$2^{nd}$ derivative

$$y_j'' = \delta^+\delta^- y_j = \frac{y_{j+1} - 2y_j + y_{j-1}}{\Delta x^2}$$

$$y_j'' = \frac{-y_{j+2} + 16y_{j+1} - 30y_j + 16y_{j-1} - y_{j-2}}{12\Delta x^2} + O(\Delta x^4)$$

3.3.

## 2.7 Integration of odes

Initial Value Problem IVP
$$\begin{cases} \vec{y}\,' = \frac{d\vec{y}}{dt} = \vec{F}(t, \vec{y}) & t \geq 0 \\ \vec{y}(0) = \vec{y}_0 \end{cases}$$

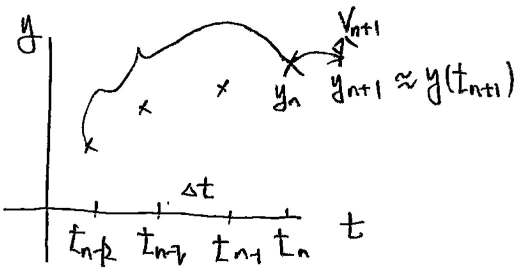
EX: Continuous bond price

$r(t)$ the interest rate

$R(t)$ coupon payment.

$$\frac{dV}{dt} = r(t)\,V - R(t) \equiv F(t, V)$$

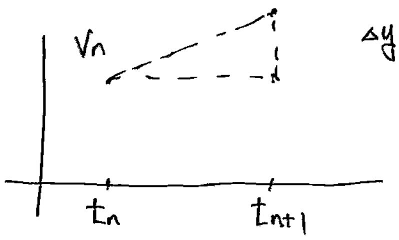
We will look at a class of step-by-step methods $t_n = n \cdot \Delta t$




if $k>0$, the method is called multistep.

if $k=0$, the method is called single step.

Example: Euler's method [Forward Euler]




$$\Delta y = \frac{dy}{dt} \cdot \Delta t = F \Delta t$$

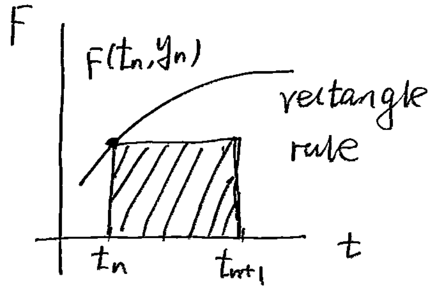
Derivative 1: $\quad y' = F(t,y)$

$$\int_{t_n}^{t_{n+1}} y'\,dt = \int_{t_n}^{t_{n+1}} F(t,y)\,dt$$

$$y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} F(t,y)\,dt$$

$$y_n \equiv y(t_n)$$

$$y_{n+1} = y_n + \underbrace{\int_{t_n}^{t_{n+1}} F(t,y)\,dt}_{\text{approx. by Quadrature.}}$$




$$F(t,y(t)) = \underbrace{F(t_n,y_n)}_{P_0} + \frac{F'(\xi)}{1!}(t$$

$$F(t,y(t)) = \underbrace{F(t_n,y_n)}_{P_0} + \frac{F'(\xi)}{1!}(t-t_n)$$

$$F'(\xi) = y''(\xi)$$

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} F(t_n,y_n)\,dt + y''(\xi)\int_{t_n}^{t_{n+1}} (t-t_n)\,dt$$

$$y_{n+1} = y_n + F(t_n,y_n)\Delta t + y''(\xi)\cdot\frac{\Delta t^2}{2}$$

Drop $O\Delta t^2$ term and let $Y_{n+1}$ satisfy

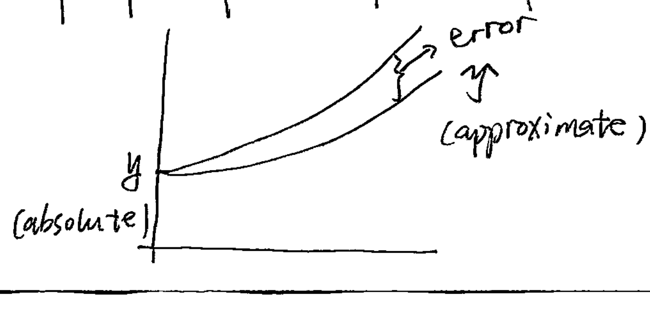
$$Y_{n+1} = Y_n + \Delta t\, F(t_n, Y_n)$$

