Documentation on $1/f$ Noise

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1 Overview of Machlup's derivation of $1/f$

Machlup's paper on $1/f$ noise shows that a convolution of purely random processes, i.e., those with autocorrelations of the form $e^{-t/\tau}$, with scale-invariant correlation time τ, will produce a $1/f$ power spectrum over a large frequency range. We explain this statement below.

It can be shown mathematically that random datasets have exponentially decaying autocorrelation, $R(t) = Ae^{-t/\tau}$, where A is the data variance and τ is the correlation time of the data. Its power spectrum is thus proportional to the Fourier transform of $e^{-t/\tau}$:

$$\begin{aligned}
S(\tau,\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} R(t)\,\mathrm{d}t && (1)\\
&\propto \int_{-\infty}^{0} e^{-i\omega t} e^{t/\tau}\,\mathrm{d}t + \int_{0}^{\infty} e^{-i\omega t} e^{-t/\tau}\,\mathrm{d}t && (2)\\
&= \frac{i\tau}{i+\tau\omega} + \frac{i\tau}{i-\tau\omega} && (3)\\
&\propto \frac{\tau}{1+\omega^2\tau^2}. && (4)
\end{aligned}$$

Here, $\omega = 2\pi f$, and we reflect $R(t)$ about the origin to extend it to the negative time domain.

Suppose we have a large assemble of random data with the same data variance, and their correlation times are scale-invariant. That is, the probability density distribution of τ behaves like

$$\rho(\tau)\,\mathrm{d}\tau \propto \frac{\mathrm{d}\tau}{\tau} \tag{5}$$

within some domain $[\tau_1, \tau_2]$. Then the power spectrum of the assemble of random data is

$$\overline{S}(\omega) = \int_{\tau_1}^{\tau_2} S(\tau,\omega)\rho(\tau)\,\mathrm{d}\tau \tag{6}$$
$$\propto \int_{\tau_1}^{\tau_2} \frac{\mathrm{d}\tau}{1+\omega^2\tau^2} \tag{7}$$
$$= \frac{\tan^{-1}(\tau_2\omega) - \tan^{-1}(\tau_1\omega)}{\omega}. \tag{8}$$

If we assume $\tau_1 \ll \tau_2$, then within the frequency range of $\tau_1 \ll 1/\omega \ll \tau_2$, $\overline{S}(\omega)$ approaches

$$\overline{S}(\omega) \propto \frac{\frac{\pi}{2} + \mathcal{O}(\frac{1}{\tau_2\omega}) - \mathcal{O}(\tau_1\omega)}{\omega} \tag{9}$$
$$= \frac{\pi}{2\omega} \tag{10}$$

to zeroth-order. Thus $\overline{S}(\omega) \propto 1/\omega$ within the region where $\tau_1 \ll 1/\omega \ll \tau_2$.

2 $1/f$ from arbitrary index α, the simple form

Equation 4 is called the Lorenzian function. In log-log scale, it is flat when $\tau\omega \ll 1$ and has a slope of -2 when $\tau\omega \gg 1$, as shown in Figure 1.

Most spectra we observe in the solar wind behave similarly, except that the slopes at high frequencies can take a range of different values. We want to show in this section that a convolution of spectra with an arbitrary slope $-\alpha$ will also produce a $1/f$ overall spectrum.

We assume, for mathematical simplicity, that each spectrum behaves like

$$S(\tau,\omega) \propto \frac{\tau}{(1+\omega\tau)^\alpha}, \tag{11}$$

where τ is the correlation time. The data assemble is assumed to have the same data variance, and a scale-invariant distribution of correlation time as in equation 5, which ranges from τ_1 to τ_2. Then the overall spectrum is

$$\overline{S}(\omega) \propto \int_{\tau_1}^{\tau_2} \frac{\mathrm{d}\tau}{(1+\omega\tau)^\alpha} \tag{12}$$
$$= \frac{(1+\tau_2\omega)^{1-\alpha} - (1+\tau_1\omega)^{1-\alpha}}{\omega(1-\alpha)}. \tag{13}$$

