An example of linear quadratic mean field game

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1 Problem setup

Let $(\Omega, \mathcal{F}_T, \mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a completed filtered probability space satisfying the usual conditions, on which

- $W = (W_t)_{0\le t\le T}$ is a standard Brownian motion taking values in $\mathbb{R}$,
- ξ is a random variable in $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R})$,
- $\mathcal{F}_t$ is generated by $\{W_s : 0 \le s \le t\}$ and ξ.

Denote by $\mathcal{A}$ the set of $\mathbb{F}$-progressively measurable $A \subset \mathbb{R}$ valued stochastic process $\alpha = (\alpha_t)_{0\le t\le T}$ that satisfy the square integrability condition

$$\mathbb{E}\left[\int_0^T |\alpha_t|^2\, dt\right] < \infty.$$

Consider the linear quadratic mean field game problem:

- (i) For each fixed deterministic function $[0,T] \ni t \mapsto \bar{\mu}_t \in \mathbb{R}$, solve the standard stochastic control problem

$$\inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\alpha_t^2 + \frac{1}{2}\left(X_t - \bar{\mu}_t\right)^2\right) dt + \frac{k}{2}\left(X_T - \bar{\mu}_T\right)^2\right]$$

subject to

$$\begin{cases} dX_t = \alpha_t\, dt + \sigma\, dW_t \\ X_0 = \xi. \end{cases}$$

- (ii) Determine a function $[0,T] \ni t \mapsto \bar{\mu}_t \in \mathbb{R}$ so that, for all $t \in [0,T]$, $\mathbb{E}[\hat{X}_t] = \bar{\mu}_t$, where $(\hat{X}_t)_{0\le t\le T}$ is the optimal path of the optimal problem in the environment $(\bar{\mu}_t)_{0\le t\le T}$.

2 Existence and uniqueness by SMP

We define the generalized Hamiltonian $H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times A \to \mathbb{R}$ by

$$H(t, x, \mu, y, z, a) = ay + \sigma z + \frac{1}{2}a^2 + \frac{1}{2}(x - \mu)^2.$$

The Hamiltonian is minimized for

$$\hat{\alpha} = \hat{\alpha}(t, x, \mu, y, z) = -y,$$

which is independent of the measure argument μ. By the Pontryagin maximum principle, for each fixed $[0,T] \ni t \mapsto \bar{\mu}_t \in \mathbb{R}$, the optimal control problem of step (i) has a unique solution if and only if we can uniquely solve the FBSDE:

$$\begin{cases} dX_t = -Y_t\, dt + \sigma\, dW_t, \\ dY_t = -(X_t - \bar{\mu}_t)\, dt + Z_t\, dW_t \end{cases} \tag{1}$$

with the initial condition $X_0 = \xi$ and the terminal condition $Y_T = k(X_T - \bar{\mu}_T)$. Assume that the fixed point step (ii) can be solved, we can substitute $\bar{\mu}_t$ for $\mathbb{E}[X_t]$ in (1) and the above FBSDE becomes the McKean-Vlasov FBSDE

$$\begin{cases} dX_t = -Y_t\, dt + \sigma\, dW_t, \\ dY_t = -(X_t - \mathbb{E}[X_t])\, dt + Z_t\, dW_t \end{cases} \tag{2}$$

with the initial condition $X_0 = \xi$ and the terminal condition $Y_T = k(X_T - \mathbb{E}[X_T])$.

Taking expectation of the both side of (2) and using the notation $\bar{x}_t$ and $\bar{y}_t$ for the expectations $\mathbb{E}[X_t]$ and $\mathbb{E}[Y_t]$ respectively, we have

$$\begin{cases} d\bar{x}_t = -\bar{y}_t\,dt, & \bar{x}_0 = \mathbb{E}[\xi], \\ d\bar{y}_t = 0\,dt, & \bar{y}_T = 0, \end{cases}$$

which implies that $\mathbb{E}[Y_t] = \bar{y}_t = 0$ and $\mathbb{E}[X_t] = \bar{x}_t = \mathbb{E}[\xi]$ for all $t \in [0,T]$. From the Theorem 3.34 of (1), we know that the existence and uniqueness of the solution to the LQ MFG problem (i)-(ii) holds.

