An example of linear quadratic mean field game

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1 Problem setup

Let $(\Omega, \mathcal{F}_T, \mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a completed filtered probability space satisfying the usual conditions, on which

- $W = (W_t)_{0\le t\le T}$ is a standard Brownian motion taking values in $\mathbb{R}$,
- ξ is a random variable in $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R})$,
- $\mathcal{F}_t$ is generated by W_t and ξ.

Denote by $\mathcal{A}$ the set of $\mathbb{F}$-progressively measurable A valued stochastic process $\alpha = (\alpha)_{0\le t\le T}$ that satisfy the square integrability condition

$$\mathbb{E}\left[\int_0^T |\alpha_t|^2\, dt\right] < \infty.$$

Consider the linear quadratic mean field game problem

- (i) For each fixed deterministic function $[0,T] \ni t \mapsto \bar{\mu}_t \in \mathbb{R}$, solve the standard stochastic control problem

$$\inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\alpha_t^2 + \frac{1}{2}(X_t - \bar{\mu}_t)^2\right) dt + \frac{k}{2}(X_T - \bar{\mu}_T)^2\right] \tag{1}$$

  subject to

$$\begin{cases} dX_t = \alpha_t\, dt + \sigma\, dW_t \\ X_0 = \xi. \end{cases}$$

- (ii) Determine a function $[0,T] \ni t \mapsto \bar{\mu}_t \in \mathbb{R}$ so that, for all $t \in [0,T]$, $\mathbb{E}[\hat{X}_t] = \bar{\mu}_t$, where $(\hat{X}_t)_{0\le t\le T}$ is the optimal path of the optimal problem in the environment $(\bar{\mu}_t)_{0\le t\le T}$.

2 Existence and uniqueness by SMP

We define the generalized Hamiltonian $H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times A \to \mathbb{R}$ by

$$H(t, x, \mu, y, z, a) = ay + \sigma z + \frac{1}{2}a^2 + \frac{1}{2}(x-\mu)^2.$$

The Hamiltonian is minimized for

$$\hat{\alpha} = \hat{\alpha}(t, x, \mu, y, z) = -y,$$

which is independent of the measure argument μ. By the Pontryagin maximum principle, for each fixed $[0, T] \ni t \mapsto \bar{\mu}_t \in \mathbb{R}$, the optimal control problem of step (i) has a unique solution if and only if we can uniquely solve the FBSDE:

$$\begin{cases} dX_t = -Y_t\, dt + \sigma\, dW_t, \\ dY_t = -(X_t - \bar{\mu}_t)\, dt + Z_t\, dW_t \end{cases} \tag{2}$$

with the initial condition $X_0 = \xi$ and the terminal condition $Y_T = k(X_T - \bar{\mu}_T)$. Assume that the fixed point step (ii) can be solved, we can substitute $\bar{\mu}_t$ for $\mathbb{E}[X_t]$ in (2) and the FBSDE becomes the McKean-Vlasov FBSDE

$$\begin{cases} dX_t = -Y_t\, dt + \sigma\, dW_t, \\ dY_t = -(X_t - \mathbb{E}[X_t])\, dt + Z_t\, dW_t \end{cases} \tag{3}$$

with the initial condition $X_0 = \xi$ and the terminal condition $Y_T = k(X_T - \mathbb{E}[X_T])$.

Taking expectation of the both side of (3) and using the notation $\bar{x}_t$ and $\bar{y}_t$ for the expectations $\mathbb{E}[X_t]$ and $\mathbb{E}[Y_t]$ respectively, we have

$$\begin{cases} d\bar{x}_t = -\bar{y}_t\, dt, & \bar{x}_0 = \mathbb{E}[\xi], \\ d\bar{y}_t = 0\, dt, & \bar{y}_T = 0, \end{cases}$$

which implies that $\mathbb{E}[Y_t] = \bar{y}_t = 0$ and $\mathbb{E}[X_t] = \bar{x}_t = \mathbb{E}[\xi]$ for all $t \in [0, T]$. From the Theorem 3.34 of [1], we know that the existence and uniqueness of a solution to the LQ MFG problem (i)-(ii) hold.

