Homework 1 Solution

Problem 1

Pf :
$$\inf_{\lambda>0} \frac{E(e^{\lambda X})}{e^{\lambda\delta}} = \inf_{\lambda>0} \frac{E\sum_{k=0}^{\infty} \frac{\lambda^k X^k}{k!}}{\sum_{k=0}^{\infty} \frac{(\lambda\delta)^k}{k!}} \quad \text{(by Taylor expansion)}.$$

$$(*) := \inf_{\lambda>0} \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \frac{\lambda^k E X^k}{k!}}{\sum_{k=0}^{n} \frac{(\lambda\delta)^k}{k!}}$$

By applying the inequality, for any n.n.g seqs. $a_k, b_k \geq 0$, $k=1,\dots,n$.

we have
$$\frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k} \geq \min_{k=1,\dots,n}\left(\frac{a_k}{b_k}\right)$$

$$(*) \geq \inf_{\lambda>0} \lim_{n\to\infty}\left(\min_{k=0,\dots,n} \frac{\lambda^k E X^k/k!}{\lambda^k \delta^k/k!}\right) = \lim_{n\to\infty}\left(\min_{k=0,\dots,n} \frac{EX^k}{\delta^k}\right)$$

$$= \inf_{k=0,1,2,\dots} \frac{EX^k}{\delta^k}$$

$\square$

Problem 2.

Pf: $a \Rightarrow b$.

Let $C_0 = \frac{1}{b}$, then $E e^{\lambda X} \leq e^{\frac{\lambda^2 v^2}{2}} \leq e^{\frac{C_0^2 v^2}{2}} < \infty$.

$b \Rightarrow c$.

By chernoff bound

$$p(X>t) \leq \inf_{\lambda \in [0, C_0]} \frac{E e^{\lambda X}}{e^{\lambda t}} \leq E(e^{\frac{C_0 X}{2}}) \cdot e^{-\frac{C_0}{2} t}$$ by taking $\lambda = C_0/2$.

Let $C_1' = E(e^{\frac{C_0}{2} X})$ and $C_2' = \frac{C_0}{2}$, then

$$p(X \geq t) \leq C_1' e^{-C_2' t}.$$

Similar arguments to another side,

$$p(X \leq -t) = p(-X \geq t) \leq \inf_{\lambda \in [0, C_0]} \frac{E e^{-\lambda X}}{e^{\lambda t}}$$

$$\leq E(e^{-\frac{C_0}{2} X}) \cdot e^{-\frac{C_0}{2} t}$$

by taking $C_1'' = E e^{-\frac{C_0}{2} X}$, $C_2'' = \frac{C_0}{2}$.

Combine two pieces together,

$$p(|X| \geq t) \leq C_1 e^{-C_2 t},$$

where $C_1 = \max(E e^{-\frac{C_0}{2} X}, E e^{\frac{C_0}{2} X})$, $C_2 = \frac{C_0}{2}$.

$\underline{c \Rightarrow a}$.

$$E e^{\lambda X} = E \sum_{k=0}^{\infty} \frac{\lambda^k X^k}{k!} = 1 + \sum_{k=2}^{\infty} \frac{|\lambda|^k E|X|^k}{k!}$$

$$E|X|^k = \int_0^{+\infty} P(|X|^k > t)\, dt = \int_0^{+\infty} k \cdot t^{k-1} P(|X| > t)\, dt$$

by applying (c).

$$\leq \int_0^{+\infty} k \cdot t^{k-1} C_1 e^{-C_2 t}\, dt = C_1 \cdot k \Gamma(k) C_2^{-k} = C_1 k! C_2^{-k}$$

plug in the $E|X|^k$ to the above summation,

$$E e^{\lambda X} \leq 1 + \sum_{k=2}^{\infty} \frac{|\lambda|^k}{k!} \cdot C_1 k! C_2^{-k} = 1 + C_1 \sum_{k=2}^{\infty} \left(\frac{|\lambda|}{C_2}\right)^k$$

$$= 1 + \frac{C_1}{C_2} \frac{\lambda^2}{C_2 - |\lambda|} \quad \text{for all } |\lambda| < C_2.$$

