with $Y_0 = 0$ and let $Z_t = e^{Y_t - \frac{1}{2}\langle Y\rangle_t}$. Applying Itô's formula to $X_t = Y_t - \frac{1}{2}\langle Y\rangle_t$ with the function e^x yields

$$Z_t = e^{Y_t - \frac{1}{2}\langle Y\rangle_t} = 1 + \int_0^t Z_s \, dY_s.$$

  This can be abbreviated by $dZ_t = Z_t \, dY_t$. Z_t is called the exponential of the martingale Y, and since Z is the stochastic integral with respect to a local martingale, it is itself a local martingale.

- Suppose Y is a continuous local martingale with $Y_0 = 0$ and $Z_t = e^{Y_t - \frac{1}{2}\langle Y\rangle_t}$. If $\langle Y\rangle_t$ is a bounded random variable for each t, then $\mathbb{E}[|Z_t|^p] < \infty$ for each $p > 1$ and each t.

- Suppose A_t is a continuous increasing process adapted to a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let X be a bounded random variable, H a bounded adapted process, $s < t$, and $B \in \mathcal{F}_s$, then

$$\mathbb{E}\left[\int_s^t X H_r \, dA_r; B\right] = \mathbb{E}\left[\int_s^t \mathbb{E}[X|\mathcal{F}_r] H_r \, dA_r; B\right].$$

- Suppose W_t is a Brownian motion with respect to $\mathbb{P}$, H is bounded and predictable,

$$M_t = \exp\left(\int_0^t H_r\,dW_r - \frac{1}{2}\int_0^t H_r^2\,dr\right),$$

and

$$\mathbb{Q}(B) = \mathbb{E}_{\mathbb{P}}[M_t; B]$$

if $B \in \mathcal{F}_t$, then

$$W_t - \int_0^t H_r\,dr$$

is a Brownian motion with respect to $\mathbb{Q}$.

14 Local time

- Let W_t be a one-dimensional Brownian motion. Then
  - there exists a non-negative increasing continuous adapted process L_t^0 such that

$$|W_t| = \int_0^t \operatorname{sgn}(W_s)\,dW_s + L_t^0.$$

  - L_t^0 increases only when W is at 0. More precisely, if $W_s(\omega) \neq 0$ for $r \leq s \leq t$, then $L_r^0(\omega) = L_t^0(\omega)$.
  - L_t^0 is called the local time at 0. The equation $|W_t| = \int_0^t \operatorname{sgn}(W_s)\,dW_s + L_t^0$ is called the Tanaka formula.
- We have exhibited reflecting Brownian motion $|W_t|$ as the sum of another Brownian motion and a continuous process that increases only when W is at zero.
- Let $M_t = \sup_{s \leq t} W_s$. The two-dimensional processes $(|W|, L^0)$ and $(M - W, M)$ have the same law.
- Just as we defined L_t^0 via the Tanaka formula, we can construct local time at the level a by the formula

$$|W_t - a| - |W_0 - a| = \int_0^t \operatorname{sgn}(W_s - a)\,dW_s + L_t^a,$$

and we can show that L_t^a is the limit in L^2 of

$$\frac{1}{2\epsilon}\int_0^t \mathbb{1}_{[a-\epsilon,a+\epsilon]}(W_s)\,ds.$$

- Let W be a one-dimensional Brownian motion and let L_t^a be the local time of W at level a. For each $a \in \mathbb{R}$ there exists a version $\tilde{L_t^a}$ of L_t^a so that with probability one, $\tilde{L_t^a}$ is jointly continuous in t and a.
- Let W_t be a Brownian motion and L_t^y the local time at the level y, where we take L_t^y to be joint continuous in t and y. If y is non-negative and Borel measurable,

$$\int f(y) L_t^y\,dy = \int_0^t f(W_s)\,ds, a.s.$$

with the null set independent of f and t.

