Generalized Multipartitioning for Multi-Dimensional Arrays

Daniel Chavarría, Alain Darte (CNRS, ENS-Lyon),
Robert Fowler, John Mellor-Crummey

Department of Computer Science
Rice University

IPDPS 2002

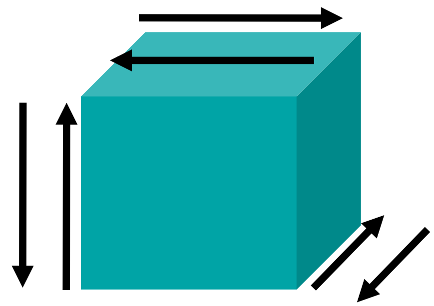
Supported by DOE/LACSI

Outline

- Line-sweep computations
  - multipartitioning, a sophisticated data distribution that enables better parallelization
- Generalized multipartitioning
  - objective function
  - find partitioning (number of cuts in each dimension)
  - map tiles to processors
- Performance results using the dHPF compiler
- Summary

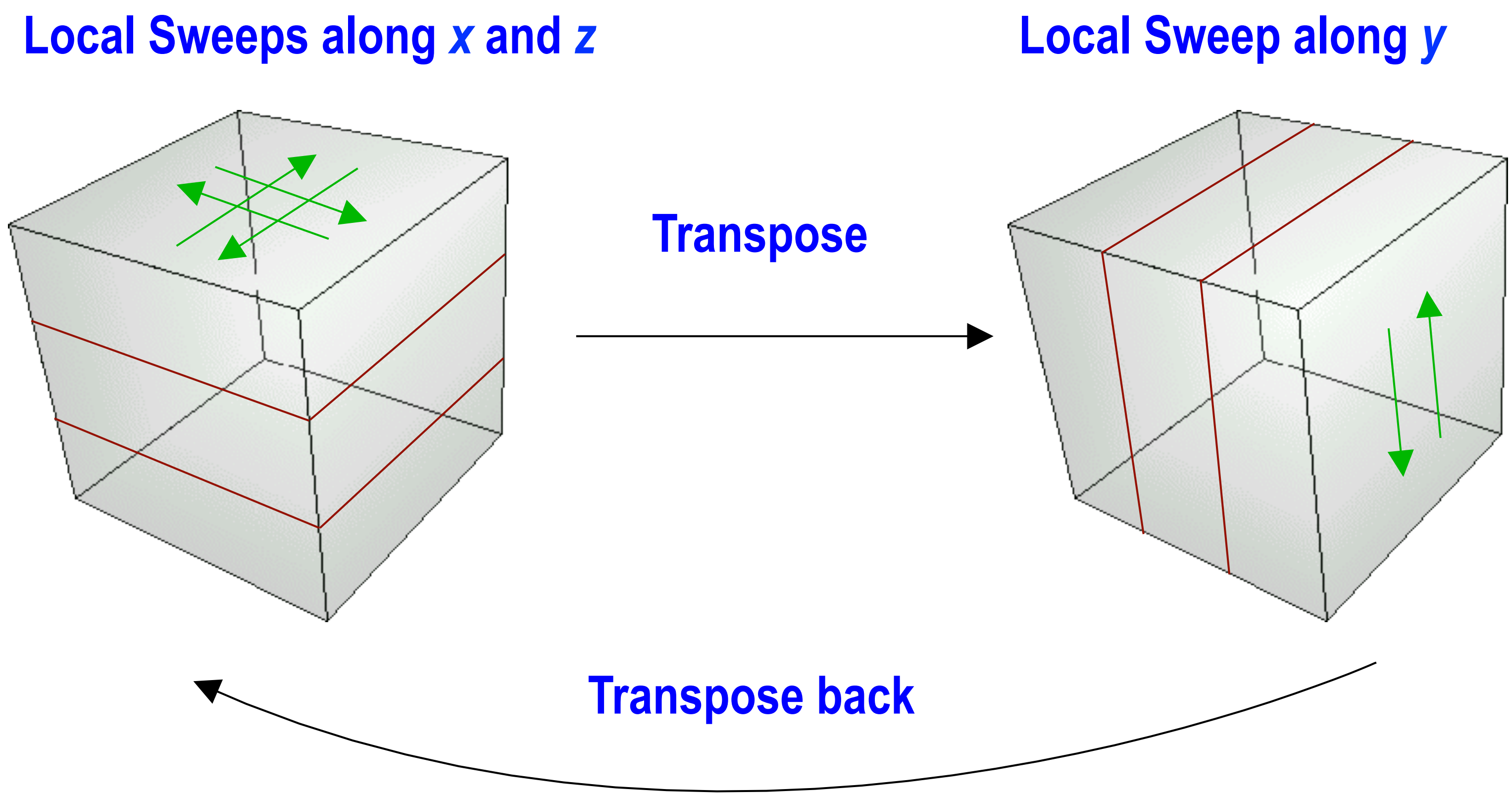
Line Sweep Computations

- 1D recurrences on a multidimensional domain



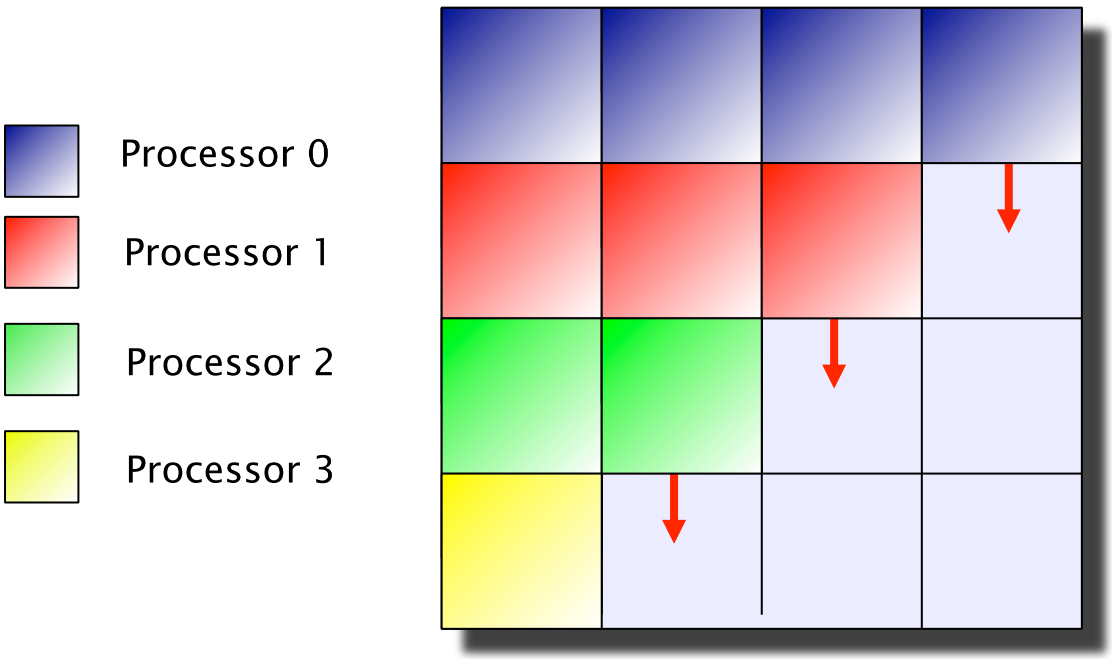
- Recurrences order computation along each dimension
- Compiler based parallelization is hard: loop carried dependences, fine-grained parallelism

Partitioning Choices: Transpose



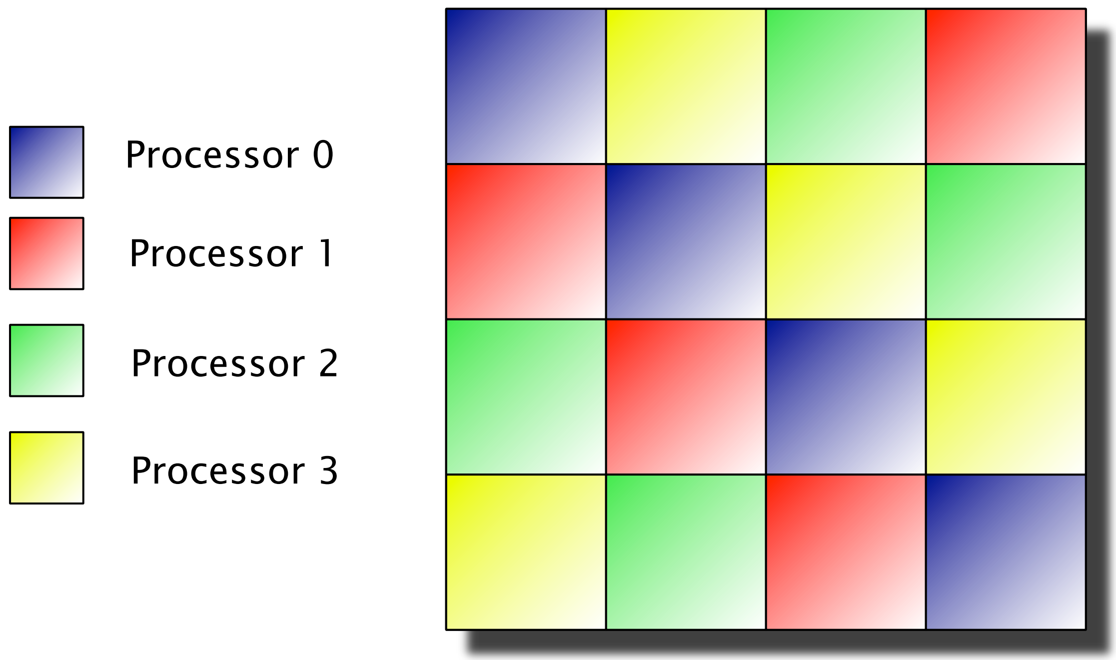
Partitioning Choices: Block + CGP

- Partial wavefront-type parallelism



Multipartitioning

- Style of skewed-cyclic distribution
- Each processor owns a tile between each pair of cuts along each distributed dimension

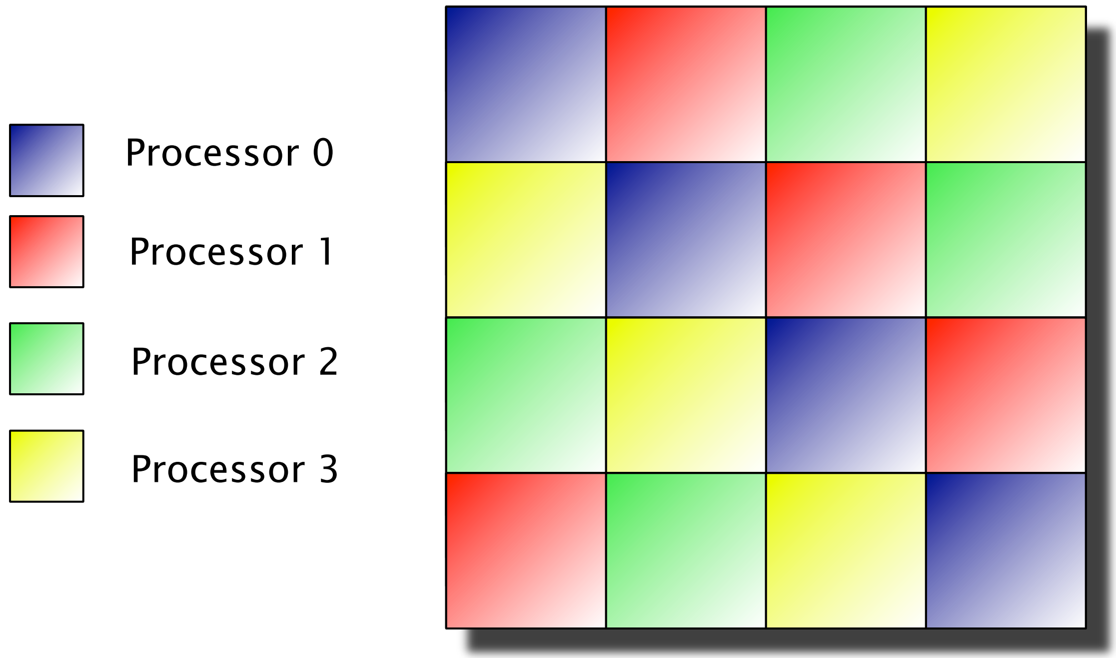


Multipartitioning

+ Full parallelism for a line-sweep computations

+ Coarse-grain communication

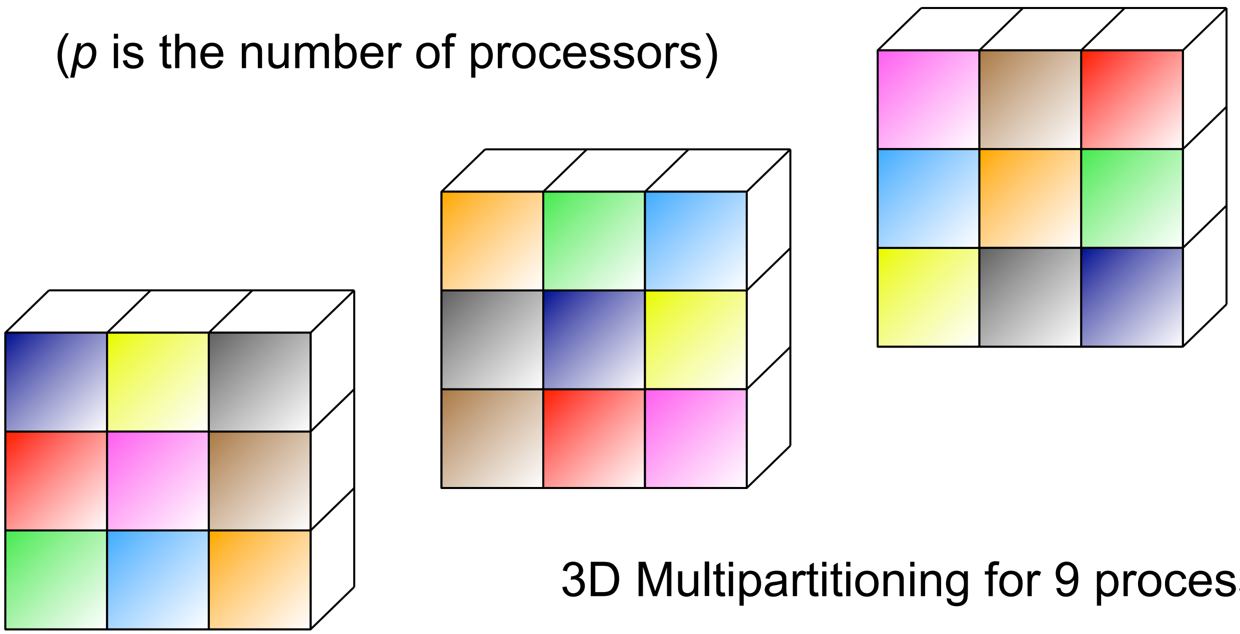
– Difficult to code by hand



Higher-dimensional Multipartitioning

- An array of k > d dimensions can be partitioned into $p^{d/(d-1)}$ tiles (diagonal multipartitioning)

