HW #5 Quadrature

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0. Executive Summary

Write adaptive quadrature which uses simpson's rule to calculate Gaussian probability function and obtain the result when the x is equal to 1.0 (at least six significant digits). Besides, use the function defined above to calculate the result of the call option via the B-S formula(in four digit numbers). Finally, I use bisection method to find out the implied volatility (in four significant digits). The result are listed as follows:

The adaptive quadrature function that use the simpson's rule is working correctly.

The integral of Gaussian probability function from minus infinity to 1 is 0.841345

The calculated Call option price is 98.4794

The calculated implied volatility is 7.609%

If I use the Tol equal to 10^{-7} , then the related error will be less than 5*10^{-5}

I. Statement of Problem

To find the value of the call option price, we should use the adaptive quadrature method to calculate the Gaussian probability which can not derive the value of the integral directly. Besides, show the algorithm is working correctly. In the end, to find the volatility of the BS model given the assumptions.

II. Description of The Mathematics

The two order of quadrature polynomial $p_2(x)$ is:

$$p_2(x) = f(x) + (x - x_0) * f[x_0, x] + (x - x_0) * (x - x_1) * f[x_0, x_1, x_2]$$

Where $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$, $f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

From this we can get the approximation of the integral as follows:

$$\tilde{I} = \int_a^b p_2(x)dx = \frac{b-a}{6}[f(a) + 4 * f(\frac{a+b}{2}) + f(b)] = \frac{\Delta x}{3}[f(a) + 4 * f(\frac{a+b}{2}) + f(b)]$$

where $\Delta x = \dfrac{b-a}{2}$

With the help of Taylor's expansion , we can get the Error E, the process is as follows:

$$\widetilde{I} = \frac{\Delta x}{3}[f_1 + \Delta x * f_1' + \frac{\Delta x^2}{2} * f_1'' + \frac{\Delta x^3}{3!} * f_1''' + \frac{\Delta x^4}{4!} * f^{(4)}(\xi)]$$
$$+4f_1 + [f_1 - \Delta x * f_1' + \frac{\Delta x^2}{2} * f_1'' - \frac{\Delta x^3}{3!} * f_1''' + \frac{\Delta x^4}{4!} * f^{(4)}(\xi)]$$
$$= \frac{\Delta x}{3}[6f_1 + \Delta x^2 * f_2'' + \frac{\Delta x^4}{2} f^{(4)}(\xi)]$$

$$I = \int_{x_0}^{x_2} f dx = \int_{x_0}^{x_2} (f_1 + (x - x_1) * f_1' + \frac{(x-x_1)^2}{2} * f_1'' + \frac{(x-x_1)^3}{3!} * f_1''' + \frac{(x-x_1)^4}{4!} * f^{(4)}(\xi)) dx$$
$$= 2\Delta x f_1 + \frac{\Delta x^3}{3} * f_1'' + \frac{\Delta x^5}{60} * f_1^{(4)}(\xi)$$

$E = I - \widetilde{I}$ such that:

$$E = \frac{\Delta x^5}{12} * f^{(4)}(\xi) * (\frac{1}{5} - \frac{1}{3}) = -\frac{\Delta x^5}{90} * f^{(4)}(\xi)$$

We also know that $E = C\Delta x^r f^m(\xi)$, for the simpson's method the r=4 and the m=4

$$E = -\frac{\Delta x^5}{90} * f^{(4)}(\xi) = -\frac{\Delta x^4}{180} * (b-a) * f^{(4)}(\xi)$$

$E \to -\frac{1}{180}(f'''(b) - f'''(a))\Delta x^4$ when $b - a \to 0$

So $E \to C * \Delta x^4$, when $\Delta x \to 0$, where $C = -\frac{1}{180}(f'''(b) - f'''(a))$ is a constant

So if we half the interval each time, and let $\widetilde{I}_c$, $\widetilde{I}_f$ keep the values of two rounds calculation.

$$I \approx \widetilde{I}_c + C * \Delta x^4$$
$$I \approx \widetilde{I}_f + C * \frac{\Delta x^4}{16}$$

In the adaptive method, we use $\widetilde{I}^L + \widetilde{I}^R$ instead of $\widetilde{I}_f$. Then we got the formula as below:

$$E_f = \frac{1}{2^r - 1}(\widetilde{I}_f - \widetilde{I}_c) = \frac{1}{15}[\widetilde{I}^L + \widetilde{I}^R - \widetilde{I}_C]$$

So we can make sure the total error less than given tolerance by checking $\widetilde{I}^L + \widetilde{I}^R - \widetilde{I}_C$

III. Description of the Algorithm

1. The simpson's rule

Suppose we have f be integrated function, [a,b] be the closed interval, then the simpson's rule is that:

```
Input f,a,b
Return (f(a)+f(a+(b-a)/2)*4+f(b))*(b-a)/6
END
```

2. Adaptive algorithm

Suppose we have f be integrated function, [a,b] be the closed interval, Tol be the given absolute error.

