STA 4103/5107 Computational Methods in Statistics II

Department of Statistics
Florida State University

Class 12
February 16, 2017

Announcement

- **Midterm Project: (Out: Thursday, 2/23)**
- **Project Report: (Due: Friday, 3/10, by noon)**
- **Project Presentation: (Tuesday 3/7 and Thursday 3/9)**
  - Required only for PhD students in Statistics.
  - Presentation Style: Slide presentation (PPT, PDF, etc).
- **Independent Topic is allowed if any of the following methods is used:**

Kalman filter

Sequential Monte Carlo

Poisson Process

Midterm Presentation Schedule (temp)

Tuesday (03/07)	Thursday (03/09)
1. Rene, Lexi	1. Al Amer, Fahad
2. Seeger, Travis	2. Chen, Yang
3. Shamp, Wright	3. Griffith, Marie
4. Shen, Jiahui	4. Hu, Guanyu
5. Steppi, Albert	5. Lee, Hwiyoung
6. Tang, Shao	6. Lee, In Koo
7. Um, Seungha	7. Li, Donghang
8. Wang, Xianbin	8. Lim, Jaehui
9. Wang, Yunfang	9. Liu, Sida
10. Xu, Zhixing	10. Qi, Kai

Email me soon if you want to make a presentation. The schedule will be finalized next Thursday.

Review: Kalman Filter Model

Definition:

System Equation:

$$\mathrm{x}_k = \mathrm{A}_k \mathrm{x}_{k-1} + \mathrm{w}_k, \qquad \mathrm{w}_k \in N(0, \mathrm{W}_k)$$

$$k = 2, 3, \cdots$$

Measurement Equation:

$$\mathrm{y}_k = \mathrm{H}_k \mathrm{x}_k + \mathrm{q}_k, \qquad \mathrm{q}_k \in N(0, \mathrm{Q}_k)$$

$$k = 1, 2, \cdots$$

$$\mathrm{A}_k \in \mathrm{R}^{d \times d}, \ \mathrm{w}_k \sim N(0, \mathrm{W}_k), \ \mathrm{W}_k \in \mathrm{R}^{d \times d}, k = 2, 3, \cdots M.$$

$$\mathrm{H}_k \in \mathrm{R}^{c \times d}, \ \mathrm{q}_k \sim N(0, \mathrm{Q}_k), \ \mathrm{Q}_k \in \mathrm{R}^{c \times c}, \ k = 1, 2, \cdots, M.$$

Review: Learning Kalman Model

- In practice, the parameters in the model need to be estimated from training data. (In training data, we know both hidden states and measurements.)
- Common simplification: $\mathrm{A}_k, \mathrm{H}_k, \mathrm{W}_k, \mathrm{Q}_k$ are constant over time (independent of k).
- The A, H, W, Q can be estimated by maximizing the joint probability $p(\mathrm{X}_M, Y_M)$. That is,

$$\{A, W, H, Q\} = \arg\max_{A,W,H,Q} p(\mathrm{X}_M, \mathrm{Y}_M)$$

Review: Closed-form Solutions

$$\mathrm{A} = \left(\sum_{k=2}^{M} \mathrm{x}_k \mathrm{x}_{k-1}^T\right)\left(\sum_{k=2}^{M} \mathrm{x}_{k-1} \mathrm{x}_{k-1}^T\right)^{-1},$$

$$\mathrm{W} = \frac{1}{M-1}\left(\sum_{k=2}^{M} \mathrm{x}_k \mathrm{x}_k^T - \mathrm{A}\sum_{k=2}^{M} \mathrm{x}_{k-1} \mathrm{x}_k^T\right),$$

$$\mathrm{H} = \left(\sum_{k=1}^{M} \mathrm{y}_k \mathrm{x}_k^T\right)\left(\sum_{k=1}^{M} \mathrm{x}_k \mathrm{x}_k^T\right)^{-1},$$

$$\mathrm{Q} = \frac{1}{M}\left(\sum_{k=1}^{M} \mathrm{y}_k \mathrm{y}_k^T - \mathrm{H}\sum_{k=1}^{M} \mathrm{x}_k \mathrm{y}_k^T\right).$$