Example: $\begin{cases} y' = y \\ y(0) = 1 \end{cases} \qquad y = e^t$

Euler: $Y_{n+1} = Y_n + \Delta t\, Y_n = (1+\Delta t)\, Y_n$

choose $\Delta t = 0.1$, $\quad Y_{n+1} = 1.1\, Y_n$

| $n$ | $t_n$ | $Y_n$ (approx. of $y_n$) | $y_n$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 1 | 0.1 | 1.1 | 1.105 |
| 2 | 0.2 | 1.210 | 1.221 |
| 3 | 0.3 | 1.331 | 1.350 |
| 4 | 0.4 | 1.464 | 1.492 |



Derivation 2.

$$y_{n+1} = y_n + y'\Delta t + y''(\xi)\cdot\frac{\Delta t^2}{2}$$

$$y' = F(t_n, y_n)$$

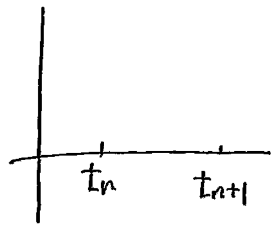
$$y_{n+1} = y_n + F(t_n, y_n)\Delta t + \frac{y''(\xi)}{2}\Delta t^2$$

$$\tilde{y}_{n+1} = \tilde{y}_n + F(t_n, \tilde{y}_n)\Delta t \qquad \text{FE.}$$

Derivation 3.

$$y' = F(t, y)$$

$$\delta_t^+ y_n = F(t_n, y_n) + \text{Error}$$




$$\frac{y_{n+1} - y_n}{\Delta t} = F(t_n, y_n) + E$$

$$y_{n+1} = y_n + \Delta t F(t_n, y_n) + \Delta t E \rightarrow \text{flow away}$$

$$\tilde{y}_{n+1} = \tilde{y}_n + \Delta t F(t_n, \tilde{y}_n)$$

we can find the error over one step a-posteriori

$$\tilde{y}_{n+1} = \tilde{y}_n + (\Delta t)F(t_n, \tilde{y}_n)$$

$$\tilde{y}_{n+1} = \tilde{y}_n + \Delta t F(t_n, \tilde{y}_n)$$

substitute exact solution.

$$y_{n+1} = y_n + \Delta t F(t_n, y_n) + \tau$$

$$y_{n+1} = y_n + y_n'\Delta t + y''(\xi)\cdot\frac{\Delta t^2}{2}$$

$$= y_n + \Delta t F(t_n, y_n) + \tau \qquad \Rightarrow \tau = y''(\xi)\cdot\frac{\Delta t^2}{2}$$

$$\downarrow$$

trancation error

Definition: The local truncation error is the error created over one time step.

Definition: A step by step method is of order $r$ if $\tau = O(\Delta t^{n+1})$

Forward Euler is order 1. $\tau \approx O(\Delta t^2)$

Definition: A method is consistent if it is at least order 1.

Remark: Euler's Method

$$y_{n+1} = y_n + \Delta t F(t_n, y_n) + y''(\xi) \cdot \frac{\Delta t^2}{2}$$

$$\frac{y_{n+1} - y_n}{\Delta t} = F(t_n, y_n) + \frac{1}{\Delta t} \tau \qquad \tau = y''(\xi) \cdot \frac{\Delta t^2}{2}$$

and limit, $\lim_{\Delta t \to 0} y' = F + 0$

Want to know.

$$e_n = y_n - Y_n$$

THM: let $\Delta t = \frac{T}{N}$, $t_n = n\Delta t$

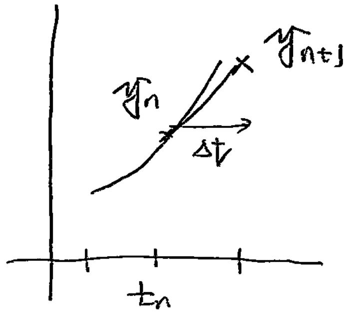
let $Y_{n+1} = Y_n + \Delta t F(t_n, Y_n)$

suppose $\left|\frac{\partial F}{\partial y}\right| \le L$ $\left|\frac{\partial F}{\partial t}\right| \le R$, $|F| \le Z$.

then $Y_N \to y(T)$ as $N \to \infty$ ($\Delta t \to 0$)

3.5. $y' = F(t, y)$

Forward E. $\tilde{y}_{n+1} = \tilde{y}_n + \Delta t\, F(t_n, \tilde{y}_n)$




Proof of THM:

$$y_{n+1} = y_n + \Delta t\, F(t_n, y_n) + \tau_n$$
$$-\ \tilde{y}_{n+1} = \tilde{y}_n + \Delta t\, F(t_n, \tilde{y}_n)$$

~~$e_{n+1} = \tilde{y}_{n+1}$~~ $e_{n+1} = y_{n+1} - \tilde{y}_{n+1}$

$$= e_n + \Delta t\,[F(t_n, y_n) - F(t_n, \tilde{y}_n)] + \tau_n$$

$$|e_{n+1}| \le |e_n| + \Delta t\,|F(t_n, y_n) - F(t_n, \tilde{y}_n)| + |\tau_n|$$

