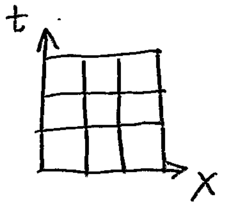
4.7 B-S.

$$\begin{cases} V_\tau = \mathcal{L}V \\ V(0,\tau) = B^L(\tau) \\ V(\infty,\tau) = B^R(\tau) \\ V(x,0) = P(x) \end{cases}$$

$$\mathcal{L}V = \frac{\sigma^2 S^2}{R} V_{xx} - rV + r x V_x$$

Put.
$$\begin{cases} B^L(\tau) = e^{-r\tau} \\ B^R(\tau) = 0 \\ P(x) = \max(1-x, 0) \end{cases}$$

Approximate: truncate



$$\begin{cases} [-\infty, \infty) \times [0,T] \to [0, X_{max}] \times [0,T] \\ x_i = i \cdot \Delta x \\ \tau_n = n \cdot \Delta\tau \end{cases}$$

$$\Delta x = X_{max} / N_x$$

$$\Delta\tau = T/N_\tau$$

$$\begin{cases} V_i(\tau) \doteq V(x_i, \tau) \\ V_i^n \doteq V(x_i, \tau_n) \end{cases}$$

$$V_\tau|_i = \mathcal{L}_h V_i$$

$$\mathcal{L}_h V_i = \frac{\sigma_i^2 x_i^2}{R} \delta_x^+ \delta_x^- V_i - V_i r_i + V_i x_i \delta_x^0 V_i + B.C's \quad i = 1, 2, \cdots, N_x - 1$$

or $\mathcal{L}_n V_i = \frac{\sigma_i^2 X_i^2}{R} \cdot \frac{V_{i+1} - 2V_i + V_{i-1}}{\Delta X^2} - r_i V_i + r_i V_i \cdot \frac{V_{i+1} - V_{i-1}}{2\Delta X}$

$$= \left(\frac{\sigma_i^2 X_i^2}{R\Delta X^2} + \frac{r_i X_i}{2\Delta X}\right) V_{i+1} + \left(V_i - \frac{2\sigma_i^2 X_i^2}{R\Delta X^2}\right) V_i + \left(\frac{\sigma_i^2 X_i^2}{R\Delta X^2} - \frac{r_i X_i}{2\Delta X}\right) V_{i-1}$$

$$\left.\begin{cases} \mathcal{L}_n V_i = c_i V_{i+1} + b_i V_i + a_i V_{i-1} & i = 1, 2, \cdots, N_x - 1 \\ V_0 = B^L(\tau) \\ V_{N_x} = B^R(\tau) \end{cases}\right\} \rightarrow \text{ODEs}$$

Let $\vec{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{N_x - 1} \end{bmatrix}$

Then, $\frac{d\vec{V}}{d\tau} = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ \cancel{a_3} & \cancel{b_3} & \cancel{c_3} & \ddots & \\ & \ddots & \ddots & \ddots & c_{N_x-2} \\ & & & a_{N_x-1} & b_{N_x} \end{bmatrix} \vec{V} + \begin{bmatrix} a_1 B^L(\tau) \\ 0 \\ 0 \\ \vdots \\ c_{N_x-1} B^R(\tau) \end{bmatrix}$

$$= A\vec{V} + \vec{g}(\tau)$$

$$A = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{N_x-2} \\ & & & a_{N_x-1} & b_{N_x-1} \end{bmatrix} \qquad \vec{g}(\tau) = \begin{bmatrix} a_1 B^L(\tau) \\ 0 \\ 0 \\ \vdots \\ 0 \\ c_{N_x-1} \cdot B^R(\tau) \end{bmatrix}$$

$$\frac{d\vec{V}}{d\tau} = A\vec{V} + \vec{g} = \mathcal{F}(\vec{V}, \tau)$$

$$\vec{V}^{n+1} = \vec{V}^n + \frac{\Delta\tau}{2}\{(A\vec{V}+J)^{n+1} + (A\vec{V}+\vec{g})^n\}$$

$$(I - \frac{\Delta\tau}{2}A)^{n+1}\vec{V}^{n+1} = \vec{V}^n + \frac{\Delta\tau}{2}A^n V^n + \frac{\Delta\tau}{2}(\vec{g}^{n+1} + \vec{g}^n)$$

$M_1 \vec{V}^{n+1} = \overrightarrow{RHS} \Rightarrow$ right-hand-side vector

$M_1 = \text{diag}(-\frac{\Delta\tau}{2}a, 1-\frac{\Delta\tau}{2}b, -\frac{\Delta\tau}{2}c)$