(*p* is the number of processors)



3D Multipartitioning for 9 processors

Multipartitioning: Restrictions

In 3D, array of blocks of size b_1, b_2, b_3

One tile per processor per slice ➔ $p = b_1 b_2 = b_2 b_3 = b_1 b_3$

Thus: $\boxed{b_1 = b_2 = b_3}$

➔ the number of processors is a square, and the number of cuts in each dimension is $\sqrt{p}$.

In 3D, (standard) multipartitioning is possible for 1, 4, 9, 16, 25, 36, … processors.

What if we have 32 processors? Must we use only 25?

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Generalized Multipartitioning: More Tiles for each Processor

Given a data domain n_1 x … x n_d and p processors

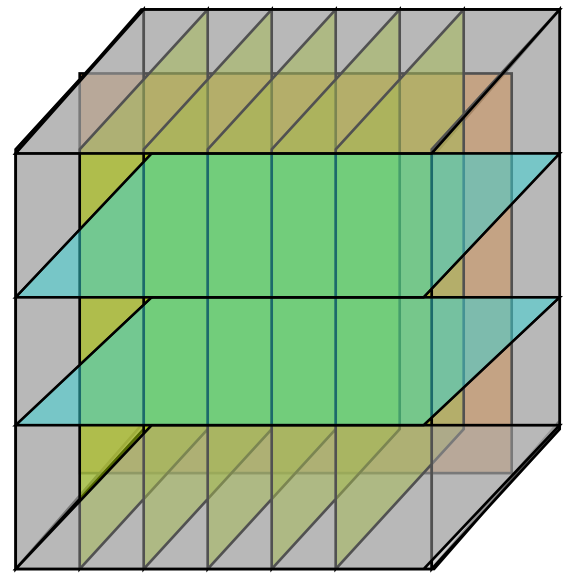
- Cut the array into b_1 x … x b_d so that, for each slice, the number of tiles is a multiple of the number of processors, i.e., for any i in [1..d], $\prod_{j \neq i} b_j$ is a multiple of p.
- Among valid partitionings, choose one that induces minimal communication overhead.
- Find a way to map tiles to processors so that:
  - in each slice, the same number of tiles is assigned to each processor (load-balancing property).
  - in any direction, the neighbor of a given processor is the same (neighbor property).

Objective Function for Multipartitioning

- Communication time depends upon the partitioning.

  Ex: p=6, array of 2x6x3 tiles.

$$\sum_{i=1}^{d} b_i k_i$$

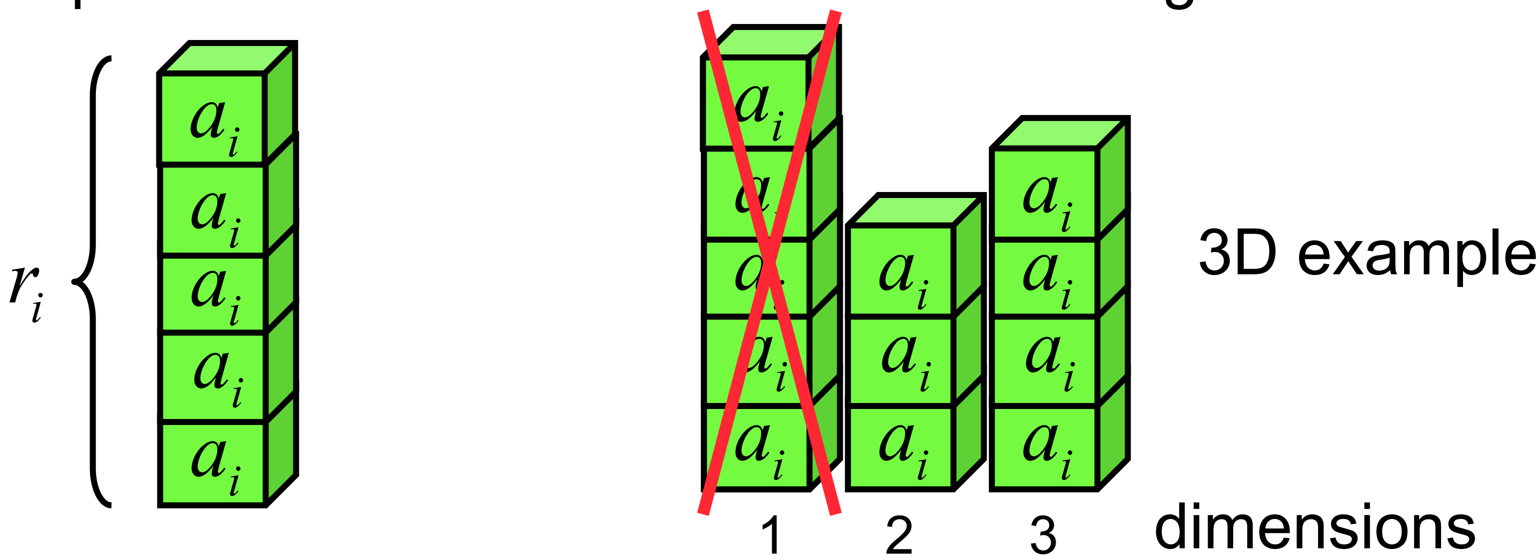


- Communication phases should be minimized.

➔The b_i's (number of cuts) should be large enough so that there are enough tiles per slice, but should be minimized to reduce the number of communication phases.

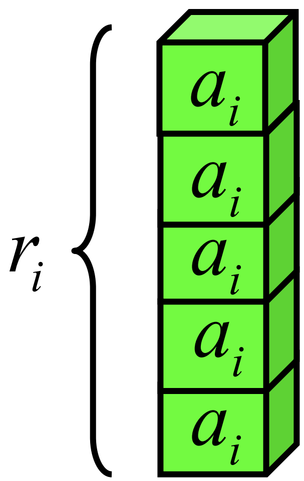
Elementary Multipartitioning

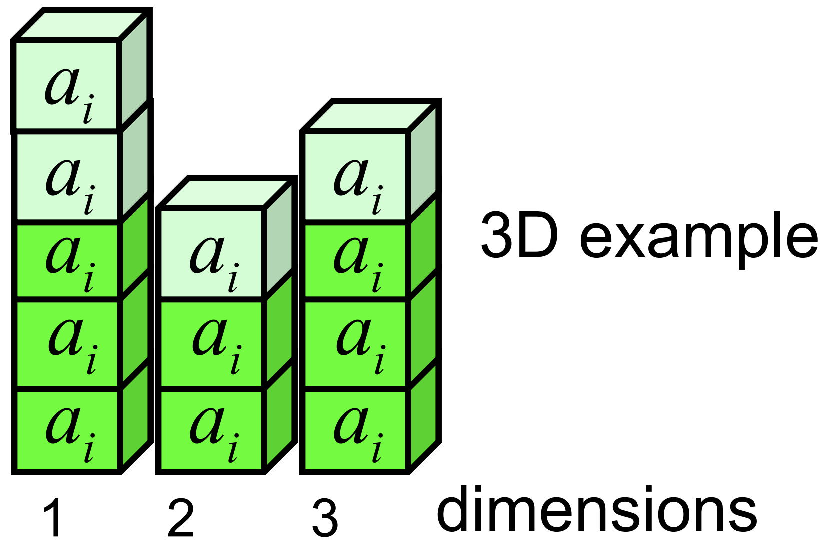
- Identify all sizes that are not "multiple" of smaller sizes. Decompose p into prime factors $p = (a_1)^{r_1} \ldots (a_s)^{r_s}$ and interpret each dimension as a bin containing such factors.



- Property 1: valid multipartitioning iff each prime factor with multiplicity r appears at least r+m times in the bins, where m is the maximal number of its occurrences in any bin.

Elementary Multipartitioning

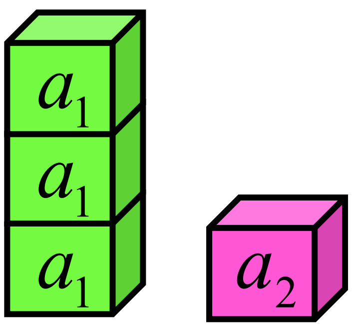


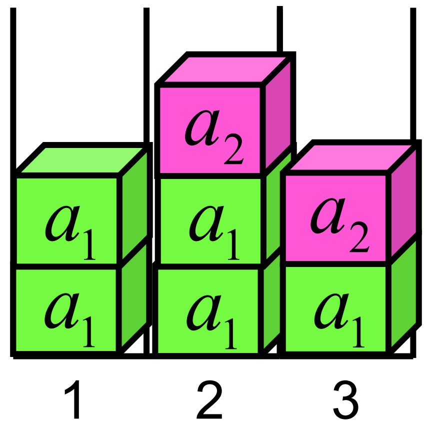


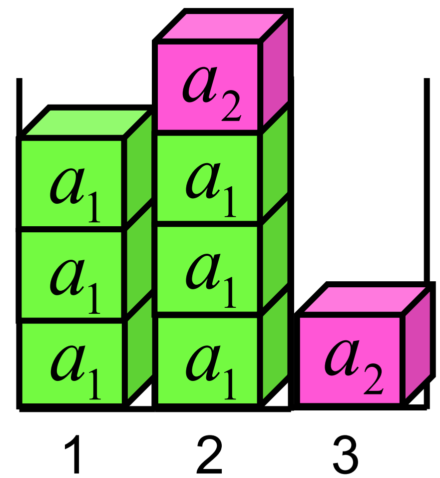
- If only one maximum, remove one in this maximal bin
  ➔ (r+m)-1 = r+(m-1) copies, thus a valid solution.
- If #elements > r+m, remove one anywhere ➔ still valid.
- Property 2: in an elementary multipartitioning, the total number of occurrences is exactly r+m, and the maximum m is reached for at least two bins.

Multipartitioning Choices

- Several elementary solutions:
  - For each factor: r+m=2m+e with 0 ≤ e ≤ (d-2)m ➔ r/(d-1) ≤m ≤ r.
  - Combine all factors.
- Example: $p = a_1^{\;3} a_2^{\;1}$







... plus all permutations

p = 8x3=24 ➔ solutions: 4x12x6, 8x24x3, 12x12x2, 24x24x1, ...

Algorithm for One Factor

(Algorithm similar to the generation of all partitions of an integer, see Euler, Ramanujam, Knuth, etc.).

```
Partitions(int r, int d) {
  for (int m= ⌈r/(d-1)⌉; m<=r; m++)     /* choose the maximal value */
    P(r+m,m,2,d);
}

P(int n, int m, int c, int d) { /* n elements in d bins, max m for at least c bins */
  if (d==1) bin[0]=n;           /* no choice for the first bin */
  else {
    for (int i=max(0,n-m(d-1)); i<=min(m-1,n-cm); i++) {
      bin[d-1]=i; P(n-i,m,c,d-1);  /* not maximum in bin number d-1 */
    }
    if (n>=m) {
      bin[d-1]=m; P(n-m,m,max(0,c-1),d-1); /* maximum in bin number d-1 */
    }
  }
}
```

Complexity of Exhaustive Search

- Naïve approach:

  For each dimension, choose a number between *1* and *p*, check that the load-balancing property holds, compute the sum, pick the best one.

  Complexity: more than p^d.

- By enumeration of elementary solutions:

  Generate only the tuples that form an elementary solution, compute the sum, pick the best one.

  Complexity:

  $$\left(\frac{d(d-1)}{2}\right)^{\frac{(1+o(1))\log p}{\log\log p}}$$

  and very fast in practice.

