

# 模型预测控制

**M**odel **P**redictive **C**ontrol

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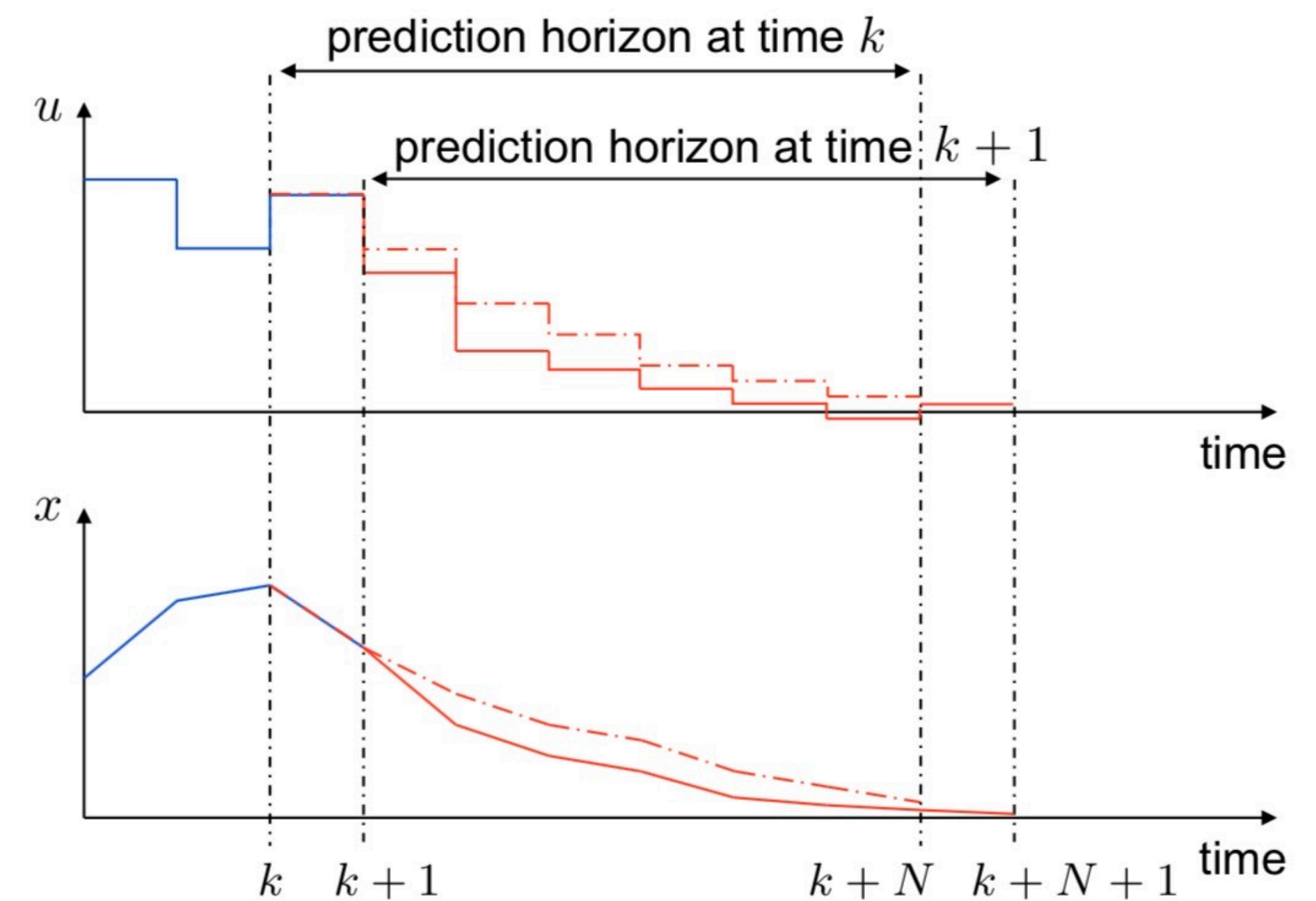
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# That is something similar to model predictive control...