$\downarrow$ diaganal

Algorithm:

1. set $\vec{V}^n$ to payoff ($V_j^0 = P(x_j)$)
2. for $n=0$ to $N_\tau - 1$
3. Compute the $\overrightarrow{RHS} = (I + \frac{\Delta\tau}{2}A)\vec{V} + \frac{\Delta\tau}{2}(\vec{g}^{n+1} + \vec{g}^n)$
4. solve $\underline{(I - \frac{\Delta\tau}{2}A)}\,\vec{V}^{n+1} = \overrightarrow{RHS}$
5. next n $\hookrightarrow$ tri-diag $\rightarrow$ store only the 3 vectors.
6. output results. Don't store V^n as [*][*], only store V[*]

In step 3.

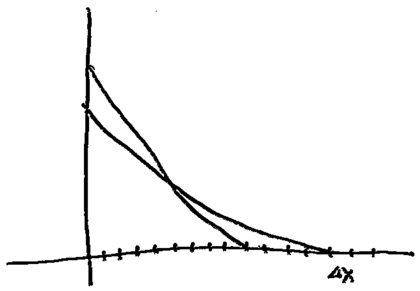
$RHS_i = V_i + \frac{\Delta\tau}{2}(a_i V_{i-1} + b_i V_i + C_i V_{i+1})$ $i = 2, \cdots, N_x (N_{x-2})$

$RHS_1 = V_1 + \frac{\Delta\tau}{2}(b_1 V_1 + C_1 V_2)$

$\vdots$

$RHS_{N_{x-1}} = V_{N_{x-1}} + \frac{\Delta\tau}{2}(a_{N_{x-1}} V_{N_{x-2}} + b_{N_{x-1}} V_{N_{x-1}}) \cancel{+ C_{N_{x-1}} V_{N_x}}$

results $V_j^{N_x - 1}$



Example :
$$\begin{cases} u_t = u_{xx} & x \in [0, L] \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

$x_i = i \cdot \Delta x \qquad \Delta x = \frac{L}{N_x}$

$\tau_n = n \cdot \Delta\tau$

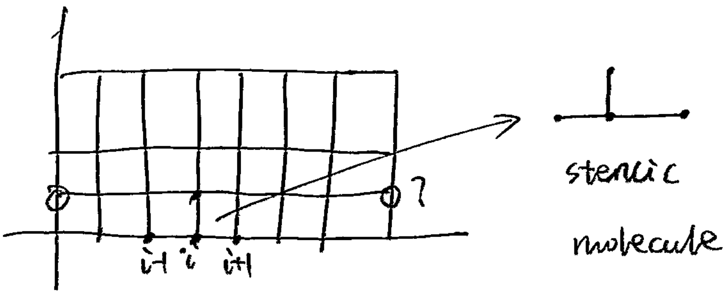
$u(x_i, t) \approx u_i(\tau)$

$$\begin{cases} u_t = \frac{du}{dt} = \delta^+ \delta^- u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} & i = 1, 2, \cdots, N_x - 1 \\ u_0(t) = 0 \\ u_{N_x}(t) = 0 \end{cases}$$

+ forward euler

$$u_i^{n+1} = u_i^n + \Delta t \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n \right) / \Delta x^2$$

$$= \frac{\Delta t}{\Delta x^2} u_{i+1}^n + \left(1 - \frac{2\Delta t}{\Delta x^2}\right) u_i^n + \frac{\Delta t}{\Delta x^2} u_{i-1}^n$$



boundary condition.

4.9 $\quad u_t = u_{xx}, \quad x \in (-\infty, \infty)$

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$\lambda = \frac{\overset{(\nu)}{\Delta t}}{\Delta x^2}$$

$$u_j^{n+1} = \lambda u_{j-1}^n + (1-2\lambda)u_j^n + \lambda u_{j+1}^n$$

Example:

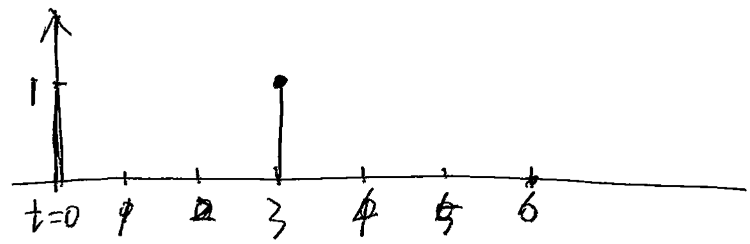
IVP.
$$\begin{cases} u_t = u_{xx} & x \in (0,6) \\ u(x,0) = \begin{cases} 1 & x=3 \\ 0 & x \neq 3 \end{cases} \\ u(0,t) = 0 \\ u(6,t) = 0 \end{cases}$$