Possible unique factors of P

Number of choices for picking a pair of dimensions to partition with a number of cuts divisible by a particular prime factor

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Tile-to-processor Mapping

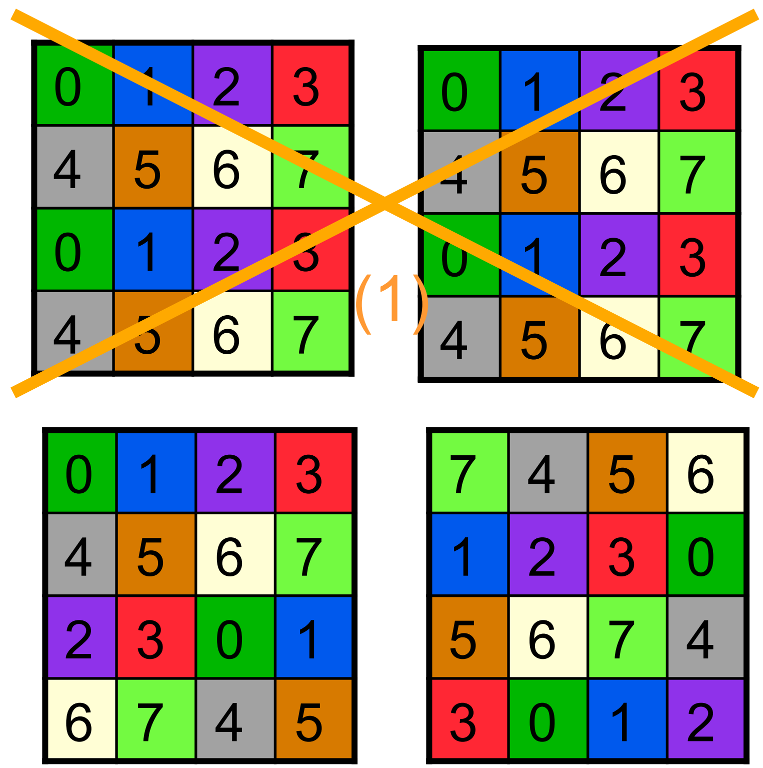
- So far, we have ensured that the number of tiles in each slice is a multiple of p. Is this sufficient? We want:

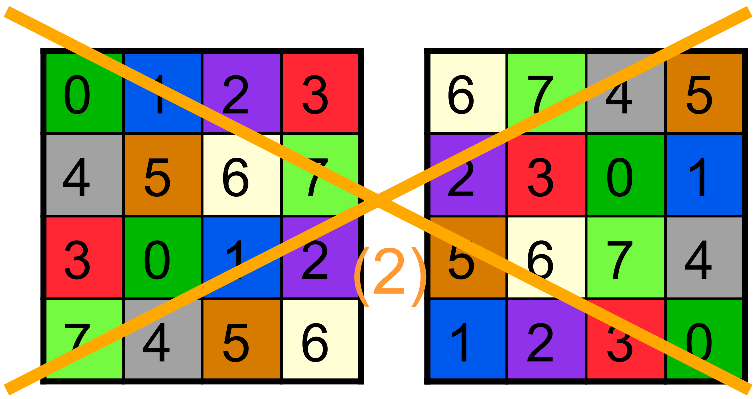
  (1) same number of tiles per processor per slice

  (2) neighbor property

- Example, 8 processors in 3D

  with 4x4x2 tiles.





Latin Squares and Orthogonality

In 2D, well-known concepts:

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

0	3	1	2
2	1	3	0
3	0	2	1
1	2	0	3

2 orthogonal diagonal latin squares.

When superimposed… magic squares!

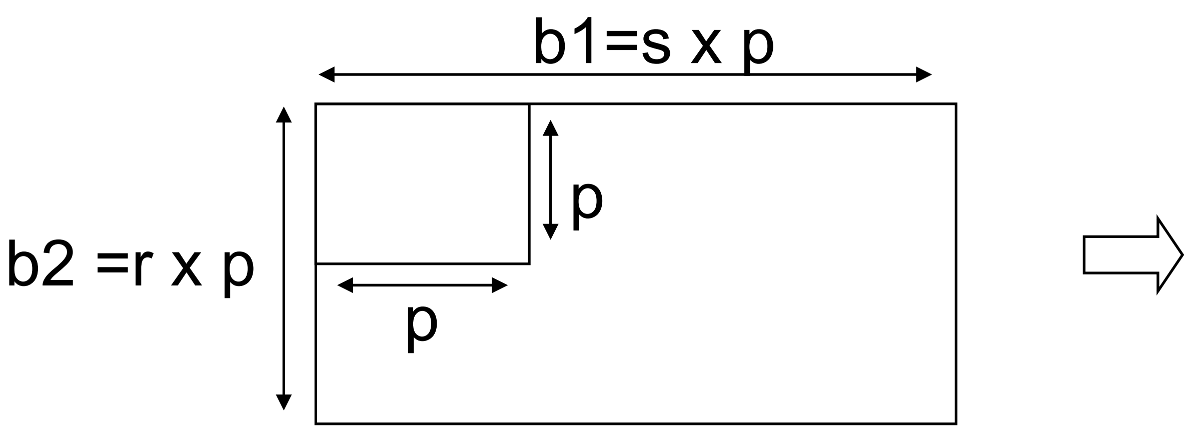
0,0	1,3	2,1	3,2
2,2	3,1	0,3	1,0
3,3	2,0	1,2	0,1
1,1	0,2	3,0	2,3

equivalent to (in base 4)

0	7	9	14
10	13	3	4
15	8	6	1
5	2	12	11

Dim ≥ 3 + Rectangles = Difficulties

- In any dimension, latin hypercubes are easy to build.
- In 2D, a latin rectangle can be built as a “multiple” of a latin square:



Pave with copies of latin squares.

- Not true in dim ≥ 3!

  Ex: for p=30, 10x15x6 is elementary, therefore it cannot be a multiple of any valid hypercube.

Mapping Tiles with Modular Mappings

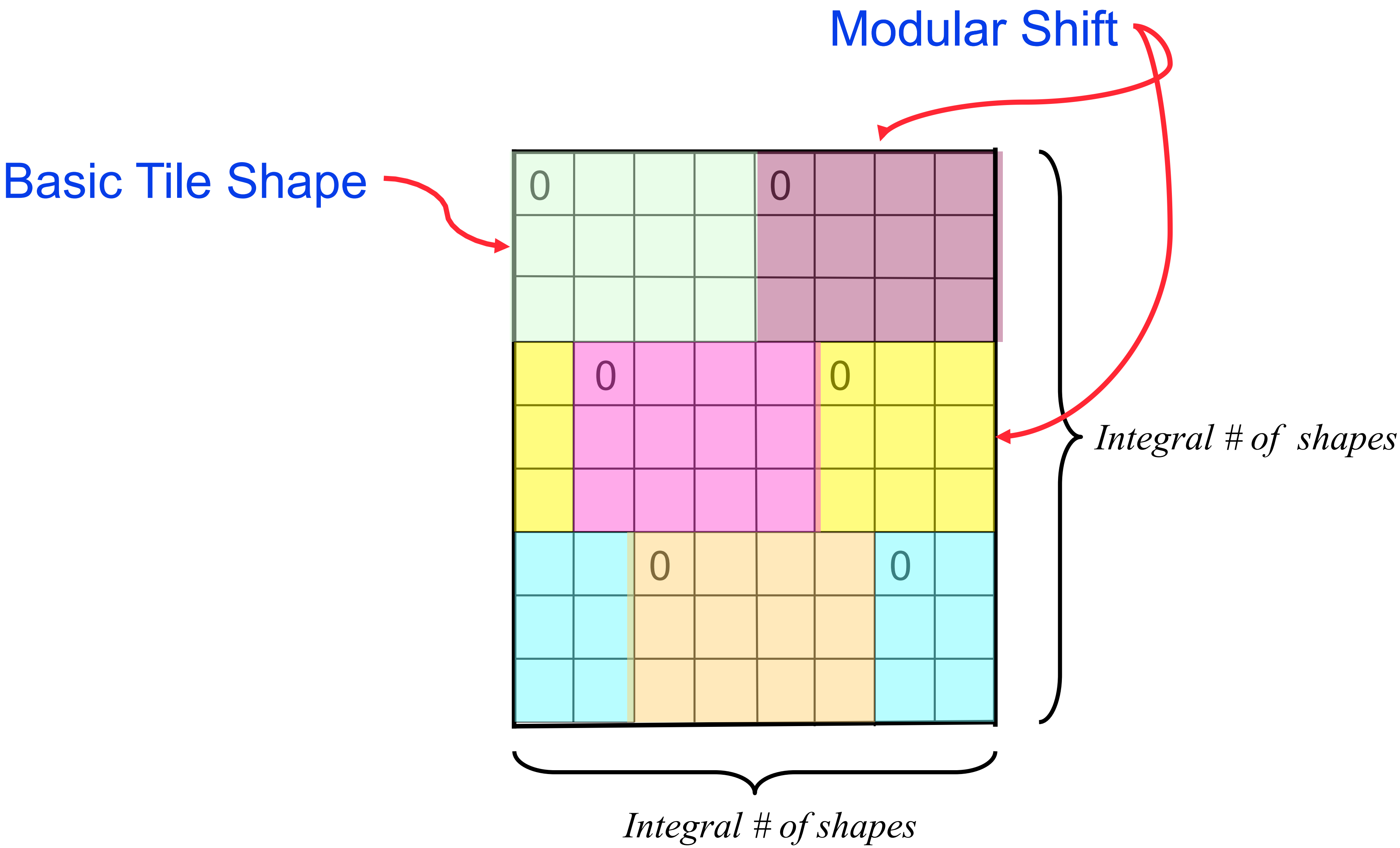
- Represent data tiles as a multi-dimensional grid
- Assign tiles with a linear mapping, modulo the grid sizes

Example: A modular mapping for a 3D multipartitioning

$$\text{Tile}\begin{pmatrix} i \\ j \\ k \end{pmatrix} \text{ on Processor } \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \end{pmatrix}\begin{pmatrix} i \\ j \\ k \end{pmatrix} \bmod \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

(details in the paper)

A Plane of Tiles from a Modular Mapping



Modular Mapping: Solution

Long proofs but …
"simple" codes! ☺

Computation of m:

```
f=1; g=p;
for (i=1; i<=d; i++) {
  f = f*b[i];
}
for (i=1; i<=d; i++) {
  m[i]=g;
  f=f/b[i];
  g=gcd(g,f);
  m[i]=m[i]/g;
}
```

Computation of M:

```
for (i=1; i<=d; i++) {
  for (j=1; j<=d; j++) {
    if ((i==1) || (i==j)) M[i][j] = 1;
    else M[i][j]=0;
  }
}
for (i=1;i<=d;i++) {
  r=m[i];
  for (j=i-1; j>=2; j--) {
    t = r/gcd(r, b[j]);
    for (k=1; k<=i-1; k++)
      M[i][k] = M[i][k] - t*M[j][k];
    r = gcd(t*m[j],r);
  }
}
```

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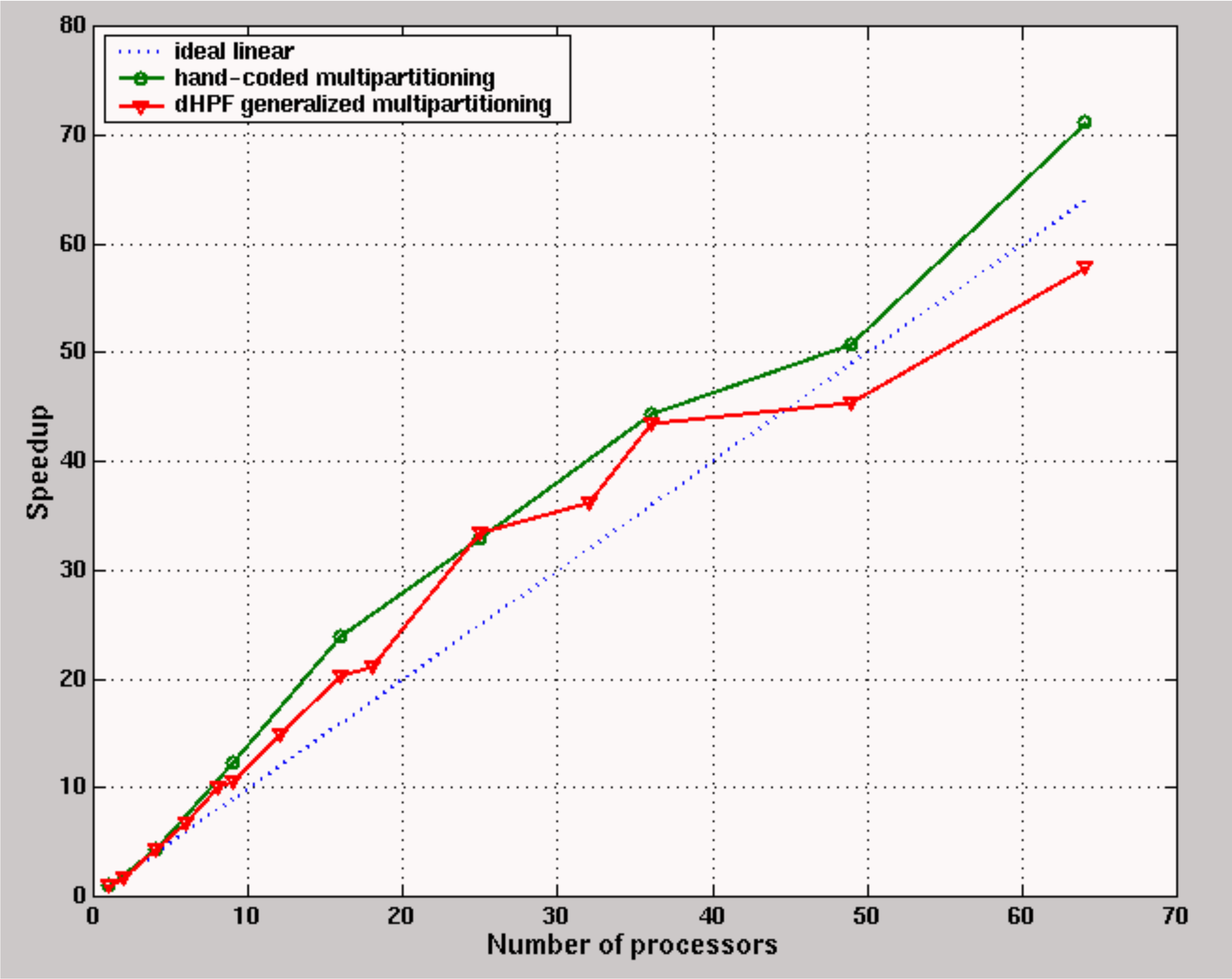
Compiler Support for Multipartitioning

