1. $\Pr(A-B) = \Pr(A\cap\bar{B})$

$\Pr(A\cup B) = \Pr(A)+\Pr(B)-\Pr(A\cap B)$

$\Rightarrow \Pr(A\cup\bar{B}) = \Pr(A)+\Pr(\bar{B})-\Pr(A\cap\bar{B})$

$\Rightarrow \Pr(A\cap\bar{B}) = \Pr(A)+\Pr(\bar{B})-\Pr(A\cup\bar{B})$

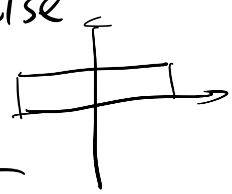
2. $\sum_{i=1}^{n} \Pr(B|A_i)\cdot\Pr(A_i) = \sum_{i=1}^{n} \Pr(B\cap A_i)$

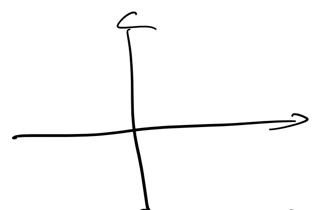
<u>mutual</u> $\Pr\left(B\cap\left(\bigcup_{i=1}^{n} A_i\right)\right) \underset{\Uparrow}{\le} \Pr(B\cap S) = \Pr(B)$

$\bigcup_{i=1}^{n} A_i \subseteq S$, where $\Pr(S)=1$

3. $f_X(x|X\in A) = \dfrac{f_X(x)\cdot I_A(x)}{\Pr(x\in A)} = \begin{cases}\dfrac{f_X(x)}{\Pr(x\in A)}, & x\in A\\ 0, & \text{else}\end{cases}$

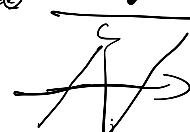
$I_A(x) = \begin{cases}1, & x\in A\\ 0, & \text{else}\end{cases}$





4. $X\sim U(-5,5)$

$q(x) = \begin{cases}-2.5 & -5\le x<0\\ 2.5 & 0\le x\le 5\end{cases}$ $\quad g(x) = X-q(X) = \begin{cases}x+2.5, & x\in[-5,0)\\ x-2.5, & x\in[0,5]\end{cases}$



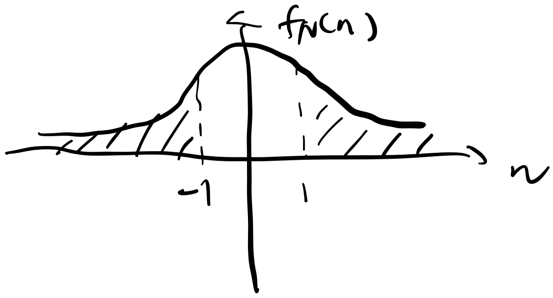
$\mathrm{Var}[X-q(X)] = \frac{1}{12}\underset{\Uparrow}{(b-a)^2} = \frac{1}{12}[2.5-(-2.5)]^2 = \frac{25}{12}$

$X-q(X)\sim U[-2.5, 2.5]$

5. $\Pr(e) = \Pr(\text{absent}\mid\text{present})\cdot\Pr(\text{present})$
$+\Pr(\text{present}\mid\text{absent})\cdot\Pr(\text{absent})$

$= \Pr(2+N\le 1)\cdot\frac{1}{2} + \Pr(N>1)\cdot\frac{1}{2}$

$= \Pr(N \le -1) \cdot \frac{1}{2} + \Pr(N > 1) \cdot \frac{1}{2}$



$= 2 \cdot \Pr(N \le -1) \cdot \frac{1}{2} = \Phi(-1)$

$= 2 \cdot \Pr(N > 1) \cdot \frac{1}{2} = 1 - \Phi(1)$

6. $\hat{x}_{MAP} = \underset{x}{\operatorname{argmax}} f_{X|Y}(x|y)$

$X \sim N(1, 1)$, $N \sim N(0, 1)$

$\begin{cases} Y = X + N \\ X = X \end{cases} \Rightarrow X, Y$ joint Gaussian

$f_{X|Y}(x|y)$ is a conditional Gaussian pdf.