3 Search for the optimal paths

Finding the optimal mean function $[0, T] \ni t \mapsto \bar{x}_t$ guarantees the existence of a solution to the MFG problem, but it does not tell much about the optimal state trajectories or the optimal control. The latter can be obtained by plugging the so-obtained $\bar{x}_t$ into the FBSDE (3) in lieu of $\mathbb{E}[X_t]$ and solving for $X = (X_t)_{0 \le t \le T}$ and $Y = (Y_t)_{0 \le t \le T}$. This search reduces to the solution of the affine FBSDE:

$$\begin{cases} dX_t = -Y_t\, dt + \sigma\, dW_t, & X_0 = \xi, \\ dY_t = -(X_t - \mu)\, dt + Z_t\, dW_t, & Y_T = k(X_T - \mu), \end{cases} \tag{4}$$

where $\mu = \mathbb{E}[\xi]$. The standard theory of FBSDEs suggests that Y_t should be given by a deterministic function of t and X_t, the so-called decoupling field. The affine structure of the FBSDE (4) suggests that this decoupling field should be affine. Suppose

$$Y_t = a_t X_t + b_t, \quad t \in [0, T],$$

where a_t and b_t are differentiable function on $[0, T]$. Then

$$Y_T = a_T X_T + b_T = k X_T - k\mu,$$

which gives $a_T = k$ and $b_T = -k\mu$. Also by calculation, we have

$$\begin{aligned} dY_t &= a_t\, dX_t + X_t\, da_t + db_t \\ &= a_t(-Y_t\, dt + \sigma\, dW_t) + X_t a_t'\, dt + b_t' dt \\ &= a_t\left((-a_t X_t - b_t)\, dt + \sigma\, dW_t\right) + X_t a_t'\, dt + b_t' dt \\ &= \left((a_t' - a_t^2)X_t - a_t b_t + b_t'\right)\, dt + \sigma a_t\, dW_t. \end{aligned}$$

Identifying term by term with the expression of dY_t given in (4), then

$$\begin{cases} a_t' - a_t^2 + 1 = 0, & a_T = k, \\ b_t' - a_t b_t - \mu = 0, & b_T = -\mu k, \\ Z_t = \sigma a_t. \end{cases} \tag{5}$$

The solution of the ODE system (5) is

$$\begin{cases} a_t = -1 + \frac{2}{1-\left(1-\frac{2}{k+1}\right)e^{2(T-t)}}, \\ b_t = -\mu k e^{\int_t^T -a_s\, ds} - \int_t^T \mu e^{\int_t^s -a_u\, du}\, ds, \\ Z_t = \sigma a_t. \end{cases} \tag{6}$$

Plugging $Y_t = a_t X_t + b_t$ back to the FBSDE (4), then we know that

$$dX_t = (-a_t X_t - b_t)\, dt + \sigma\, dW_t,$$

which gives the optimal trajectory

$$X_t = X_0 e^{\int_0^t -a_s\, ds} + \int_0^t -b_t e^{\int_s^t -a_u\, du}\, ds + \sigma \int_0^t e^{\int_s^t -a_u\, du}\, dW_s,$$

and $Y_t = a_t X_t + b_t$, where a_t and b_t are given in (6).

3.1 A special case: k = 1

Consider $k = 1$, the solution of the coefficient functions is as follows

$$\begin{cases} a_t = 1, \\ b_t = -\mu, \\ Z_t = \sigma. \end{cases} \tag{7}$$

All the coefficient function are constants, and we know that $Y_t = -X_t + \mu$. Plugging it back to the FBSDE (4), we have

$$dX_t = (\mu - X_t)\, dt + \sigma\, dW_t,$$

thus $X = (X_t)_{0 \le t \le T}$ is a OrnsteinUhlenbeck process and the explicit form of X is given by

$$X_t = X_0 e^{-t} + \mu(1 - e^{-t}) + \sigma \int_0^t e^{-(t-s)}\, dW_s.$$

4 Analytical approach

Denote

$$v(X,t)=\inf_{\alpha\in\mathcal{A}}\mathbb{E}_t\left[\int_t^T\left(\frac{1}{2}\alpha_s^2+\frac{1}{2}(X_s-\bar{\mu}_s)^2\right)ds+\frac{k}{2}(X_T-\bar{\mu}_T)^2\middle|X_t=X\right].$$