Review: Recursive Estimation

$$p(\mathrm{x}_k|\mathrm{Y}_k) = \kappa\, p(\mathrm{y}_k|\mathrm{x}_k)\int p(\mathrm{x}_k|\mathrm{x}_{k-1})p(\mathrm{x}_{k-1}|\mathrm{Y}_{k-1})d\mathrm{x}_{k-1}$$

Time update:

posterior at previous step:

$$p(\mathrm{x}_{k-1}|\mathrm{Y}_{k-1})$$

temporal prior:

$$p(\mathrm{x}_k|\mathrm{x}_{k-1})$$

prior distribution:

$$p(\mathrm{x}_k|\mathrm{Y}_{k-1}) = \int p(\mathrm{x}_k|\mathrm{x}_{k-1})p(\mathrm{x}_{k-1}|\mathrm{Y}_{k-1})d\mathrm{x}_{k-1}$$

Measurement update:

prior distribution:

$$p(\mathrm{x}_k|\mathrm{Y}_{k-1})$$

likelihood:

$$p(\mathrm{y}_k|\mathrm{x}_k)$$

posterior distribution:

$$p(\mathrm{x}_k|\mathrm{Y}_k) = \kappa\, p(\mathrm{y}_k|\mathrm{x}_k)p(\mathrm{x}_k|\mathrm{Y}_{k-1})$$