15 Skorokhod embedding

- Suppose Y is a random variable with $\mathbb{E}[Y] = 0$ and $\mathbb{E}[Y^2] < \infty$. There exists a Brownian motion N and a stopping time T with respect to the minimal augmented filtration of N such N_T is equal in law to Y. Moreover, $\mathbb{E}[T] = \mathbb{E}[Y^2]$.
- In the above theorem, we started with a Brownian motion W, constructed a new Brownian motion N, and then defined our stopping time T in terms of N. We can actually start with a Brownian motion W and define a stopping time that is a stopping time with respect to the minimal augmented filtration of W.

- Let W be a Brownian motion and let $\{\mathcal{F}_t\}$ be the minimal augmented filtration for W. Let Y be a random variable with $\mathbb{E}[Y] = 0$ and $\mathrm{Var}(Y) < \infty$. There exists a stopping time V with respect to $\{\mathcal{F}_t\}$ such that W_V has the same law as Y.

- The application of Skorokhod embedding: we can find a Brownian motion that is relatively close to a random walk. Suppose $Y_1, Y_2, \cdots$ is an i.i.d. sequence of real-valued random variables with mean zero and variance one. Given a Brownian motion W_t we can find a stopping time T_1 such that W_{T_1} has the same law as Y_1. We use the strong Markov property at time T_1 and find a stopping time T_2 for $W_{T_1+t} - W_{T_1}$ so that $W_{T_1+T_2} - W_{T_1}$ has the same distribution as Y_2 and is independent of $\mathcal{F}_{T_1}$. We continue and see that T_i are i.i.d. and $\mathbb{E}[Y_i] = \mathbb{E}[Y_i^2] = 1$. Let $U_k = \sum_{i=1}^k T_i$. Then for each n, $S_n = \sum_{i=1}^n Y_i$ has the same law as W_{U_n}.

- For the $U_i, 1 \le i \le n$ defined above, we have
$$\sup_{i \le n} \frac{|W_{U_i} - W_i|}{\sqrt{n}}$$
tends to 0 in probability as $n \to \infty$.

16 Stochastic differential equations

16.1 Pathwise solutions of SDEs

- We consider SDEs of the form
$$dX_t = \sigma(X_t)\, dW_t + b(X_t)\, dt, \quad X_0 = x_0. \tag{1}$$
This means that X_t satisfies the equation
$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b(X_s)\, ds, \quad t \ge 0. \tag{2}$$
Here σ and b are Borel measurable functions.

- A Stochastic process X will be a pathwise solution to (1) if X is adapted to the filtration $\{\mathcal{F}_t\}$ and (2) holds almost surely, where the null set does not depend on t. We say the solution to (1) is pathwise unique if whenever X_t' is another solution, then
$$\mathbb{P}(X_t \ne X_t' \text{ for some } t \ge 0) = 0.$$

- Suppose σ and b are bounded Lipschitz functions. Then there exists a pathwise solution to (1) and this solution is pathwise unique.

- Suppose σ and b are Lipschitz functions and satisfies $|\sigma(x)| \le c(1 + |x|)$ and $|b(x)| \le c(1 + |x|)$. Then there exists a pathwise solution to (1) and this solution is pathwise unique.

16.2 One-dimensional SDEs

- Suppose b is bounded and Lipschitz. Suppose there exists a continuous function $\rho : [0, \infty) \to [0, \infty)$ such that $\rho(0) = 0$,
$$\int_0^\epsilon \rho^{-2}(u)\, du = \infty$$
for all $\epsilon > 0$, and σ is bounded and satisfies
$$|\sigma(x) - \sigma(y)| \le \rho(|x - y|)$$
for all x and y. Then the solution to (1) is pathwise unique.

- Suppose σ satisfies the conditions above, X_t satisfies (2) with b a Lipschitz function. Suppose Y_t is a continuous semimartingale satisfying
$$Y_t \ge Y_0 + \int_0^t \sigma(Y_s)\, dW_s + \int_0^t B(Y_s)\, ds,$$
where B is a Borel measurable function and $B(z) \ge b(z)$ for all z. If $Y_0 \ge x_0$, a.s., then $Y_t \ge X_t$ almost surely for all t.