Note that the above linear quadratic mean field game model can be characterized by Hamilton-Jacobian-Bellman equationcoupled by Fokker-Planck-Kolmogorov equation:

$$\begin{cases}\partial_t v+\frac{1}{2}\sigma^2\partial_{xx}v-\frac{1}{2}(\partial_x v)^2+\frac{1}{2}(x-\bar{\mu})^2=0, & (t,x)\in[0,T]\times\mathbb{R},\\ \partial_t m-\frac{1}{2}\sigma^2\partial_{xx}m-\partial_x(m\partial_x v)=0, & (t,x)\in[0,T]\times\mathbb{R},\\ m_0\sim\mathcal{L}(\xi),v(x,T)=\frac{k}{2}(x-\bar{\mu}_T), & x\in\mathbb{R}.\end{cases}$$

Inspired by the linear quadratic structure, we use the ansatz of value function

$$v(x,t)=f_1(t)x^2+f_2(t)x+f_3(t),\quad t\in[0,T],$$

Plugging it back to the HJB equation, then we can get the ODE system for the coefficient functions as follows

$$\begin{cases}f_1'-2f_1^2+\frac{1}{2}=0, & f_1(T)=\frac{k}{2},\\ f_2'-2f_1f_2-\bar{\mu}=0, & f_2(T)=-k\bar{\mu}_T,\\ f_3'+\sigma^2f_1-\frac{1}{2}f_2^2+\frac{1}{2}\bar{\mu}^2=0, & f_3(T)=\frac{k}{2}\bar{\mu}_T^2.\end{cases}$$

We can see that the coefficient function f_2 depends on the expectation of X. Note that the optimal strategy $\alpha_t^*=-\partial_x v=-2f_1(t)x-f_2(t)$. Hence under the optimal control

$$\mathbb{E}\left[X_t\right]=\mathbb{E}[X_0]+\int_0^t\left(-2f_1(s)\mathbb{E}[X_s]-f_2(s)\right)ds,$$

Since $\bar{\mu}(t)=\mathbb{E}[X_t]$, which follows

$$\bar{\mu}'=-2f_1\bar{\mu}-f_2.$$

Therefore we can obtain the forward-backward Riccati ODE system

$$\begin{cases}f_1'-2f_1^2+\frac{1}{2}=0, & f_1(T)=\frac{k}{2},\\ f_2'-2f_1f_2-\bar{\mu}=0, & f_2(T)=-k\bar{\mu}_T,\\ f_3'+\sigma^2f_1-\frac{1}{2}f_2^2+\frac{1}{2}\bar{\mu}^2=0, & f_3(T)=\frac{k}{2}\bar{\mu}_T^2,\\ \bar{\mu}'+2f_1\bar{\mu}+f_2=0, & \bar{\mu}_0=\mathbb{E}[\xi]:=\mu.\end{cases}\tag{8}$$

We find that the solution of (8) is coincide with the solution of ODE system (5) by

$$f_1(t)=\frac{1}{2}a_t,\ f_2(t)=b_t,\ \bar{\mu}_t=\mathbb{E}[\xi]=\mu,\quad t\in[0,T].$$

When $k=1$, we know that the solution of the Ricatti ODE system (8) is

$$f_1(t)=\frac{1}{2},\ f_2(t)=-\mu,\ f_3(t)=\frac{\sigma^2}{2}(T-t)+\frac{1}{2}\mu,\ \bar{\mu}_t=\mu$$

for all $t\in[0,T]$.

There is a similar model in the appendix of [2], where also gives an ODE for the variance of X. Therefore, the analytical method can give more information, but at the cost of more complex forward-backward ODE system.

References

[1] Carmona René, and François Delarue. *Probabilistic theory of mean field games with applications I*, volume 83 of Probability Theory and Stochastic Modelling. 2018.

[2] Jiamin Jian, Peiyao Lai, Qingshuo Song, and Jiaxuan Ye. Regime Switching Mean Field Games with Quadratic Costs. *arXiv preprint arXiv:2106.04762.* 2021.