$\Delta x = 1, \quad N_x = 6$

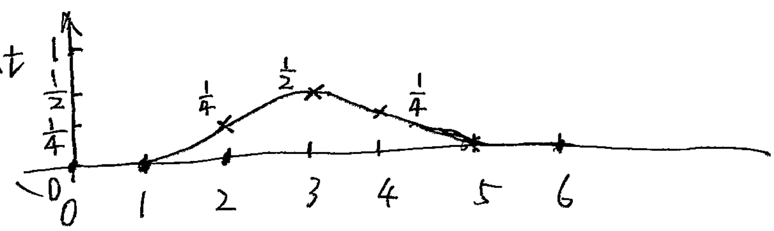
$\Delta t = \frac{1}{4},$

then
$$u_j^{n+1} = \frac{1}{4}u_{j+1}^n + \frac{1}{2}u_j^n + \frac{1}{4}u_{j-1}^n$$
$$= \frac{1}{4}\left(u_{j+1}^n + 2u_j^n + u_{j-1}^n\right)$$

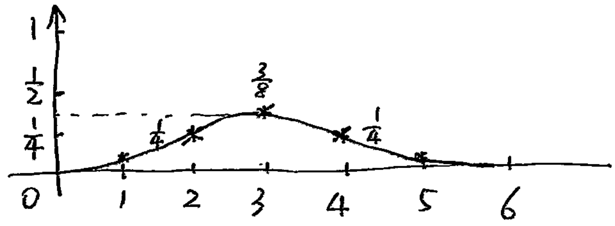
$t=0$



$t=\Delta t$

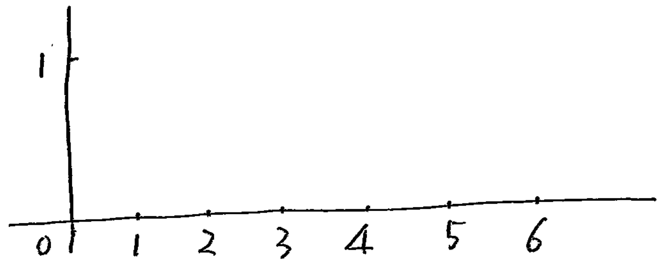


$t = 2\Delta t$

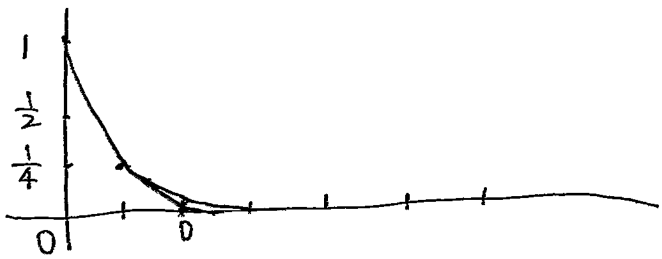


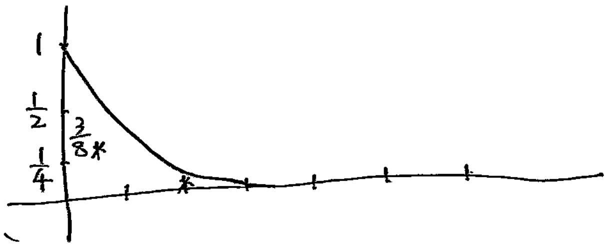
Example 2.

$$\begin{cases} u(x,0) = 0 \\ u(0,t) = 1 \\ u(6,t) = 0 \end{cases}$$



Δt





Looking at $u_j^{n+1} = \lambda u_{j+1}^n + (1-2\lambda)u_j^n + \lambda u_{j-1}^n$

in matrix form.

$$u(0,t) = \beta^L$$
$$u(6,t) = \beta^R$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}^{n+1} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}^{n} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \\ & \ddots & \ddots & \ddots \end{bmatrix} + \begin{bmatrix} \frac{1}{4}B^L \\ 0 \\ 0 \\ \vdots \\ \frac{1}{4}B^R \end{bmatrix}$$

$\cancel{u_6}$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}^{n+1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & & & & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & \\ & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \\ & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \\ & & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}^{n} + \begin{bmatrix} \frac{1}{4}B^L \\ 0 \\ 0 \\ 0 \\ \frac{1}{4}B^R \end{bmatrix}$$

3.5 Some Basic Theory

1. has $\Delta x, \Delta t \to 0$

   does $u_j^n \to u(x_j, t_n)$?

