STA 4103/5107 Computational Methods in Statistics II

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Special Topic 3

Laplace Approximation and Point Process Filter

S3.1 Laplace Approximation

Basic Idea

- The idea behind the Laplace approximation is simple.

- We assume that the probability density $P(x) = P^*(x)/Z_P$, where the normalizing constant
$$Z_P = \int P^*(x)dx.$$

- We also assume that the unnormalized density $P^*(x)$ has a peak at a point x_0.

- We Taylor-expand the logarithm of $P^*(x)$ around this peak:
$$\log P^*(x) \approx \log P^*(x_0) - \frac{c}{2}(x - x_0)^2 + \ldots,$$
where
$$c = -\frac{\partial^2}{\partial x^2} \log P^*(x)\Big|_{x=x_0}.$$

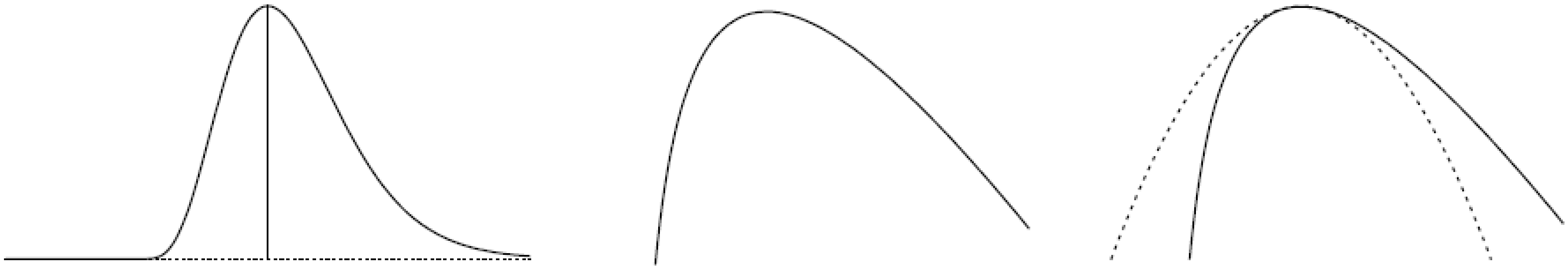
Basic Idea

- We then approximate $P^*(x)$ by an unnormalized Gaussian,

$$Q^*(x) \approx P^*(x_0)\exp[-\frac{c}{2}(x-x_0)^2],$$

and we approximate the normalizing constant Z_P by the normalizing constant of this Gaussian,

$$Z_Q \approx \int P^*(x_0)\exp[-\frac{c}{2}(x-x_0)^2]dx = P^*(x_0)\sqrt{\frac{2\pi}{c}}$$



$P^*(x)$ log $P^*(x)$ log $P^*(x)$ & log $Q^*(x)$

General Case

- We can generalize this integral to approximate Z_P for a density $P^*(x)$ over a K-dimensional space x.

- If the matrix of second derivatives of - log $P^*(x)$ at the maximum x_0 is A, defined by:

$$A_{ij} = -\frac{\partial^2}{\partial x_i \partial x_j} \log P^*(x)\Big|_{x=x_0}.$$

- Therefore,

$$\log P^*(x) \approx \log P^*(x_0) - \frac{1}{2}(x - x_0)^T A(x - x_0) + \ldots,$$

- We let

$$Q^*(x) \approx P^*(x_0) \exp[-\frac{1}{2}(x - x_0)^T A(x - x_0)],$$

General Case

- Then, the normalizing constant is

$$Z_Q = \int P^*(x_0)\exp[-\frac{1}{2}(x-x_0)^T A(x-x_0)]dx$$

$$= P^*(x_0)\int \exp[-\frac{1}{2}(x-x_0)^T A(x-x_0)]dx$$

- Note that for k-dimensional random vector $x \sim N(\mu, \Sigma)$

$$f(x) = \frac{1}{(2\pi)^{k/2}(\det \Sigma)^{1/2}}\exp[-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)]$$

- Therefore,

$$Z_Q = P^*(x_0)\sqrt{\frac{(2\pi)^k}{\det A}}.$$

Property

- The Laplace approximation is a Gaussian based method.
- That is, the logarithm of a Gaussian function is a quadratic function, and this is the basic idea for the approximation.
- This method works well when the logarithm of the object function is concave.