$\hat{x}_{MAP} = m_{X|Y}(y) = \bar{X} + \frac{\sigma_X}{\sigma_Y} \rho_{XY} (y - \bar{Y})$

$= 1 + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} (y - 1) = \frac{1}{2} y + \frac{1}{2}$

$Var[Y] = Var[X + N] \overset{\text{indep}}{=} Var[X] + Var[N] = 2$

$E[Y] = \bar{Y} = E[X + N] = E[X] + E[N] = 1$

$\rho_{XY} = \frac{Cov[X, Y]}{\sigma_X \sigma_Y} = \frac{\underbrace{Cov[X, X+N]}_{= Cov[X,X] + Cov[X,N]}}{\sigma_X \sigma_Y} = \frac{Var[X] + Cov[X, N]}{\sigma_X \sigma_Y}$

$$= \frac{Var[X]}{\sigma_X \sigma_Y} = \frac{1}{\sigma_Y} = \frac{1}{\sqrt{2}}$$

7. Memoryless

8. $Cov[U,V] = Cov[X+Y, X-Y]$

$$= Cov[X,X] - Cov[X,Y] + Cov[Y,X] - Cov[Y,Y]$$

$$= Var[X] - Var[Y]$$

9. $\begin{cases} U = X+Y \\ V = X-Y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(U+V) \\ y = \frac{1}{2}(U-V) \end{cases}$

$$J(x,y) = \left\| \begin{matrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{matrix} \right\| = \left\| \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right\| = 2$$

$$f_{U,V}(u,v) = \frac{1}{J(x,y)} \cdot f_{X,Y}\left(\tfrac{1}{2}(u+v), \tfrac{1}{2}(u-v)\right)$$

10. $\begin{cases} ① \ E[X(t)] = const. \\ ② \ E[X(t)X(t+\tau)] = f(\tau) \end{cases}$

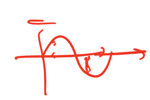
① $E[X(t)] = E[A\cos(\omega t + \Theta)]$

$\Theta \sim U[0,\pi)$, indep of A.

$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi}, & \theta \in [0,\pi) \\ 0, & \text{Else} \end{cases}$

$$= E[A] \cdot E[\cos(\omega t + \Theta)]$$

$$= \bar{A} \cdot \underbrace{\int_0^\pi \cos(\omega t + \theta) \cdot \frac{1}{\pi} \, d\theta}_{f(t)}$$



$\sin(\theta + \omega t) \Big|_0^\pi$

$\sin(\pi + \omega t) - \sin \omega t \neq const.$

11. $E[X(t)] = E[\sum_{k=-\infty}^{\infty} Y_k \cdot p(t-k\cdot\Delta)]$

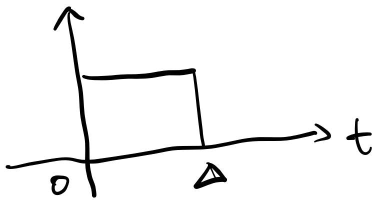
$$= \sum_{k=-\infty}^{\infty} E[Y_k \cdot p(t-k\cdot\Delta)]$$

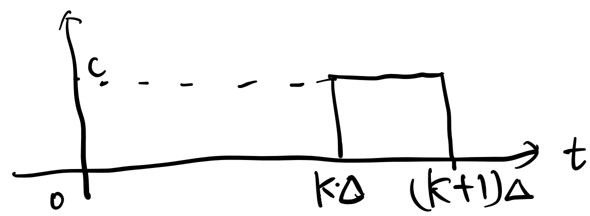
$$\overset{\text{indep}}{=} \sum_{k=-\infty}^{\infty} E[Y_k] \cdot E[p(t-k\cdot\Delta)]$$

Y_k binary sequence, can be 0 or 1

$E[Y_k]$ can be 0 or 1.

$p(t)$ $\qquad\qquad$ $p(t-k\cdot\Delta)$





$$E[p(t-k\cdot\Delta)] = \begin{cases} c, & t \in [k\cdot\Delta, (k+1)\Delta) \\ 0, & \text{else} \end{cases}$$

$$\sum_{k: Y_k=1} E[p(t-k\cdot\Delta)] \neq \text{const.}$$

$$= \begin{cases} c, & t \in [k\cdot\Delta, (k+1)\Delta), \quad k: Y_k=1 \\ 0, & t \notin [k\cdot\Delta, (k+1)\Delta), \quad k: Y_k=1 \end{cases}$$

12. $E[x(t)] = \text{const}_1 = c_1$

$Var[x(t)] = E[X^2(t)] - c_1^2 = R_X(0) - c_1^2 = \text{const}_2.$

$E[X^2(t)] = E[x(t)\cdot x(t+0)] = \underline{R_X(0)}$

$R_X(\tau) = E[X(t)\cdot x(t+\tau)]$

13. Lecture 35. Page 202

14. Lecture 35. page 203

15. $X(t) \longrightarrow \boxed{LTI} \longrightarrow Y(t)$

LTI property: Lec 37. page 223

if $X(t)$ is WSS $\Rightarrow \begin{cases} Y(t) \text{ is WSS. r.p.} \\ X(t), Y(t) \text{ are J WSS. r.p.} \end{cases}$