By MVT. $$|F(t_n, y_n) - F(t_n, \tilde{y}_n)| \le \underbrace{\frac{\partial F}{\partial y}}_{\le L} \overbrace{|y_n - \tilde{y}_n|}^{e_n}$$

$\tau \sim$ trancation error

$$|e_{n+1}| \le \underbrace{|e_n| + \Delta t L\,|e_n|}_{(1+\Delta t L)|e_n|} + |\tau_n| \qquad \tau = \max_n |\tau_n|$$

$$|e_{n+1}| \le (1+\Delta t L)|e_n| + \tau$$
$$\le (1+\Delta t L)\{(1+\Delta t L)(e_{n-1}) + \tau\} + \tau$$
$$\le (1+\Delta t L)^2 |e_{n-1}| + (1+\Delta t L)\tau + \tau$$
$$\cdots \le \underbrace{(1+\Delta t L)^{n+1}|e_0|}_{0} + \tau \sum_{j=0}^{n-1} (1+\Delta t L)^j$$

$$|e_N| \le \tau \sum_{j=0}^{N-1} (1+\Delta t L)^j$$

$T = N\Delta t$

Recall $\sum_{j=0}^{N-1} \xi^j = \frac{\xi^N - 1}{\xi - 1}$

So $|e_N| \leq \tau \frac{(1+\Delta t L)^N - 1}{\Delta t L - 1}$

~~$1+\Delta t L$~~ $1+\Delta t L \leq e^{\Delta t L}$

$|e_N| \leq \frac{\tau}{\Delta t}(e^{LT} - 1)/L \Rightarrow |e_N| \leq \frac{\tau}{\Delta t} \frac{e^{LT}-1}{L}$

But $\tau = \max \frac{1}{2} |y''(\xi)| \Delta t^2$

$|e_N| \leq \frac{1}{2} \max |y''(\xi)| \Delta t \cdot \frac{e^{LT}-1}{L}$

But $y'' = F' = \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt}$ $\quad (F = \frac{dy}{dt})$

$= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} \cdot F$

So, $y'' = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} F \leq K + LZ$ $\quad$ just a number

So. $|e_N| \leq \Delta t \left( \frac{1}{2}(K + LZ) \frac{e^{LT}-1}{L} \right)$

$\lim_{\substack{N \to \infty \\ \Delta t \to \frac{T}{N}}} |e_N| = 0$

Also, $|e| = O(\Delta t)$

$(\tau = O(\Delta t^2))$

RMK: consistancy is a necessary condition for convergence.

Except: we assume that

$$(1+\Delta t L)^N \cdot |e_0| \equiv 0$$

so, really, $|e_N| \leq e^{LT}|e_0| + O(\Delta t)$

if $LT$ is large, this doesn't tell us much.
need condition on growth of errors.

Test problem:

$$y' = \lambda y \qquad \lambda \in \mathbb{C}$$

$(\vec{y}' = \vec{F}(t, \vec{y}))$

$\downarrow$ linearize

$\vec{y} = A(t) * \vec{y}$

$\downarrow$ matrix

$\downarrow$

freeze $t$ $\quad \vec{y}' = A\vec{y}$

$\downarrow$ diagnolize $\quad \cancel{A = S \Lambda S^{-1}}$ $\downarrow$ ~~matrix~~

$$A = S \Lambda S^{-1}$$

$$\vec{y}' = S \Lambda S^{-1} \vec{y}$$

$$S^{-1} \vec{y}' = \Lambda S^{-1} \vec{y}$$

Refine $\vec{w} = S^{-1} \vec{y} \Rightarrow \vec{w} = \Lambda \vec{w}$

$$w_i' = \lambda w_i$$

Now let $\quad u =$ solution with no error initial

$v =$ solution with initial error

$$u' = \lambda u$$
$$v' = \lambda v$$

$$\hat{y}' = (u - v)' = \lambda \hat{y}$$

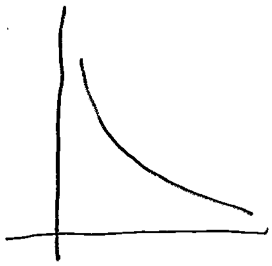
$\uparrow$

$u - v$ error

$$\hat{y}(0) = e_0$$

and scale $\quad y = \dfrac{\hat{y}}{e_0}$

$$\begin{cases} y' = \lambda y \\ y(0) = 1 \end{cases}$$ $\rightarrow$ the problem we have to test.