3 Search for the optimal paths

Finding the optimal mean function $[0,T] \ni t \mapsto \bar{x}_t$ guarantees the existence of a solution to the MFG problem, but it does not tell much about the optimal state trajectories or the optimal control. The latter can be obtained by plugging the so-obtained $\bar{x}_t$ into the FBSDE (2) in lieu of $\mathbb{E}[X_t]$ and solving for $X = (X_t)_{0\le t\le T}$ and $Y = (Y_t)_{0\le t\le T}$. This search reduces to the solution of the affine FBSDE:

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t, & X_0 = \xi, \\ dY_t = -(X_t - \mu)\,dt + Z_t\,dW_t, & Y_T = k(X_T - \mu), \end{cases} \tag{3}$$

where $\mu = \mathbb{E}[\xi]$. The standard theory of FBSDEs suggests that Y_t should be given by a deterministic function of t and X_t, the so-called decoupling field. The affine structure of the FBSDE (3) suggests that this decoupling field should also be affine. Suppose

$$Y_t = a_t X_t + b_t, \quad t \in [0,T],$$

where a_t and b_t are differentiable function on $[0,T]$. Then

$$Y_T = a_T X_T + b_T = kX_T - k\mu,$$

which gives $a_T = k$ and $b_T = -k\mu$. Also by calculation, we have

$$\begin{aligned} dY_t &= a_t\,dX_t + X_t\,da_t + db_t \\ &= a_t(-Y_t\,dt + \sigma\,dW_t) + X_t a_t'\,dt + b_t' dt \\ &= a_t\left((-a_t X_t - b_t)\,dt + \sigma\,dW_t\right) + X_t a_t'\,dt + b_t' dt \\ &= \left((a_t' - a_t^2)X_t - a_t b_t + b_t'\right)\,dt + \sigma a_t\,dW_t. \end{aligned}$$

Identifying term by term with the expression of dY_t given in (3), then

$$\begin{cases} a_t' - a_t^2 + 1 = 0, & a_T = k, \\ b_t' - a_t b_t - \mu = 0, & b_T = -\mu k, \\ Z_t = \sigma a_t. \end{cases} \tag{4}$$

The solution of the ODE system (4) is

$$\begin{cases} a_t = -1 + \frac{2}{1-\left(1-\frac{2}{k+1}\right)e^{2(T-t)}}, \\ b_t = -\mu k e^{\int_t^T -a_s\,ds} - \int_t^T \mu e^{\int_t^s -a_u\,du}\,ds, \\ Z_t = \sigma a_t. \end{cases} \tag{5}$$

Plugging $Y_t = a_t X_t + b_t$ back to the FBSDE (3), then we know that

$$dX_t = (-a_t X_t - b_t)\,dt + \sigma\,dW_t,$$

which gives the optimal trajectory

$$X_t = X_0 e^{\int_0^t -a_s\,ds} + \int_0^t -b_t e^{\int_s^t -a_u\,du}\,ds + \sigma \int_0^t e^{\int_s^t -a_u\,du}\,dW_s,$$

and $Y_t = a_t X_t + b_t$, where a_t and b_t are given in (5).

3.1 A special case: k = 1

Consider $k = 1$, the solution of the coefficient functions is as follows

$$\begin{cases} a_t = 1, \\ b_t = -\mu, \\ Z_t = \sigma. \end{cases} \tag{6}$$

All the coefficient function are constants, and we know that $Y_t = X_t - \mu$. Plugging it back to the FBSDE (3), we have

$$dX_t = (\mu - X_t)\, dt + \sigma\, dW_t,$$

thus $X = (X_t)_{0\le t\le T}$ is a Ornstein–Uhlenbeck process and the explicit form of X is given by

$$X_t = X_0 e^{-t} + \mu(1 - e^{-t}) + \sigma \int_0^t e^{-(t-s)}\, dW_s.$$

4 Analytical approach

Denote

$$v(X,t) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}_t \left[\int_t^T \left(\frac{1}{2}\alpha_s^2 + \frac{1}{2}(X_s - \bar{\mu}_s)^2 \right) ds + \frac{k}{2}(X_T - \bar{\mu}_T)^2 \middle| X_t = X \right].$$