2. How fast does it converge ?

3. Are there practical limits on Δt & Δx ?

Def. The local truncation error of a finite difference approx. is the amount by which a smooth enough soln. fails to satisfy the approximation.

EX : $u_t = u_{xx}$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

Substitute in u.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \tau$$

$$\delta_t^+ u_j^n = \frac{\left(u_j^n + \Delta t \cdot \partial_t u_j^n + \frac{\Delta t^2}{2} \partial_t^2 \cdot u_j^n + \cdots\right) - u_j^n}{\Delta t}$$

$$= \partial_t \cdot u_j^n + \frac{\Delta t}{2} \partial_t^2 u_j^n$$

$$\Delta x^2 \delta_x^+ \delta_x^- u_j^n = u_j^n + \Delta x \cdot \partial_x u_j^n + \frac{\Delta x^2}{2} \partial_t^2 u_j^n + \frac{\Delta x^3}{3!} \partial_t^3 u_j^n + \frac{\Delta x^4}{4!} \partial_t^4 u(\xi, t_n)$$
$$- 2u_j^n$$
$$+ u_j^n - \Delta x \partial_x u_j^n + \frac{\Delta x^2}{2} \partial_t^2 u_j^n - \frac{\Delta x^3}{3!} \partial_t^3 u_j^n + \frac{\Delta x^4}{4!} \partial_t^4 u(\xi, t_n)$$

$$= \Delta x^2 \partial_t^2 u_j^n + \frac{2}{4!} \cdot \Delta x^4 \partial_t^4 u(\xi, t_n)$$

so

$$\frac{\partial}{\partial t} u_j^n + \frac{\Delta t}{2} \partial_t^2 u(x_j, u) = \partial_x^2 u_j^n + \frac{2}{4!} \Delta x^2 \partial_x^4 u(\xi, t_n) + \tau$$

$$\Rightarrow \tau_j^n = \underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u_j^n}_{0} + \frac{\Delta t}{2} \cdot \frac{\partial^2 u(x_j, u)}{\partial t^2} - \frac{2}{4!} \Delta x^2 \partial_x^4 u(\xi, t_n)$$

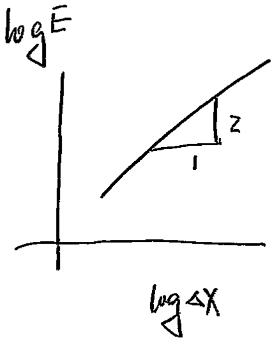
$$\tau_j^n = O(\Delta t, \Delta x^2)$$

Def. A FDE is consistent if $\tau = O(\Delta t^p, \Delta x^q)$

with $p, q \geq 1$

Thm: A necessary condition for convergence is consistency.

4.12



$$K^* x = S^*$$

$$\Delta t = O(\Delta t)$$

change strike price $\quad V = \frac{V^*}{K^*}$

we need stability.

we can write the FDE as

$$\vec{u}^{n+1} = Q\vec{u}^n + \vec{g}$$

for example.

$$\frac{du_j}{dt} = a_j u_{j+1} + b_j u_j + c_j u_{j-1}$$

for the heat equ.

$$\frac{du_j}{dt} = \frac{1}{\Delta x^2}(u_{j-1} - 2u_j + u_{j+1})$$

or $\quad \frac{d\vec{u}}{dt} = \frac{1}{\Delta x^2} A\vec{u} + \vec{g}, \qquad A = \mathrm{diag}(1, -2, 1)$

if we use forward euler

$$\vec{u}^{n+1} = \vec{u}^n + \frac{\Delta t}{\Delta x^2} A\vec{u}^n + \Delta t \vec{g}^n$$

$$= \underbrace{(I + \frac{\Delta t}{\Delta x^2})}_{Q} u^n + (\Delta t \vec{g}^n)$$

if we use trapizoidal rule.

$$u^{n+1} = u^n + \frac{\Delta t}{2\Delta x^2}\left(A(u^{n+1} + u^n) + \frac{\Delta t}{2}(\vec{g}^{\,n+1} + \vec{g}^{\,n})\right)$$

$$\left(I - A\frac{\Delta t}{\Delta x^2}\right) u^{n+1} = \left(I + A\Delta t/\Delta x^2\right) \vec{u}^{\,n} + \frac{\Delta t}{2}\left(\vec{g}^{\,n+1} + \vec{g}^{\,n}\right)$$

$$\vec{u}^{\,n+1} = \underbrace{\left(I - \left(\frac{\Delta t}{2\Delta x^2}\right)A\right)^{-1}\left(I + \frac{\Delta t}{2\Delta x^2}A\right)}_{Q} \vec{u}^{\,n} + \vec{\hat{g}}$$