S3.2 Point Process Filter

Filtering Estimation

- Let $x_1, x_2, \ldots, x_t, x_{t+1}, \ldots,$ form a process of interest, and instead of measuring x_ts directly, one observes variables $y_1, y_2, \ldots, y_t, y_{t+1}, \ldots$
- The goal is to use the observations, and joint probability models of x and y to estimate the unknown x_ts.
- Let $f(x_t \mid y_1, y_2, \ldots, y_t)$ be the posterior density function of x_t given a set of observations $(y_1, y_2, \ldots, y_t)$.
- We are interested in the filtering estimation (posterior mean):

$$\hat{x}_t = E_f(x_t \mid y_1, y_2, \ldots, y_t).$$

Nonlinear Filtering Problem

- Let the *state vector* $x_t \in \mathbf{R}^d$ and the observation vector $y_t \in \mathbf{R}^C$.

- **State equation:** $x_{t+1} = \mathrm{F}(x_t) + w_t$

  Observation equation: $y_t = \mathrm{G}(x_t) + q_t$

- Estimation:

  update equation

$$f(x_t \mid y_1, y_2, \ldots, y_{t-1}) = \int_{x_{t-1}} f(x_t \mid x_{t-1}) f(x_{t-1} \mid y_1, y_2, \ldots, y_{t-1}) dx_{t-1}$$

  prediction equation

$$f(x_t \mid y_1, y_2, \ldots, y_t) = \frac{f(y_t \mid x_t) f(x_t \mid y_1, y_2, \ldots, y_{t-1})}{f(y_t \mid y_1, y_2, \ldots, y_{t-1})}$$

Point Process Observation

- Assume the state vector $x_k \in \mathbf{R}^d$
- Assume the observation is a **point process** in time interval [0, T].
- The time interval is discretized to time bins $t_1, t_2, \ldots, t_M$.
- For $c = 1, \ldots, C$, let the c-th observation component

  $$y_{k,c} = \text{number of events in the } k\text{-th time bin.}$$

- This number is either 0 or 1 if the bin size is sufficiently small.
- Therefore, the observation at the k-th time bin is

  $$y_k = \{y_{k,c}\}_c \in \mathbf{R}^C$$

State-Space Model

- Assume x_k follow a simple linear Gaussian transition. That is,

$$x_k = Ax_{k-1} + w_k, \qquad w_k \in N(0, W)$$

  where the A and W can be fitted in closed-form using **Maximum likelihood Estimation (MLE).**

- For y_k, we assume a generalized linear model (GLM) with an inhomogeneous Poisson process condition on x_k. That is,

$$y_{k,c} \sim Poisson(\lambda_{k,c})$$

  where

$$\lambda_{k,c} = \exp(\mu_c + \alpha_c^T x_k)$$

- $\{\mu_c, \alpha_c\}$ can be identified using **MLE.**

System Identification

- For each $c = 1, \ldots, C$, assume the observations are $\{x_k, y_{k,c}\}$.