- We have implemented automatic support for generalized multipartitioning in the dHPF compiler
  - Compiler takes High Performance FORTRAN (HPF) as input
    - » + MULTI data distribution directive
  - Outputs FORTRAN77 + MPI calls
- Support for generalized multipartitioning
  - Computes the optimal multipartitioning and tile mapping for p processors
  - Aggregates communication for multiple tiles & multiple arrays (by exploiting the neighbor property)

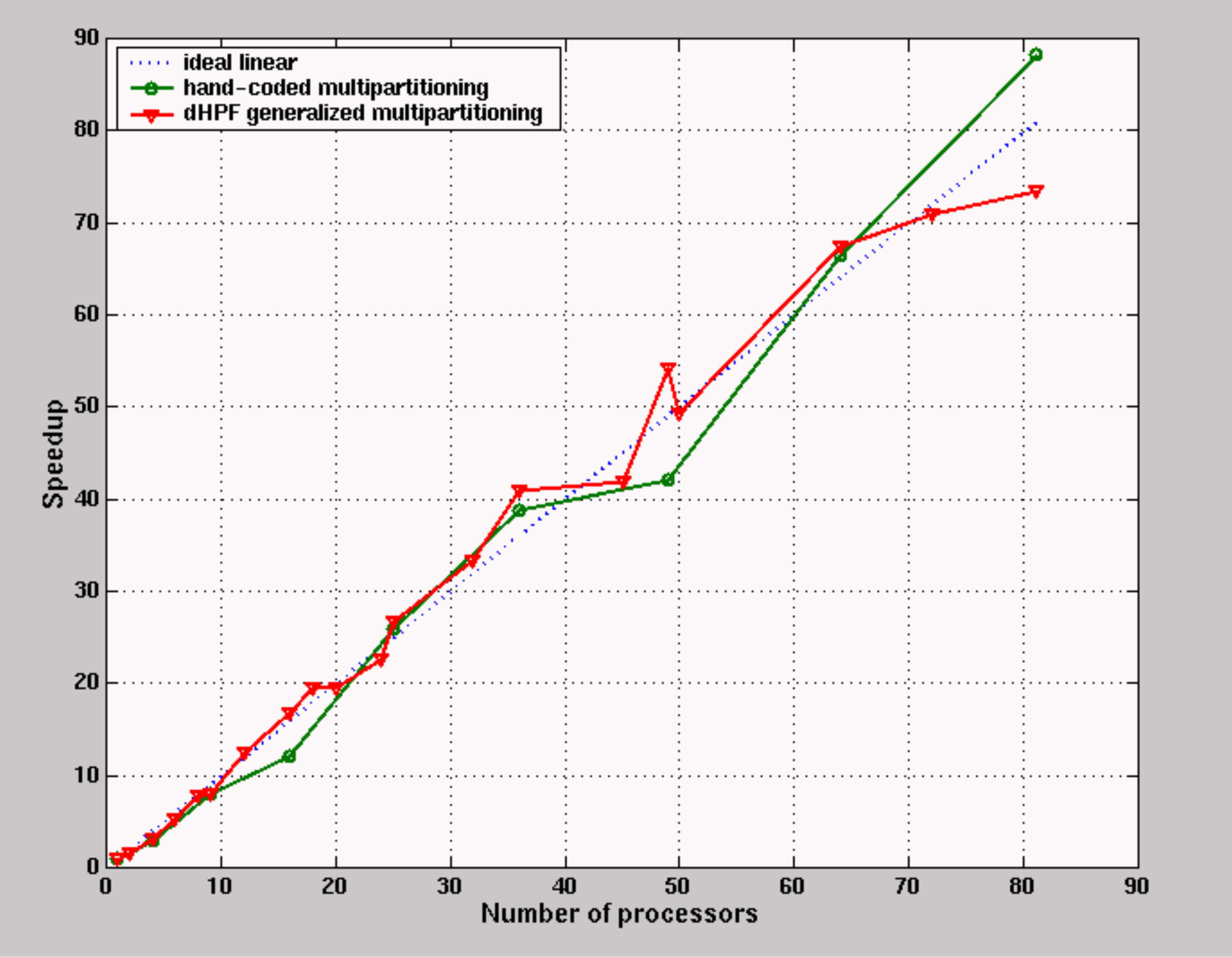
NAS SP Parallelization

- NAS SP benchmark
  - Uses ADI to solve the Navier-Stokes equation in 3D
  - Forward & backward line sweeps on each dimension, for each time step
- 2 versions from NASA, each written in FORTRAN77
  - Parallel MPI hand-coded version
  - Sequential version
- dHPF input: sequential version + HPF directives (including MULTI)
- Experimental platform: SGI Origin 2000, SGI's MPI implementation

NAS SP Speed-up (Class A)



NAS SP Speed-up (Class B)



Summary

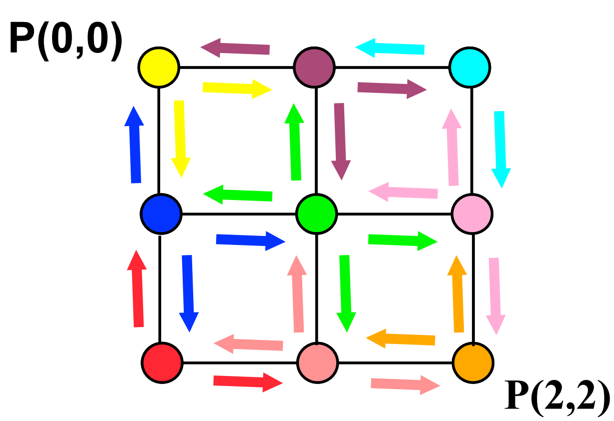
- A generalization of multipartitioning to any number of processors.
- A fast algorithm for selecting the best multipartitioning.
- A constructive proof for a suitable mapping (i.e., a "multi-dimensional latin hyper-rectangle") provided that the size of each slice is a multiple of p.
- New results on modular mappings.
- Complete implementation in the Rice dHPF compiler.
- Many open questions for mathematicians, related to the extension of Hajós theorem to "many-to-one" direct sums and "magic" squares, and to combinatorial designs.

HPF Program: Example

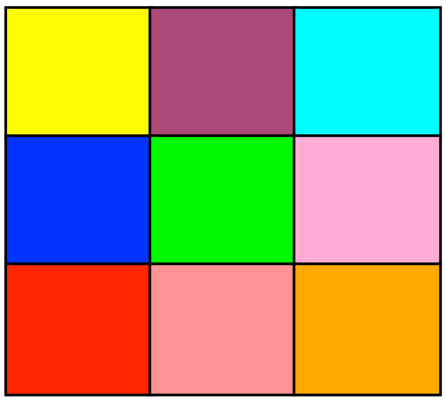
```
CHPF$ processors P(3,3)
CHPF$ distribute A(block, block) onto P
CHPF$ distribute B(block, block) onto P

    DO i = 2, n - 1
     DO j = 2, n - 1
        A(i,j) = .25 *(B(i-1,j) + B(i+1,j)+
                       B(i,j-1) + B(i,j+1))
```

High Performance Fortran

- Data-parallel programming style
- Implicit parallelism
- Communications generated by the compiler



Processors



Data for A, B

(BLOCK,BLOCK)distribution

dHPF Compiler at a Glance...

- Select computation partitionings
  - determine where to perform each statement instance
  - replicate computations to avoid communications
- Analyze and optimize communications
  - determine where communication is required
  - optimize communication placement
  - aggregate messages
- Generate SPMD code for partitioned computation
  - reduce loop bounds and insert guards
  - insert communication
  - transform references

Multipartitioning in the dHPF Compiler

- New directive for multipartitioning. Tiles are manipulated as virtual processors. Directly fits the mechanisms used for block distribution:
  - analyze messages with Omega Library.
  - vectorize both carried and independent communications.
  - aggregate communications.
    - » for multiple tiles (exploit "same neighbor" property)
    - » for multiple arrays
  - partial replication of computation to reduce communication.
- Carefully control code generation.
- Careful (painful) cache optimizations.

Objective Function

- One phase, several steps:

$$\text{computation} \longleftarrow \boxed{\tau_1 \frac{n}{p}} + (b_i - 1)\boxed{\left(\tau_2 + \tau_3 \frac{n}{p n_i}\right)} \longrightarrow \text{communication}$$

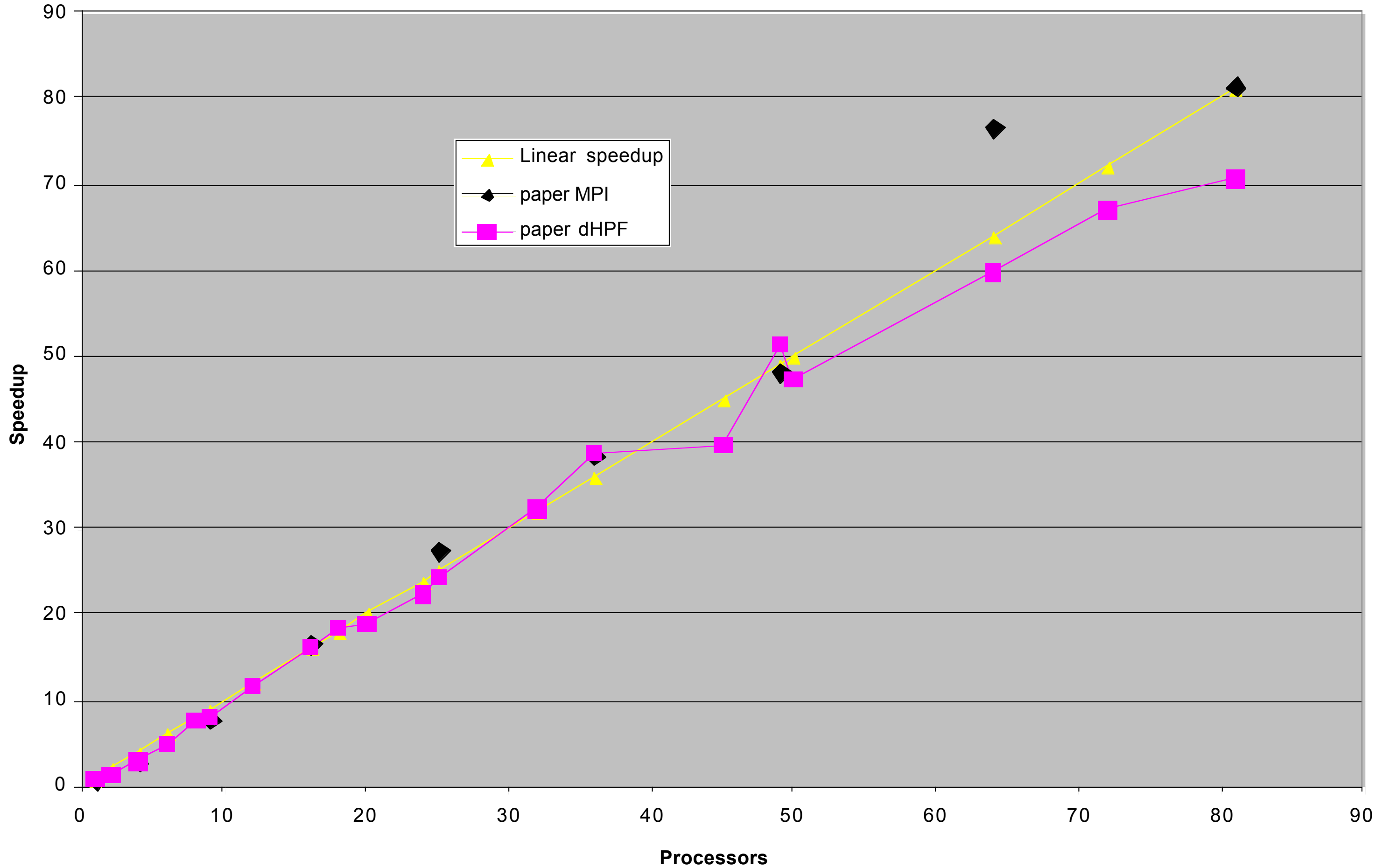
- All phases (one per dimension):

$$\underbrace{\sum_{i=1}^{d}\left(\left(\tau_1 \frac{n}{p}\right) - \left(\tau_2 + \tau_3 \frac{n}{p n_i}\right)\right)}_{\text{depends on p only}} + \underbrace{\sum_{i=1}^{d} b_i \left(\tau_2 + \tau_3 \frac{n}{p n_i}\right)}_{\text{objective function}}$$