- **Predict** from current information
- Calculate **optimization** based on prediction
- Implement the **first step optimal action**
- **Repeat** the above procedures at the next time step



## In detail...

- Plant:  $x(k + 1) = f(x(k), u(k))$

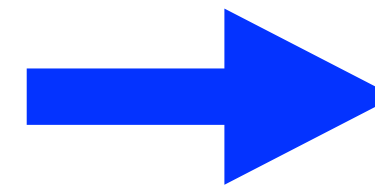
The model can be linear, nonlinear, continuous, discrete, hybrid, deterministic, stochastic, or even data-based ...

- Prediction:

Initial state:  $x(k)$

Input series:  $\{u(k), u(k + 1), u(k + 2), \dots\}$

Determines



State series:  $x(k + 1), x(k + 2), \dots$

- Optimization:

$l(x(i), u(i))$ : Stage cost

$$U^*(k) = \arg \min_{u(k), u(k+1), \dots} \sum_{i=k}^{\infty} l(x(i), u(i)) = \{u^*(k), u^*(k + 1), \dots\} \quad \text{s.t. } x \in \mathcal{X}, u \in \mathcal{U}$$

- Implement the first step optimal action:  $u(k) = u^*(k)$

MPC provides feedback,  
since  $u^*(k)$  is a function of  $x(k)$ .

- Repeat the above procedures at time  $k + 1$

- **Advantages**

- **Handles constraints on control inputs and states**
  - e.g., actuator limits; safety, environmental, or economic constraints
- **Approximately optimal control**

- **Disadvantages**

- **Requires online optimization**
  - e.g., large computation for nonlinear and uncertain systems

**TABLE 1 A list of the survey results in order of industry impact as perceived by the committee members.**

Rank and Technology	High-Impact Ratings	Low- or No-Impact Ratings
PID control	100%	0%
Model predictive control	78%	9%
System identification	61%	9%
Process data analytics	61%	17%
Soft sensing	52%	22%
Fault detection and identification	50%	18%
Decentralized and/or coordinated control	48%	30%
Intelligent control	35%	30%
Discrete-event systems	23%	32%
Nonlinear control	22%	35%
Adaptive control	17%	43%
Robust control	13%	43%
Hybrid dynamical systems	13%	43%

**T. Samad, "A Survey on Industry Impact and Challenges Thereof [Technical Activities]," in IEEE Control Systems Magazine, vol. 37, no. 1, pp. 17-18, Feb. 2017.**



## New Survey on the Impact of Advanced Control of IFAC's Industry Committee (2018)



Current Impact	Future Impact
1. PID	1. <b>Model-predictive</b>
2. System Identification	2. PID
3. Estimation and Filtering	3. Fault Detection and Identification
4. <b>Model-predictive control</b>	4. System Identification
5. Fault Detection and Identification	5. Process data analytics
6. Process data analytics	6. Estimation and Filtering
7. Decentralized and/or coordinated control	7. Decentralized and/or coordinated control
8. Robust control	8. Intelligent control
9. Intelligent control	9. Adaptive control
10. Adaptive control	10. Robust control
11. Nonlinear control	11. Nonlinear control
12. Discrete-event systems	12. Discrete-event systems
13. Other advanced control technologies	13. Hybrid dynamical systems
14. Hybrid dynamical systems	14. Other advanced control technologies
15. Repetitive control	15. Repetitive control
16. Game theory	16. Game theory

Respondents were asked to rate each technology on a 5-point rating scale ("high impact—multiple sectors" to "no impact").

**Frank Allgower, "From Stabilizing to Economic Model Predictive Control: A Paradigm Shift towards Increased Control Performance," Planary speech at 2019 Asian Control Conference.**

# Outline

- Unconstrained linear MPC
- Constrained linear MPC
- Output feedback MPC
- Robust MPC
- MPC toolbox in MATLAB

# Unconstrained linear MPC

## Lecture 1



- LTI system:

$$x(k+1) = Ax(k) + Bu(k) \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$

Suppose  $(A, B)$  is stabilizable

Suppose there are no constraints on states or control inputs

- Define:

$$x(i|k), u(i|k) : \text{ Prediction of } i \text{ steps ahead from time } k \quad x(0|k) = x(k), \quad u(0|k) = u(k)$$

