Preliminary Exam Syllabus

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Committee:

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Topic: Stochastic Analysis
Professor Sturm

Reference: Bass, Richard F. *Stochastic Processes*. Vol. 33. Cambridge University Press, 2011. Chapter 1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 24, 25, 39, 40, 41.

1. Basic notions
1.1 Processes and σ-fields
1.2 Laws and state spaces

2. Brownian motion
2.1 Definition and basic properties

3. Martingales
3.1 Definition and examples
3.2 Doob's inequalities
3.3 Stopping times
3.4 The optimal stopping theorem
3.5 Convergence and regularity
3.6 Some applications of martingales

4. Markov properties of Brownian motion
4.1 Markov properties
4.2 Applications

7. Path properties of Brownian motion

8. The continuity of paths

9. Continuous semimartingales
9.1 Definitions
9.2 Square integrable martingales
9.3 Quadratic variation
9.4 The Doob-Meyer decomposition

10. Stochatic integrals
10.1 Construction
10.2 Extensions

11. Itô's formula

12. Some applications of Itô's formula
12.1 Lévy's Theorem
12.2 Time changes of martingals
12.3 Quadratic variation
12.4 Martingale representation
12.5 The Burkholder-Davis-Gundy inequalities
12.6 Stratonovich integrals

13. The Girsanov theorem
13.1 The Brownian moton case
13.2 An example

24. Stochastic differential equations
24.1 Pathwise solutions of SDEs
24.2 One-dimensional SDEs
24.3 Examples of SDEs

25. Weak solution of SDEs

39. Markov processes and SDEs
39.1 Markov properties
39.2 SDEs and PDEs
39.3 Martingale problems

40. Solving partial differential equations
40.1 Poisson's equation
40.2 Dirichlet problem
40.3 Cauchy problem
40.4 Schrödinger operators

41. One-dimensional diffusions
41.1 Regularity
41.2 Scale functions
41.3 Speed measures
41.4 The uniqueness theorem
41.5 Time change
41.6 Examples

Topic: Stochastic Control
Professor Wang

Reference: Pham, Huyên. *Continuous-time Stochastic Control and Optimization with Financial Applications.* Vol.61. Springer Science & Business Media, 2009. Chapter 2, 3, 4, 5.

2. Stochastic optimization problems. Example in finance
2.1 Introduction
2.2 Examples
- 2.2.1 Portfolio allocation
- 2.2.2 Production-consumption model
- 2.2.3 Irreversible investment model
- 2.2.4 Quadratic hedging of options
- 2.2.5 Superreplication cost in uncertain volatility
- 2.2.6 Optimal selling of an asset
- 2.2.7 Valuation of natural resources

2.3 Other optimization problems in finance
- 2.3.1 Ergodic and risk-sensitive control problems
- 2.3.2 Superreplication under gamma constraints
- 2.3.3 Robust utility maximization problem and risk measures
- 2.3.4 Forward performance criterion

3. The classical PDE approach to dynamic programming
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3.2 Controlled diffusion processes
3.3 Dynamic programming principle
3.4 Hamilton-Jacobi-Bellman equation
- 3.4.1 Formal derivation of HJB
- 3.4.2 Remarks and extensions

3.5 Verification theorem
3.6 Applications
- 3.6.1 Merton portfolio allocation problem in finite horizon
- 3.6.2 Investment-consumption problem with random time horizon
- 3.6.3 A model of production-consumption on infinite horizon

3.7 Example of singular stochastic control problem

4. The viscosity solutions approach to stochastic problems
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4.2 Definition of viscosity solutions
4.3 From dynamic programming to viscosity solutions of HJB equations
- 4.3.1 Viscosity properties inside the domain
- 4.3.2 Terminal condition

4.4 Comparison principles and uniqueness results
- 4.4.1 Classical comparison principle
- 4.4.2 Strong comparison principle

4.5 An irreversible investment model
- 4.5.1 Problem
- 4.5.2 Regularity and construction of the value function
- 4.5.3 Optimal strategy

4.6 Supperreplication cost in uncertain volatility model
- 4.6.1 Bounded volatility
- 4.6.2 Unbounded volatility

5. Optimal switching and free boundary problems

5.1 Introduction

5.2 Optimal stopping

- 5.2.1 Dynamic programming and viscosity property
- 5.2.2 Smooth-fit principle
- 5.2.3 Optimal strategy
- 5.2.4 Methods of solution in the one-dimensional case
- 5.2.5 Examples of applications

5.3 Optimal switching

- 5.3.1 Problem formulation
- 5.3.2 Dynamic programming and system of variational inequalities
- 5.3.3 Switching regions
- 5.3.4 The one-dimensional case
- 5.3.5 Explicit solution in the two-regime case

Topic: Mean Field Games
Professor Song

Reference: Cardaliaguet, Pierre. *Notes on mean field games.* Technical report, 2013.
https://www.ceremade.dauphine.fr/~cardaliaguet/MFG20130420.pdf

1. Introduction

2. Nash equilibria in games with a large number of players
2.1 Symmetric functions of many variables
2.2 Limits of Nash equilibria in pure strategies
2.3 Limits of Nash equilibria in mixed strategies
2.4 A uniqueness result
2.5 Example: potential games
2.6 Comments

3. Analysis of second order MFEs
3.1 On the Fokker-Planck equation
3.2 Proof of the existence theorem
3.3 Uniqueness
3.4 Application to games with finitely many players
3.5 Comments

4. Analysis of first order MFEs
4.1 Semi-concavity estimates
4.2 On the continuity equation
4.3 Proof of the existence theorem
4.4 The vanishing viscosity limit
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5. The space of probability measures
5.1 The Monge-Kantorovich distances
5.2 The Wasserstein space of probability measures on $\mathbb{R}^d$
5.3 Polynomials on $\mathcal{P}(Q)$
5.4 Hewitt and Savage Theorem
5.5 Comments

6. Hamilton-Jacobi equations in the space of probability measures
6.1 Derivative in the Wasserstein space
6.2 First order Hamilton-Jacobi equations
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7. Heuristic derivation of the mean field equation
7.1 The differential game
7.2 Derivation of the equation in $\mathcal{P}_2$
7.3 From the equation in $\mathcal{P}_2$ to the mean field equation

8. Appendix
8.1 Nash equilibria in classical differential games
8.2 Desintegration of a measure
8.3 Ekeland's and Stegall's variational principles