$$\lambda_A = -2 + 2\cos\left(\frac{j\pi}{\Delta j_x}\right) \quad j = 1, 2, \cdots, N-1 \quad = -4\sin^2\left(\frac{j\pi}{2N_x}\right)$$

$$\Rightarrow \lambda_Q = \left(1 - \frac{4\Delta t}{\Delta x^2}\sin^2\left(\frac{j\pi}{2N}\right)\right)$$

$$\|Q\| = \max_j \left| 1 - \frac{4\Delta t}{\Delta x^2}\sin^2\left(\frac{j\pi}{2N_x}\right)\right| \Rightarrow \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \cdot \frac{1}{\sin^2\left(\frac{j\pi}{2N}\right)}$$

$$\leq 1$$

$j = 1, 2, \cdots, N-1$

$$\Rightarrow \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

4.14. $\|u^{n+1}\| \leq \|Q^n\| \cdot \|u^0\|$ $\qquad n = \frac{T}{\Delta t}$

$\|Q^n\| \leq C_T$ $\qquad \Delta x = \frac{L}{N_x} \Rightarrow N_x = \frac{L}{\Delta x}$

BS & Heat $\|Q^n\| \leq 1 \Rightarrow \|Q\| \leq 1$

Heat Equ. $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$, $\Delta t = O(\Delta x^2)$

Conditionally stable.

EX: Crank - Nicolson

$$(2I - \alpha A)\vec{u}^{n+1} = (2I + \alpha A)\vec{u}^n \qquad \alpha = \frac{\Delta t}{\Delta x^2}$$

$A = \text{Diag}(1, -2, 1)$

$$Q = (2I - \alpha A)^{-1}(2I + \alpha A)$$

show $\|Q\| \leq 1$

Q is symmetric

so $\|Q\|_2 = \rho(Q)$

Q is tri-diagonal $\Rightarrow$ can find EV's

To show Q is symmetric use two Lemmas:

Lemma 1: Suppose $A = A^T$, $B = B^T$,

then $(AB)^T = AB$ iff $AB = BA$.

Proof:
Suppose $AB = BA$,

then $(AB)^T = B^T A^T = BA$

similarily $(BA)^T = A^T B^T = AB$

"= equal"

suppose $(AB)^T = AB$

$(AB)^T = B^T A^T = BA = AB$

Lemma 2: If $A = A^T$, then A^{-1} is symmetric.

Proof: $AA^{-1} = I$

$(AA^{-1})^T = I$

$(A^{-1})^T A^T = I$

$(A^{-1})^T A = I \Rightarrow (A^{-1})^T = A^{-1}$

~~so, $Q = 2I$~~

Now $Q = \underbrace{(2I - \alpha A)^{-1}}_{\text{symm.}} \underbrace{(2I + \alpha A)}_{\text{symm.}}$

$A = \text{diag}(1, -2, 1)$

Since $(2I - \alpha A)^{-1}$ & $(2I + \alpha A)$ have same $\overrightarrow{EV}$

$\Rightarrow Q$ is symmetric

$$\|Q\|_2 = \rho(Q) = \frac{\rho(2I + \alpha A)}{\rho(2I - \alpha A)}$$

$$ABX = A(\lambda_B X) = \lambda_B (AX) = \lambda_B \lambda_A X$$

The EV's are

$2I + \alpha A$: $2 - 4\alpha \sin^2\left(\frac{j\pi}{2N}\right)$

$2I - \alpha A$: $2 + 4\alpha \sin^2\left(\frac{j\pi}{2N}\right)$

$$AX = \lambda X$$
$$X = \lambda A^{-1} X$$
$$\frac{1}{\lambda} X = A^{-1} \cdot X$$

$$\|Q\|_2 = \max_j \left| \frac{2 - 4\alpha \sin^2\left(\frac{j\pi}{2N}\right)}{2 + 4\alpha \sin^2\left(\frac{j\pi}{2N}\right)} \right|$$

≤ 1 for all $\alpha = \frac{\Delta t}{\Delta x^2}$

unconditionally stable

The same is true for B-S Equation.