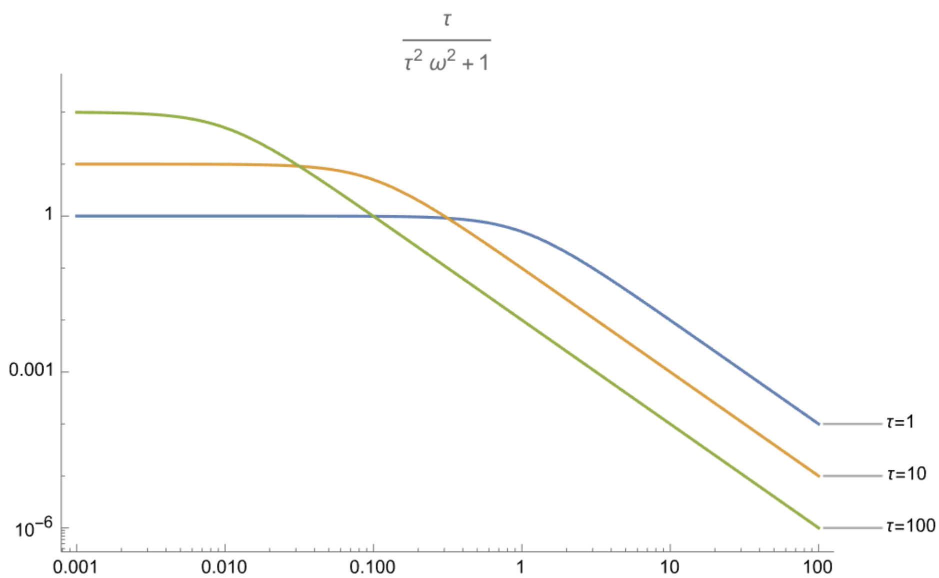

Figure 1:

In the frequency domain where $\tau_1 \ll 1/\omega \ll \tau_2$,

$$\overline{S}(\omega) \propto \frac{\mathcal{O}\left[(\tau_2\omega)^{1-\alpha}\right] - \left[1 + \mathcal{O}(\tau_1\omega)\right]}{\omega(1-\alpha)} \tag{14}$$

$$= \frac{1}{\omega(\alpha-1)} \tag{15}$$

to zeroth order, given $\alpha > 1$. Thus an assemble of data of spectral index $\alpha > 1$ may produce an overall spectrum that behaves like $\overline{S}(\omega) \propto 1/\omega$ within the region $\tau_1 \ll 1/\omega \ll \tau_2$.

3 $1/f$ from arbitrary index α, the more complicated form

Now, instead of equation 11, we consider individual spectrum of the following form:

$$S(\tau,\omega) \propto \frac{\tau}{(1+\omega^2\tau^2)^{\alpha/2}}. \tag{16}$$

Equation 16 is plotted and compared with equation 11 in Figure 2. We see that it is more characteristic of being flat at small frequencies and having a slope of $-\alpha$ at large frequencies. A convolution of these spectra also produces an overall $1/\omega$ behavior, although showing this requires some mathematical complexity.

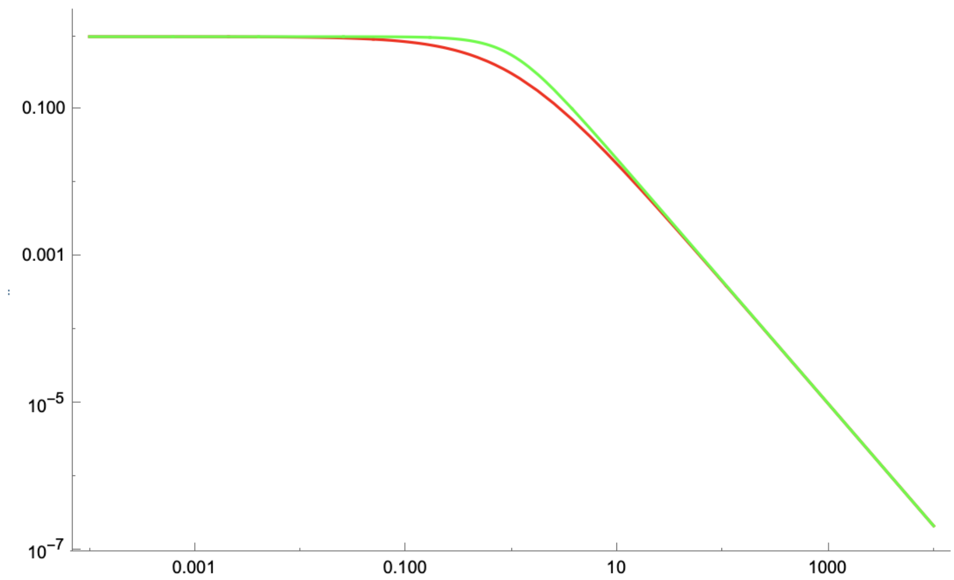

Figure 2: Red curve shows equation 11 and green curve shows equation 16. Here α is assumed to be 5/3.