Review: Kalman Filter Algorithm

Time Update

Prior estimate

$$\hat{x}_k^- = A_k \hat{x}_{k-1}$$

Error covariance

$$P_k^- = A_k P_{k-1} A_k^T + W_k$$

Measurement Update

Posterior estimate

$$\hat{x}_k = \hat{x}_k^- + K_k (y_k - H_k \hat{x}_k^-)$$

Error covariance

$$P_k = (I - K_k H_k) P_k^-$$

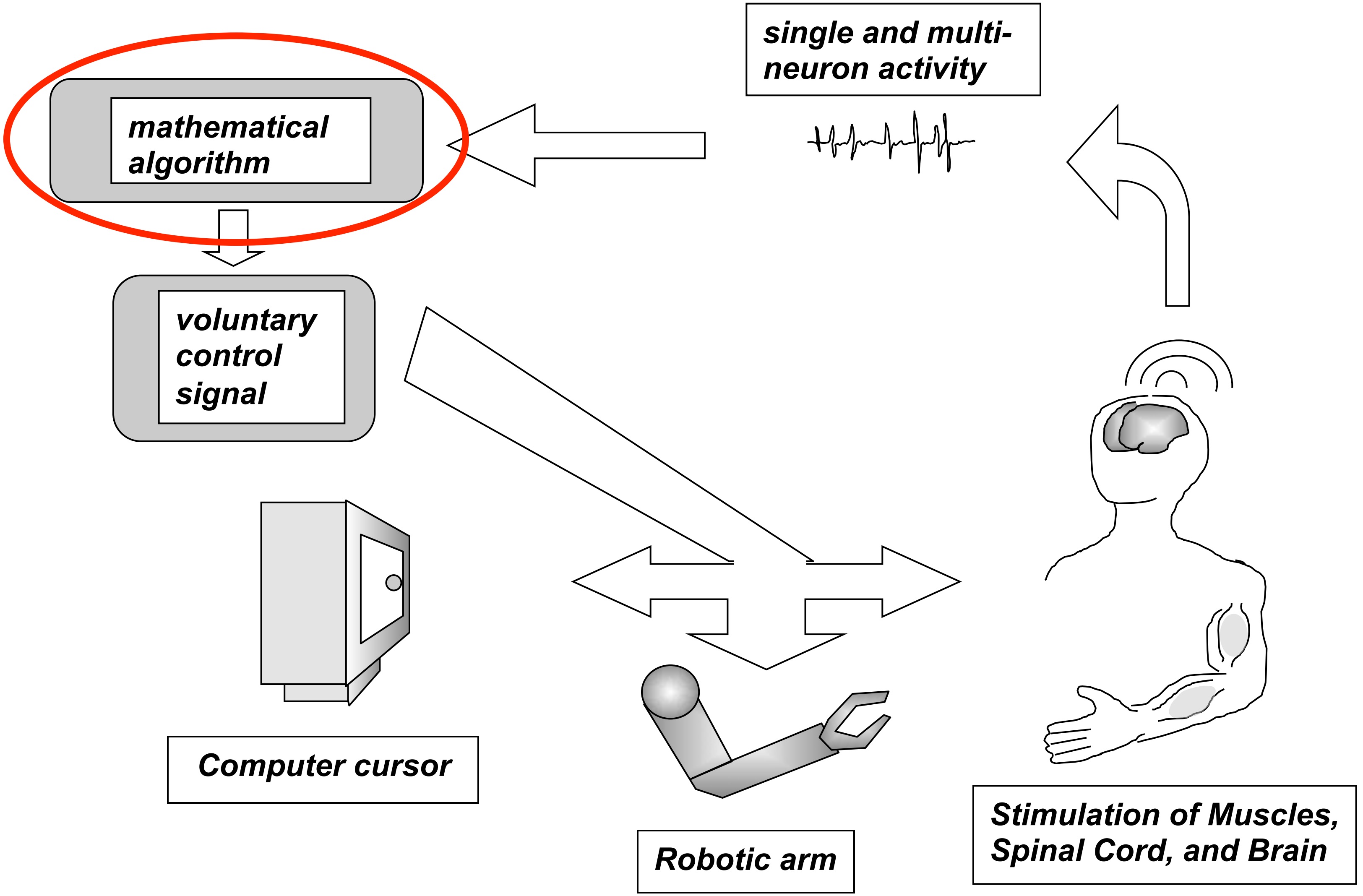
Kalman gain

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + Q_k)^{-1}$$

previous estimate of $\hat{x}_{k-1}$ and P_{k-1}