- Prediction:

$$x(1|k) = Ax(0|k) + Bu(0|k)$$

$$\begin{aligned} x(2|k) &= Ax(1|k) + Bu(1|k) = A[Ax(0|k) + Bu(0|k)] + Bu(1|k) \\ &= A^2x(0|k) + ABu(0|k) + Bu(1|k) \end{aligned}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\begin{aligned} x(i|k) &= Ax(i-1|k) + Bu(i-1|k) = \dots \\ &= A^i x(0|k) + A^{i-1} Bu(0|k) + A^{i-2} Bu(1|k) + \dots + Bu(i-1|k) \end{aligned}$$

- In compact form:

$$X(k) = Fx(k) + \Phi U(k)$$

$N$  : Control/predictive horizon

控制/预测时域

$$X(k) \triangleq \begin{bmatrix} x(1|k) \\ x(2|k) \\ \vdots \\ x(N|k) \end{bmatrix} \quad U(k) \triangleq \begin{bmatrix} u(0|k) \\ u(1|k) \\ \vdots \\ u(N-1|k) \end{bmatrix} \quad F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \quad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}$$

\* Recall in the previous page:

$$x(1|k) = Ax(0|k) + Bu(0|k)$$

$$x(2|k) = A^2x(0|k) + ABu(0|k) + Bu(1|k)$$

$$\vdots \quad \vdots \quad \vdots$$

$$x(i|k) = A^i x(0|k) + A^{i-1}Bu(0|k) + A^{i-2}Bu(1|k) + \cdots + Bu(i-1|k)$$

- Cost function (with finite control/predictive horizon)

$$J(k) = \sum_{i=1}^N \|x(i | k)\|_Q^2 + \|u(i - 1 | k)\|_R^2 = X^T(k) \mathcal{Q} X(k) + U(k)^T \mathcal{R} U(k)$$

$$\mathcal{Q} = \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix} \quad \mathcal{R} = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix} \quad X(k) = Fx(k) + \Phi U(k)$$

$$\begin{aligned} J(k) &= (Fx(k) + \Phi U(k))^T \mathcal{Q} (Fx(k) + \Phi U(k)) + U^T(k) \mathcal{R} U(k) \\ &= x^T(k) F^T \mathcal{Q} F x(k) + 2x^T(k) F^T \mathcal{Q} \Phi U(k) + U^T(k) (\Phi^T \mathcal{Q} \Phi + \mathcal{R}) U(k) \end{aligned}$$

- Minimizing the cost function by predictive control series

$$\nabla_U \Big|_{U=U^*} = \frac{\partial J}{\partial U} \Big|_{U=U^*} = 0$$

- Cost function

$$J(k) = x^T F^T Q F x + 2x^T F^T Q \Phi U + U^T (\Phi^T Q \Phi + \mathcal{R}) U$$

- Minimizing the cost function by predictive control:

$$\nabla_U \Big|_{U=U^*} = \frac{\partial J}{\partial U} \Big|_{U=U^*} = 0$$

$$2x(k)^T F^T Q \Phi + 2U(k)^T (\Phi^T Q \Phi + \mathcal{R}) \Big|_{U(k)=U^*} = 0$$

$$U^*(k) = - (\Phi^T Q \Phi + \mathcal{R})^{-1} \Phi^T Q F x(k)$$

$(R > 0, Q \geq 0; \text{ or } R \geq 0, Q > 0, \text{ and } \Phi \text{ is fully ranked})$

$$u^*(k) = - [I_{p \times p} \ 0 \ \cdots \ 0] (\Phi^T Q \Phi + \mathcal{R})^{-1} \Phi^T Q F x(k) = - K_{mpc} x(k)$$

**\* The unconstrained linear MPC is actually linear feedback control !!**

- Example

$$x(k+1) = Ax(k) + Bu(k) \quad x \in \mathbb{R}^2, \quad u \in \mathbb{R}^1 \quad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