Note that the above linear quadratic mean field game model can be characterized by Hamilton-Jacobian-Bellman equationcoupled by Fokker-Planck-Kolmogorov equation:

$$\begin{cases} \partial_t v + \frac{1}{2}\sigma^2 \partial_{xx} v - \frac{1}{2}(\partial_x v)^2 + \frac{1}{2}(x - \bar{\mu})^2 = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ \partial_t m - \frac{1}{2}\sigma^2 \partial_{xx} m - \partial_x (m \partial_x v) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ m_0 \sim \mathcal{L}(\xi), v(x,T) = \frac{k}{2}(x - \bar{\mu}_T)^2, & x \in \mathbb{R}. \end{cases}$$

Inspired by the linear quadratic structure, we use the ansatz of value function

$$v(x,t) = f_1(t)x^2 + f_2(t)x + f_3(t), \quad t \in [0,T].$$

Plugging it back to the HJB equation, then we can get an ODE system for the coefficient functions as follows

$$\begin{cases} f_1' - 2f_1^2 + \frac{1}{2} = 0, & f_1(T) = \frac{k}{2}, \\ f_2' - 2f_1 f_2 - \bar{\mu} = 0, & f_2(T) = -k\bar{\mu}_T, \\ f_3' + \sigma^2 f_1 - \frac{1}{2} f_2^2 + \frac{1}{2}\bar{\mu}^2 = 0, & f_3(T) = \frac{k}{2}\bar{\mu}_T^2. \end{cases}$$

We can see that the coefficient functions f_2 and f_3 depend on the expectation of X. Note that the optimal strategy is given by $\alpha_t^* = -\partial_x v(x,t) = -2f_1(t)x - f_2(t)$. Hence under the optimal control,

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] + \int_0^t (-2f_1(s)\mathbb{E}[X_s] - f_2(s))\, ds.$$

Since $\bar{\mu}(t) = \mathbb{E}[X_t]$, taking derivative to t for $\bar{\mu}(t)$ on the both sides, which follows

$$\bar{\mu}' = -2f_1\bar{\mu} - f_2.$$

Therefore we can obtain the forward-backward Riccati ODE system

$$\begin{cases} f_1' - 2f_1^2 + \frac{1}{2} = 0, & f_1(T) = \frac{k}{2}, \\ f_2' - 2f_1 f_2 - \bar{\mu} = 0, & f_2(T) = -k\bar{\mu}_T, \\ f_3' + \sigma^2 f_1 - \frac{1}{2} f_2^2 + \frac{1}{2}\bar{\mu}^2 = 0, & f_3(T) = \frac{k}{2}\bar{\mu}_T^2, \\ \bar{\mu}' + 2f_1\bar{\mu} + f_2 = 0, & \bar{\mu}_0 = \mathbb{E}[\xi] := \mu. \end{cases} \tag{7}$$

We find that the solution of (7) is coincide with the solution of ODE system (4) by

$$f_1(t) = \frac{1}{2}a_t,\ f_2(t) = b_t,\ \bar{\mu}_t = \mathbb{E}[\xi] = \mu, \quad t \in [0,T].$$

When $k = 1$, we know that the solution of the Ricatti ODE system (7) is

$$f_1(t) = \frac{1}{2},\ f_2(t) = -\mu,\ f_3(t) = \frac{\sigma^2}{2}(T - t) + \frac{1}{2}\mu,\ \bar{\mu}_t = \mu$$

for all $t \in [0, T]$.

There is a similar model in the appendix of (2), which also gives an ODE for the variance of X. Therefore, the analytical method can give more information, but at the cost of more complex forward-backward ODE system.

References

[1] Carmona René, and François Delarue. *Probabilistic theory of mean field games with applications I*, volume 83 of Probability Theory and Stochastic Modelling. 2018.

[2] Jiamin Jian, Peiyao Lai, Qingshuo Song, and Jiaxuan Ye. Regime Switching Mean Field Games with Quadratic Costs. *arXiv preprint arXiv:2106.04762.* 2021.