Computational Finance Homework 3 Rootfinding

Name: Jian Wang **FSUID: Jw09r**

0. Executive Summary

Use the Bisect and Newton method and combine those two methods together to find the effective annual interest rate if $1000 per month, compounded monthly and invested over a 15 year period, grows to $1 million at the end of t that period.

My result is that the compounded interest rate is 20.692869% and the speed of the Newton method convergence is faster than Bisection and is nearly equal to $\dfrac{\log\left|\dfrac{e_{n+1}}{p}\right|}{\log\left|\dfrac{e_n}{p}\right|} \approx 2$

I. Statement of Problem

The accumulated savings in an account into which regular periodic payments are made is given by:

$$A = \frac{P}{r}[(1 + r)^n - 1]$$

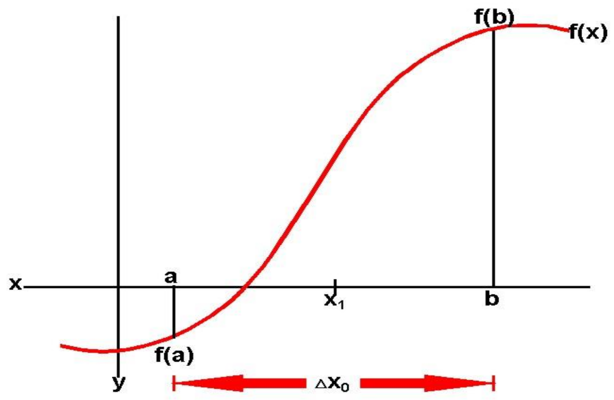
Where A is the amount in the account. P is the deposit amount, and r is the interest rate per period. The main purpose for me is to find the effective annual interest rate. I firstly use the bisect method to find the root in one digits and then use the Newton method to find the result in the **IEEE single precision machine accuracy**.

II. Description of The Mathematics

1. Bisection method:

The bisection method requires two initial points a and b such that f(a) and f(b) have opposite signs.

Firstly, chose two ending points that the function values at these two points have opposite signs.
Secondly, divided the interval in two by computing the midpoint p = (a+b) / 2 of the interval.
Thirdly, if F(a)*F(P) >0 then a=p else b=p
We can see the chart as follows:



The properties of the convergence is as follows:

We can see that

$$p_n = a_n + \frac{b_n - a_n}{2}$$

$$|p_n - p| \le \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^{n+1}} \Rightarrow \lim_{n\to\infty} |p_n - p| = 0$$

When $|P_n| \approx 0$, then we have Abs error test

$|P_n| \gg 0$ then we have the Rel error test.

The Tol should be:

Rel Tol $\le$ $5 \times 10^{-(m+1)}$

Abs Tol $\approx$ smallest distance. Which is equal to the 10^{-38} for the single perception

In addition, since $|p_n - p| \le |p_n - p_{n-1}| = \frac{b_n - a_n}{2}$, we may choose $|p_n - p_{n-1}|$ as the upper bound of error. So the final criterion to check if it is converge is:

$$|p_n - p_{n-1}| \le absTol + relTol * |p_n|$$

At last, we may calculate the maximum round number when it converges given two starting points, absolute tolerance and relative tolerance.

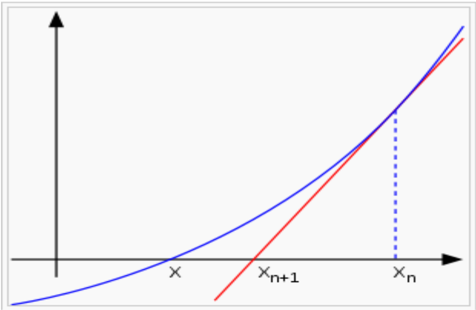
$$|p_n - p| \le \frac{b-a}{2^{n+1}} \le absTol + relTol * |p_n| \Rightarrow N \le \log_2 \frac{b-a}{absTol + \min(|a|,|b|) * relTol} - 1$$

2. Newton's method:

Find roots of successive linear approximations of F start with the initial point P_0

follow the irritation until it converges. $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$

It can be seen clearly from the following chart:



From the Taylor expansion:

$f(x) = f(p) + f'(p)(p-x) + \frac{f''(\xi)}{2}(p-x)^2$, where $f(p)$ =p, $f'(p)$ =0, choose x= p_{n+1}, then: $f(p_{n+1}) = p + \frac{f''(\xi)}{2}(p-p_{n+1})^2$.

$$e_n = p_n - p$$
$$e_{n+1} = p_{n+1} - p$$
$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = \lim_{n\to\infty} \frac{f''(\xi)}{2} = \frac{f''(p)}{2}$$

$$\log|e_{n+1}| \approx 2\log|e_n| + \log(\frac{f''(p)}{2})$$
$$\Rightarrow \log\left|\frac{e_{n+1}}{p}\right| \approx 2 * \log\left|\frac{e_n}{p}\right| + \log(\frac{f''(p)}{2})$$

We can see that Newton's method can converge more quickly than the Bisection method.