- Try to minimize a linear function with positive parameters

$$\sum_{i=1}^{d} b_i k_i \quad \text{or} \quad \sum_{i=1}^{d} b_i$$

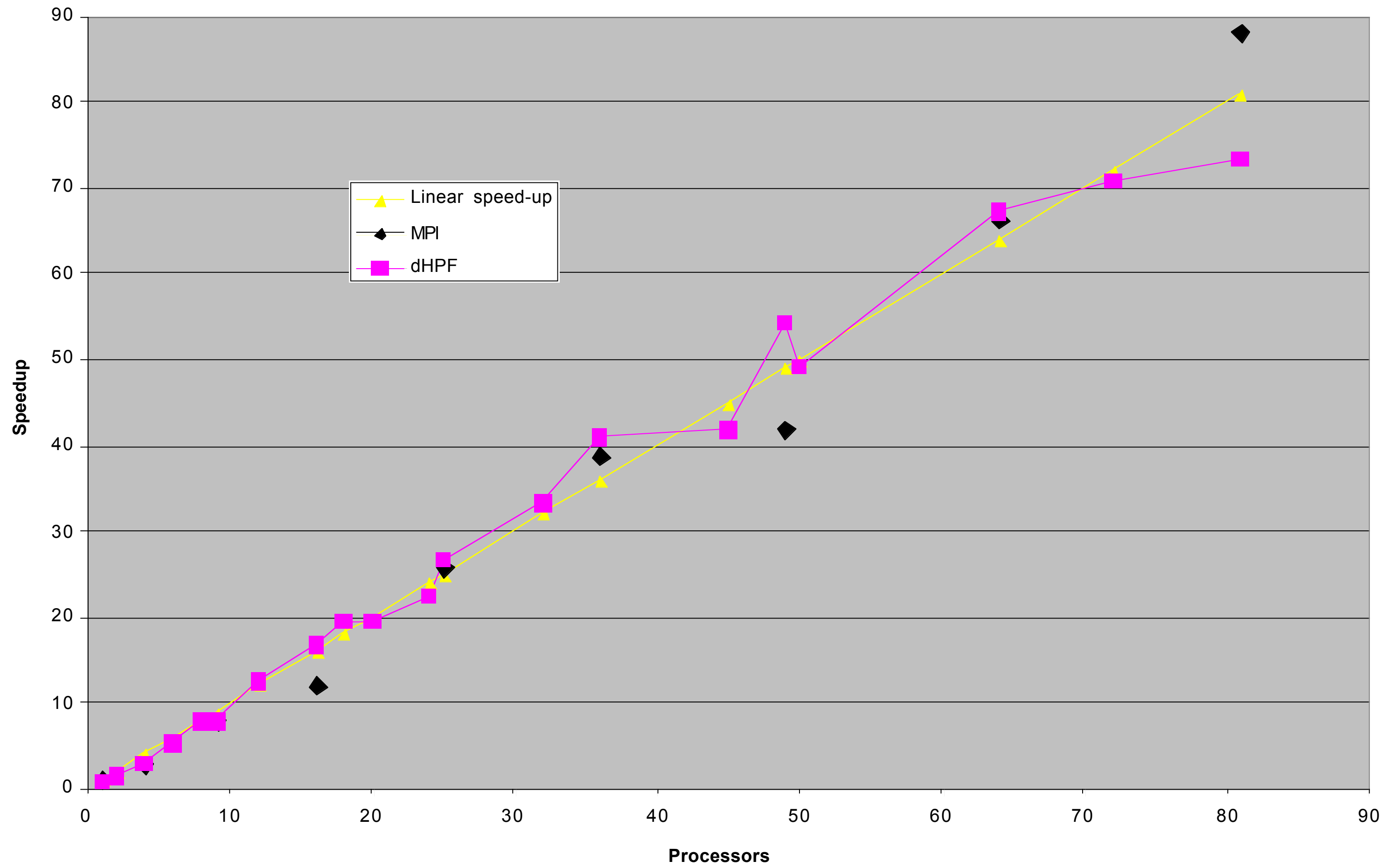
NAS SP Speed-up (Class B) using Generalized Multipartitioning



(June 2001)

SGI Origin 2000

NAS SP Speedups (Class B) using Generalized Multipartitioning



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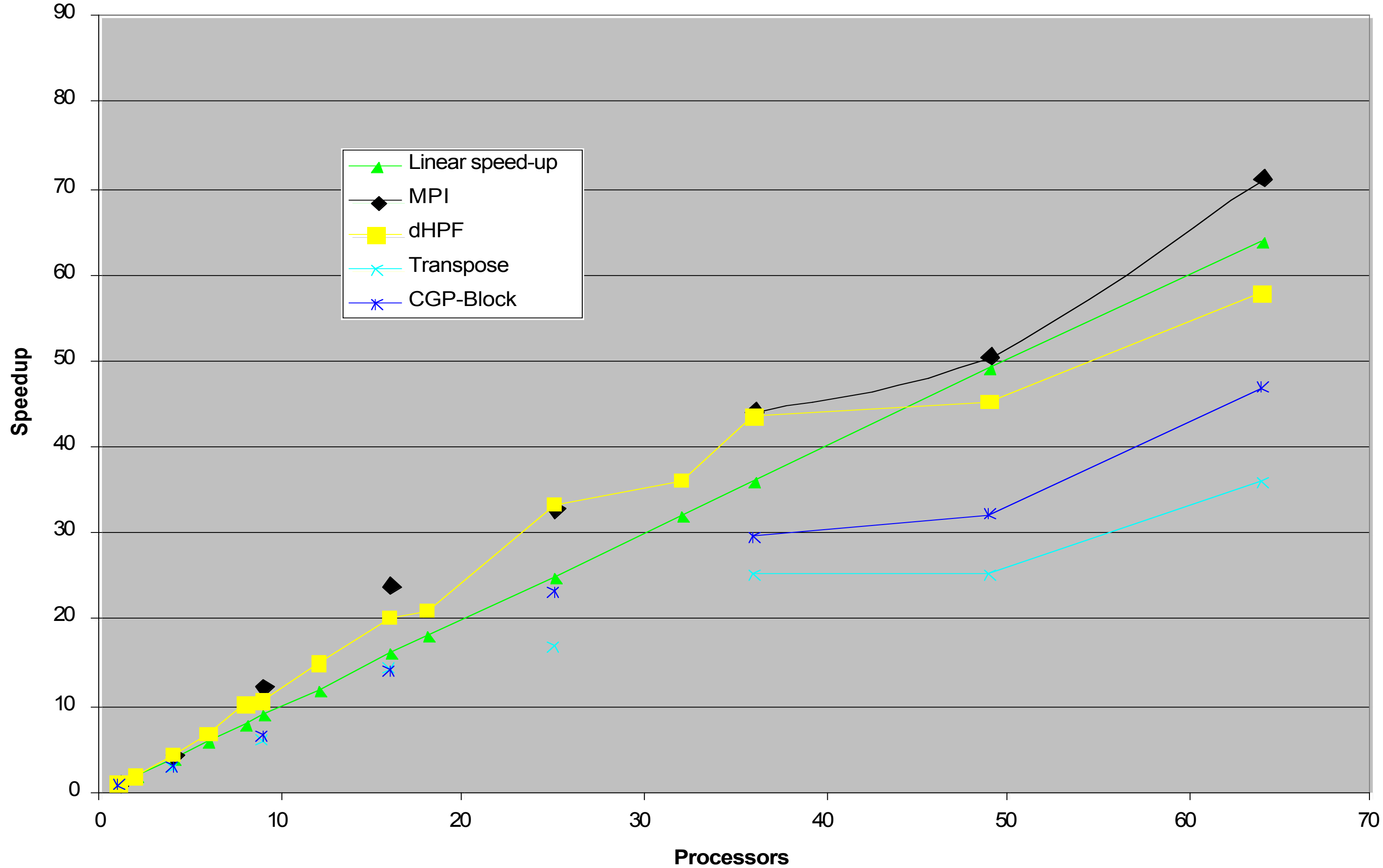
SGI Origin 2000

NAS SP Speedups (Class A) using Generalized Multipartitioning



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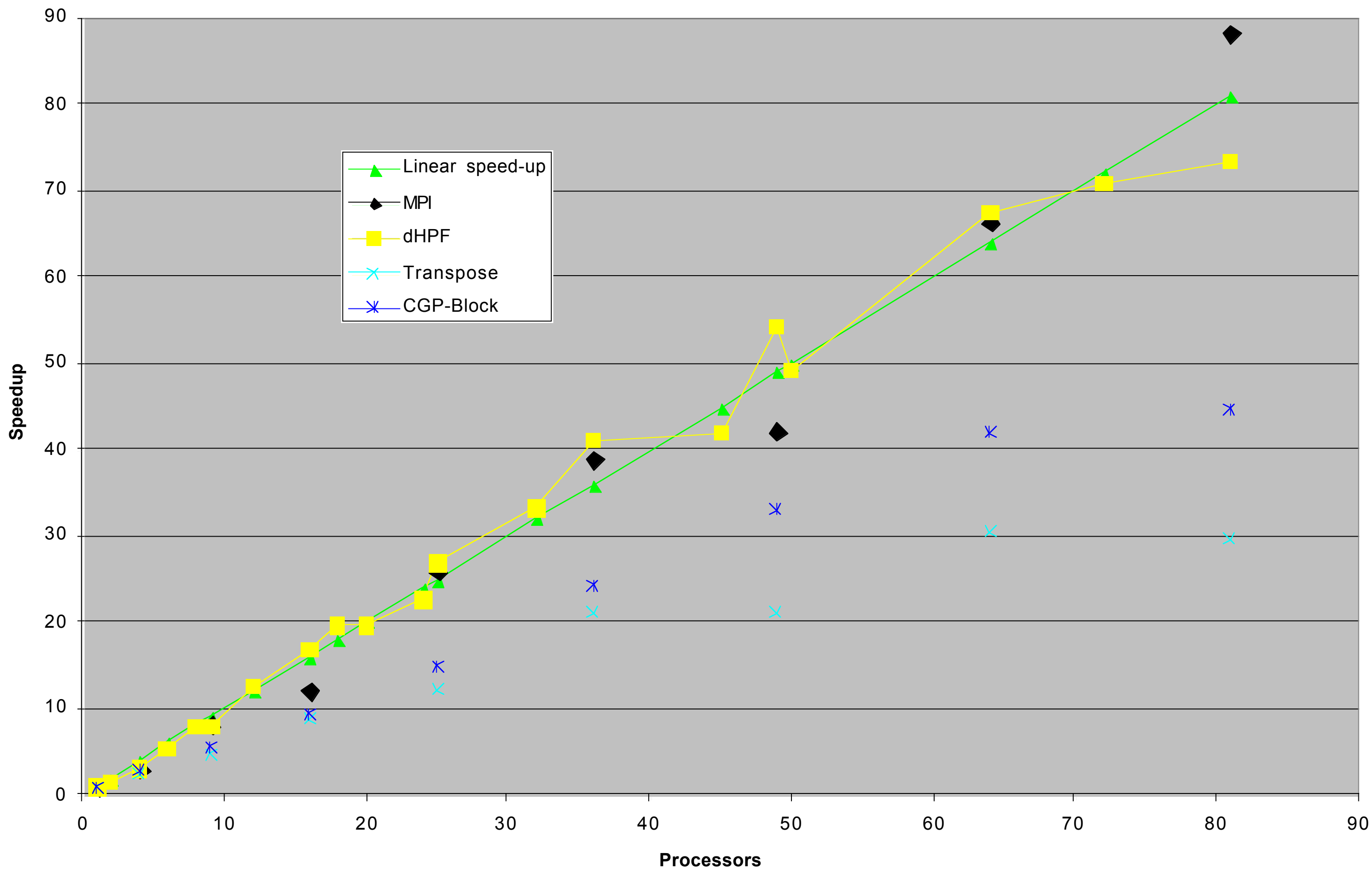
NAS SP Speedups (Class B) using Generalized Multipartitioning