Compare the term $e^{\frac{\lambda^2 v^2}{2}} = 1 + \frac{\lambda^2 v^2}{2} + \sum_{k=2}^{\infty} \frac{\left(\frac{\lambda^2 v^2}{2}\right)^k}{k!}$

There exist n.n.g numbers b and ν s.t.

$$\frac{C_1}{C_2} \frac{1}{C_2 - |\lambda|} \leq \frac{C_1}{C_2} \frac{1}{C_2 - |\frac{1}{b}|} \leq \frac{\nu^2}{2}, \quad \text{for } |\lambda| \leq \frac{1}{b} < C_2.$$

$$\Rightarrow \quad E e^{\lambda X} \leq e^{\frac{\lambda^2 \nu^2}{2}}, \quad \text{for } |\lambda| \leq \frac{1}{b}$$

$\square$

Problem 3.

(a) Define random variables $g_i = \frac{X_i}{\sigma} \overset{iid}{\sim} N(0,1) \quad i=1,\cdots,n.$

It's equivalent to show $\lim_{n\to\infty} E \max_{i=1,\cdots,n} |g_i| / \sqrt{2\log n} = 1.$

It suffice to show $\limsup_{n\to\infty} \frac{E \max_{i=1,\cdots,n} |g_i|}{\sqrt{2\log n}} \leq 1$ and

$$\liminf_{n\to\infty} \frac{E \max |g_i|}{\sqrt{2\log n}} \geq 1.$$

First show the upper bound,

$$E \max_{i=1,\cdots,n} |g_i| = \int_0^{+\infty} P\left(\max_{i=1,\cdots,n} |g_i| > t \right) dt$$

$$= \int_0^{\sqrt{2\log n}} P\left(\max_{i=1,\cdots,n} |g_i| > t \right) dt + \int_{\sqrt{2\log n}}^{+\infty} P\left(\max_{i=1,\cdots,n} |g_i| > t \right) dt.$$

by union bound

$$\leq \sqrt{2\log n} + n \int_{\sqrt{2\log n}}^{+\infty} P(|g_i| > t)\, dt$$

$$= \sqrt{2\log n} + n \cdot \sqrt{\frac{2}{\pi}} \int_{\sqrt{2\log n}}^{+\infty} \int_t^{+\infty} e^{-\frac{u^2}{2}}\, du\, dt$$

$$= \sqrt{2\log n} + n \sqrt{\frac{2}{\pi}} \int_{\sqrt{2\log n}}^{+\infty} e^{-\frac{u^2}{2}} \int_{\sqrt{2\log n}}^{u} dt\, du$$

$$\leq \sqrt{2\log n} + n\sqrt{\frac{2}{\pi}}\, e^{-\frac{2\log n}{2}} - n\sqrt{\frac{2}{\pi}} \frac{2\log n}{2\log n + 1} e^{-\frac{2\log n}{2}} = \sqrt{2\log n} + n\sqrt{\frac{2}{\pi}} \frac{1}{2\log n+1} \cdot \frac{1}{n}$$

Thus $E \max_{1\le i\le n} |g_i| \le \sqrt{2\log n} + \sqrt{\frac{2}{\pi}} \Big/ (1+2\log n)$

$$\Rightarrow \limsup_{n\to\infty} \frac{E \max_{i=1,\cdots,n} |g_i|}{\sqrt{2\log n}} \le 1$$

Now to check the other direction,

For $t \le \sqrt{(2-\delta)\log n}$, for $0<\delta<2$,

$$P(|g|>t) \ge \sqrt{\frac{2}{\pi}} \frac{t}{t^2+1} e^{-\frac{t^2}{2}} \ge \sqrt{\frac{2}{\pi}} \frac{\sqrt{(2-\delta)\log n}}{(2-\delta)\log n+1} n^{-(2-\delta)/2}$$

$$P(\max_{i=1,\cdots,n} |g_i|>t) = 1-(1-P(|g|>t))^n \quad \text{by apply } 1-x \le e^{-x} \text{ for all } x$$

$$\ge 1-e^{-nP(|g|>t)} \to 1, \text{ as } n\to\infty \text{ for all } \delta\in(0,2)$$