$E[Y^2(t)] = R_Y(0) = \int_{-\infty}^{\infty} e^{i2\pi f\tau} S_Y(f)df$ where $\tau=0$

$R_Y(\tau) = \mathcal{F}^{-1}(S_Y(f))$

$$\| \quad \int_{-\infty}^{\infty} S_Y(f)df$$

$\underbrace{S_Y(f) = |H(f)|^2 \cdot S_X(f)}_{\text{proved in Lec 37}}$

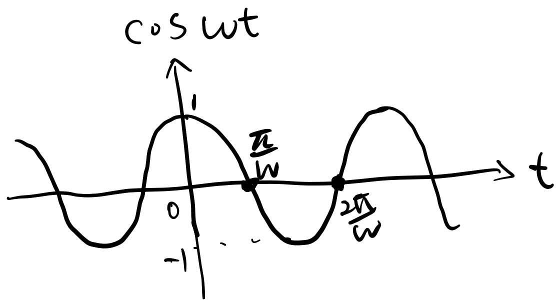
$$\| \quad \int_{-\infty}^{\infty} |H(f)|^2 \cdot S_X(f)df$$

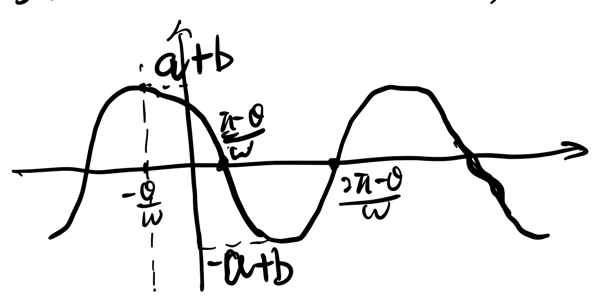
17. $\Theta \sim U[0, 2\pi)$, $B \sim U[-a, a]$

B is indep. of Θ

(a) $X(t) = a\cos(\omega t + \theta) + b$ $\quad t \in (-\infty, \infty)$

$a\cos(\omega t + \theta)^{+b} = a\cos(\omega(t + \frac{\theta}{\omega}))^{+b}$





(b) $E[X(t)] = E[a\cos(wt+\Theta)+B]$

$= aE[\cos(wt+\Theta)] + E[B]$ $\overset{\shortparallel}{0}$

$= a\int_0^{2\pi}\cos(wt+\theta)\cdot\frac{1}{2\pi}d\theta + 0$

$f_\Theta(\theta) = \begin{cases}\frac{1}{2\pi}, & \theta\in[0,2\pi) \\ 0, & \text{else}\end{cases}$

$\Downarrow$ 0

$E[X(t)] = 0+0=0$

$E[X(t)X(t+\tau)] = E[(\underbrace{a\cos(wt+\Theta)}_{①}+\underbrace{B}_{②})(\underbrace{a\cos(w(t+\tau)+\Theta)}_{③}+\underbrace{B}_{④})]$

$= E[a^2\cos(wt+\Theta)\cos(wt+w\tau+\Theta)]$ ⑤

$+ E[B\cdot \cancel{a\cos(wt+\Theta)}] = E[B]\cdot E[a\cos\ldots] = 0$

$+ E[B\cdot \cancel{a\cos(wt+w\tau+\Theta)}] = 0$

$+ E[B^2]$ $\frac{a^2}{3}$

$B\sim U(-a,a)$

$E[B^2] = Var[B]+E^2[B]$

$= Var[B]$

$= \frac{1}{12}(a-(-a))^2$

$= \frac{a^2}{3}$

$(\cos x)\cdot(\cos y) = \frac{1}{2}(\cos(x+y)+\cos(x-y))$

⑤ $= E\left[\frac{1}{2}a^2[\cos(w\tau)+\cos(2wt+2\Theta+w\tau)]\right]$

$\Downarrow$

$\Theta\sim U[0,2\pi)$ $E[\cos(2wt+2\Theta+w\tau)]=0$

$E[\cos(w\tau)] = \int_0^{2\pi}\cos(w\tau)\frac{1}{2\pi}d\theta$ → pdf of Θ

$$\frac{1}{2\pi}\cos(\omega\tau)\theta\Big|_0^{2\pi} = \cos(\omega\tau).$$

$$E[X(t)X(t+\tau)] = \frac{1}{2}a^2\cos\omega\tau + \frac{a^2}{3} = f(\tau)$$

$$\begin{cases} m_X = E[X(t)] = 0 \\ R_X(\tau) = E[X(t)X(t+\tau)] = \frac{1}{2}a^2\cos\omega\tau + \frac{a^2}{3} \end{cases}$$