suppose $\vec{\hat{g}} = 0$

$$\vec{u}^{\,n+1} = Q\vec{u}^{\,n} = Q^2\vec{u}^{\,n-1} = \cdots = Q^{n+1}\vec{u}^{\,0}$$

$$\vec{u}^{\,n} = Q^n \vec{u}^{\,0} \qquad n = \frac{T}{\Delta t}$$

In norm

$$\|\vec{u}^{\,n}\| = \|Q^n \vec{u}^{\,0}\| \leq \|Q^n\| * \|\vec{u}^{\,0}\|$$

Def: The FDE is stable if $\exists G$ independent of Δt & Δx.

But may be depend on time T. $\Rightarrow$

$$\|Q^n\| \leq C_T \quad \forall n.$$

THM: A necessary condition for convergence is stability.

The PBE itself satisfies a simulator property. Assume dirichlet B.C's $\beta^L = \beta^R = 0$. for heat eqn.

$$\begin{cases} u_t = u_{xx} \\ u(R,t) = u(L,t) = 0 \qquad x \in [0, L] \\ u(x,t) = u(x) \end{cases}$$

$$\int_0^L u_t u = \int_0^L u \cdot u_{xx}$$

$$\int_0^L u_t u \, dx = \int_0^L u u_{xx} \, dx$$

$$\frac{d}{dt} \int_0^L \frac{u^2}{2} dx = \int_0^L u u_{xx} \, dx$$

call $\quad \|u\|_2 \equiv \sqrt{\int_0^L u^2 dx}$

so, $\quad \frac{1}{2} \cdot \frac{d}{dt} \|u\|_2^2 = \underbrace{\int_0^L u \cdot u_{xx} \, dx}_{} = \underbrace{\underline{\underline{u \cdot u_x \Big|_0^L}}}_{\text{Boundary Conditions this } = 0.} - \underbrace{\int_0^L (u_x)^2 dx}_{\|u_x\|^2}$

$$\frac{d}{dt} \|u\|_2^2 = -2 \|u_x\|^2 \leq 0$$

$$\Rightarrow \|u\|_2^2 \leq \|u_0\|_2^2$$

$$\Rightarrow \|u\|_2 \leq \|u_0\|_2$$

Def: An initial value problem is well-posed if the solution satisfies.

$$\|u\| \leq C_T \cdot \|u_0\|$$

Note: Suppose $\begin{cases} V_t = V_{xx} \\ V(x,0) = u_0(x) + \delta \end{cases}$

then $V_t - u_t = (V-u)_{xx} = (V-u)_t$

call $w = v - u$

$$\begin{cases} w_t = w_{xx} \\ w(x,0) = \delta \end{cases}$$

then

$$\|w\| \leq \|\delta\|$$

$$\|v-u\| \leq \|\delta\|$$

Def: The approximation u_j^n converges to $u(x_j, t_n)$ if $\forall t < T$

$$\|u_j^n - u(x_j, t_n)\| \to 0$$

as $\Delta t, \Delta x \to 0$

THM: (LAX-Richtmyer equivalence)

For a consistent approximation to a well-posed initial value problem, stability is necessary and sufficient for convergence.

Stability + Consistency $\Leftrightarrow$ Convergence.

$\uparrow$ $\|Q^n\| \leq C_T$ (C_T: const.)

$\uparrow$ $\tau = O(\Delta t^p, \Delta x^q)$, $p, q \geq 1$

For heat equ.

$$\|u\| \leq \|u_0\|$$

$$\Rightarrow \quad \|u\| \leq \underbrace{\|Q^n\|}_{\leq 1}\|u_0\|$$

we'll require

$$\|Q^n\| \leq 1$$

$$\|Q^n\| \leq \|Q\|^n$$

and therefore require $\|Q\| \leq 1$

The $\|A\|_2 = \sqrt{\rho(A^T, A)}$ matrix

$$\rho = |\lambda|_{max}$$

if $A = A^T$,

then $\|A\|_2 = \rho(A)$

EX: heat equation. forward euler in time.

$$\vec{u}^{n+1} = \underbrace{\left(I + \frac{\Delta t}{\Delta x^2} A\right)}_{Q} \vec{u}^n \qquad \left(\begin{array}{l} Ax = \lambda x \\ Ix = x \end{array}\right.$$

$$A = \text{diagonal}(1, -2, 1)$$

$$\lambda_Q = 1 + \frac{\Delta t}{\Delta x^2} \lambda_A \qquad \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$