Convergence :

THM: If the solution $u(x,t)$ is sufficiently smooth and

$\tau = O(\Delta t^p, \Delta x^q)$

then the FD approx. converges as $\|e\| = O(\Delta t^p, \Delta x^q)$

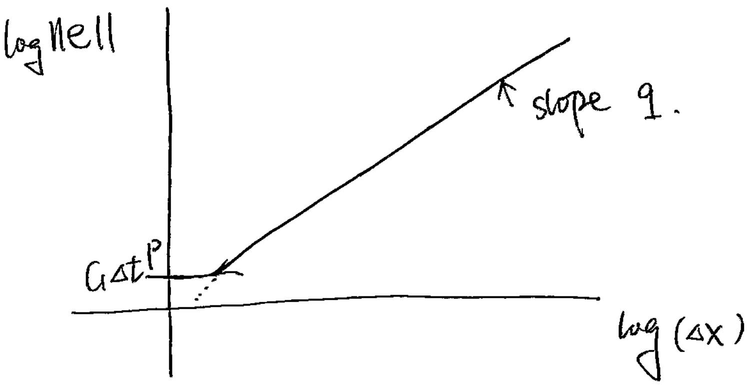
Also,

suppose $\Delta t = O(\Delta x^r)$ eg. $\Delta t = \frac{\Delta x^2}{2}$ for F.E. $r=2$

$\Delta t = C \cdot \Delta x$ (C-N) $r=1$

then error is $O(\Delta x^{\min(rp, q)})$

$\|e\| \to C_1 \Delta t^p + C_2 \Delta t^q$



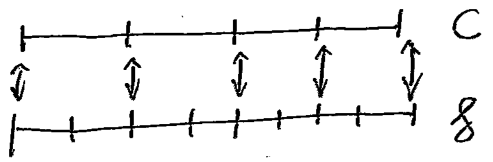
Norm?

Easiest : $\|\vec{u}\|_\infty = \max_j |u_j|$

Error Estimation.

With $\Delta t = O(\Delta x)$ & C-N $\|e\| = O(\Delta x^2)$

$$u_j^n \doteq u_{j,c}^n + C\cdot\Delta x^2$$



$$u_{2j}^n \doteq u_{2j,f}^n + C\cdot(\Delta x/2)^2$$

so $$u_{j,c}^n - u_{2j,f}^n = \underbrace{C\cdot\Delta x^2}_{e_c|_j^n}\left(1-\tfrac{1}{4}\right) \to 0$$

$$\Rightarrow e_c|_j^n \to \frac{u_{2j,f}^n - u_{j,c}^n}{1-\frac{1}{4}}$$

$\|e\|_\infty \le 0,01$

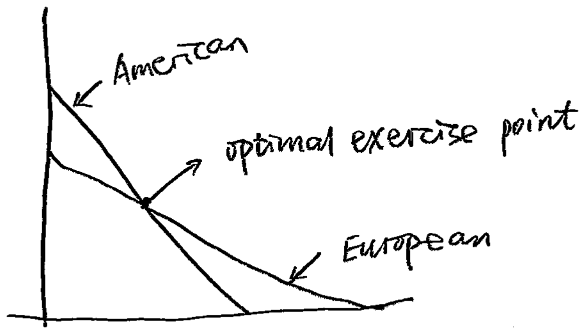
4.16 American Options

Can exercised at any time up to expiry.

Q's : When to exercise?

How do we price the extra benefit?

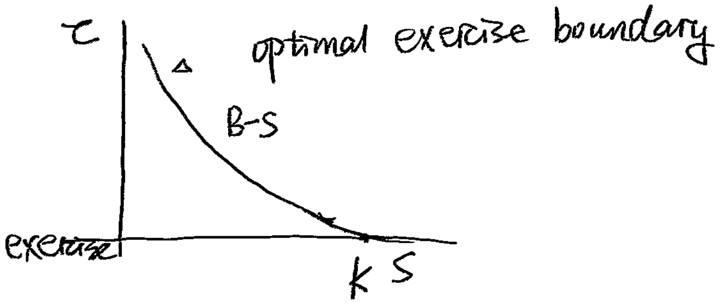
The American option must be larger than the European.