Set control horizon  $N = 4$ . Set weight matrices  $Q = I_{2 \times 2}$ ,  $R = 0.1$ . Then,

$$F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \\ 1.21 & 4.1 \\ 0 & 0.902 \\ 1.331 & 6.315 \\ 0 & 0.857 \\ 1.464 & 8.661 \\ 0 & 0.814 \end{bmatrix} \quad \Phi = \begin{bmatrix} B & 0 & 0 & 0 \\ AB & B & 0 & 0 \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.0790 & 0 & 0 & 0 \\ 0.1580 & 0 & 0 & 0 \\ 0.0750 & 0.0790 & 0 & 0 \\ 0.3239 & 0.1580 & 0 & 0 \\ 0.0713 & 0.0750 & 0.0790 & 0 \\ 0.4989 & 0.3239 & 0.1580 & 0 \\ 0.0677 & 0.0713 & 0.0750 & 0.0790 \end{bmatrix}$$

$$u^*(k) = -[I_{p \times p} \ 0 \ \dots \ 0](\Phi^T Q \Phi + \mathcal{R})^{-1} \Phi^T Q F x(k) = -K_{mpc} x(k) = -[2.51 \ 12.29]x(k)$$



- Closed-loop system

$$x(k + 1) = (A - BK_{mpc})x(k)$$

Check whether  $|\text{eig}(A - BK_{mpc})| < 1$  to determine its stability

A necessary condition is that  $(A, B)$  is stabilizable, such that  $K_{mpc}$  exists.

- Optimality does NOT necessarily indicate stability !!

### \* Criterion for stability

#### Continuous-time systems

#### Discrete-time systems

**Lyapunov indirect method:**

$$\text{Re}(\text{eig}(A - BK)) < 0$$

$$|\text{eig}(A - BK)| < 1$$

**Lyapunov direct method:**

$$V(x(t)) > 0, \quad \dot{V}(x(t)) < 0$$

$$V(x(k)) > 0, \quad V(x(k + 1)) - V(x(k)) < 0$$

- Example:

$$x(k+1) = Ax(k) + Bu(k) \quad x \in \mathbb{R}^2, \quad u \in \mathbb{R}^1 \quad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

Set weight matrices  $Q = I_{2 \times 2}$ ,  $R = 0.1$ .

	N=4	N=3	N=2	N=1
$K$	[2.51 12.29]	[2.41 10.90]	[1.40 5.76]	[0 0.70]
$\text{eig}(A - BK)$	$0.5394 \pm 0.2872j$	$0.5944 \pm 0.3543j$	$0.7974 \pm 0.3601j$	1.10, 0.89
	Stable	Stable	Stable	Unstable

## Can we guarantee the closed-loop stability rather than check it by trial ??

**Yes we can !!**

- Try cost function with infinite control/predictive horizon:  $N = \infty$

$$J(k) = \sum_{i=1}^{\infty} \|x(i | k)\|_Q^2 + \|u(i-1 | k)\|_R^2$$

- Suppose at time  $k$ , the optimal solution is  $u^*(i | k)$  for  $i = 0, 1, \dots$

At time  $k+1$ , there exists at least one feasible solution

$$u(i | k+1) = u^*(i+1 | k) \quad \text{such that} \quad x(i | k+1) = x^*(i+1 | k) \quad \text{for all } i = 0, 1, \dots,$$

- Use the optimal cost function  $J^*(k)$  as the Lyapunov candidate

$$J^*(k+1) - J^*(k) \leq J(k+1) - J^*(k) = -\|x(1 | k)\|_Q^2 - \|u(0 | k)\|_R^2 < 0 \quad \text{Asymptotically stable}$$

## The remaining problem is... How to solve the infinite-horizon optimization ??

It seems like everything is almost the same

$$\begin{aligned}
 J(k) &= \sum_{i=1}^{\infty} \|x(i|k)\|_Q^2 + \|u(i-1|k)\|_R^2 = X^T(k) \mathcal{Q} X(k) + U(k)^T \mathcal{R} U(k) \\
 &= (Fx + \Phi U)^T \mathcal{Q} (Fx + \Phi U) + U^T \mathcal{R} U = x^T F^T \mathcal{Q} F x + 2x^T F^T \mathcal{Q} \Phi U + U^T (\Phi^T \mathcal{Q} \Phi + \mathcal{R}) U
 \end{aligned}$$

So,

$$u^*(k) = - [I_{p \times p} \ 0 \ \cdots \ 0] (\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F x(k) = - K_{mpc} x(k)$$

Still linear state feedback, **except...** they are with **infinite** dimensions !!

$$\mathcal{Q} = \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix} \quad \mathcal{R} = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix} \quad F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ \vdots \end{bmatrix} \quad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & \ddots & \\ \vdots & \cdots & \cdots & \cdots \end{bmatrix}$$

