Viscosity solution and comparison principle

Exercise 1:

Consider the ODE

$$\begin{cases} |u'(x)| - 1 = 0, \text{ on } x \in (-1,1) \\ u(\pm 1) = 0 \end{cases} \tag{1}$$

(1) Prove that $u(x) = |x| - 1$ is a viscosity solution.

(2) Can you prove comparison principle?

Solution:

(1) For $|u'(x)| - 1 = 0$, we denote

$$F(x,u,p,X) = |p| - 1. \tag{2}$$

By the definition of semi-jets, for $u(x) = |x| - 1$, when $x \in (0,1)$, we have

$$J^{2,+}u(x) = \{1\} \times [0,+\infty), \quad J^{2,-}u(x) = \{1\} \times (-\infty, 0].$$

And when $x \in (-1,0)$,

$$J^{2,+}u(x) = \{-1\} \times [0,+\infty), \quad J^{2,-}u(x) = \{-1\} \times (-\infty, 0].$$

Thus for any $x \in (-1,1) \setminus \{0\}$, for any $(p,X) \in J^{2,+}u(x)$, we have

$$F(x,u,p,X) = |p| - 1 = 1 - 1 = 0,$$

and for any $(p,X) \in J^{2,-}u(x)$, we also can get

$$F(x,u,p,X) = |p| - 1 = 1 - 1 = 0.$$

For $x = 0$, by the definition of the closure of semi-jets, for any $(p,X) \in \bar{J}^{2,+}u(0)$, we have $p = 1$ or $p = -1$, then $F(x,u,p,X) = |p| - 1 = 0$. Similarly, for any $(p,X) \in \bar{J}^{2,-}u(0)$, $F(x,u,p,X) = |p| - 1 = 0$. Hence we can conclude that for any $x \in (-1,1)$ and $(p,X) \in \bar{J}^{2,+}u(x)$,

$$F(x,u,p,X) = |p| - 1 \leq 0,$$

then $u(x) = |x| - 1$ is a viscosity subsolution. And for any $(p,X) \in \bar{J}^{2,-}u(x)$,

$$F(x,u,p,X) = |p| - 1 \geq 0,$$

then $u(x) = |x| - 1$ is a viscosity supersolution. For $x = 1$ or $x = -1$, $u(x) = |x| - 1 = 0$, so we know that $u(x) = |x| - 1$ is a viscosity solution of the ODE (1).

(2) No, we can not get the comparison principle for the ODE (1). For $v(x) = 1 - |x|$, by the same method, we can prove that $v(x)$ is also a viscosity solution of (1). As the comparison principle yields to the uniqueness of the viscosity solution, we can not prove the comparison principle for the ODE (1).