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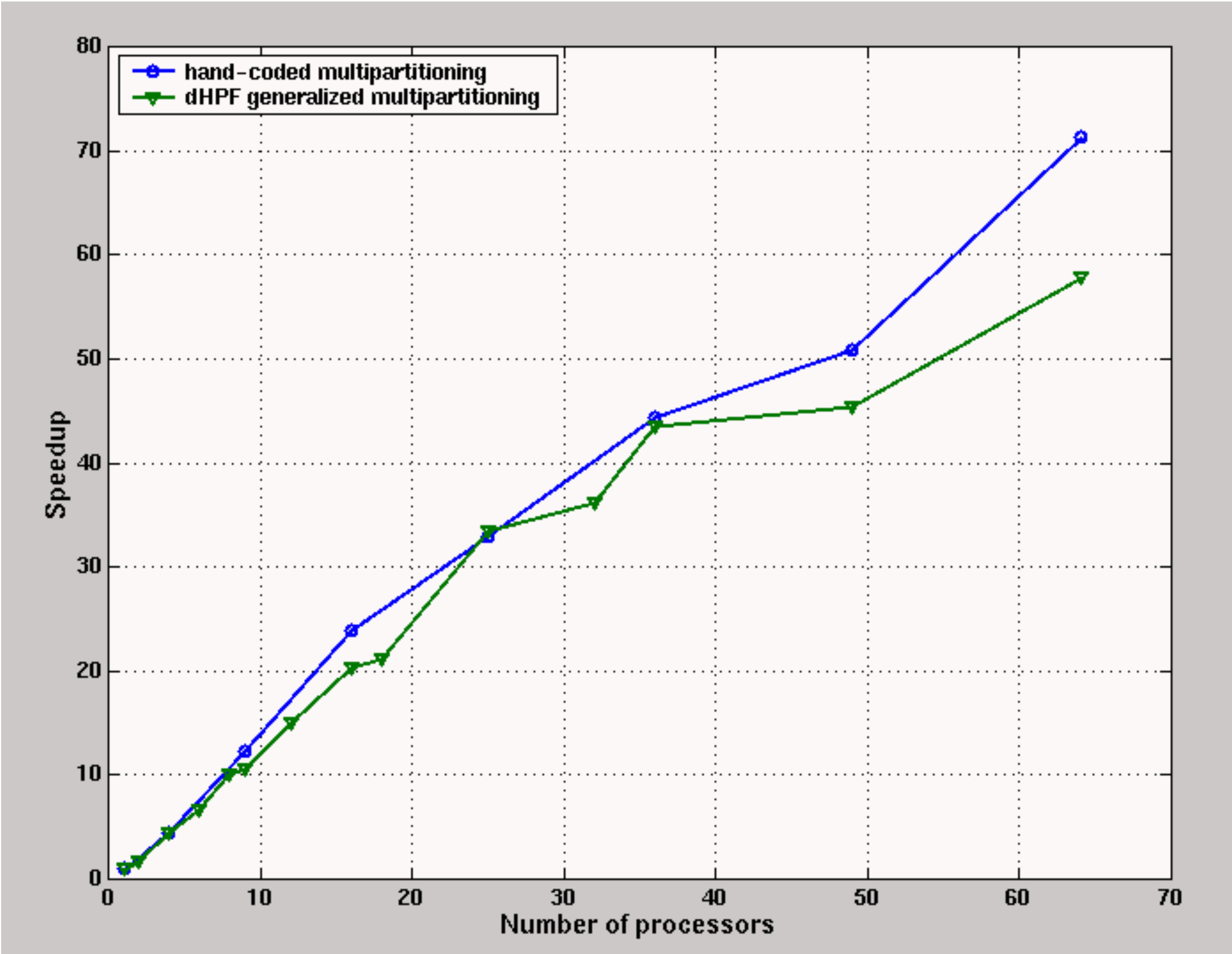
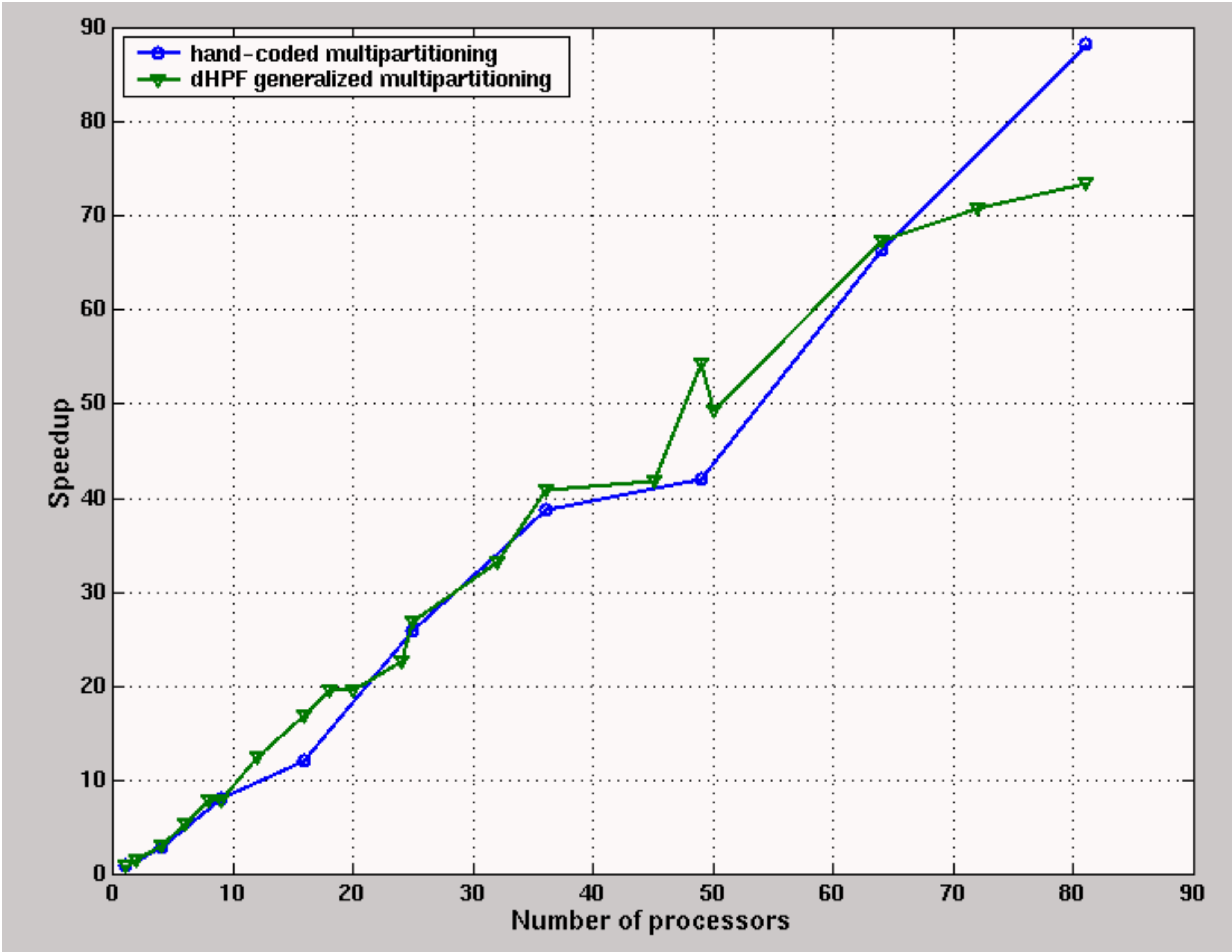
SGI Origin 2000


hand-coded multipartitioning
dHPF generalized multipartitioning
Speedup
Number of processors
0
10
20
30
40
50
60
70
80


hand-coded multipartitioning
dHPF generalized multipartitioning
Speedup
Number of processors
0
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20
30
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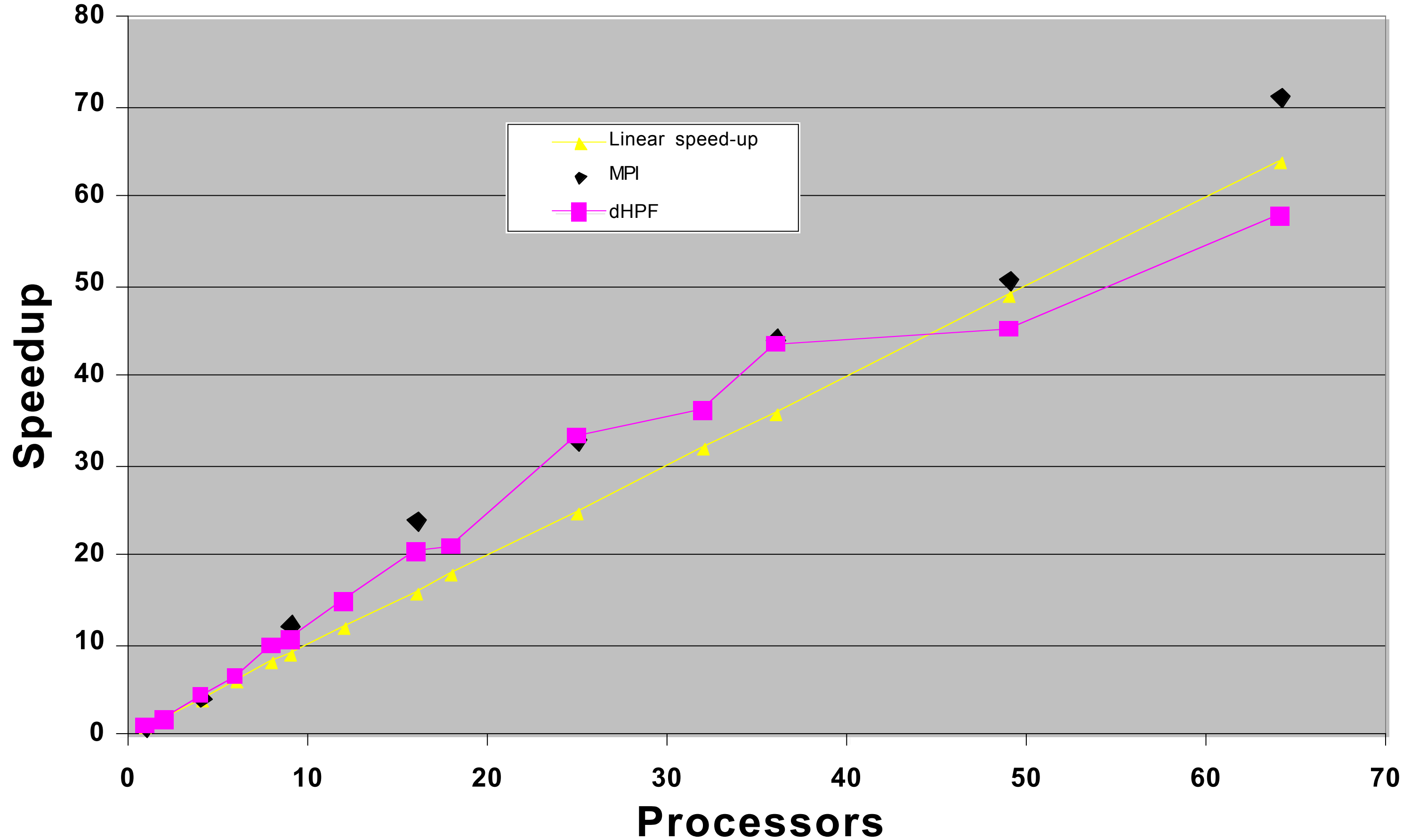
60
70
80
90

NAS SP Speed-up (Class A) using Generalized Multipartitioning



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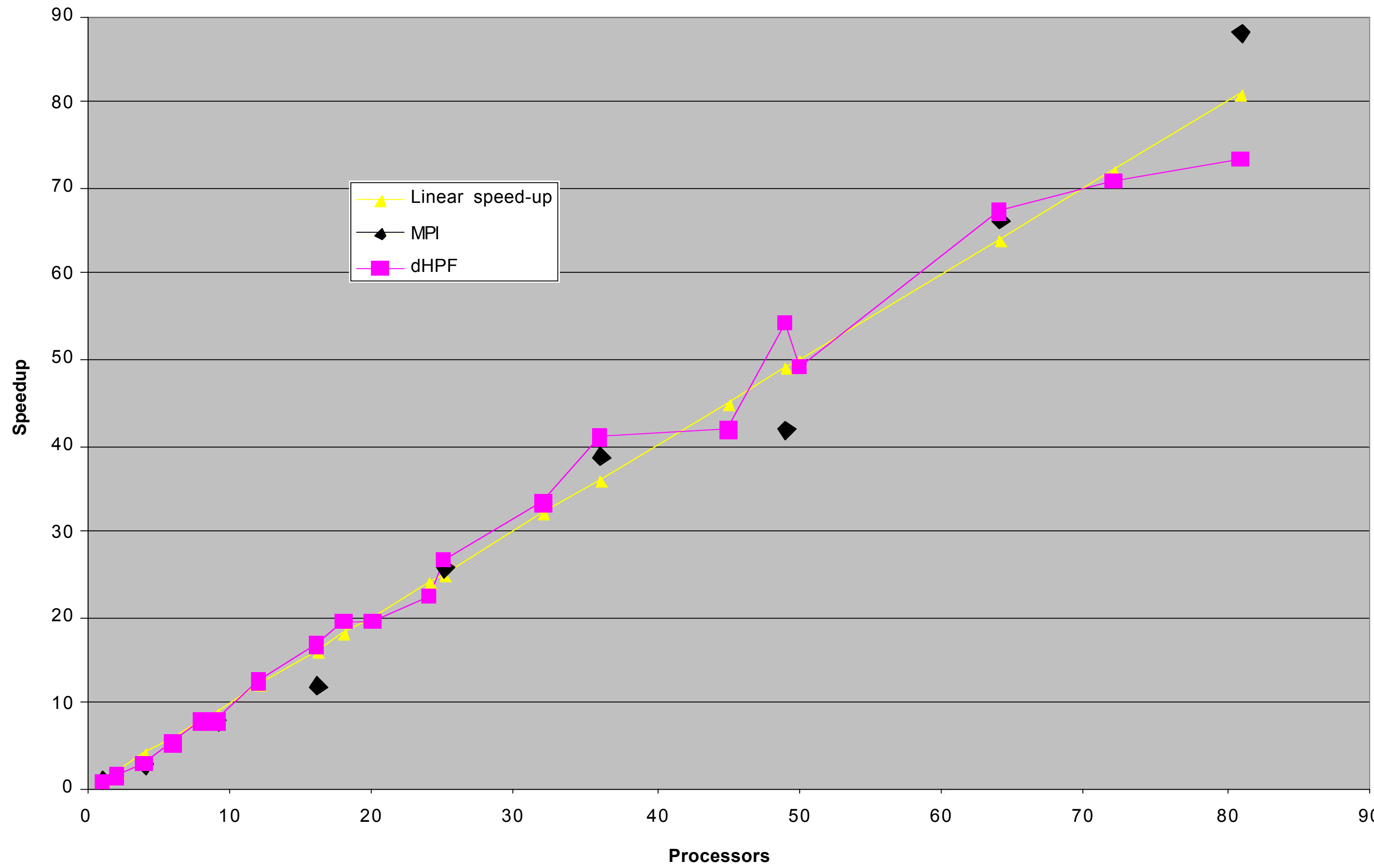
NAS SP Speedups (Class B) using Generalized Multipartitioning



