July 27, 2021

Content: Solvability of McKean-Vlasov FBSDEs by Schauder's theorem

- The equivalence of two norms in Hölder space
- The full-fledged McKean-Vlasov FBSDE model (4.32)
- Assumptions: Nondegenerate MKV FBSDE
- Idea to show the existence
- Some preparation results:
  - Lemma 4.33: Existence result of FBSDE; the regularity of decoupling field
  - Lemma 4.34 and remark 4.35: the senstivity of the FBSDE; the bound of W_2 distance between two given measures

Questions:

- Prove the inequality: for $x, y > 0$ and $\gamma > 1$, we have $(x+y)^\gamma \geq x^\gamma + y^\gamma$.
- (Carmona, Delarue), Page 245. Assumption (Nondegenerate MKV FBSDE): why we need the growth condition with respect to the square root of the second moment of the measure? And why the growth condition of F is independent to the law of X?
- (Carmona, Delarue), Page 247. Lemma 4.33: why the decoupling field u is Lipschitz to x? In the assumption (A2), F is linear growth w.r.t. y, z.

August 03, 2021

Content: Solvability of McKean-Vlasov FBSDEs by Schauder's theorem. Fixed point argument in some space under the bounded case.

- The assumptions and spaces.
- Main result: Lemma 4.35 and Proposition 4.37 (Carmona, Delarue).
- The proof of Lemma 4.35.

Questions:

- For any bounded continuous function $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ and for any flow of probability measures $\boldsymbol{\mu} = (\mu_t)_{0 \leq t \leq T} \in C([0,T]; \mathcal{P}_2(\mathbb{R}))$. Show that the map $[0,T] \ni t \mapsto \varphi(t, \cdot) \diamond \mu_t \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ is continuous.
- Why we need to use $\mathcal{M}_f^1(\mathbb{R})$ rather than $\mathcal{P}_1(\mathbb{R})$? In the Schauder's fixed point theorem, $(V, \|\cdot\|)$ is a normed linear vector space.
- The distance induced by the Kantorovich-Rubinstein norm $\|\cdot\|_{KR*}$ coincides with the 1-Wasserstein metric W_1.
- Let
$$\mathcal{E} = \left\{ \boldsymbol{\mu} \in C([0,T]; \mathcal{P}_4(\mathbb{R})) : \sup_{t \in [0,T]} \int_{\mathbb{R}} |x|^4 \, d\mu_t(x) \leq \gamma \right\},$$
where γ is a constant. Let $(\boldsymbol{\mu}^n)_{n \geq 1}$ be a sequence in $\mathcal{E}$. Show that the measures $(\mu_t^n)_{n \geq 1}$ are tightness for any $t \in [0,T]$.

September 03, 2021

Content: The working paper: Regime Switching Mean Field Games with Quadratic Costs. How to solve the problem in original paper. We used two auxiliary processes and let the initial value of the mean to be zero. Then one ODE system can be derived and we can prove the existence and uniqueness of the solution for the ODE system. What's more, we can implement a numerical experiment to see if there is any convergence between the N-player system and the mean field game system.

September 07, 2021

Content: The master equation and the paper(Mean Field Games Master Equations with Non-separable Hamiltonians and Displacement Monotonicity)

- Calculus on $\mathcal{P}(\mathbb{R}^d)$
  - $U : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ is C^1
  - Definition of intrinsic derivative
  - Definition of $D_v D_m U(m, v)$ and $D_m^2 U(M, v, v')$
- Itô's formula for $F(X_t, \mu_t)$
- Feynman-Kac formula, verification theorem
- The master equation for mean field games
- Simplification of the master equation: drift control and constant volatility; a linear quadratic model
- Paper: Mean Field Games Master Equations with Non-separable Hamiltonians and Displacement Monotonicity

Questions:

- Why the decoupling FBSDE (2.24) of the paper holds?
- How to understand the notation $(d_x d)_\xi$ in (1.6) and (2.16)?
- Page 12 of the paper: when $\xi \in \mathbb{L}^2(\mathcal{F}_t^1)$ is independent of $\mathcal{F}_t^0$, then $J(t, x, \xi; \alpha, \alpha')$ is deterministic and

$$\xi' \in \mathbb{L}^2(\mathcal{F}_t^1),\ \mathcal{L}_{\xi'} = \mathcal{L}_\xi \implies J(t, x, \xi'; \alpha, \alpha') = J(t, x, \xi; \alpha, \alpha') \quad \forall x, \alpha, \alpha'.$$

- Page 30 of the paper: theorem 6.3, why assumption 3.1, 3.2, 3.5 holds, then we will have the unique classical solution to the master equation? What is the connection of the displacement monotonicity with the solution of the master equation?