STA 6448 Homework 3

Due March 20

20 pts for each problem below (in total, 100pts):

1. Prove the following statements:

a. The variance $\mathrm{Var(Q)} = \mathbb{E}[\mathrm{Q}^2] - (\mathbb{E}[\mathrm{Q}])^2$ of a symmetric random matrix Q is positive semidefinite.

b. Consider the random matrix $Q = gB$, where g is a zero-mean sub-Gaussian random variable with parameter σ^2, and B is a fixed matrix. If g has a symmetric distribution around zero, then Q is sub-Gaussian random matrix with parameter $c^2\sigma^2 B$ for some universal constant c.

2. (Tail bounds for non-symmetric random matrices) Suppose $A_1, \ldots, A_n$ are independent random matrices in $\mathbb{R}^{d_1 \times d_2}$, each satisfies the matrix Bernstein condition with parameter $b > 0$. Let

$$\sigma^2 = \max\Big\{ \Big\|\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[A_i A_i^T]\Big\|_{\mathrm{op}},\ \Big\|\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[A_i^T A_i]\Big\|_{\mathrm{op}} \Big\}.$$

Show:

a. For each i, define a symmetric $(d_1 + d_2) \times (d_1 + d_2)$ matrix via

$$Q_i = \begin{bmatrix} 0_{d_1 \times d_2} & A_i \\ A_i^T & 0_{d_2 \times d_1} \end{bmatrix}.$$

b. We have $\|n^{-1}\sum_{i=1}^{n} \mathrm{Var(Q_i)}\|_{\mathrm{op}} \le \sigma^2$.

c. Conclude that for all $\delta > 0$,

$$\mathbb{P}\Big[\Big\|\sum_{i=1}^{n} A_i\Big\|_{\mathrm{op}} \ge \delta\Big] \le 2\,(d_1 + d_2)\ \exp\Big\{ -\frac{n\delta^2}{2(\sigma^2 + b\delta)} \Big\}.$$

3. (Relation between matrix operator norms) For a matrix $A \in \mathbb{R}^{m\times n}$ and any $q \in [1, \infty]$, the ℓ_q-operator norm is defined by

$$|||A|||_q = \sup_{\|x\|_q=1} \|Ax\|_q.$$

a. Derive explicit expressions for the operator norms when $q = 1,\ 2,\ \infty$ in terms of elements and/or singular values of A.

b. Prove that $|||AB|||_q \leq |||B|||_q |||B|||_q$ for any size-compatible matrices A and B.

c. Show that if $\frac{1}{p} + \frac{1}{q} = 1$, then $|||A|||_p = |||A^T|||_q$.

d. Show that if A is symmetric, then $|||A|||_2 \leq |||A|||_q$ for any $q \in [1, \infty]$.

e. Show that if $\frac{1}{p} + \frac{1}{q} = 1$, then $|||A|||_2^2 \leq |||A|||_p |||A|||_q$.

4. Let G be a graph with maximum degree $s - 1$ that contains a s-clique. Letting A denote its adjacency matrix, show that $|||A|||_{\text{op}} = s - 1$.

5. (Diagonal covariance matrix estimation) Let $\{x_i\}_{i=1}^n$ be i.i.d. sequence of zero-mean d-dim vectors with diagonal covariance matrix $\Sigma = D$. Consider $\widehat{D} = \text{diag}(\widehat{\Sigma})$ where $\widehat{\Sigma}$ is the usual sample covariance matrix. Show that when each x_i is sub-Gaussian with parameter at most σ^2, there exist universal constants (c_0, c_1, c_2) such that

$$\mathbb{P}\Big[|||\widehat{D} - D|||_{\text{op}}/\sigma^2 \geq c_0\sqrt{\frac{\log d}{n}} + \delta\Big] \leq c_1 e^{-c_2 n \min\{\delta, \delta^2\}}, \quad \text{for all } \delta > 0.$$