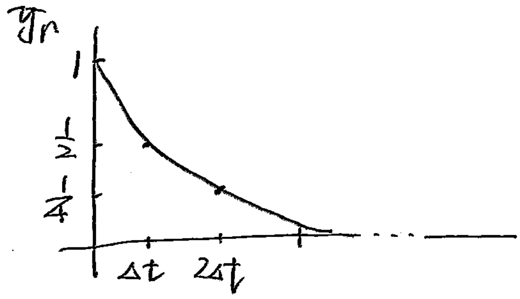
Apply Euler's Method

$$y_{n+1} = y_n + \Delta t(\lambda y_n)$$
$$= (1+\lambda\Delta t)\, y_n$$

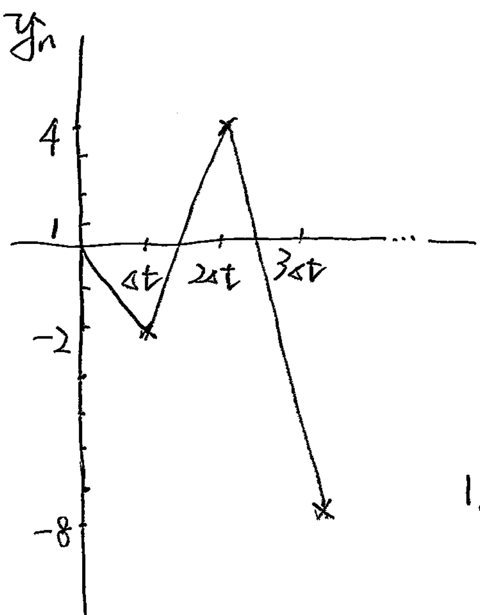
Assume $\lambda$ is real and negative ($\lambda < 0$)

CASE I: choose $\lambda\Delta t$, such that $1+\lambda\Delta t = \frac{1}{2}$

CASE II: choose $\Delta t$, $\Rightarrow$ $1+\lambda\Delta t = -2$




(I)




(II)

$n \to \infty$

$|y_n| \to \infty$ blows up.

To Not blow up,

$$|1+\lambda\Delta t| \le 1$$

Definition: A step-by-step method is absolutely stable if

$$|y_{n+1}| \le |y_n|$$

Forward Euler is absolutely stable if $|1+\lambda\Delta t| \le 1$

in 3.15 time

$$\frac{dy}{dt} = F(t, y)$$

$t_n = n \cdot \Delta t$

$y_{n+1} = y_n + \Delta t \cdot F(t_n, y_n)$

$|y_N - y(T)| = O(\Delta t)$

But $\Delta t$ must satisfy $|1+\lambda\Delta t| \le 1$ for $y' = \lambda y$

$|1+\lambda\Delta t| \le 1 \iff |1+\lambda\Delta t|^2 \le 1$

Let $x \equiv Re(\lambda\Delta t)$ Real

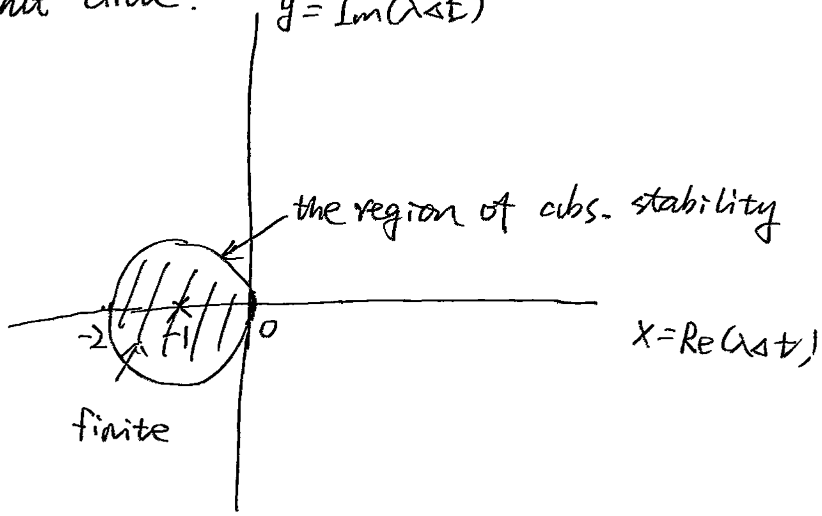
$y \equiv Im(\lambda\Delta t)$ Imagine

Then for absolute stability

$$|1+x+iy|^2 \le 1$$

$$(1+x)^2 + y^2 \le 1$$

ie. $\lambda\Delta t$ must be inside the unit circle.