Welch & Bishop, *An Introduction to the Kalman Filter*, 2006

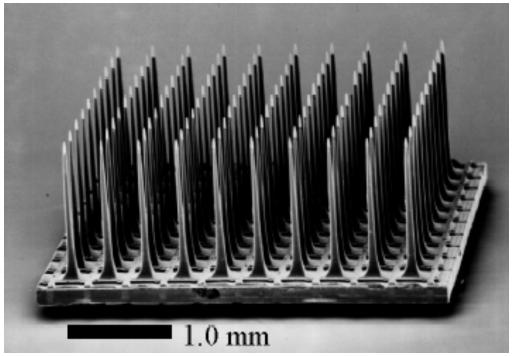
Application in Brain-Machine Interfaces

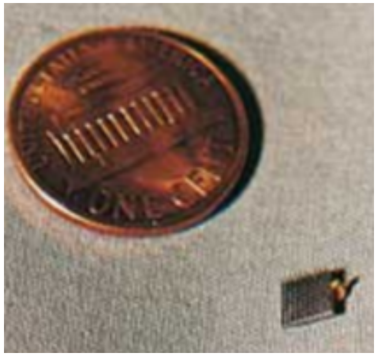


Key Questions

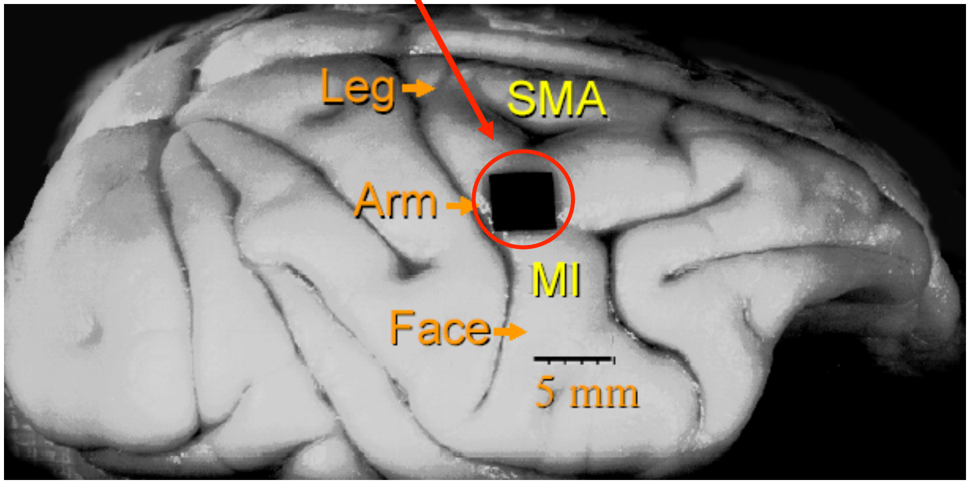
- **Encoding:** How to model the information in the brain? **(hand movement → neural signals)**
- **Decoding:** What mathematical algorithms can we use to predict behavior from neural activity?
  (neural signals → hand movement)