(c) $E[X(t_1)X(t_2)] = E[X(t_1)]\,E[X(t_2)]$

$R_X(t_1-t_2) = m_X \cdot m_X$

$$\frac{1}{2}a^2\cos\omega(t_1-t_2) + \frac{a^2}{3} = 0$$

$\Downarrow$

$\cos\omega(t_1-t_2) = -\frac{2}{3}$ $\Rightarrow$ solvable (have solution)

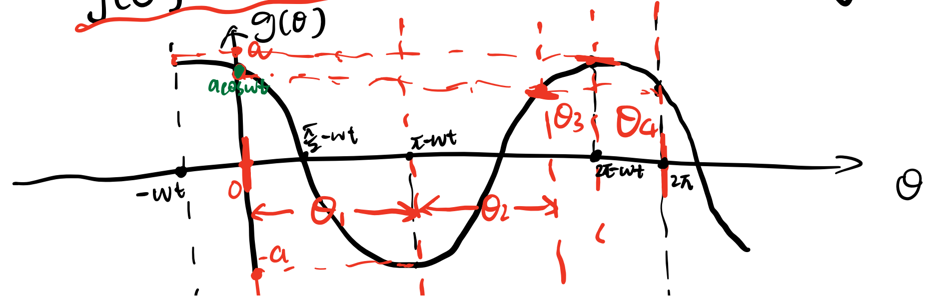
$\downarrow$

$[-1,1]$

(d) find $f_{Y(t)}(y)$ $\quad f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & \theta\in(0,2\pi) \\ 0, & \text{else.} \end{cases}$

$Y(t) = a\cos(\omega t + \Theta) = g(\Theta)$

$g(\theta) = a\cos(\omega t + \theta)$ $\qquad$ Density method



$$f_{Y(t)}(y) = \begin{cases} \sum_{i=1,2} f_{\Theta}(\theta_i) \cdot \frac{1}{|g'(\theta_i)|} & y \in (-a, a\cos\omega t) \\ \sum_{i=3,4} f_{\Theta}(\theta_i) \cdot \frac{1}{|g'(\theta_i)|}, & y \in (a\cos\omega t, a) \end{cases}$$

$$a\sin(\omega t+\theta) = \sqrt{a^2-y^2}$$

$$g'(\theta) = -a\sin(\omega t+\theta) = -\sqrt{a^2-y^2}$$

$$f_{Y(t)}(y) = \begin{cases} 2 \cdot \frac{1}{2\pi} \cdot \frac{1}{\sqrt{a^2-y^2}}, & y \in (-a, a\cos\omega t) \\ 2 \cdot \frac{1}{2\pi} \cdot \frac{1}{\sqrt{a^2-y^2}}, & y \in (a\cos\omega t, a) \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{a^2-y^2}}, & y \in (-a, a) \\ 0, & \text{else} \end{cases}$$

$$X(t) = Y(t) + B$$

$$\begin{cases} B, \Theta \text{ indep} \\ Y(t) = g(\Theta) \end{cases}$$

$\Rightarrow$ B, $g(\Theta)$ indep

$\Downarrow$

$Y(t)$, B indep

$$f_{X(t)}(x) = f_B * f_{Y(t)}(x) = \int_{-\infty}^{\infty} f_B(b) \cdot f_{Y(t)}(x-b)\,db$$

$$\underline{\qquad\qquad\qquad\qquad}$$
$$\|$$

$$f_B(b)=\begin{cases}\frac{1}{2a}, & b\in(-a,a)\\ 0, & \text{else}\end{cases}$$

$$\Downarrow$$

$$\int_{-a}^{a}\frac{1}{2a}\cdot f_{Y(t)}(x-b)\,db$$

$$\int_{-a}^{a}\frac{1}{\pi\sqrt{a^2-(x-b)^2}}\cdot\frac{1}{2a}\,db$$

def: $c=b-x$

$$=\int_{-a-x}^{a-x}\frac{1}{2a\pi\sqrt{c^2-a^2}}\,dc$$

$$=\frac{1}{2a\pi}\int_{-a-x}^{a-x}\frac{1}{\sqrt{c^2-a^2}}\,dc$$

$$=\frac{1}{2a\pi}\left(\sin^{-1}\left(\frac{c}{a}\right)\right)\Big|_{-a-x}^{a-x}$$

$$=\frac{1}{2a\pi}\left(\sin^{-1}\left(\frac{a-x}{a}\right)-\sin^{-1}\left(\frac{-a-x}{a}\right)\right)$$

$$=\frac{1}{2a\pi}\left(\sin^{-1}\left(\right.\right.$$