STA 4103/5107 Computational Methods in Statistics II

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Class 2
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Review: Stochastic Process

- **Definition 15** A **stochastic process** is an indexed collection of random variables.
- Let $X = \{X_t\}$ be a stochastic process. For a fixed time t, X_t is just a real-valued random variable.
- In case the indexing variable t is continuous, the stochastic process called a **continuous-time process**; else, it is called a **discrete-time process**.
- Similarly, if the space in which X_t takes values is continuous, then the process is called a **continuous-valued process**. Otherwise, it is called a **discrete-valued process**.

Review: Wiener Process

- A **random walk** is one of the simplest examples of a discrete time, discrete state stochastic process.

- At every time nT (T: unit time), for $n = 1, 2, \ldots$, toss a coin and depending on the outcome, add $+s$ for a head or $-s$ for a tail to the current value of the process.

- When the number of heads in n tosses is k, the process value is
$$X(nT) = ks - (n-k)s = (2k - n)s$$

- Another interpretation of $X(nT)$ is given by the following:
$$X(nT) = X_1 + X_2 + \ldots + X_n,$$
where X_is are i.i.d random variables with $E[X_i] = 0$, $var(X_i) = s^2$.

Review: Limiting Situation

- Letting $\boldsymbol{s} = \boldsymbol{\alpha T}^{1/2}$ for some $\alpha > 0$. When $T \rightarrow 0$, the random walk becomes a continuous time, continuous state process.
- This limiting process is called the **Wiener process (**also called **Brownian motion)**.
- **(Formal Definition) Wiener Process** is a continuous-time stochastic process $X(t)$ for $t \geq 0$ with $X(0) = 0$ and such that
  - the increment $X(t) - X(s)$ is Gaussian with mean 0 and variance $\alpha^2(t - s)$ for any $0 \leq s < t$.
  - increments for non-overlapping time intervals are independent.
- For $\boldsymbol{\alpha} = \mathbf{1}$, it is called the **standard Wiener process**.

Example

- **Simulate a random walk**

  For a given value of T and $s = \alpha T^{1/2}$, plot the sample paths of the X_t for $\alpha = 1.0$ and $T = 1, 0.1, 0.01$, and 0.001.

  Choose the total number of steps n to be $10/T$.

  So the total time is constant 10.

Gaussian Process

- A **Gaussian process** $X(t)$ is a collection of random variables, any finite number of which have a joint Gaussian distribution. That is, for any finite indices $t_1, \ldots, t_k$,

$$(X(t_1), \ \ldots, X(t_k))$$

is a multivariate Gaussian random variable.

- We define mean function $m(t)$ and the covariance function $k(t, s)$ of a real Gaussian process $X(t)$ as

$$m(t) = E[X(t)]$$
$$k(t,s) = E[(X(t) - m(t))(X(s) - m(s))],$$

and will write the Gaussian process as

$$X(t) \sim \mathrm{GP}(m(t), k(t, s)).$$

Stationarity

- A key fact of Gaussian processes is that they can be completely defined by their first and second-order statistics.

- If a Gaussian process is assumed to have mean zero, defining the covariance function completely defines the process' behavior.

- **Stationarity** refers to the process's behavior regarding the separation of any two points t and s.

- If the process is **stationary**, it depends on their separation, $t - s$, while if **non-stationary** it depends on the actual position of the points t and s.

Relation to Wiener Process

- **The standard Wiener process is a Gaussian process that has mean $m(t) = 0$ and a non-stationary covariance function $k(t, s) = \min(t, s)$.**

- A standard Wiener process is also the integral of a white noise Gaussian process. That is,

$$X(t) = \int_0^t w(s)\,ds$$

where $w(s)$ is a Gaussian Process with zero mean, delta covariance (i.e. $\mathrm{cov}(w(t), w(s)) = 1_{t=s}\,\infty$).

- More detail on Gaussian Process can be found at the website:

http://www.gaussianprocess.org

One-Dimensional Poisson Process

- Assume that we are interested in the occurrences of an event, and that event occurs at random times.

- This event, for example, can be an automobile accident at a particular crossing, or ringing of a particular telephone.

- We are interested in recording and modeling the times of occurrences, also called the **arrival times**. The time lapses between the arrival times are called **inter-arrival times**.

- Denote the arrival times by $\{t_i\}$ and the inter-arrival times by $\{\tau_i\}$. They are related according to:

$$t_i = t_{i-1} + \tau_{i-1} \ .$$

Homogeneous Poisson Process

- In case of a homogeneous Poisson process with intensity λ, these τ_is are assumed to be independent and identically distributed according to the **exponential** density with intensity λ, i.e.

$$\tau_i \sim \lambda \exp(-\lambda\tau) \qquad \text{(note: } E\tau_i = 1/\lambda\text{)}$$

- The resulting Poisson counting process can be defined as follows:

$$N(t) = \sum_{i=1}^{\infty} 1_{[0,t)}(t_i), \quad N(0) = 0.$$

$N(t)$ counts the number of arrivals, or the occurrences, till time t.

- It can be shown that on any interval (s_1, s_2), the number of arrival times, i.e. $N(s_2) - N(s_1)$, is given by a Poisson random variable with mean $\lambda(s_2 - s_1)$.

Properties

- Properties of a homogeneous Poisson process:

  1. If (s_1, s_2) and (s_3, s_4) are non-overlapping intervals, then $N(s_2)-N(s_1)$ and $N(s_4)-N(s_3)$ are statistically independent random variables.

  2. The incremental count, given by the random variable $N(s_2)-N(s_1)$, is independent of the process value at s_1 or any other previous time.

  3. For any two times $s_1 < s_2$, we have:

$$
\begin{aligned}
E[N(s_1)N(s_2)] &= E[N(s_1)(N(s_2)-N(s_1)+N(s_1))] \\
&= E[N(s_1)^2]+E[N(s_1)(N(s_2)-N(s_1))] \\
&= \lambda s_1 + \lambda^2 s_1^2 + \lambda s_1 \lambda (s_2 - s_1) = \lambda s_1 (1+\lambda s_2)
\end{aligned}
$$