- We maximize the likelihood

$$L = p(\{y_{k,c}\} \mid \{x_k\}) = \prod_{k=1}^{M} p(y_{k,c} \mid x_k) = \prod_{k=1}^{M} \frac{e^{-\lambda_{k,c}} (\lambda_{k,c})^{y_{k,c}}}{y_{k,c}!}$$

- The log-likelihood is:

$$LL = \sum_{k=1}^{M} y_{k,c} \log(\lambda_{k,c}) - \lambda_{k,c} + const$$

$$= \sum_{k=1}^{M} y_{k,c} (\theta_c^T X_k) - \exp(\theta_c^T X_k) + const$$

where

$$\theta_c = (\mu_c, \alpha_c^T)^T, X_k = (1, x_k^T)^T$$

System Identification

- We use a Newton-Raphson method.

$$\frac{\partial LL}{\partial \theta_c} = \sum_{k=1}^{M} y_{k,c} X_k - \exp(\theta_c^T X_k) X_k$$

$$\frac{\partial^2 LL}{\partial \theta_c \partial \theta_c^T} = -\sum_{k=1}^{M} \exp(\theta_c^T X_k) X_k X_k^T$$

- Recursive update:

$$(\theta_c)_{i+1} = (\theta_c)_i - \left(\frac{\partial^2 LL}{\partial \theta_c \partial \theta_c^T} \right)_i^{-1} \left(\frac{\partial LL}{\partial \theta_c} \right)_i$$

Point Process Filter

- To estimate the posterior $f(x_k \mid y_1, y_2, \ldots, y_k)$, we can use a sequential Monte Carlo (SMC) method.
- However, the method depends on number of sample points at each time step. (Note: these sample points are also called "particles" and the SMC is also called "particle filtering").
- A large number of particles often leads to inefficient computation.
- Here we introduce an efficient, deterministic estimation method, called **point process filter**.
- This method is based on Laplace approximation by approximate the posterior at each time using a Gaussian distribution.

Estimation Process

- We use the recursive formula

$$f(x_k \mid y_1, y_2, \ldots, y_k) \propto f(y_k \mid x_k) f(x_k \mid y_1, y_2, \ldots, y_{k-1})$$

- Assume that conditioned on x_k, all components in y_k are independent. Therefore

$$f(y_k \mid x_k) = \prod_{c=1}^{C} f(y_{k,c} \mid x_k)$$

$$= \prod_{c=1}^{C} \exp(-\lambda_{k,c})) \frac{(\lambda_{k,c})^{y_{k,c}}}{y_{k,c}!}$$

- We use the following notation to simplify the sub-index

$$\boldsymbol{a_{1:n} = a_1, a_2, \ldots, a_n.}$$

Time Update

- We approximate the posterior using a Gaussian distribution at each time k. That is, let

$$x_{k|k} = E(x_k \mid y_{1:k}) \qquad W_{k|k} = Var(x_k \mid y_{1:k})$$

- Then

$$f(x_k \mid y_{1:k-1}) = \int f(x_k \mid x_{k-1}) f(x_{k-1} \mid y_{1:k-1}) dx_{k-1}$$

is also normally distributed.

- The mean is computed as:

$$\begin{aligned} x_{k|k-1} &= E(x_k \mid y_{1:k-1}) = E(Ax_{k-1} + w_k \mid y_{1:k-1}) \\ &= AE(x_{k-1} \mid y_{1:k-1}) \\ &= Ax_{k-1|k-1} \end{aligned}$$

Time Update

- The covariance is computed as:

$$W_{k|k-1} = Var(x_k \mid y_{1:k-1}) = Var(Ax_{k-1} + w_k \mid y_{1:k-1})$$
$$= Var(Ax_{k-1} \mid y_{1:k-1}) + Var(w_k)$$
$$= AW_{k-1|k-1}A^T + W$$

- Therefore,

$$f(x_k \mid y_{1:k}) \propto f(y_k \mid x_k) f(x_k \mid y_{1:k-1})$$
$$= \left(\prod_{c=1}^{C} \exp(-\lambda_{k,c}) \frac{(\lambda_{k,c})^{y_{k,c}}}{y_{k,c}!} \right) \cdot \exp(-\frac{1}{2}(x_k - x_{k|k-1})^T W_{k|k-1}^{-1} (x_k - x_{k|k-1}))$$