- For the infinite horizon optimization, the closed-loop system is asymptotically stable, so the infinite horizon cost function actually has a **finite** limit

$$J(k) = \sum_{i=1}^{\infty} \|x(i|k)\|_Q^2 + \|u(i-1|k)\|_R^2 = \|x(k)\|_P^2 - \|x(k)\|_Q^2$$

where

$$P - (A - BK)^T P (A - BK) = Q + K^T R K \quad \text{with} \quad |\text{eig}(A - BK)| < 1$$

Discrete-time Lyapunov function

**Proof:**

$$x^T [P - (A - BK)^T P (A - BK)] x = x^T [Q + K^T R K] x$$

$$x^T P x - x^T (A - BK)^T P (A - BK) x = x^T Q x + x^T K^T R K x$$

$$x(k)^T P x(k) - x^T(k+1)^T P x(k+1) = x^T(k) Q x(k) + u^T(k) R u(k)$$

$$\|x(k)\|_P^2 = \sum_{i=k}^{\infty} \|x(i)\|_Q^2 + \|u(i)\|_R^2$$



- **What we need to do:**

**A necessary condition is that  $(A, B)$  is stabilizable, such that the stabilizing  $K$  exists.**

- find a linear feedback  $u = -Kx$  that can asymptotically stabilize the system
- solve the discrete-time Lyapunov equation according to  $Q, R, K$

$$P - (A - BK)^T P (A - BK) = Q + K^T R K$$

s.t. the equivalent finite horizon optimization is given by

$$J(k) = \sum_{i=1}^{\infty} \|x(i|k)\|_Q^2 + \|u(i-1|k)\|_R^2$$

$$\mathcal{Q} = \text{diag}[Q, Q, \dots, Q, P]$$

$$\mathcal{R} = \text{diag}[R, R, \dots, \dots, R]$$

$$= \|x(N|k)\|_P^2 + \|u(N-1|k)\|_R^2 + \sum_{i=1}^{N-1} \|x(i|k)\|_Q^2 + \|u(i-1|k)\|_R^2$$

$$= X^T(k) \mathcal{Q} X(k) + U(k)^T \mathcal{R} U(k)$$

- calculate  $u^*(k) = -[I_{p \times p} \ 0 \ \dots \ 0](\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F x(k) = -K_{mpc} x(k)$

**Shall we calculate the optimal  $K$  ?? Not necessary !! Its existence matters**

- Theoretically, however you choose  $K$  (s.t.  $|\text{eig}(A - BK)| < 1$ ), the following still holds:

$$\begin{aligned}
 J(k) &= \sum_{i=1}^{\infty} \|x(i|k)\|_Q^2 + \|u(i-1|k)\|_R^2 \\
 &= \|x(N|k)\|_P^2 + \|u(N-1|k)\|_R^2 + \sum_{i=1}^{N-1} \|x(i|k)\|_Q^2 + \|u(i-1|k)\|_R^2
 \end{aligned}$$

$$\begin{aligned}
 u(i|k+1) &= u^*(i+1|k) & \text{for all } i = 0, 1, \dots, N-1 \\
 u(i|k+1) &= -Kx(i|k+1) & \text{for all } i = N, N+1, \dots
 \end{aligned}
 \quad \text{such that} \quad x(i|k+1) = x^*(i+1|k)$$

**“State shift”**

$$J^*(k+1) - J^*(k) \leq J(k+1) - J^*(k) = -\|x(1|k)\|_Q^2 - \|u(0|k)\|_R^2 < 0$$

**Asymptotically stable**

- Example

$$x(k+1) = Ax(k) + Bu(k) \quad x \in \mathbb{R}^2, \quad u \in \mathbb{R}^1 \quad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

Set weight matrices  $Q = I_{2 \times 2}$ ,  $R = 0.1$ . Set control horizon  $N = 4$ .