16.3 Examples of SDEs

- Ornstein-Uhlenbeck (OU) process: the OU process is the solution to the SDE

$$dX_t = dW_t - \frac{X_t}{2}\, dt, \quad X_0 = x.$$

The equation can be solved explicitly. Multiplying by $e^{t/2}$, and using the product rule, then we have

$$X_t = xe^{-t/2} + \int_0^t e^{(s-t)/2}\, dW_s.$$

X_t is a Gaussian process and the distribution of X_t is that of a normal random variable with mean $e^{-t/2}x$ and variance equal to $e^{-t}\int_0^t e^s\, ds = 1 - e^{-t}$.

- Let $Y_t = \int_0^t e^{s/2}\, dW_s$ and $V_t = Y_{\log(t+1)}$, then Y_t is a mean zero continuous Gaussian process with independent increments, and hence so is V_t. Since

$$\mathrm{Var}(V_u - V_t) = \int_{\log(t+1)}^{\log(u+1)} e^s\, ds = u - t,$$

then V_t is a Brownian motion. Hence

$$X_t = xe^{-t/2} + e^{-t/2} V(e^t - 1).$$

This representation of an OU process in term of a Brownian motion is useful.

- Linear equations: the unique pathwise solution to

$$dX_t = AX_t\, dW_t + BX_t\, dt$$

is

$$X_t = X_0 e^{AW_t + (B - A^2/2)t}.$$

- Bessel process: a Bessel process of order $v \geq 2$ is defined to be a solution of the SDE

$$dX_t = dW_t + \frac{v-1}{2X_t}\, dt, \quad X_0 = x. \tag{3}$$

Bessel processes of order $0 \leq v \leq 2$ can also be defined using (3), but only up until the first time the process X reaches 0. The square of a Bessel process of order $v \geq 0$ is defined to be the solution to the SDE

$$dY_t = 2\sqrt{|Y_t|}\, dW_t + v\, dt, \quad Y_0 = y. \tag{4}$$

- If X_t is a Bessel process of order v started at x, then $aX_{a^{-2}t}$ is a Bessel process of order v started at ax. In fact,

$$d(aX_{a^{-2}t}) = a\, dW_{a^{-2}t} + a^2 \frac{v-1}{2aX_{a^{-2}t}}\, d(a^{-2}t).$$

- Suppose Y_t is the square of a Bessel process of order v. Suppose $Y_0 = y$. The following hold with probability one.
  - If $v > 2$ and $y > 0$, Y_t never hits 0.
  - If $v = 2$ and $y > 0$, Y_t hits every neighborhood of 0, but never hits the point 0.
  - If $0 < v < 2$, Y_t hits 0.
  - If $v = 0$, then Y_t hits 0. If started at 0, then Y_t remains at 0 forever.

- When we say that Y_t hits 0, we consider only times $t > 0$. We define $T_0 = \inf\{t > 0 : Y_t = 0\}$ and say that Y_t hits 0 if $T_0 < \infty$.

17 Weak solution of SEDs

- A weak solution $(X, W, \mathbb{P})$ to (1) exists if there exists a probability measure $\mathbb{P}$ and a pair of processes (X_t, W_t) such that W_t is a Brownian motion under $\mathbb{P}$ and (1) holds. There is weak uniqueness holding for (1) if whenever $(X, W, \mathbb{P})$ and $(X', W', \mathbb{P}')$ are two weak solutions, then the joint law of (X, W) under $\mathbb{P}$ and the joint law of (X', W') under $\mathbb{P}'$ are equal. When this happens, we also say that the solution to (1) is unique in law.
- Suppose σ and b are bounded Lipschitz functions and $x_0 \in \mathbb{R}$, then the weak uniqueness holds for (1).
- Consider
$$dX_t = \sigma(X_t)\, dW_t, \quad X_0 = x_0. \tag{5}$$
If σ is a Borel measurable function and there exists $c_2 > c_1 > 0$ such that $c_1 \leq \sigma(x) \leq c_2$ for all x, then weak existence and weak uniqueness hold for (5).
- Suppose σ and b are measurable and bounded above and σ is bounded below by a positive constant. Then the weak existence and uniqueness holds for (1).