Measurement Update

- Then, the logarithm of the posterior is

$$\log f(x_k \mid y_{1:k})$$
$$= \left(\sum_{c=1}^{C} y_{k,c} \log \lambda_{k,c} - \lambda_{k,c} \right) - \frac{1}{2}(x_k - x_{k|k-1})^T W_{k|k-1}^{-1}(x_k - x_{k|k-1}) + const$$

- We approximate this posterior by a Gaussian distribution

$$\log f(x_k \mid y_{1:k}) = -\frac{1}{2}(x_k - x_{k|k})^T W_{k|k}^{-1}(x_k - x_{k|k}) + const$$

- Then,

$$\frac{1}{2}(x_k - x_{k|k})^T W_{k|k}^{-1}(x_k - x_{k|k})$$
$$= \frac{1}{2}(x_k - x_{k|k-1})^T W_{k|k-1}^{-1}(x_k - x_{k|k-1}) - \left(\sum_{c=1}^{C} y_{k,c} \log \lambda_{k,c} - \lambda_{k,c} \right) + const$$

Measurement Update

- Differentiate w.r.t. to x_k, we have

$$
\begin{aligned}
&W_{k|k}^{-1}(x_k - x_{k|k}) \\
&= W_{k|k-1}^{-1}(x_k - x_{k|k-1}) - \sum_{c=1}^{C}[y_{k,c}\frac{\partial \log \lambda_{k,c}}{\partial x_k} - \frac{\partial \lambda_{k,c}}{\partial x_k}] \\
&= W_{k|k-1}^{-1}(x_k - x_{k|k-1}) - \sum_{c=1}^{C}[y_{k,c}\alpha_c - \lambda_{k,c}\alpha_c]
\end{aligned}
$$

- Differentiate again,

$$
W_{k|k}^{-1} = W_{k|k-1}^{-1} + \sum_{c=1}^{C}\frac{\partial \lambda_{k,c}}{\partial x_k}\alpha_c = W_{k|k-1}^{-1} + \sum_{c=1}^{C}\alpha_c \lambda_{k,c}\alpha_c^T
$$

Measurement Update

- Let $x_k = x_{k|k-1}$ after the second differentiation, we have

$$W_{k|k} = \left(W_{k|k-1}^{-1} + \sum_{c=1}^{C} \alpha_c \exp(\mu_c + \alpha_c^T x_{k|k-1}) \alpha_c^T \right)^{-1}$$

- Let $x_k = x_{k|k-1}$ after the first differentiation, we have

$$W_{k|k}^{-1}(x_{k|k-1} - x_{k|k}) = -\sum_{c=1}^{C} [y_{k,c}\alpha_c - \exp(\mu + \alpha_c^T x_{k|k-1})\alpha_c]$$

- Therefore,

$$x_{k|k} = x_{k|k-1} + W_{k|k} \sum_{c=1}^{C} [y_{k,c} - \exp(\mu_c + \alpha_c^T x_{k|k-1})]\alpha_c$$

Algorithm

- **Point Process Filter Algorithm:**

Update from time *k*-1 to *k*:

$$x_{k-1} \mid y_{1:k-1} \sim N(x_{k-1|k-1}, W_{k-1|k-1}) \rightarrow x_k \mid y_{1:k} \sim N(x_{k|k}, W_{k|k})$$

Time update: $W_{k|k-1} = AW_{k-1|k-1}A^T + W$

$$x_{k|k-1} = Ax_{k-1|k-1}$$

Measurement update:

$$W_{k|k} = \left(W_{k|k-1}^{-1} + \sum_{c=1}^{C} \alpha_c \exp(\mu_c + \alpha_c^T x_{k|k-1}) \alpha_c^T \right)^{-1}$$

$$x_{k|k} = x_{k|k-1} + W_{k|k} \sum_{c=1}^{C} [y_{k,c} - \exp(\mu_c + \alpha_c^T x_{k|k-1})] \alpha_c$$