The overall spectrum, with the same correlation time distribution as in the sections above, is

$$\overline{S}(\omega) \propto \int_{\tau_1}^{\tau_2} \frac{\mathrm{d}\tau}{(1+\omega^2\tau^2)^{\alpha/2}} \tag{17}$$

$$= \frac{1}{\omega} \int_{x_1}^{x_2} \frac{\mathrm{d}x}{(1+x^2)^{\alpha/2}} \tag{18}$$

$$= \frac{1}{\omega} \left[\int_0^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\alpha/2}} - \int_0^{x_1} \frac{\mathrm{d}x}{(1+x^2)^{\alpha/2}} - \int_{x_2}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\alpha/2}} \right], \tag{19}$$

where we apply the substitution $x \equiv \omega\tau$, and by definition $x_1 \ll 1$ and $x_2 \gg 1$. The first term in the bracket has a definite solution

$$\int_0^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\alpha/2}} = \frac{\sqrt{\pi}\,\Gamma(\frac{\alpha-1}{2})}{2\Gamma(\frac{\alpha}{2})}. \tag{20}$$

The third term can be approximated as

$$\int_{x_2}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\alpha/2}} \approx \int_{x_2}^{\infty} \frac{\mathrm{d}x}{x^{\alpha}} = \frac{x_2^{1-\alpha}}{\alpha - 1}. \tag{21}$$

The solution to the second term is the hypergeometric function

$$\int_0^{x_1} \frac{\mathrm{d}x}{(1+x^2)^{\alpha/2}} = x_1\,{}_2F_1\left(\frac{1}{2}, \frac{\alpha}{2}; \frac{3}{2}; -x_1^2\right) = x_1 - \frac{\alpha x_1^3}{6} + \mathcal{O}(x_1)^5. \tag{22}$$

Here ${}_2F_1$ is the hypergeometric function. The hypergeometric function is originally defined as

$$ {}_2F_1\left(a, b; c; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \qquad |z| < 1, \tag{23} $$

where $(p)_n$ is the Pochhammer symbol of values

$$ (p)_n = \begin{cases} 1 & n = 0 \\ \frac{\Gamma(p+n)}{\Gamma(p)} = p(p+1)\cdots(p+n-1) & n = 1, 2, \cdots . \end{cases} \tag{24} $$

Substituting equations 20 22, and 21 into equation 19 gives the overall scaling relationship

$$ \overline{S}(\omega) \propto \frac{1}{\omega} \left[\frac{\sqrt{\pi}\Gamma(\frac{\alpha-1}{2})}{2\Gamma(\frac{\alpha}{2})} - \mathcal{O}(x_2)^{1-\alpha} - x_1 + \mathcal{O}(x_1)^3 \right]. \tag{25} $$

If we only care about the least order term, then $\overline{S}(\omega) \propto 1/\omega$ only if $\alpha > 1$, that is, the order of the second term in the bracket is less than that of the first term. The constraint of $\alpha > 1$ also makes sure that $\Gamma[(\alpha - 1)/2]$ is positive. In conclusion, superposition of spectra of a fixed slope $-\alpha$ with $\alpha > 1$ will produce a $1/\omega$ spectrum in the range $\tau_1 \ll 1/\omega \ll \tau_2$.

4 Alternative derivation of $1/f$, the superposition of slopes

In this section, we present an alternative derivation based on mean slope evaluation, showing that the superposition of signals with power-law index $-\alpha$ gives an overall $1/\omega$ spectrum. The outline is as follows: (1) Given a distribution of τ_c, we can determine the distribution of $f_c = 1/2\pi\tau_c$, (2) given equal total energy (variance) of individual dataset, we can determine the distribution of S_c, defined as the power at the flat part of the spectrum, (3) the distribution of S_c provides a weighting function for the ensemble of power-law indices at a given frequency f, which is either 0 if $f_c \leq f$ or $-\alpha$ if $f_c > f$, (4) we can then find the expected power-law index at any given f.

Note that we are switching notation for the correlation times from τ to τ_c, and we are working in frequency space instead of angular frequency space.

Given a correlation time τ_c for a specific dataset, we define the break frequency $f_c \equiv 1/2\pi\tau_c$, representing the point before which the power spectrum is flat, and after which the power spectrum is a power-law of index $-\alpha$. The reciprocal of an inversely distributed random variable also follows an inverse distribution. From $\rho(\tau_c) = 1/\ln(\tau_2/\tau_1)\tau_c$, we get

$$ \rho(f_c)\,\mathrm{d}f_c = \rho(\tau_c) \left| \frac{\mathrm{d}\tau_c}{\mathrm{d}f_c} \right| \mathrm{d}f_c = \frac{1}{\ln(\frac{f_1}{f_2})} \frac{\mathrm{d}f_c}{f_c}, \tag{26} $$

where $f_1 = 1/2\pi\tau_1$, $f_2 = 1/2\pi\tau_2$, and $f_c \in [f_2, f_1]$.