The quadrature rule is $Q(f,a,b) = \int_a^b f dx\ and\ the\ adeptive\_Q(f,a,b,tol)$

```
Input f, a, b, Tol
Ic = Q(f, a, b)
If =Q (f, a, (a+b)/2) + Q (f, (a+b)/2, b)
E = (If-Ic)/15
IF abs(E) > Tol
      IL=adaptive_Q ( f, a, (a+b)/2, Tol/2 )
      IR=adaptive_Q ( f, (a+b)/2, b, Tol/2)
      Return IL+IR
Otherwise
      Return If+E
End if
END
```

Since we only need at least 6 significant digits, so I used the 10^{-7}as the Tol.

To show the algorithm is working correctly, I test the routine on some integrals that I know the exact value. The interval of the integration is [0,1]

The result is as follows:

Function	x	x^2	x^3	x^4	x^(6)	sin(x)	e^x
True Value	0.5	0.3333	0.25	0.2	0.142857142857	0.4596976941319	1.71828182845905000
The approximate value	0.5	0.3333	0.25	0.2	0.142857143227	0.4596976940738	1.71828182849763000
ABS Error	0	0	0	0	0.000000000370	-0.000000000058	0.000000000039

From the above result, we can see that since the error polynomial is of 4th order, so the error of the first three functions are Zero. Besides the abs error of the other functions are very small, which showed that the routing can work well to be the approximation of the integration.

IV. Results

a. The integral of Gaussian probability function from minus infinity to 1 is 0.841345 with six significant numbers.
I integrate the function from 0 to 1 and the result is 0.341345.According to the property of symmetric, the Gaussian probability function integrated from minus infinity to 0 is 0.5. so the probability from minus infinity to 1 is 0.841345, with six significant digits.

Since the absolute error is bounded by the parameter Tol which is chosen as 10E-7, and P>0.5, so the relative error $\frac{|\Delta p|}{|p|} < \frac{10^{-7}}{0.5} = 5*10^{-7}$ and the answer has six significant digits.

b. The calculated Call option price is 98.4794 with four significant numbers.
The process is as follows:
With the formula: $d_1 = \frac{\log\left(\frac{s}{k}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, d_2 = \frac{\log\left(\frac{s}{k}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ and the given assumptions,
S= 3441.0
K=3425.0
r=6.75%
T=54/365
t=0
delta=0.1350

I calculated the result as follows:

d1	0.25726166
d2	0.20533574
P(d1)	0.60151161
P(d2)	0.58134509

I also check the result with the table of the standard normal distribution, the result is similar.

So the call price is 98.4794

Besides, $\frac{|\Delta C|}{|C|} = \frac{|S\Delta P - Ke^{-r(T-t)}\Delta P|}{|C|} \leq \frac{|S + Ke^{-r(T-t)}||\Delta P|}{|C|} = \frac{6831.97}{94}*10^{-7} < 5*10^{-5}$

Which answer the figures that I think I shall have

c. The calculated implied volatility is 7.609% with four significant numbers.

To calculate the implied volatility, I used bisection method which was written before and the volatility rate is 7.609% with 4 significant numbers. The assumption are as follows:
S= 3441.0
K=3425.0
r=6.75%
T=54/365
t=0
c=94

V. Conclusions

My conclusions are as follows:

The adaptive quadrature function that use the simpson's rule is working correctly.
The integral of Gaussian probability function from minus infinity to 1 is 0.841345
The calculated Call option price is 98.4794
The calculated implied volatility is 7.609%

If I use the Tol equal to 10^{-7} , then the related error will be less than 5*10^{-5}

VI. Code Listing

```
Private Sub simpson()

Dim a As Double
Dim b As Double
Dim tol As Double
a = 0
b = 0.25726165943415
tol = 10 ^ (-7)
Worksheets("sheet1").Range("a1").Value = simp(a, b)
Worksheets("sheet1").Range("a2").Value = adaptive(a, b, tol)
End Sub

Function simp(a As Double, b As Double) As Double
simp = (f(a) + f(a + (b - a) / 2) * 4 + f(b)) * (b - a) / 6
End Function

Function f(c As Double) As Double
'
'f = c ^ 1
'f = c ^ 2
'f = c ^ 3
'f = c ^ 4
'f = c ^ 6
'f = Sin(c)
'f = Exp(c)
f = Exp(-c ^ 2 / 2) / Sqr(3.14159265358979 * 2)
End Function

Function adaptive(d As Double, e As Double, tol As Double) As Double
Dim ic, ig, ee, il, ir As Double
    ic = simp(d, e)
    ig = simp(d, d + (e - d) / 2) + simp(d + (e - d) / 2, e)
    ee = (ig - ic) / 15
    If Abs(ee) > tol Then
        il = adaptive(d, d + (e - d) / 2, tol / 2)
        ir = adaptive(d + (e - d) / 2, e, tol / 2)
        adaptive = il + ir
    Else
        adaptive = ig + ee
            End If

End Function
```