STA 4103/5107 Computational Methods in Statistics II

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Class 4
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Review: Homogeneous Poisson Process

- The resulting Poisson **counting process** can defined as follows:

$$N(t) = \sum_{i=1}^{\infty} 1_{[0,t)}(t_i), \quad N(0) = 0.$$

$N(t)$ counts the number of arrivals, or the occurrences, till time t.

- **Theorem 1:** The inter-arrival time $\tau \sim \lambda \exp(-\lambda\tau)$ if and only if for any $s_1 < s_2$, and $k = 0, 1, 2, \ldots$

$$P\{N(s_2) - N(s_1) = k\} = \frac{\exp(-\lambda(s_2 - s_1))(\lambda(s_2 - s_1))^k}{k!}$$

- **Definition 1:** A **homogeneous Poisson process** is a process with a constant intensity λ. That is, for any time interval $[t, t+\Delta t]$,

$$P\{k \text{ events in } [t, t + \Delta t]\} = \frac{\exp(-\lambda\Delta t)(\lambda\Delta t)^k}{k!}$$

Inhomogeneous Poisson Process

- **Definition 2:** An **inhomogeneous Poisson process** is a process with a rate function $\lambda(t)$. That is, for any time interval $[t, t+\Delta t]$,

$$P\{k \text{ events in } [t, t+\Delta t]\} = \frac{\exp(-s)(s)^k}{k!}$$

where $s = \int_t^{t+\Delta t} \lambda(t)dt.$

- Note: when $\lambda(t)$ is a constant λ, this simplifies to a homogeneous case where

$$s = \lambda \Delta t,$$

and

$$P\{k \text{ events in } [t, t+\Delta t]\} = \frac{\exp(-\lambda \Delta t)(\lambda \Delta t)^k}{k!}.$$

Time Rescaling Theorem

- **Time Rescaling Theorem** Suppose $\lambda(t) > 0$ in $[0, T]$. If $\{s_1, \ldots, s_n\}$ is a random sample from a Poisson process with rate $\lambda(t)$, then $\{t_1, \ldots, t_n\}$, with

$$t_k = F(s_k),$$

and

$$F(s) = \int_0^s \lambda(t) dt$$

is a Poisson process with constant rate 1 from $[0, F(T)]$.

Proof: At first, the mapping F maps $[0, T]$ to $[0, F(T)]$.

For any time interval $[t, t+\Delta t]$ in $[0, F(T)]$, we compute the probability

$$P\{k \text{ of } t_i \text{ in } [t, t+\Delta t]\}$$

Simulation

which is equal to, $P\{k \text{ of } s_i \text{ in } [F^{-1}(t), F^{-1}(t+\Delta t)]\}$

$$= Poisson\left(\int_{F^{-1}(t)}^{F^{-1}(t+\Delta t)} \lambda(t) dt\right)$$

$$= Poisson(F(F^{-1}(t+\Delta t)) - F(F^{-1}(t)))$$

$$= Poisson(\Delta t)$$

This is a Poisson process with rate 1.

- **Simulation** of a Poisson process with rate function $\lambda(t)$ on $[0, T]$:

  Step 1: Sample $\{t_1, \ldots, t_n\}$ from Poisson with constant rate 1 on $[0, F(T)]$.

  Step 2: Output $\{F^{-1}(t_1), \ldots, F^{-1}(t_n)\}$.

Example

- Simulate an inhomogeneous Poisson process over the interval [0, 10] where the rate function

$$\lambda(t) = 1.5 + \sin(2t)$$

1. Plot the rate function versus time t.
2. Generate 30 sample paths for this process.

$$F(s) = \int_0^s \lambda(t)dt = \int_0^s [1.5 + \sin(2t)]dt = 1.5s - (\cos(2s) - 1)/2$$

Simulation by Thinning

- **Theorem 3** Suppose that $s_1, \ldots, s_n$ are random variables representing event times from a Poisson process with rate function $\lambda_u(t)$, $t \geq 0$, in the interval $[0, T]$. Let $\lambda(t)$ be a rate function such that $0 \leq \lambda(t) \leq \lambda_u(t)$ for all $t \in [0, T]$. If the i-th event time s_i is independently deleted with probability

$$1 - \frac{\lambda(s_i)}{\lambda_u(s_i)}$$

for $i = 1, 2, \ldots, n$, then the remaining event times form a inhomogeneous Poisson process with rate function $\lambda(t)$ in $[0, T]$.

Simulation by Thinning

- Simulation of a Poisson process with rate function $\lambda(t)$ on $[0, T]$:

  Step 1: Sample $\{s_1, \ldots, s_n\}$ from Poisson process with constant rate $M = \max(\lambda(t))$ on $[0, T]$.

  Step 2: For each s_i, we delete it with probability $1 - \lambda(s_i)/M$.

  Step 3: Output the remaining sample.

- **Example:** Simulate an inhomogeneous Poisson process over the interval [0, 10] where the rate function

$$\lambda(t) = 3 + 3\sin(2t)$$

1. Plot the rate function versus time t.
2. Generate 30 sample paths for this process.