EX: $y' = -100y$

$y(0) = 1$

Compute $t \in [0, 100]$

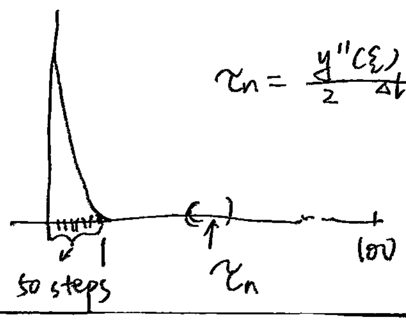
$\Delta t$ is limited : $\lambda\Delta t \ge -2$

$-100\Delta t \ge -2$

$\Delta t \le \frac{1}{50}$

Need 5000 steps.

But

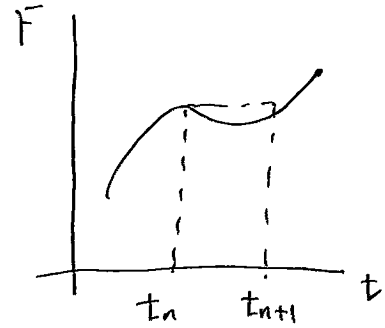


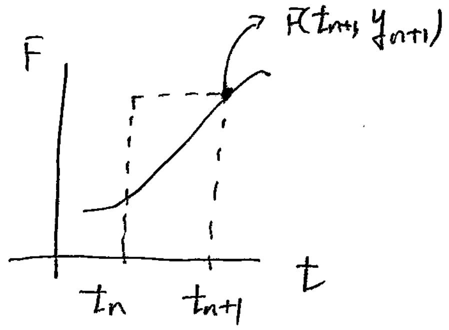

$\tau_n = \frac{y''(\xi)}{2}\Delta t^2$ on $[t_n, t_{n+1}]$

→ very inefficient.

# 2.7.2 Implicit Methods

Forward Euler :




altimately (alternately) choose right value

$$\left.\begin{aligned} y_{n+1} &= y_n + \int_{t_n}^{t_{n+1}} F(t,y)\,dt + \tau_n \\ y_{n+1} &= y_n + \Delta t F(t_{n+1}, y_{n+1}) \end{aligned}\right\}$$

Backwark Euler : Example of Implicit method where

$$y_{n+1} = \underset{?}{\underbrace{G(y_{n+1})}}$$

Forward Euler : Example of Explicit method.

$$y_{n+1} = \hat{G}(y_n)$$

write as.

$$\mathcal{G}(y) = y - G(y) = 0$$

ie. A (non-linear) root-findings problem.

$$y_{n+1} = y_n + \Delta t F(t_{n+1}, y_{n+1})$$

$$\underbrace{y_{n+1} - y_n - \Delta t F(t_{n+1}, y_{n+1}) = 0}_{\text{rootfinding}}$$

A common approx. is 1 step of Newton-Raphson

$$F(t_{n+1}, y_{n+1}) = F(t_{n+1}, y_n) + \cancel{\Delta t} \cdot \underbrace{\frac{\partial F}{\partial y}}_{J} \underbrace{(y_{n+1} - y_n)}_{\Delta y_n} + O(\Delta t^2)$$

$$y_{n+1} - y_n = \Delta y_n$$

$$\Delta y_n - \Delta t \left( F(t_{n+1}, y_n) + \cancel{\Delta t} \cdot J \cdot \Delta y_n \right) = 0$$

$$(I - \Delta t J) \Delta y_n = \Delta t F(t_{n+1}, y_n)$$

$\uparrow$
solve for & update $y_{n+1} = y_n + \Delta y_n$

why do the extra work?

(error formula for interpolation

formula for error for interpolant.

3.1.

bring calculator
bring paper.

3.1) $y_{n+1} = y_n + \Delta t\, F(t_n, y_n)$ explicit

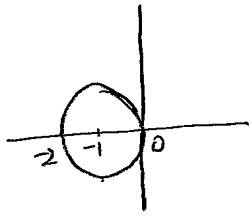
$E = O(\Delta t)$
$\tau = O(\Delta t^2)$

$y_{n+1} = y_n + \Delta t F(t_{n+1}, y_{n+1})$ implicit

$\tau = O(\Delta t^2)$ exercise
$E = O(\Delta t)$

absolute stability.

