STA 6448 Homework 1

Due January 25

20 pts for each problem below (in total, 100pts):

1. Suppose random variable $X \geq 0$ admits a moment generating function in an interval near zero. Given any $\delta > 0$, show that

$$\inf_{k=0,1,2,\ldots} \frac{\mathbb{E}[|X|^k]}{\delta^k} \leq \inf_{\lambda>0} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda\delta}}.$$

Consequently, an optimized bound based on polynomial moments is always at least as good as the Chernoff upper bound.

2. Assume X to be a zero-mean random variable. Show the following statements are equivalent. Therefore, they provide equivalent characterizations of sub-exponential random variables.

a. There are nonnegative numbers (ν^2, b) such that

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2\nu^2}{2}} \quad \text{for all } |\lambda| \leq \frac{1}{b}.$$

b. There is some positive number c_0, such that $\mathbb{E}[e^{\lambda X}] < \infty$ for all $|\lambda| \leq c_0$.

c. There are constants $c_1, c_2 > 0$ such that

$$\mathbb{P}[|X| \geq t] \leq c_1 e^{-c_2 t} \quad \text{for all } t > 0.$$

3. Prove the following statements concerning properties of the sub-Gaussian maxima.

a. Suppose X_i are i.i.d. sequence of $\mathcal{N}(0, \sigma^2)$. Then

$$\lim_{n\to\infty} \frac{\mathbb{E}[\max_{i=1,\ldots,n} |X_i|]}{\sqrt{2\sigma^2 \log n}} = 1.$$

b. Let X_i be a sequence of zero-mean sub-Gaussian variables with parameter σ^2 (no independence assumptions are needed). Then

$$\mathbb{E}[\max_{i=1,\ldots,n} X_i] \leq \sqrt{2\,\sigma^2 \log n} \quad \text{for all } n \geq 1.$$

(Hint: Apply Jensen's inequality and the convexity of the exponential function.)

Consequently, the upper bound $\sqrt{2\,\sigma^2 \log n}$ is sharp for the sub-Gaussian maxima.

4. Let $X_1, \ldots, X_n$ be i.i.d. samples of random variable with density f on the real line. A standard estimator of f is the kernel density estimator

$$\widehat{f}_n = \frac{1}{nh} \sum_{i=1}^{n} K\Big(\frac{x - X_i}{h}\Big),$$

where $K : \mathbb{R} \to [0, \infty)$ is a kernel function satisfying $\int_{-\infty}^{\infty} K(t)\, dt = 1$, and h is a bandwidth parameter. Suppose we choose the loss function as the L^1 norm $\|\widehat{f} - f\|_1 = \int_{-\infty}^{\infty} |\widehat{f}(t) - f(t)|\, dt$. Prove that

$$\mathbb{P}\big[\|\widehat{f} - f\|_1 \geq \mathbb{E}[\|\widehat{f} - f\|_1] + \delta\big] \leq e^{-n\delta^2/8} \quad \text{for all } \delta > 0.$$

5. Suppose X_1 and X_2 are zero-mean sub-Gaussian variables with parameters σ_1^2 and σ_2^2 respectively.

a. If X_1 and X_2 are independent, show that $X_1 + X_2$ is sub-Gaussian with parameter $\sigma_1^2 + \sigma_2^2$.

b. Show that in general (without the independence assumption), $X_1 + X_2$ is sub-Gaussian with parameter $4\sigma_1^2 + 4\sigma_2^2$.

c. If X_1 and X_2 are independent, show that $X_1 X_2$ is sub-exponential with parameters $(2\sigma_1^2\sigma_2^2, \frac{1}{\sqrt{2}\sigma_2\sigma_2})$.