STA 4103/5107 Computational Methods in Statistics II

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5.4 Simulating Stochastic Processes

Stochastic Process

- We can extend the ideas of simulating random variables to simulating stochastic processes.
- To start with, we briefly introduce the notion of **stochastic process.**
- So far we have dealt only with real-valued, i.e. scalar, random variables but now we extend to a collection of random variables.
- **Definition 15** A stochastic process is an indexed collection of random variables.
- Most often, the indexing variable is time.

Continuous and Discrete Processes

- For example let X be a stochastic process, indexed by time, such that X_t is a real-valued random variable. For a fixed time t, X_t is just a random variable.
- Each realization, or sample, of X_t is now a function of time, also called a sample path.
- In case the indexing variable t is continuous, the stochastic process is called a **continuous-time process**; else, it is called a **discrete-time process**.
- Similarly, if the space in which X_t takes values is continuous, then the process is called a **continuous-valued process**. Otherwise, it is called a **discrete-valued process**.

Discrete-Time White Noise

- A **discrete white noise process** is defined as a process ε_t which is identically and independently distributed (i.i.d) with zero expectation and σ^2 variance. Hence,

$$E(\varepsilon_t) = 0$$

$$E(\varepsilon_s \varepsilon_t) = \begin{cases} \sigma^2 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$$

- In discrete-time models, a white noise process is often normally distributed (**Gaussian white noise**).

- A white noise process can also be distributed by any other distribution as long as the i.i.d. assumption is valid.

Random Walk

- A **random walk** is one of the simplest examples of a discrete time, discrete state stochastic process.
- The process transitions only at times separated by a unit time T, and takes values only in the set of integer multiples of s.
- At every transition time nT, for $n = 1, 2, \ldots,$ toss a coin and depending on the outcome, add $+s$ for a head or $-s$ for a tail to the current value of the process.
- Assume we start with process at time zero with the process value being zero. At a transition time nT, the value of the process is

$$X(nT) = ks - (n-k)s$$

where the number of heads in n tosses is k.

Probability Formula

- That is,

$$X(nT) = ms, \text{ where } m = (2k - n) \text{ or } k = (m + n)/2.$$

- Since $k \in \{0, 1, 2, \ldots, n\}$, the possible values for m are $\{-n, \ldots, 0, \ldots, n\}$.

- Assuming a fair coin, the probability that $X(nT) = ms$ is given by

$$P\{X(nT) = ms\} = \binom{n}{(m+n)/2}\left(\frac{1}{2}\right)^n$$

- Given this probability, one can analyze this random walk for any time nT (e.g. computing the mean and variance of $X(nT)$).

Sum of Random Variables

- Another interpretation of $X(nT)$ is given by the following:
$$X(nT) = X_1 + X_2 + \ldots + X_n,$$
where X_is are i.i.d random variables.

- Each X_i takes on the value either $+s$ or $-s$ with equal probabilities; it is easy to show that $E[X_i] = 0$ and $var(X_i) = s^2$.

- $X(nT)$ is a sum of independent and identically distributed random variables with mean zero and variance s^2, or **a sum of discrete white noise with variance s^2.**

- Therefore, $X(nT)$ is a random variable with mean zero and variance ns^2.

Limiting Situation

- Setting $s = \alpha T^{1/2}$ for some positive number α. In the limit $T \to 0$, the random walk becomes a continuous time, continuous state process.

- Let $t = nT$ denote the time index of the process; t is a continuous random variable in the limiting case.

- Consider the random process at time t. $X(t)$, through its relation with $X(nT)$, has the following statistics:

$$E[X(t)] = E[X(nT)] = 0$$

$$var(X(t)) = var(X(nT)) = ns^2 = \alpha^2 T\, t/T = \alpha^2 t.$$

Wiener Process

- That is, $X(t)$ is a continuous random variable with mean zero and variance $\alpha^2 t$.

- This limiting process is called the **Wiener process (**also called **Brownian motion)**.

- **(Formal Definition) Wiener Process** is a continuous-time stochastic process $X(t)$ for $t \geq 0$ with $X(0) = 0$ and such that
  - the increment $X(t) - X(s)$ is Gaussian with mean 0 and variance $\alpha^2(t - s)$ for any $0 \leq s < t$.
  - increments for non-overlapping time intervals are independent.

- For $\boldsymbol{\alpha = 1}$, it is called the **standard Wiener process**.

Properties

- 1. If (t_1, t_2) and (t_3, t_4) are non-overlapping intervals, then $X(t_2)-X(t_1)$ and $X(t_4)-X(t_3)$ are statistically independent random variables.
- 2. The increment $X(t_2)-X(t_1)$ is independent of the value $X(t_1)$ or any past value of the process. In fact, this increment is a Gaussian random variable with mean zero and variance $\alpha^2(t_2 - t_1)$.
- 3. For any two times $t_1 < t_2$, we have:

$$\begin{aligned} E[X(t_1)X(t_2)] &= E[X(t_1)(X(t_2) - X(t_1) + X(t_1))] \\ &= E[X(t_1)^2] + E[X(t_1)(X(t_2) - X(t_1))] \\ &= \alpha^2 t_1 + E[X(t_1)]E[X(t_2) - X(t_1)] \\ &= \alpha^2 t_1. \end{aligned}$$

Example

- **Simulate a random walk**

  For a given value of T and $s = \alpha T^{1/2}$, plot the sample paths of the X_t for $\alpha = 1.0$ and $T = 1$, 0.1, 0.01, and 0.001.

  Choose the total number of steps n to be $10/T$.

  So the total time is constant 10.