18 Markov processes and SDEs

18.1 Markov properties

- Let $\mathbb{P}$ be a probability measure and W be a d-dimensional Brownian motion with respect to $\mathbb{P}$. Consider the SDE
$$dX_t = \sigma(X_t)\, dW_t + b(X_t)\, dt. \tag{6}$$
Here σ is a $d \times d$ matrix-valued function and b is a vector-valued function, both Borel measurable and bounded. This can be written in terms of components as
$$dX_t^i = \sum_{j=1}^{d} \sigma_{ij}(X_t)\, dW_t^j + b_i(X_t)\, dt, \quad i = 1, 2, \cdots, d,$$
where $W = (W^1, W^2, \cdots, W^d)$. Let X_t^x be the solution to (6) when $X_0 = x$. Let $\mathbb{P}^x$ be the law of X_t^x.
- Let $\Omega = C([0, \infty))$, let $\mathcal{F}$ be the cylindrical subsets of Ω, and define $Z_t(\omega) = \omega(t)$. The main result of this section is that if weak existence and weak uniqueness hold for (6) for every starting point x, then the solutions $(Z_t, \mathbb{P}^x)$ form a strong Markov process.
- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{E}$ be a σ-field contained in $\mathcal{F}$. A regular conditional probability for $\mathbb{E}[\cdot|\mathcal{E}]$ is a kernel $Q(\omega, d\omega')$ such that
  - (1) $Q(\omega, \cdot)$ is a probability measure on $(\Omega, \mathcal{E})$ for each ω;
  - (2) for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is a random variable that is measurable with respect to $\mathcal{F}$;
  - (3) for each $A \in \mathcal{F}$ and each $B \in \mathcal{E}$,
$$\int_B Q(\omega, A)\mathbb{P}(d\omega) = \mathbb{P}(A \cap B)$$
- Suppose weak existence and weak uniqueness hold for the SDE (6) whenever X_0 is a random variable that is in L^2 and is measurable with respect to $\mathcal{F}_0$. Suppose the matrix $\sigma(y)$ is invertible for each y. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be defined as above. Let $\mathbb{P}^x$ be the law of the weak solution when X_0 is identically equal to x. Let $\{\mathcal{F}_t\}$ be the minimal augmented filtration generated by Z. Then $(\mathbb{P}^x, Z_t)$ is a strong Markov process.

18.2 SDEs and PDEs

- Let $\mathcal{L}$ be the operator on functions in C^2 defined by

$$\mathcal{L}f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x).$$

  Suppose X_t is a solution to (6), σ and b are bounded and Borel measurable, and $a = \sigma\sigma^\top$. Suppose $f \in C^2$, then

$$f(X_t) = f(X_0) + M_t + \int_0^t \mathcal{L}f(X_s)\, ds,$$

  where

$$M_t = \int_0^t \sum_{i,j=1}^{d} \frac{\partial f}{\partial x_i}(X_s)\sigma_{ij}(X_s)\, dW_s^j$$

  is a local martingale.