Buy stock @ S

Buy option @ P

sell @ K and make $K-S-P>0$ risk free



finally $\quad V_A(S,\tau) \geq \begin{cases} \max(S-K, 0) & \text{call} \\ \max(K-S, 0) & \text{put} \end{cases}$

3.6.2 Approximation of the American option Problem.

Explicit Approx:

B-S: $\cancel{\delta_\tau^+ \delta}$

$$\delta_\tau^+ V_j^n = \frac{\sigma^2 x_j^2}{R} \delta_x^+ \delta_x^- V_j^n + r x_j \delta_x^0 V_j^n - r V_j^n$$

$$\equiv L_n V_j^n$$

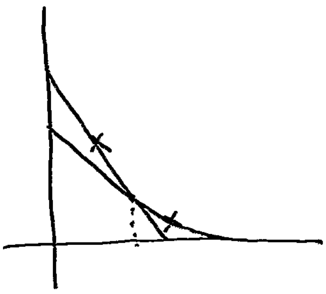
$$\overline{V_j^{n+1}} = V_j^n + \Delta t L_n V_j^n$$

Then correct

$$V_j^{n+1} = \begin{cases} \overline{V_j^{n+1}} & \text{if } \overline{V_j^{n+1}} \geq \text{payoff} \\ \text{payoff} & \end{cases}$$

Error $O(\Delta t)$, $\quad \dfrac{\left(\dfrac{\sigma^2 x_{max}}{R}\right)\Delta t}{\Delta x^2} \leq \dfrac{1}{2}$

for large $\frac{1}{R}$



Also error is $O(\Delta x)$

An forward substitution until $j = N-1, \cdots, 2, 1$

$$V_j^{n+1} < \text{payoff}$$

Then set $V_j^{n+1} = \text{payoff}$

For $j = 1$ to $N-1$

$\hat{b}_j = b_j$

Next j.

For $j = N-1$ to 2 step -1

$\hat{b}_{j-1} = b_{j-1} - c_{j-1} * a_j / \hat{b}_j$

$$\hat{y}_{j-1} = y_{j-1} - \frac{c_{j-1} * y_j}{\hat{b}_j}$$

Next j.

Backward elimination

Forward Substitution

$$V_1^{n+1} = \frac{\hat{y}_1}{\hat{b}_1}$$

for $j = 2$ to $N-1$

$V^{n+1} = \hat{y}_j - a_j V_{j-1} / \hat{b}_j$

$V_j^{n+1} = \max(V^{n+1}, \text{payoff})$ ← the only difference between the algorithms of European & American Option

next j.

Implicit :

$$\delta_t^+ V_j^n = L_h \left(\frac{V_j^{n+1} + V_j^n}{2} \right)$$

L_h

$$\left(I - \frac{\Delta t}{2} L_h\right) V_j^{n+1} = \left(I + \frac{\Delta t}{2} L_h\right) V_j^n$$

In Matrix Vector Form. this is

$$M_1 \vec{V}_j^{n+1} = \vec{y}^n = (\overrightarrow{RHS^n})$$

$$M_1 = \begin{bmatrix} b_1 & c_2 & & \\ a_2 & \ddots & \ddots & \\ & \ddots & \ddots & c_{n-2} \\ & & a_{n-1} & b_{n-1} \end{bmatrix}$$

Brennan and Schwartz (1977)

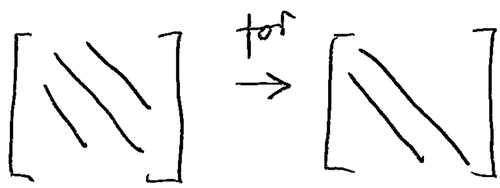
J. Finance Vol 32 P449

uses a modification of Thomas Algorithm.

$$\begin{bmatrix} b & c & & 0 \\ a & \ddots & \ddots & \\ & \ddots & \ddots & \\ 0 & & & \end{bmatrix}$$

usually forward elimination and backward substitution.

Instead do backward elimination. so as to

change  start from the right to the left