F. Euler




for Backward Euler

$$y_{n+1} = y_n + \lambda \Delta t\, y_{n+1}$$

$$(1-\lambda\Delta t) y_{n+1} = y_n$$

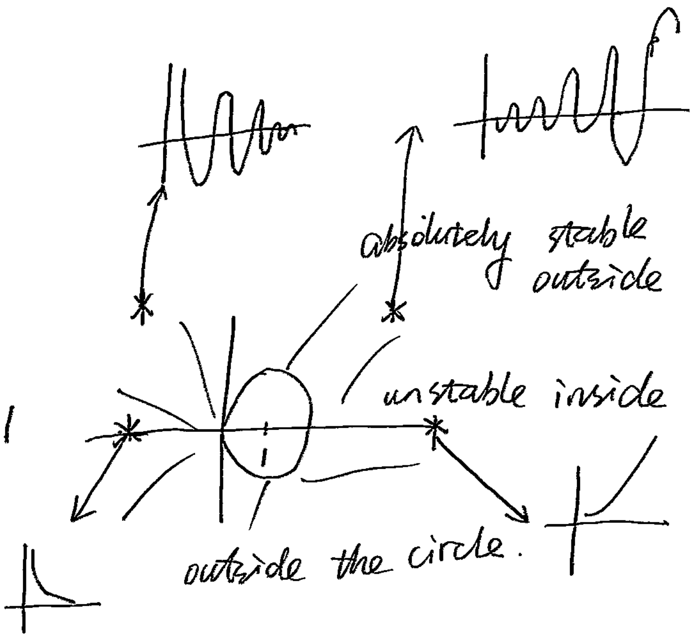
$$y_{n+1} = \frac{y_n}{1-\lambda\Delta t}$$

then $\left|\frac{1}{1-\lambda\Delta t}\right| \le 1$

$x = \mathrm{Re}(\lambda\Delta t)$ real
$y = \mathrm{Im}(\lambda\Delta t)$ imagine

$$\frac{1}{|1-x-iy|^2} \le 1 \Rightarrow \frac{1}{(1-x)^2+y^2} \le 1$$

solution looks like.



Def: A method is A-stable if it's region of absolute stability includes the entire left half of the complex plane.

EX:

Backward Euler is A-stable.

Another implicit method is the trapizoidal rule.

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} F(t, y_t)\, dt$$

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left( F(t_n, y_n) + F(t_{n+1}, y_{n+1})\right)$$

$\Updownarrow$

EX (1) second order method

note: $\delta^0 y_{n+\frac{1}{2}} = \frac{1}{2}\left(F(t_n, y_n) + F(t_{n+1}, y_{n+1})\right)$

(2) A-stable

2.7.3 Higher Order Methods and Runge Kutta

Forward & Backward Euler are the $1^{st}$ order

$$E = O(\Delta t)$$

$$\Rightarrow E_{rel} = O(\Delta t)$$

one significant figures requires 10x more work.

Need higher order, ie. match the Taylor series for solution to higher order

$$y_{n+1} = \underbrace{\underbrace{y_n + \Delta t \overset{F(t_n, y_n)}{F_n}}_{1^{st}\text{ order}} + \frac{\Delta t^2}{2} F_n'}_{2^{nd}\text{ order}} + \cdots$$

$$y_{n+1} = y_n + \Delta t F_n + \frac{\Delta t^2}{2} F_n' \quad \rightarrow \text{ } 2^{nd} \text{ order Tayler Method.}$$

instead, we approximate the higher derivatives

eg. $F_n' = \delta^- F_n$

$$y_{n+1} = y_n + \Delta t F_n + \frac{\Delta t^2}{2} \delta^- F_n$$

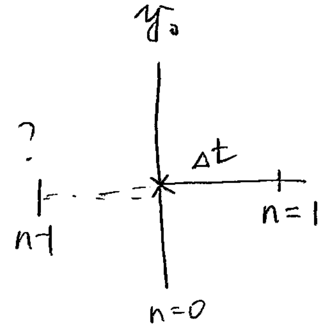
$$y_{n+1} = y_n + \Delta t \left(F_n + \frac{1}{2}(F_n - F_{n-1})\right)$$

$$\boxed{y_{n+1} = y_n + \Delta t \left(\frac{3}{2} F_n - \frac{1}{2} F_{n-1}\right)}$$

$2^{nd}$ order
adams-bashforth

requires step $\underbrace{n \text{ 和 } n-1}_{2 \text{ steps}}$ to get $n+1$

Example of a multistep method.
↓
but it's not self starting




Runge-Kutta Methods Approximate

The derivations using the F values only in $[t_n, t_{n+1}]$.

General formula: (explicit)

$$y_{n+1} = y_n + \Delta t \, \Psi(t_n, y_n, \Delta t)$$

where $\Psi(t_n, y_n, \Delta t) = \sum_{j=0}^{s-1} \gamma_j k_j$

and $k_0 = F(t_n, y_n)$

$$k_j = F\left(t_n + \eta_j \Delta t, y_n + \Delta t \sum_{i=1}^{j-1} \alpha_{j,i} k_i\right) = F\left(t_n + \eta_j \Delta t, \ y_n + \Delta t \sum_{i=1}^{j-1} \alpha_{j,i} k_i\right)$$