Since $nP(|g|>t) \ge \sqrt{\frac{2}{\pi}} \frac{\sqrt{(2-\delta)\log n}}{(2-\delta)\log n+1} n^{\delta/2} \to +\infty$ as $n\to\infty$. for all $\delta\in(0,2)$

Then lower-bound the expectation

$$E \max_{i=1,\cdots,n} |g_i| \ge \int_0^{\sqrt{(2-\delta)\log n}} P(\max_{i=1,\cdots,n} |g_i|>t)\, dt$$

$$\ge \int_0^{\sqrt{(2-\delta)\log n}} 1-e^{-nP(|g|>t)}\, dt$$

$$= \left(1-e^{-nP(|g|>t)}\right) \sqrt{(2-\delta)\log n}$$

$$\Rightarrow \liminf_{n\to\infty} \frac{E \max_{i=1,\cdots,n} |g_i|}{\sqrt{(2-\delta)\log n}} \ge 1$$, Letting $\delta\to 0$ Complete the proof. $\square$

Problem 3

(b) for any $s > 0$,

$$E[\max_{i=1,\dots,n} X_i] = \frac{1}{s} E\left(\log e^{s \max_{i=1,\dots,n} X_i}\right)$$

$$\leq \frac{1}{s} \log E\left(e^{s \max_{i=1,\dots,n} X_i}\right) \quad \text{by Jensen's } \neq$$

$$= \frac{1}{s} \log E\left(\max_{i=1,\dots,n} e^{sX_i}\right)$$

$$\leq \frac{1}{s} \log\left(\sum_{i=1}^{n} E e^{sX_i}\right)$$

$$\leq \frac{1}{s} \log\left(n \cdot e^{\frac{\sigma^2 s^2}{2}}\right) = \frac{\log n}{s} + \frac{\sigma^2 s}{2} \quad \text{for all } s > 0.$$

By taking $s = \sqrt{2\log n/\sigma^2}$ to minimize $\frac{\log n}{s} + \frac{\sigma^2 s}{2} = \sigma\sqrt{2\log n}$.

$$\Rightarrow E\left(\max_{i=1,\dots,n} X_i\right) \leq \sigma\sqrt{2\log n}. \qquad \square$$

Problem 4.

Pf: To apply bounded difference inequality, first check

for points $X=(x_1,\cdots,x_k,\cdots x_n)$ and $X'=(x_1,\cdots,x_k',\cdots,x_n)$ where x_k' is independent

copy of x_k, and consider $F(x_1,\cdots,x_n) = \|\hat{f}-f\|_1$

$$|F(x_1,\cdots,x_k,\cdots,x_n) - F(x_1,\cdots,x_k',\cdots,x_n)|$$

$$= \Big| \int \Big| \frac{1}{nh}\Big(\sum_{i\neq k} K\Big(\frac{x-X_i}{h}\Big) + K\Big(\frac{x-X_k}{h}\Big)\Big) - f(x)\Big| dx$$

$$- \int \Big| \frac{1}{nh}\Big(\sum_{i\neq k'} K\Big(\frac{x-X_i}{h}\Big) + K\Big(\frac{x-X_k'}{h}\Big) - f(x)\Big| dx \Big|$$

by applying $||a|-|b|| \le |a-b| \le |a|+|b|$

$$\le \int \frac{1}{nh}\Big|K\Big(\frac{x-X_k}{h}\Big) - K\Big(\frac{x-X_k'}{h}\Big)\Big| dx$$

$$\le \int \frac{1}{nh} K\Big(\frac{x-X_k}{h}\Big) dx + \int \frac{1}{nh} K\Big(\frac{x-X_k'}{h}\Big) dx \quad \text{since } K(x)\ge 0.$$

$$\le \frac{2}{h} = L_k \quad \text{by change of variable}$$

$$\sum_{k=1} L_k^2 = n\Big(\frac{2}{n}\Big)^2 = \frac{4}{n}.$$

$$P\big(\|\hat{f}-f\|_1 \ge E(\|\hat{f}-f\|_1)+\delta\big) \le e^{-\frac{2\delta^2}{4/n}} = e^{-n\delta^2/2} \le e^{-n\delta^2/8} \qquad \square$$