Besides, we can also use the criterion just like in the bisection method:

$$|p_n - p| \leq | p_n - p_{n-1} | = \Delta = absTol + relTol * |p_n|$$

III. Description of the Algorithm

1. The Algorithm of the Bisection Method:

```
For i = 1 to N Do
P=(a+b)/2
If converged Exit
   If F(a)*F(P)>0 then
      a=P
   Else
      b=P
next i
```

Here the convergence is:

$$|p_n - p| \le \frac{b-a}{2^{n+1}} \le absTol + relTol * |p_n|$$

2. The algorithm of the Newton Method

```
Given P=P0
For i = 1 to N Do
P=P-F(P)/F'(P)
If converged Exit
Next i
```

Here the convergence is as follows:

$$|p_n - p| \le | p_n - p_{n-1} | \le absTol + relTol * |p_n|$$

IV. Results

The Result of the Bisection Method under the one significant digit is as follows:
I choose the result of the Newton method as the true value, which is 0.0157967077420546 to derive the related errors.

The result of the Bisection Method is:

iteration Number	Int Rate	Rel Error
1	0.5005000000000	3.07E+01
2	0.2507500000000	1.49E+01
3	0.1258750000000	6.97E+00
4	0.0634375000000	3.02E+00
5	0.0322187500000	1.04E+00
6	0.0166093750000	5.14E-02
7	0.0088046875000	4.43E-01
8	0.0127070312500	1.96E-01
9	0.0146582031250	7.21E-02
10	0.0156337890625	1.03E-02

Use the result of the Bisection method (0.0156337890625) as the initial value in the Newton method, the result can be seen from the following table:

The Newton Method(Single Perception)

Iteration Number	Int Rate	Rel Error
1	0.0157999202047	2.03E-04
2	0.0157966825922	1.59E-06
3	0.0157967079447	1.28E-08
4	0.0157967077404	1.03E-10

The Newton Method (Double Perception)

Iteration Number	Int Rate	Rel Error
1	0.0157999202046632000	2.03E-04
2	0.0157966825922434000	1.59E-06
3	0.0157967079447340000	1.28E-08
4	0.0157967077404214000	1.03E-10
5	0.0157967077420679000	8.38E-13
6	0.0157967077420546000	2.86E-15

From the above table, we can see that the convergent speed of Newton method is faster than the Bisection Method. Just as I listed above in the description of the mathematics, $\dfrac{\log\left|\dfrac{e_{n+1}}{p}\right|}{\log\left|\dfrac{e_n}{p}\right|} \approx 2$. **My results are in accordance with the prediction, which can be the proof that they are right.**

Finally, using the formula $R = (1+r)^{12} - 1$ to get effective annual return rate: 20.692869%

V. Conclusions

The final result of the annual interest rate is 20.692869% (pretty good return☺) and we can find the speed of the convergence of Newton Method is faster than the Bisection Method.

The speed of the related error of Newton method is $\dfrac{\log\left|\dfrac{e_{n+1}}{p}\right|}{\log\left|\dfrac{e_n}{p}\right|} \approx 2$

VI. Code Listing

1. The method of Bisection

```
Sub Bisection()
Dim p, n, reltol, abstol, f, fb As Double
Dim a As Double
Dim b As Double
Dim int_rate As Double
n = 180
a = 0.001
b = 1
abstol = 10 ^ (-38)
reltol = 0.5 * 10 ^ (-1)
Do
int_rate = (b + a) / 2
fa = Amount(a)
fp = Amount(int_rate)
If fa * fp > 0 Then
a = int_rate
Else
b = int_rate
End If
i = i + 1
Worksheets("sheet2").Cells(i, "a").Value = int_rate
Loop While (b - a) / 2 > abstol + reltol * int_rate
End Sub
```

2. The Method of Newton

```
Sub Newton()
Dim p, n, reltol, abstol, f, fb As Double
Dim a As Double
Dim b As Double
Dim int_rate As Double
Dim condition As Double
int_rate = 0.0156337890625
n = 180
a = 0.001
b = 1
abstol = 10 ^ (-38)
reltol = 0.5 * 10 ^ (-7)
Do
fa = Amount(int_rate)
fp = Amount_derivative(int_rate)
```

```
int_rate = int_rate - fa / fp
i = i + 1
Worksheets("sheet2").Cells(i, "b").Value = int_rate
If fa / fp > 0 Then
condition = fa / fp
Else: condition = -fa / fp
End If
Loop While condition > abstol + reltol * int_rate
End Sub
```

3. The function of the Total Amount

```
Function Amount(x As Double) As Double
Amount = 1000 / x * ((1 + x) ^ 180 - 1) - 1000000
End Function
```

4. The function of the Derivative of the total amount

```
Function Amount_derivative(x As Double) As Double
Amount_derivative = 1000 * (180 * ((1 + x) ^ 179) * x - (1 + x) ^ 180 + 1) / (x ^ 2) - 1000000
End Function
```