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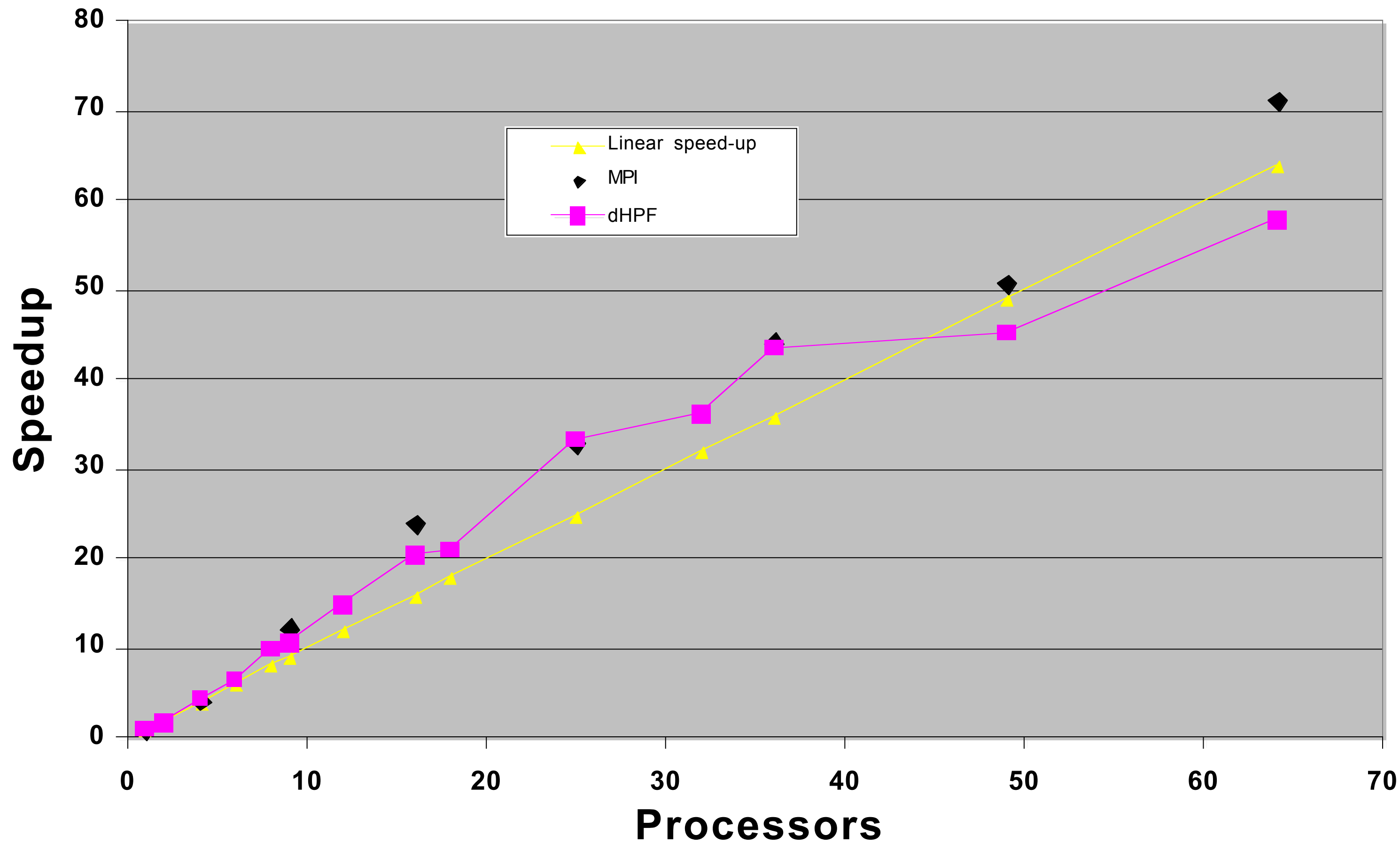
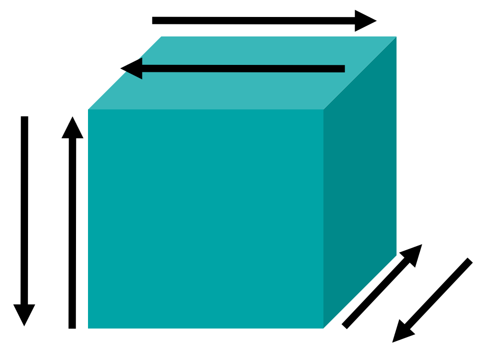


(April 2002)

SGI Origin 2000

Context

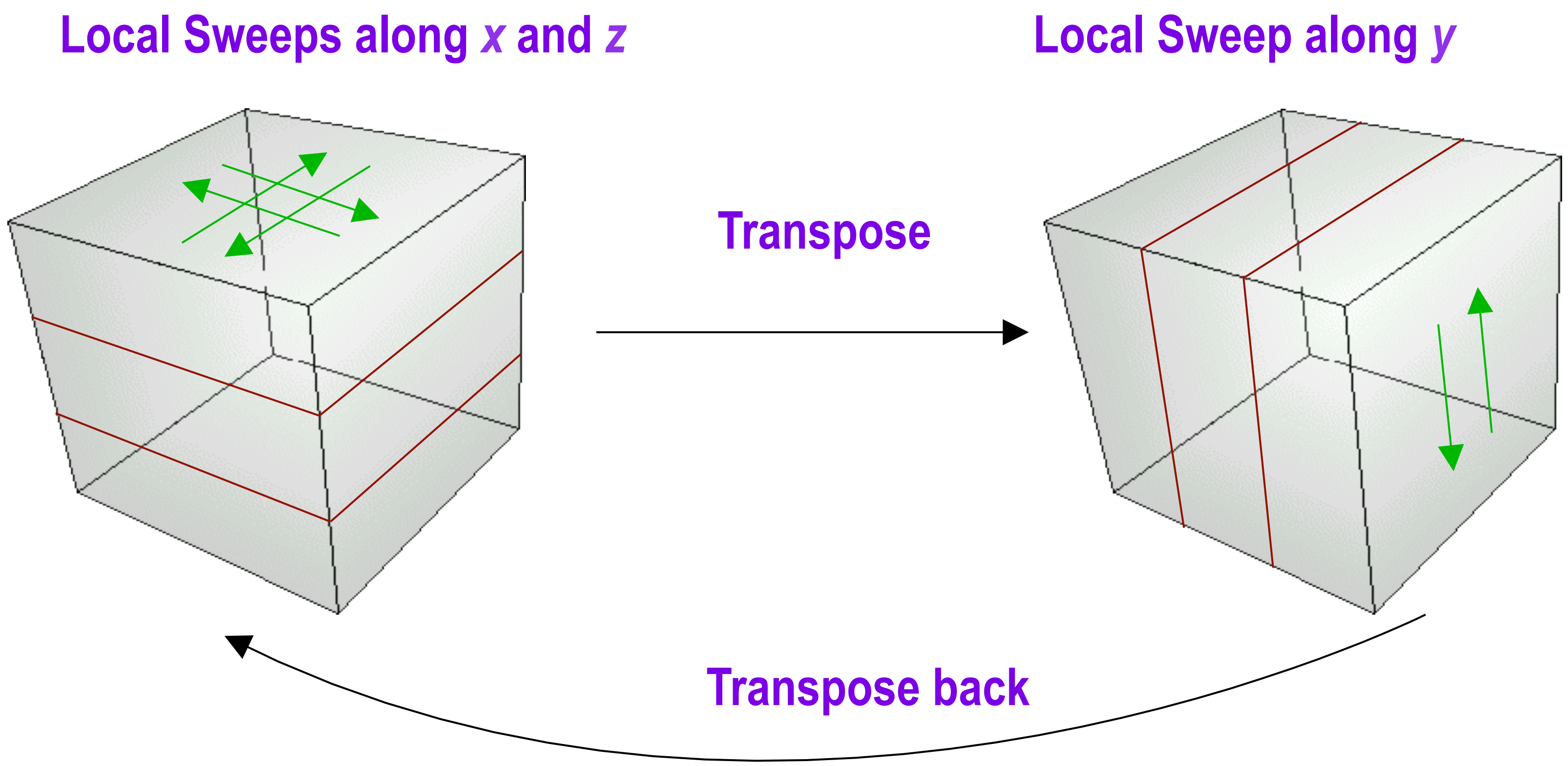
- Alternating Direction Implicit (ADI) Integration widely used for solving the Navier-Stokes equation in parallel and a variety of other computational methods [Naik'93].
- Structure of computation: line sweeps, i.e., 1D recurrences



- Compiler-based parallelization is hard: tightly coupled computations (dependences), fine-grain parallelism.
- Challenge: achieve hand-coded performance with dHPF (Rice HPF compiler).

Parallelizing Line Sweeps

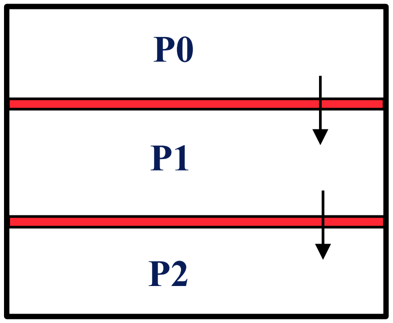
Approach 1: Avoid computing along partitioned dimensions



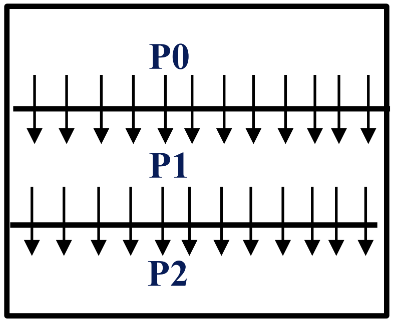
+ Fully parallel computation
– High communication volume: transpose ALL data

Parallelizing Line Sweeps

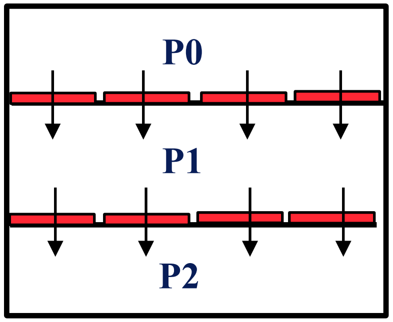
Approach 2: Compute along partitioned dimensions



- Loop carrying dependence in an outer position
  - ➔ Full serialization
  - ➔ Minimal communication overhead



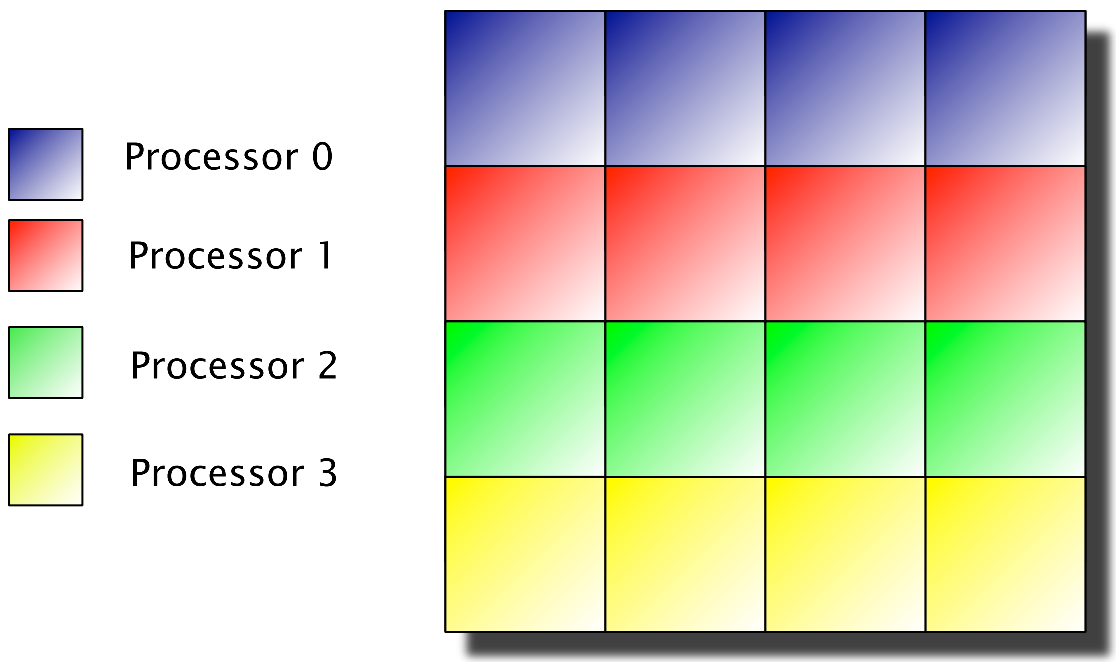
- Loop carrying dependence in innermost position
  - ➔ Fine-grained wavefront parallelism
  - ➔ High communication overhead