Problem 5

(a) pf: Since X_1, X_2 are indep.

$$E e^{\lambda(X_1+X_2)} = E_{X_1} e^{\lambda X_1} E_{X_2} e^{\lambda X_2} \le e^{\frac{\lambda^2 \sigma_1^2}{2}} e^{\frac{\lambda \sigma_2^2}{2}} = e^{\frac{\lambda}{2}(\sigma_1^2+\sigma_2^2)}$$

$\Rightarrow (X_1+X_2)$ is $(\sigma_1^2+\sigma_2^2)$ - sub gaussian.

(b), Applying Hölder's ≠ for $p, q \ge 1$ $\frac{1}{p}+\frac{1}{q}=1$

$$E e^{\lambda(X_1+X_2)} \le \left(E_{X_1}(e^{\lambda X_1})^p\right)^{\frac{1}{p}} \cdot \left(E_{X_2}(e^{\lambda X_2})^q\right)^{\frac{1}{q}}$$

$$\le \left(e^{\frac{\sigma_1^2}{2}\lambda^2 p^2}\right)^{\frac{1}{p}} \left(e^{\frac{\sigma_2^2}{2}\lambda^2 q^2}\right)^{\frac{1}{q}}$$

$$= e^{\frac{\lambda^2}{2}(p\sigma_1^2 + q\sigma_2^2)} \qquad \text{Since } q = \frac{p}{p-1}$$

Then choose $p \in [1, +\infty)$ such that minimize $f(p) = p\sigma_1^2 + \frac{p}{p-1}\sigma_2^2$

let $f'(p) = \sigma_1^2 - \frac{\sigma_2^2}{(p-1)^2} = 0 \Rightarrow p_* = 1 + \frac{\sigma_2}{\sigma_1}$

Easy to check p_* is global minima, therefore $q_* = 1 + \frac{\sigma_1}{\sigma_2}$

plug in p_* and q_*, and obtain

$$E e^{\lambda(X_1+X_2)} \le e^{\frac{\lambda^2}{2}(\sigma_1+\sigma_2)^2} \le e^{\frac{\lambda^2}{2}(4\sigma_1^2+4\sigma_2^2)}$$

□

Problem 5

(c). Since X_1, X_2 independent.

$$E(e^{\lambda X_1 X_2}) = E_{X_2}(E_{X_1}(e^{\lambda X_1 X_2} \mid X_2))$$

$$\leq E \exp\left(\frac{(\lambda X_2)^2}{2}\sigma_1^2\right) \qquad \text{by sub-gaussian property}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{\lambda\sigma_1}{2}\right)^2 \cdot E X_2^{2k}}{k!}$$

$$\leq 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\lambda\sigma_1\right)^2 (2\sigma_2^2)^k \cdot 2k \cdot \Gamma(k)}{k!}$$

$$= 1 + 2\sum_{k=1}^{\infty}\left(2 \cdot \frac{1}{2}\lambda^2\sigma_1^2\sigma_2^2\right)^k.$$

$$= 1 + 2\frac{\lambda^2\sigma_1^2\sigma_2^2}{1-\lambda^2\sigma_1^2\sigma_2^2} \leq 1 + 4\lambda^2\sigma_1^2\sigma_2^2 \qquad \text{for } \lambda^2\sigma_1^2\sigma_2^2 \leq \frac{1}{2}$$

$$\leq e^{\frac{\lambda^2}{2}(8\sigma_1^2\sigma_2^2)} \qquad \text{for } |\lambda| \leq \frac{1}{\sqrt{2}\sigma_1\sigma_2}$$

$\Rightarrow X_1 X_2$ is sub-exponential with parameters $(8\sigma_1^2\sigma_2^2, \sqrt{2}\sigma_1\sigma_2)$.

$\square$