EX: $s=1$, $\gamma_0=1$

$y_{n+1} = y_n + \Delta t F(t_n, y_n)$ i.e. forward Euler

3.27

$$y_{n+1} = y_n + \Delta t \cdot \Psi(t_n, y_n, \Delta t)$$

$$\Psi(t_n, y_n, \Delta t) = \sum_{j=0}^{s-1} \gamma_j k_j$$

$$k_0 = F(t_n, y_n)$$

$$k_j = F(t_n + n_j \cdot \Delta t,\ y_n + \Delta t \sum_{i=1}^{j-1} \alpha_j \cdot k_i)$$

Euler $\quad \Psi = F(t_n, y_n)$

$s=1$
$n_0=0$
$\alpha_0=0$

<u>second order RK</u> $(s=2)$

$$\Psi_n = \gamma_0 k_0 + \gamma_1 k_1$$

$$k_1 = F(t_n + n \cdot \Delta t,\ y_n + \Delta t \cdot \alpha \cdot k_0)$$

i.e. $y_{n+1} = y_n + \Delta t \{ \gamma_0 F(t_n, y_n) + \gamma_1 F( t_n + n\Delta t, \alpha \Delta t\, F(t_n, y_n)\}$

find $\gamma_0$ & $\gamma_1$, so that $\tau_n = O(\Delta t^3)$

exact: $\quad y_{n+1} = y_n + \Delta t F_n + \frac{\Delta t^2}{2} F_n' + O(\Delta t^3) \qquad F' = \frac{dF(t, y(t))}{dt}$

$$= y_n + \Delta t F_n + \frac{\Delta t^2}{2}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} \cdot F\right) \qquad \frac{dy}{dt} = F$$

$$y_{n+1} = y_n + \Delta t \left\{ \gamma_0 F_n + \gamma_1 \left( F_n + n \Delta t \frac{\partial F_n}{\partial t} + \alpha \Delta t \cdot \frac{\partial F_n}{\partial y} * F_n + O(\Delta t^2) \right) \right\}$$

$$+ \tau$$

$$= y_n + (\gamma_1 + \gamma_0) \Delta t F_n + \Delta t^2 \left( \frac{\partial F_n}{\partial t} \cdot n \gamma_1 + \gamma_1 \alpha \cdot \frac{\partial F_n}{\partial y} \right) + \tau + O(\Delta t^3)$$

match
$$\begin{cases} \gamma_0 + \gamma_1 = 1 \\ \gamma_1 n = \frac{1}{2} \\ \gamma_1 \alpha = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} \gamma_0 = 1 - \gamma_1 \\ n = \frac{1}{2\gamma_1} \\ \alpha = \frac{1}{2\gamma_1} \end{cases}$$

some old favorites

(1) $\gamma_1 = 1$, modified Euler
$\gamma_0 = 0$,
$n = \alpha = \frac{1}{2}$

$$y_{n+1} = y_n + \Delta t F\left(t_n + \frac{\Delta t}{2}, \; y_n + \frac{\Delta t}{2} F_n\right)$$

(2) Heun $\quad \gamma_1 = \frac{1}{2} \Rightarrow \gamma_0 = \frac{1}{2}, \; \alpha = n = 1$

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left( F(t_n + \Delta t, \; y_n + \Delta t F_n) + F_n \right)$$

$K_0 = F(t_n, y_n)$

$K_1 = F(t_n + \Delta t, y_n + \Delta t K_0)$

$\Phi_n = \frac{1}{2} K_0 + \frac{1}{2} K_1$

3.24

second order RK

$$\begin{cases} k_0 = F(t_n, y_n) \\ k_1 = F(t_n + \frac{\Delta t}{2\gamma_1}, y_n + \frac{\Delta t}{2\gamma_1} k_0) \\ \Phi_n = (1-\gamma_1)k_0 + \gamma_1 k_1 \\ y_{n+1} = y_n + \Delta t\, \Phi_n \end{cases}$$

$$\tau_n = O(\Delta t^3)$$

$$E = O(\Delta t^2)$$

classic 4th order RK

$$k_0 = F(t_n, y_n)$$

$$k_1 = F(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_0)$$

$$k_2 = F(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_1)$$

$$k_3 = F(t_n + \frac{\Delta t}{1}, y_n + \Delta t\, k_2)$$

$$y_{n+1} = y_n + \frac{\Delta t}{6}(k_0 + 2k_1 + 2k_2 + k_3)$$

$y' = F(t)$ Solve $\begin{cases} y' = F(t, y) & t \in (0, T) \\ y(0) = y_0 \end{cases}$

$$N = \frac{T}{\Delta t}$$

procedure integrate

$$N = \frac{T}{\Delta t}$$

$y$ = initial value

$N = \frac{T}{\Delta t}$

$y$ = initial value

for $n=0$ to $N-1$

  $t = n \cdot \Delta t$

  $y$ = take one step $(t, y, F, \Delta t)$

next $n$  prime ?

end procedure

take one step $(t, y, F, \Delta t)$  (HEUN)

  $k_0 = F(t, y)$

  $k_1 = F(t+\Delta t, y + k_0 \Delta t)$

  $y = y + \frac{\Delta t}{2}(k_0 + k_1)$

return $y$

end

Error Estimate.