- Tiled loop nest, dependence at mid-level
  - ➔ Coarse-grained wavefront parallelism
  - ➔ Moderate communication overhead

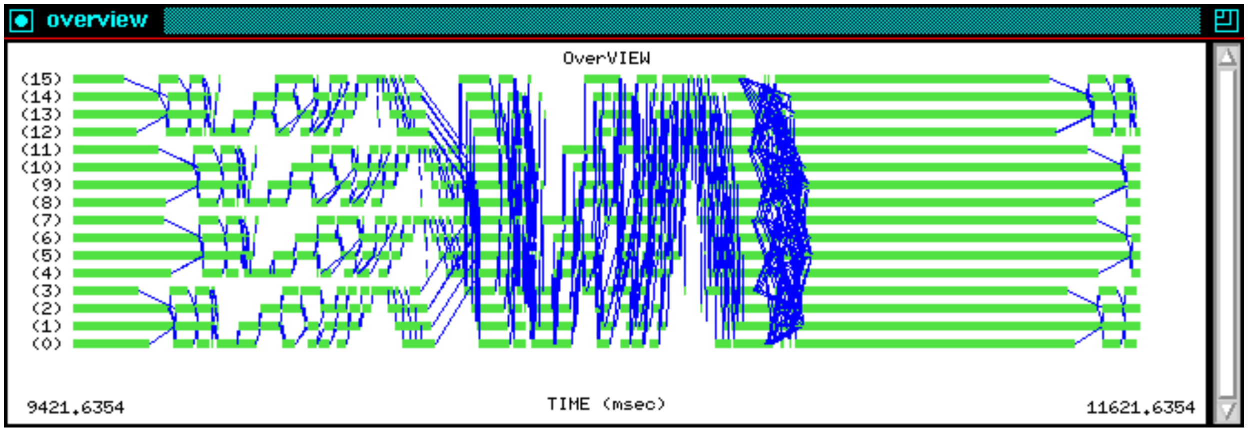
Coarse-grain Pipelining with Block Partitionings

- Wavefront parallelism
- Coarse-grain communication

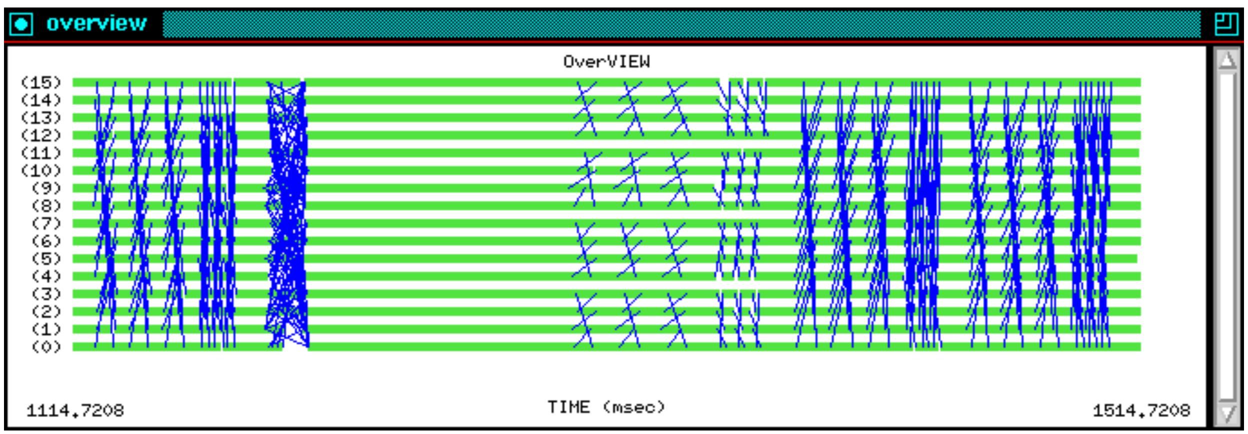


- Better performance than transpose-based parallelizations [Adve et al. SC98]

Parallelizing Line Sweeps



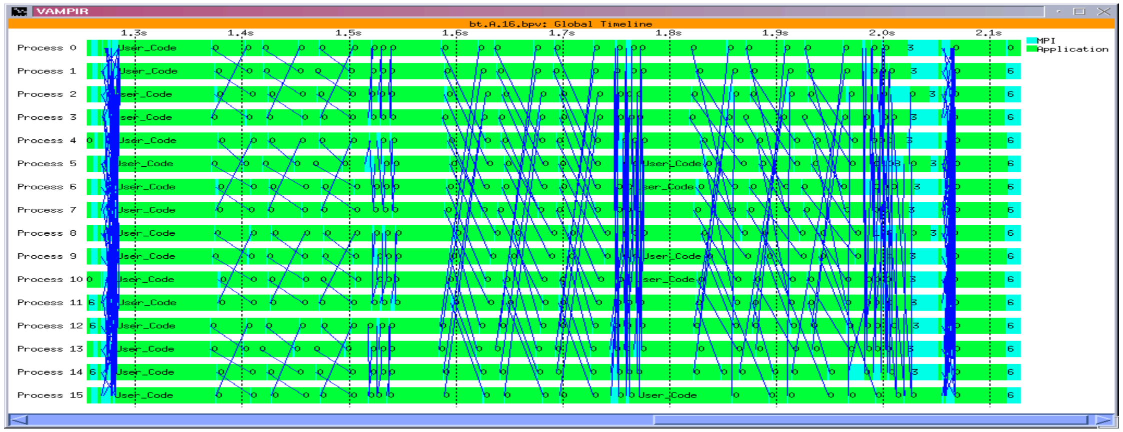
Compiler-generated coarse-grain pipelining



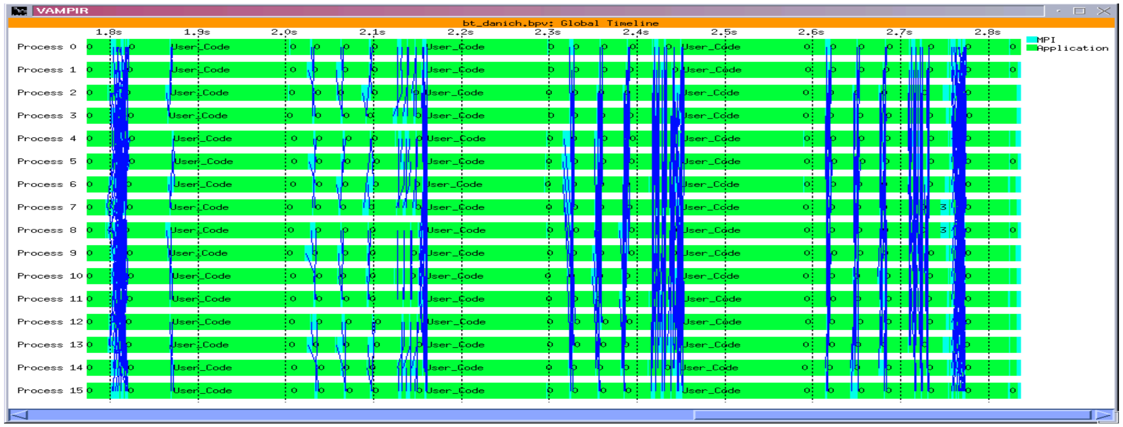
Hand-coded multipartitioning

NAS BT Parallelizations

Hand-coded
3D Multipartitioning



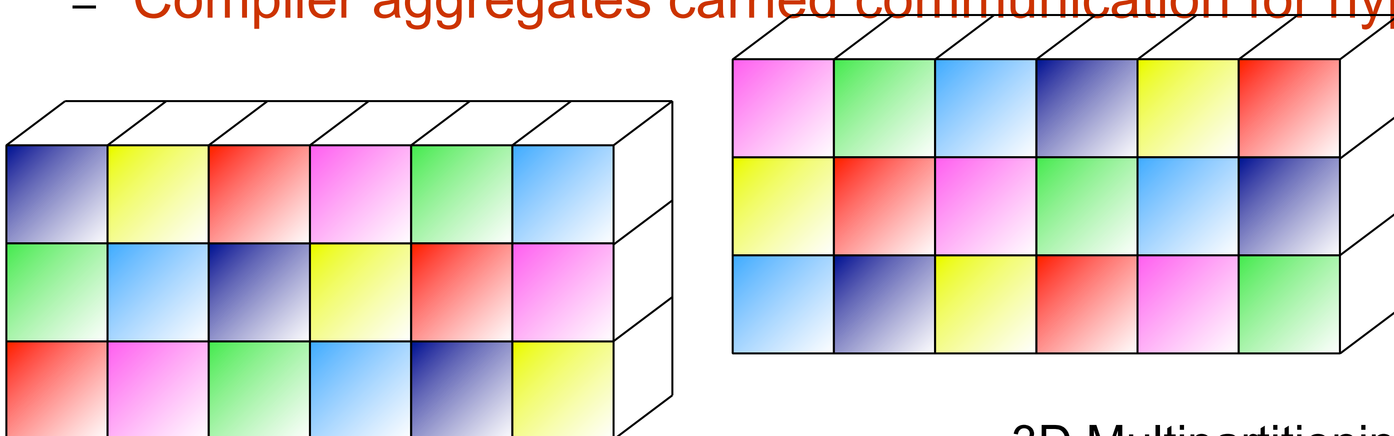
Compiler-generated
3D Multipartitioning
(dHPF, Jan. 2001)



Execution Traces for NAS BT Class 'A' - 16 processors, SGI Origin 2000

Generalized Multipartitioning

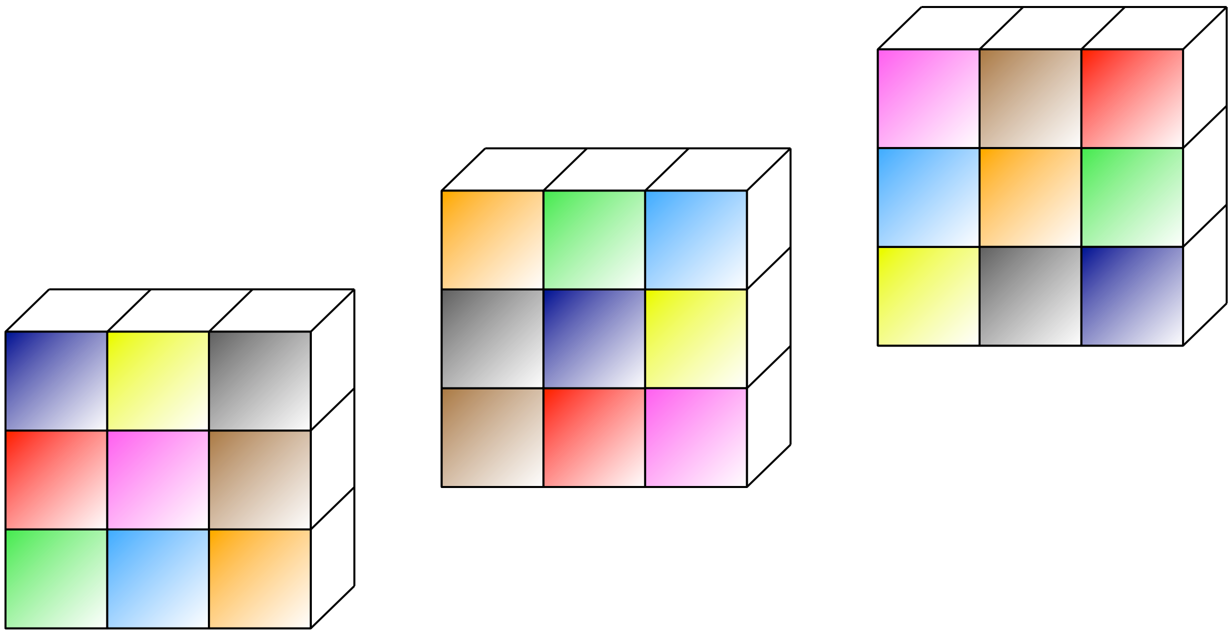
- Higher dimensional multipartitionings for arbitrary numbers of processors
  - Optimal overpartitionings (more than one tile per processor per hyperplane) + modular mappings
  - Compiler aggregates carried communication for hyperplanes



3x6x2

3D Multipartitioning for 6 processors

Higher-dimensional Multipartitioning



Diagonal 3D Multipartitioning for 9 processors