Now, for each power spectrum normalized to a total variance of 1, the power density before the break frequency is $S_c = (\alpha+1)/2\alpha f_c$. The derivation is in Section 3 of my notes *Documentation on Random Time Series, Fast Fourier Transform, and Power Spectral Density.* Thus S_c also follows the inverse distribution

$$\rho(S_c)\,\mathrm{d}S_c = \frac{1}{\ln\left(\frac{f_1}{f_2}\right)}\frac{\mathrm{d}S_c}{S_c}. \tag{27}$$

Instead of equation 16, we use a more straightforward and general definition of each power spectrum, uniquely described by S_c or f_c, as

$$S(f) = S_c\begin{cases}1 & f \le f_c\\ \left(\frac{f}{f_c}\right)^{-\alpha} & f > f_c.\end{cases} \tag{28}$$

Equation 28 omits the negative frequency domain, with the assumption that $S(f)$ is symmetric around the origin. With this definition of $S(f)$, at any given frequency f, the power-law index, denoted as β, is either 0 if the chosen spectrum has $f_c > f$, or $-\alpha$ if $f_c < f$. Now we have two relevant pieces of information to calculate the expected value of the slope of $S(f)$ on a loglog plot: (1) the distribution of S_c, and (2) the magnitude of S at frequency f. Intuitively, these cover all information needed, where (1) is the probability density of $S(f)$ and (2) is the weighting function of its slope at f such that the mean slope is

$$\overline{\beta}(f) = \frac{\int_{S_1}^{S_2}\beta(f)S(f)\rho(S_c)\,\mathrm{d}S_c}{\int_{S_1}^{S_2}S(f)\rho(S_c)\,\mathrm{d}S_c} \tag{29}$$

Why is $S(f)$ considered the weighting function of $\beta(f)$ under the context of spectrum superposition? To show this, we prove the following equivalent statement: Let $S_1(f) = A(f/f_a)^{\mu}$ and $S_2(f) = B(f/f_b)^{\nu}$, where A, B, f_a, f_b are arbitrary positive constants, and μ, ν are the power-law indices of S_1 and S_2. The slope of the spectrum S_1+S_2 on the loglog plot is $(\mu S_1+\nu S_2)/(S_1+S_2)$.

The slope on the loglog plot can be directly calculated as

$$\frac{\partial}{\partial \ln(f)}\ln(S_1+S_2) = \frac{\mathrm{d}f}{\mathrm{d}\ln(f)}\frac{\partial}{\partial f}\ln(S_1+S_2) \tag{30}$$

$$= f\left(\frac{\partial_f S_1 + \partial_f S_2}{S_1+S_2}\right) \tag{31}$$

$$= \frac{\mu S_1 + \nu S_2}{S_1+S_2}. \tag{32}$$

This means that although the power spectrum is not additive on a loglog plot (i.e., $\ln(S_1+S_2) \neq \ln S_1 + \ln S_2$), their power indices are.

We now continue our calculation of $\overline{\beta}(f)$ following equation 29, keeping in mind that $\beta = -\alpha$ when $S_c > S(f)$, and $\beta = 0$ otherwise:

$$\overline{\beta}(f) = \frac{-\alpha \int_S^{S_2} S_c (f/f_c)^{-\alpha} \rho(S_c)\,\mathrm{d}S_c}{\int_{S_1}^{S} S_c \rho(S_c)\,\mathrm{d}S_c + \int_S^{S_2} S_c (f/f_c)^{-\alpha} \rho(S_c)\,\mathrm{d}S_c} \tag{33}$$

$$= -\alpha \left[1 + f^{\alpha} \frac{\int_{S_1}^{S} S_c \rho(S_c)\,\mathrm{d}S_c}{\int_S^{S_2} S_c^{1-\alpha} \rho(S_c)\,\mathrm{d}S_c} \right]^{-1} \tag{34}$$

$$= -\alpha \left[1 + f^{\alpha} \frac{\int_{S_1}^{S} \mathrm{d}S_c}{\int_S^{S_2} S_c^{-\alpha}\,\mathrm{d}S_c} \right]^{-1} \tag{35}$$

$$= -\alpha \left[1 + f^{\alpha}(1-\alpha) \frac{(1/f - 1/f_1)}{(1/f_2^{1-\alpha} - 1/f^{1-\alpha})} \right]^{-1} \tag{36}$$

$$= -\alpha \left[1 - (1-\alpha) \frac{f/f_1 - 1}{(f_2/f)^{\alpha} - 1} \right]^{-1}. \tag{37}$$

Within the frequency region of $f_2 \ll f \ll f_1$ and with $\alpha > 1$, if we consider the least order term,

$$\overline{\beta}(f) = -1. \tag{38}$$