Spike Recording from Monkeys

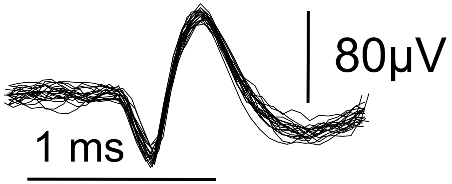




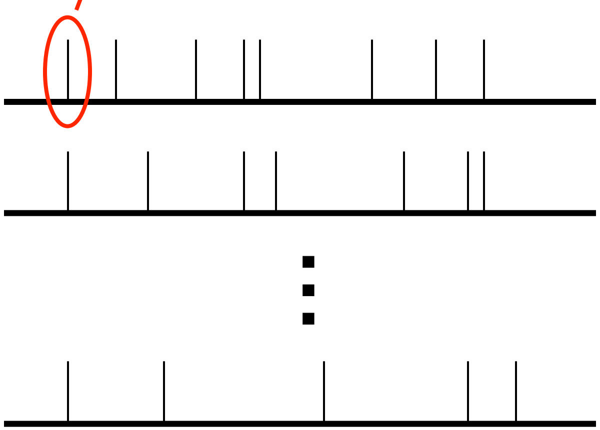
Utah Array (100 electrodes, 4x4mm)



MI: Primary Motor Cortex



spike wave form

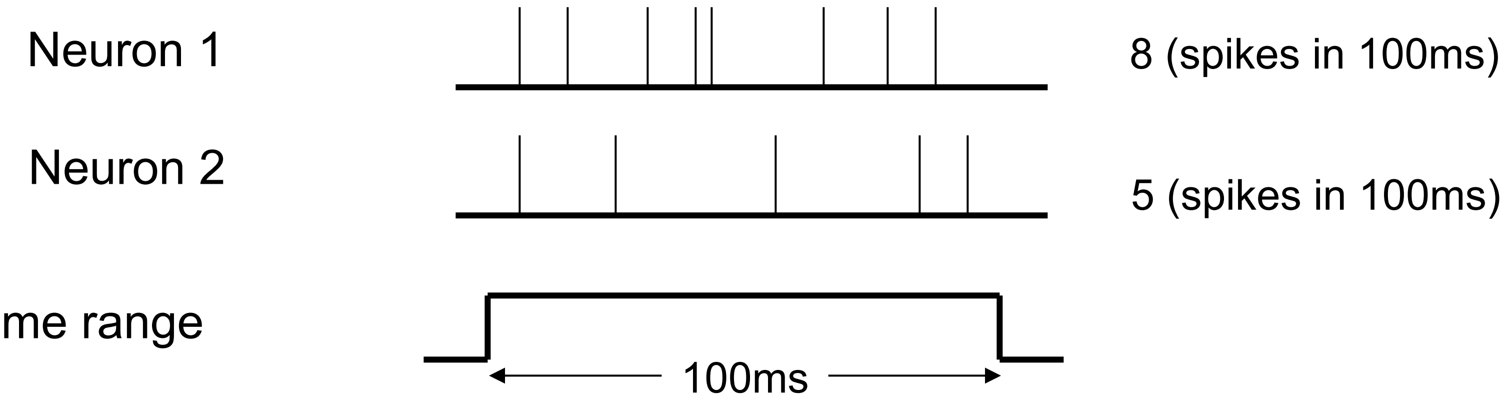


spike trains

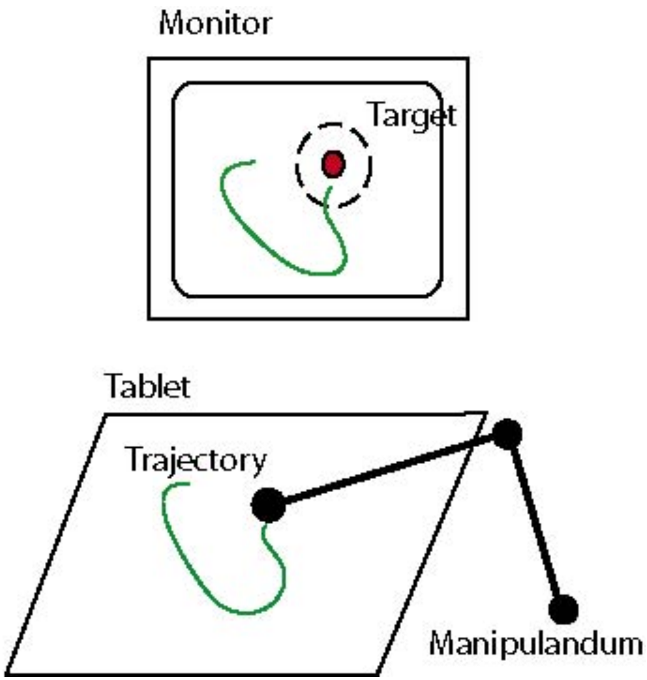
Rate Code

Rate code: models the *firing rate* of the spike train, where

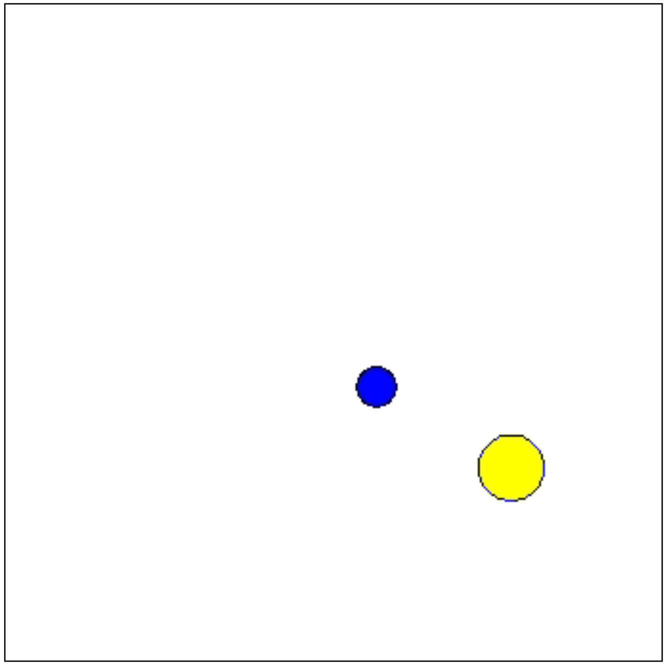
$$\text{firing rate} = \text{number of spikes within certain time range.}$$