N steps w/ $\Delta t$

$$y_N(\Delta t) = y(T) + O(\Delta t^r)$$

redo

$$y_{2N}\left(\frac{\Delta t}{2}\right) = y(T) + O\left(\left(\frac{\Delta t}{2}\right)^r\right)$$

Assume: $E_{\Delta t} \approx C \Delta t^r$

$$\begin{cases} y_N(\Delta t) = y(T) + C\Delta t^r \\ y_{2N}\left(\frac{\Delta t}{2}\right) = y(T) + C\left(\frac{\Delta t}{2}\right)^r \end{cases}$$

$$y_N - y_{2N} \approx \left(1 - \frac{1}{2^r}\right)(C\Delta t^r)$$

$$E_{\Delta t} \approx \frac{y_N - y_{2N}}{2^r - 1} \cdot 2^r$$

$$E_{\frac{\Delta t}{2}} \approx \frac{y_N - y_{2N}}{2^r}$$

# Chapter 3: Introduction to Option Pricing.

Simplest options are of value a european type. Two types.

(1) European Call

give the buyer the right but not the obligation to buy an asset $S$, at time $T$ for price $K$.

Payoff: $C(T) = (S(T) - K)^+ = \max(S(T) - K, 0)$

(2) European Put

sell $S$ at $T$ for price $K$.

payoff: $P(T) = \max(K - S(T), 0)$

the American version

can exercise at any time up to $T$.

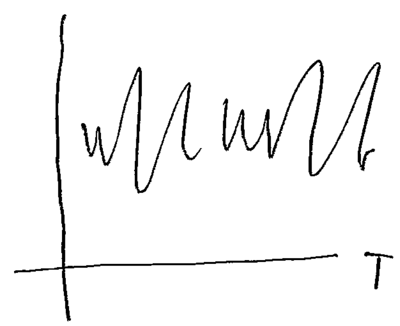
Want to know how to price, ie. value of option today.

American adds: when to exercise.

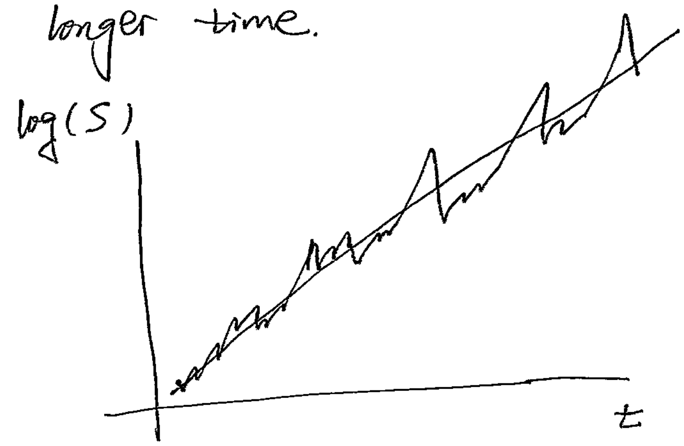
## 3.1 The Black-Scholes Model

Problem is we don't know $s(t)$.

but we know that over short time :




over longer time.




exponential + random
drift

without random, $\frac{ds}{s} = u(t)dt$
↑ drift rate

if $u$ is constant, $s(t) = e^{ut}$

B.S. Model.

$$\frac{ds}{s} = \underset{\substack{\uparrow \\ \text{constant}}}{u}dt + \underset{\substack{\uparrow \\ \text{volatility} = \text{const}}}{\sigma} dz$$

$dz$ is an increment of wiener process.

Def: A scalar standard Brownian Motion AKA a standard wiener process over $[0, T]$ is a random variables $W(t)$ that depends continuously on $t \in [0,T]$ and satisfies

1. $W(0)=0$ w/probability 1
2. for $0 \leq s \leq t \leq T$

   $W(t)-W(s)$ is normally distributed.

   w/ mean 0 & variance $t-s$

   $W(t)-W(s) \sim \sqrt{t-s}\, N(0,1)$

3. $0 \leq s < t < u \leq v \leq T$

   $W(t)-W(s)$ and $W(u)-W(v)$ are independent.

property # 2. $\Rightarrow dz = \sqrt{dt}\, \phi$

$\frac{ds}{s} = \mu dt + \sigma \sqrt{dt}\, \phi$ is a stochastic D.E.

Lets find $E[\frac{1}{s}\frac{ds}{dt}]$ and $Var[\frac{1}{s}\cdot\frac{ds}{dt}]$