18.3 Martingale problems

- We assume that the coefficient a_{ij} and b_i are bounded and measurable and that $a_{ij}(x) = a_{ji}(x)$ for all $i, j = 1, 2, \cdots, d$ and all $x \in \mathbb{R}^d$. The coefficients a_{ij} are called the diffusion coefficients and the b_i are called the drift coefficients. We also assume that the operator $\mathcal{L}$ is uniformly elliptic, which means that there exists $\Lambda > 0$ such that

$$\sum_{i,j=1}^{d} y_i a_{ij}(x) y_j \geq \Lambda |y|^2, \quad y \in \mathbb{R}^d, x \in \mathbb{R}^d.$$

  If X_t is a solution to (6), $a = \sigma\sigma^\top$ and $f \in C^2$, then

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\, ds$$

  is a local martingale under $\mathbb{P}$. Let Ω consists of all continuous functions ω mapping $[0, \infty)$ to $\mathbb{R}^d$. Let $X_t(\omega) = \omega(t)$ and given a probability $\mathbb{P}$, let $\{\mathcal{F}_t\}$ be the minimal augmented filtration generated by X. A probability measure $\mathbb{P}$ is a solution to the martingale problem for $\mathcal{L}$ started at x_0 if

$$\mathbb{P}(X_0 = x_0) = 1$$

  and

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\, ds$$

  is a local martingale under $\mathbb{P}$ whenever $f \in C^2(\mathbb{R}^d)$. The martingale problem is well posed if there exists a solution $\mathbb{P}$ and this solution is unique.

- Suppose $a = \sigma\sigma^\top$ and suppose the matrix $\sigma(x)$ is invertible for each x. Weak uniqueness for (6) holds if and only if the solution for the martingale problem for $\mathcal{L}$ started at x in unique. Weak existence for (6) holds if and only if there exists a solution to the martingale problem for $\mathcal{L}$ started at x.

19 Solving partial differential equations

19.1 Poisson's equation

- Suppose $\lambda > 0$ and f is a C^1 function with compact support. Poisson's equation is

$$\mathcal{L}u(x) - \lambda u(x) = -f(x), \quad x \in \mathbb{R}^d. \tag{7}$$

- Suppose u is a C^2 solution to (7) such that u and its first and second partial derivatives are bounded. Then

$$u(x) = \mathbb{E}^x\left[\int_0^\infty e^{-\lambda t} f(X_t)\, dt\right].$$

- Let D be a bounded domain, and $\mathcal{L}u - \lambda u = -f$ in D and $u = 0$ on ∂D. Suppose u is a solution to Poisson's equation in D that is C^2 in D and continuous on $\bar{D}$. Then

$$u(x) = \mathbb{E}^x \left[\int_0^{\tau_D} e^{-\lambda t} f(X_t)\, dt \right].$$

19.2 Dirichlet problem

- Let D be a ball (or other nice bounded domain) and consider the solution to the Dirichlet problem: given a continuous function f on ∂D, find $u \in C(\bar{D})$ such that u is C^2 in D and

$$\mathcal{L}u = 0 \text{ in } D, \quad u = f \text{ on } \partial D.$$

- Suppose u is a solution to the above Dirichlet problem. Then u satisfies

$$u(x) = \mathbb{E}^x \left[f(X_{\tau_D}) \right].$$

19.3 Cauchy problem

- The related parabolic differential equation

$$\frac{\partial u}{\partial t} = \mathcal{L}u$$

is often of interest. Here u is a function of $x \in \mathbb{R}^d$ and $t \in [0, \infty)$. When we write $\mathcal{L}u$, we mean

$$\mathcal{L}u(x,t) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i}(x,t).$$

- Suppose for simplicity that the function f is a continuous function with compact support. The Cauchy problem is to find u such that u is bounded, u is C^2 with bounded first and second partial derivatives in x, u is C^1 in t for $t > 0$, and

$$\begin{aligned} u_t(x,t) &= \mathcal{L}u(x,t), \quad t > 0, x \in \mathbb{R}^d \\ u(x,0) &= f(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{8}$$

- Suppose there exists a solution to (8) that is C^2 in x and C^1 in t for $t > 0$. Then u satisfies

$$u(x) = \mathbb{E}^x[f(X_t)].$$

19.4 Schrödinger operators

- Consider the operator

$$\mathcal{L}u(x) + q(x)u(x),$$

which is known as the Schrödinger operator, and $q(x)$ is known as the potential.