Take  $K = [1.4 \quad 5.76]$ , such that  $|\text{eig}(A - BK)| < 1$

The solution of discrete Lyapunov equation

$$P - (A - BK)^T P (A - BK) = Q + K^T R K$$

can be calculated by

$$P = \begin{bmatrix} 5.2471 & 12.8188 \\ 12.8188 & 67.1313 \end{bmatrix}$$

Use  $\mathcal{Q} = \text{diag}[Q, Q, Q, P]$   $\mathcal{R} = \text{diag}[R, R, R, R]$

to calculate  $u^*(k) = -[1 \ 0 \ \dots \ 0](\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F x(k) = -[2.6167 \quad 12.9286]x(k)$

$$K_{mpc} = [2.6167 \quad 12.9286]$$

Take  $K = [1.4 \ 5.76]$ , such that  $|\text{eig}(A - BK)| < 1$

$$P - (A - BK)^T P (A - BK) = Q + K^T R K \quad P = \begin{bmatrix} 5.2471 & 12.8188 \\ 12.8188 & 67.1313 \end{bmatrix}$$

	N=4	N=3	N=2	N=1
$K$	[2.6167 12.9286]	[2.7320 13.2108]	[2.6607 13.0972]	[2.1465 13.6108]
$\text{eig}(A - BK)$	$0.5143 \pm 0.2654j$	$0.5032 \pm 0.2747j$	$0.5077 \pm 0.2637j$	0.6776, 0.2972
	Stable	Stable	Stable	Stable

# Unconstrained MPC versus LQR ??

$$J(k) = \sum_{i=1}^{\infty} \|x(i | k)\|_Q^2 + \|u(i - 1 | k)\|_R^2$$

- **Unconstrained MPC:**

- Find the optimal control series  $\{u(k), u(k + 1), \dots\}$  to optimize the cost function
- It is proved to be a linear feedback control  $u = -K_{mpc}x$

- **Linear quadratic regulator (LQR):**

- Find the optimal control gain  $K_{lqr}$ , such that  $u = -K_{lqr}x$  optimizes the cost function



- If MPC is superior, or namely  $J_{mpc}^* < J_{lqr}^*$

then, for LQR, there exists at least one better feedback gain  $K = K_{mpc}$  !!

This is a contradictory, since  $K_{lqr}$  is the optimal feedback gain.

- If LQR is superior, or namely  $J_{lqr}^* < J_{mpc}^*$

then, for MPC, there exists at least one better control series  $\{K_{lqr}x(k), K_{lqr}x(k+1), \dots\}$  !!

This is a contradictory, since  $\{K_{mpc}x(k), K_{mpc}x(k+1), \dots\}$  is the optimal control series.

- Consequently, it has to be  $J_{mpc}^* = J_{lqr}^*$  and therefore,  $K_{mpc} = K_{lqr}$

- Example

$$x(k+1) = Ax(k) + Bu(k) \quad x \in \mathbb{R}^2, \quad u \in \mathbb{R}^1 \quad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

Set weight matrices  $Q = I_{2 \times 2}$ ,  $R = 0.1$ . Set control horizon  $N = 4$ .

Take  $K = K_{lqr} = [2.4950 \quad 12.5106]$ , and it is proved  $|\text{eig}(A - BK_{lqr})| < 1$

The solution of discrete Lyapunov equation

$$P - (A - BK)^T P (A - BK) = Q + K^T R K$$

can be calculated by

$$P = \begin{bmatrix} 4.0402 & 8.5265 \\ 8.5265 & 31.5654 \end{bmatrix}$$

Use  $\mathcal{Q} = \text{diag}[Q, Q, Q, P]$   $\mathcal{R} = \text{diag}[R, R, R, R]$  **Exactly the same with  $K_{lqr}$  !!**

to calculate  $u^*(k) = -[1 \ 0 \ \dots \ 0](\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F x(k) = -[2.4950 \quad 12.5106]x(k)$

$$K_{mpc} = [2.4950 \quad 12.5106]$$

It seems unconstrained MPC makes no sense, since it is the same with LQR ??

- **Yes, indeed.**
  - We can simply solve the Riccatti equation, and simply get  $K_{mpc} = K_{lqr}$
- **Except for**
  - Sometimes the finite-horizon performance is emphasized.
  - The closed-loop stability of constrained MPC is based on unconstrained MPC.

# Summary

- **Predict** “future” performance using plant model
  - e.g., linear or nonlinear, discrete or continuous time
- **Optimize** “future” (open loop) control series
  - computationally much easier than optimizing over feedback laws
- Implement the **first** optimal control action
  - provides feedback to reduce effects of uncertainty
- **Repeat** the above process at the next sampling time