"Pinball" Task



Tablet: 30cm x 30cm

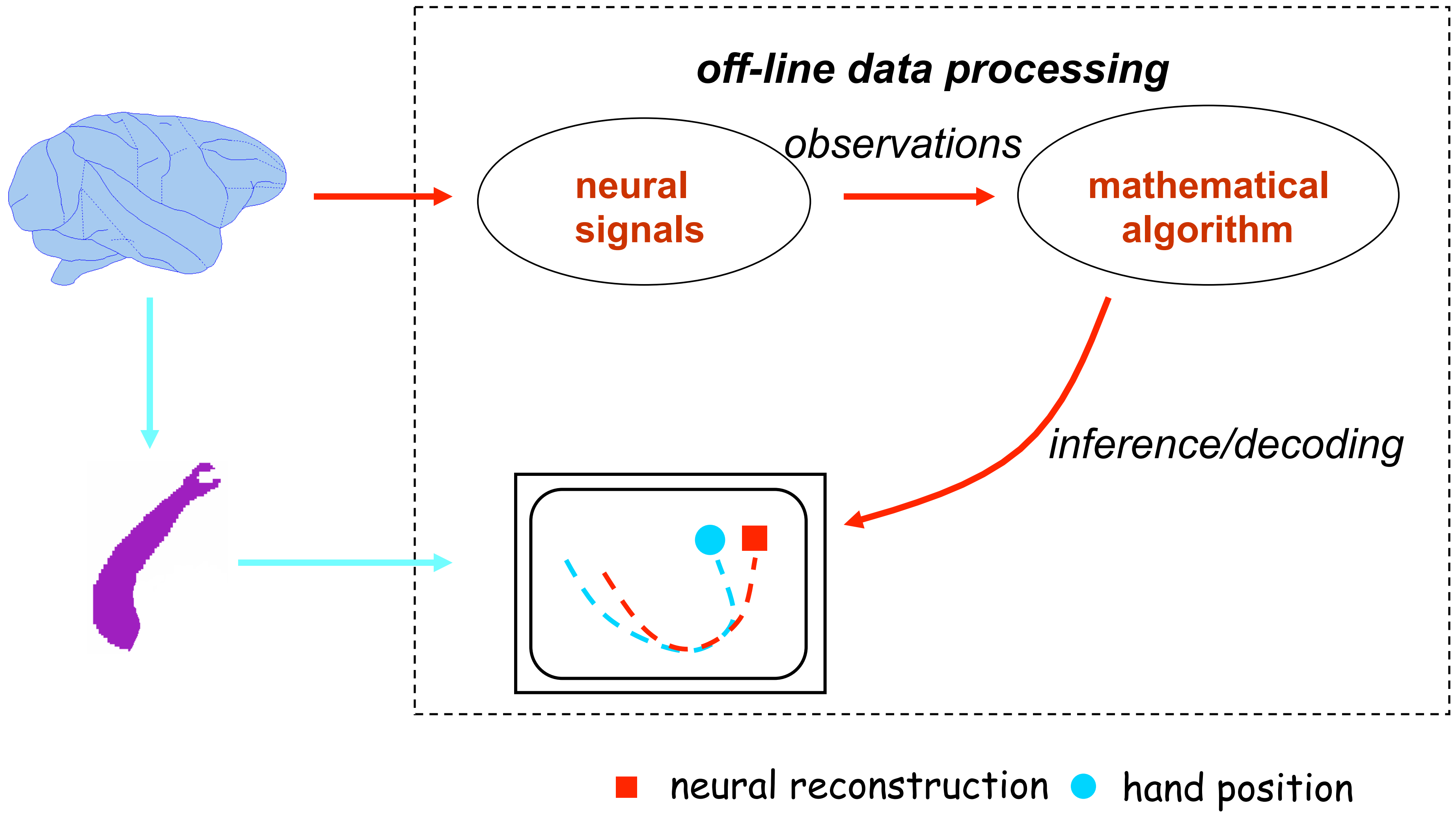


Target   Hand position

Time Series Data

- Position (Velocity, Acceleration)
- Firing rate (42 neurons, non-overlapping 70*ms* bins)

Experimental Paradigm



Kalman Filter Model

Measurement Equation:
(encoding summary, likelihood model)

system state vector $\begin{pmatrix} x_k \\ y_k \\ v_{x_k} \\ v_{y_k} \end{pmatrix}$

firing rate vector $\begin{pmatrix} z_k^1 \\ z_k^2 \\ \vdots \\ z_k^C \end{pmatrix}$

$$\vec{z}_k = H \vec{x}_k + \vec{q}_k \qquad \vec{q}_k \in N(0, Q)$$

C X C matrix

C X d matrix

System Equation:
(prior model)

$$\vec{x}_k = A \vec{x}_{k-1} + \vec{w}_k \qquad \vec{w}_k \in N(0, W)$$

d X d matrix

d X d matrix

Reconstruction on Test Data