- Let D be a nice bounded domain, q a C^2 function on $\bar{D}$, and f a continuous function on ∂D, q^+ denotes the positive part of q. Let u be a C^2 function on $\bar{D}$ that agrees with f on ∂D and satisfies $\mathcal{L}u + qu = 0$ in D. If

$$\mathbb{E}^x \left[\exp\left(\int_0^{\tau_D} q^+(X_s)\, ds \right) \right] < \infty,$$

then

$$u(x) = \mathbb{E}^x \left[f(X_{\tau_D}) e^{\int_0^{\tau_D} q(X_s)\, ds} \right].$$

20 One-dimensional diffusions

20.1 Regularity

- One-dimensional diffusion: suppose that we have a continuous process $(X_t, \mathbb{P}^x)$ defined on an interval I contained in $\mathbb{R}$. We further suppose that $(X_t, \mathbb{P}^x)$ is a strong Markov process with respect to a right-continuous filtration $\{\mathcal{F}_t\}$ such that each $\mathcal{F}_t$ contains all the sets that are $\mathbb{P}^x$-null for every x. We call such a process a one-dimensional diffusion.
- Regular: Let
$$T_y = \inf\{t : X_t = y\},$$
the first time that the process X hits the point y. We also assume that every point can be hit from every other point: for all x, y,
$$\mathbb{P}^x(T_y < \infty) = 1.$$
- Natural scale: for any interval J, define
$$\tau_J = \inf\{t : X_t \notin J\},$$
the first time the process leaves J. When X_t is a Brownian motion, we know that the distribution of X_t upon exiting $[a, b]$ is
$$\mathbb{P}^x\left(X(\tau_{[a,b]} = a\right) = \frac{b-x}{b-a}, \quad \mathbb{P}^x\left(X(\tau_{[a,b]} = b\right) = \frac{x-a}{b-a}. \tag{9}$$
We say a regular diffusion X_t is on natural scale if (9) holds for every interval $[a, b]$. We also say a regular diffusion X defined on an interval I properly contained in $\mathbb{R}$ is on natural scale if (9) holds whenever $[a, b] \subset I$ and $x \in (a, b)$.
- If X is regular, then the process started at x must leave x immediately. That is, if $S = \inf\{t > 0 : X_t \neq x\}$, then $\mathbb{P}^x(S = 0) = 1$.

20.2 Scale functions

- Suppose X_t is given as the solution to (1), where we assume σ and b are real-valued, continuous and bounded above and σ is bounded below by a positive constant. Let $a(x) = \sigma^2(x)$. The scale function $s(x)$ is a solution to
$$\frac{1}{2}a(x)s''(x) + b(x)s'(x) = 0,$$
and for some constant c_1, c_2, and x_0 is given by
$$s(x) = c_1 + c_2 \int_{x_0}^{x} \exp\left(-\int_{x_0}^{y} \frac{2b(w)}{a(w)}\, dw\right) dy.$$
- Let J be an interval $[a, b]$. Define
$$p(x) = p_J(x) = \mathbb{P}^x(X_{\tau_J} = b).$$
Then $p(X_{t\wedge\tau_J})$ is a regular diffusion on $[0, 1]$ on natural scale.
- There exists a continuous strictly increasing function s such that $s(X_t)$ is on natural scale on $s(\mathbb{R})$.

20.3 Speed measures

- Definition: suppose that $(\mathbb{P}^x, X_t)$ is a regular diffusion on $\mathbb{R}$ on natural scale. For each finite interval (a, b), define
$$G_{ab}(x, y) = \begin{cases} \frac{2(x-a)(b-y)}{b-a}, & a < x \le y < b, \\ \frac{2(y-a)(b-x)}{b-a}, & a < y \le x < b, \end{cases}$$
and set $G_{ab}(x, y) = 0$ if x or y is not in (a, b). A measure $m(dx)$ is the speed measure for the diffusion $(X_t, \mathbb{P}^x)$ if
$$\mathbb{E}^x[\tau_{(a,b)}] = \int G_{ab}(x, y)\, m(dy) \tag{10}$$
for each finite interval (a, b) and each $x \in (a, b)$.

- The speed measure for Brownian motion is a Lebesgue measure.
- Suppose that $(\mathbb{P}^x, X_t)$ is a regular diffusion on $\mathbb{R}$. If $[a, b]$ is a finite interval, then $\sup_x \mathbb{E}^x \left[\tau^k_{(a,b)}\right] < \infty$ for each positive integer k.
- If $(X_t, \mathbb{P}^x)$ has a speed measure m and $[a, b]$ is a non-empty finite interval, then $0 < m(a, b) < \infty$.
- A regular diffusion on natural scale on $\mathbb{R}$ has one and only one speed measure.
- Suppose X_t is a diffusion on natural scale on $\mathbb{R}$. If f is a bounded and measurable, for each $a < b$,

$$\mathbb{E}^x \left[\int_0^{\tau(a,b)} f(X_s)\, ds\right] = \int G_{ab}(x, y) f(y)\, m(dy).$$

20.4 The uniqueness theorem

- If $(X_t, \mathbb{P}^x_i), i = 1, 2$, are two diffusions on natural scale with the same speed measure m, then $\mathbb{P}^x_1 = \mathbb{P}^x_2$. (The speed measure characterizes the law of a diffusion).

20.5 Time change

- In this subsection we want to show that if m is a speed measure such that $0 < m(a, b) < \infty$ for all intervals $[a, b]$, then there exists a regular diffusion on natural scale on $\mathbb{R}$ having m as a speed measure. If $m(dx)$ has a density, say $m(dx) = r(x)\, dx$, let W_t be a one-dimensional Brownian motion and let

$$A_t = \int_0^t r(W_s)\, ds, \quad B_t = \inf\{u : A_t > u\}, \quad X_t = W_{B_t}.$$

In other words, we let X_t be a certain time change of Brownian motion. In general, where $m(dx)$ does not have a density, we make use of the local time L^x_t of Brownian motion. Let

$$A_t = \int_0^t L^x_t\, m(dx), \quad B_t = \inf\{u : A_t > u\}, \quad X_t = W_{B_t}. \tag{11}$$

- Let $(W_t, \mathbb{P}^x)$ be a Brownian motion and m a measure on $\mathbb{R}$ such that $0 < m(a, b) < \infty$ for every finite interval (a, b). Then, under $\mathbb{P}^x$, X_t as defined by (11) is a regular diffusion on natural scale with speed measure m.

20.6 Examples

- Suppose X is the solution to the SDE $dX_t = \sigma(X_t)\, dW_t$. Suppose $c_1 < \sigma(x) < c_2$ for all x and σ is continuous. The speed measure of X_t is given by

$$m(dx) = \frac{1}{a(x)}\, dx.$$

- Suppose X is the solution to the SDE $dX_t = \sigma(X_t)\, dW_t + b\, dt$. Then $s(x) = e^{-2bx}$ is the scale function. If $Y_t = s(X_t)$, then $(s'\sigma)(s^{-1}(y)) = -2by$, or Y_t corresponds to the operator $2b^2y^2f''$, and the speed measure is $(4b^2y^2)^{-1}dx$.
- Suppose X is the solution to the Bessel process (3). If $v \neq 2$, we have $s(x) = x^{2-v}$ is the scale function. Then $Y_t = s(X_t)$ satisfies

$$dY_t = (2 - v)Y_t^{(1-v)/(2-v)}\, dW_t,$$

and the speed measure is

$$m(dx) = (2 - v)^{-2} x^{(2v-2)/(2-v)}\, dx, \quad x > 0.$$