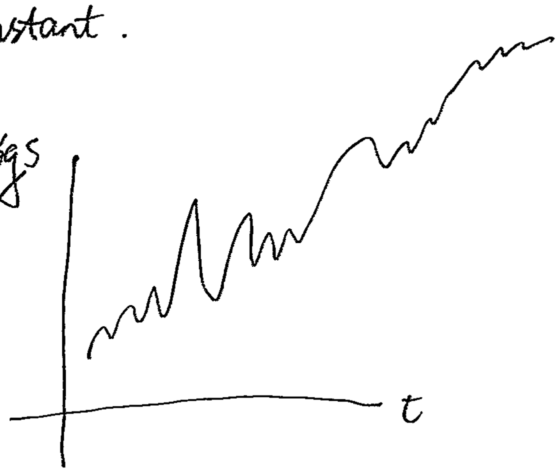
3.26

$$\begin{cases} \frac{ds}{s} = u\,dt + \sigma\,dz \\ dz = \sqrt{t}\,\phi \\ u, \sigma \text{ are constant.} \end{cases}$$

To understand s



we look at expectation s.

let $y = f(\phi)$ be random variable. then

$$E(y) = E[f(\phi)] = \int_\Omega f(\phi)\, p(\phi)\, d\mu(\phi)$$

$p(\phi)$: probability density function. μ: measure

Note: E is linear

EX: $\Omega = \mathbb{R}^1$, $d\mu = d\phi$, $p(\phi)$ is the Gaussian pdf

$$p(\phi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}$$

$$E[1] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\phi^2} d\phi = 1$$

$$E[\phi] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi e^{-\frac{1}{2}\phi^2} d\phi = 0$$

$$E[\phi^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi^2 e^{-\frac{1}{2}\phi^2} d\phi = 1$$

Next, the variance is

$$Var(y) = E(y^2) - (E(y))^2$$

EX: $Var(\phi) = \underbrace{E[\phi^2]}_{1} - \underbrace{(E(\phi))^2}_{=0} = 1$

then

$$E\left(\frac{ds}{s}\right) = E(udt + \sigma dz)$$

$\phi \to$ normal distn.

$$= udt\, E(1) + \sigma\sqrt{t}E(\phi)$$

$$= udt + 0 \quad [E(\phi)=0] \quad (dz = \sqrt{t}\phi)$$

$$= udt$$

$$\Rightarrow E\left(\frac{1}{s}\cdot\frac{ds}{dt}\right) = u$$

Also, $E(dz) = 0$

$$Var\left(\frac{ds}{s}\right) = E\left(\left(\frac{ds}{s}\right)^2\right) - \left(E\left(\frac{ds}{s}\right)\right)^2$$

$$= E\left(\left(\frac{ds}{s}\right)^2\right) - (udt)^2$$

$$E\left(\left(\frac{ds}{s}\right)^2\right) = E\left((udt + \sigma dz)^2\right) = E\left(u^2dt^2 + 2u\sigma dt\,dz + \sigma^2 dz^2\right)$$

$$= u^2dt^2 + 2u\sigma E(dt\,dz) + \sigma^2 E(dz^2)$$

$$= u^2dt^2 + 2u\sigma dt \underbrace{E(dz)}_{0} + \sigma^2 \underbrace{E(dz^2)}_{dtE(\phi^2)} = u^2dt^2 + dt\underbrace{E(\phi^2)}_{1}\sigma^2$$

$$Var\left(\frac{ds}{s}\right) = \sigma^2 dt$$

$$Var\left(\frac{1}{s}\cdot\frac{ds}{dt}\right) = \sigma^2$$

we now converge $\frac{ds}{s} = udt + \sigma dz(t)$ to a partial differentiation Equation. (PDE)

Ito's Lemma: Suppose $G = G(s, t)$, where s follows the stochastic process $\frac{ds}{s} = udt + \sigma dz$

then

~~$dG = (us \frac{dG}{ds}$~~

$$dG = \left(us \frac{\partial G}{\partial s} + \frac{\sigma^2 s^2}{2} \frac{\partial^2 G}{\partial^2 s} + \frac{\partial G}{\partial t} \right) dt + \sigma s \cdot \frac{\partial G}{\partial s} \cdot dz$$

EX: let $G(s) = \log(s)$

$$\frac{\partial G}{\partial s} = \frac{1}{s},$$

$$\frac{\partial^2 G}{\partial^2 s} = -\frac{1}{s^2}$$

$$\frac{\partial G}{\partial t} = 0$$

$$dG = \frac{\partial G}{\partial s} \cdot s \cdot \sigma \cdot dz + \left(u \cdot s \cdot \frac{\partial G}{\partial s} + \frac{\sigma^2 s^2}{2} \cdot \frac{\partial^2 G}{\partial^2 s} + \frac{\partial G}{\partial t} \right) dt$$

$$= \sigma dz + \left(u - \frac{\sigma^2}{2} \right) dt$$

$$\int_0^t dG = \sigma \int_0^t dz + \left(u - \frac{\sigma^2}{2} \right) \int_0^t dt$$

$$G(t) = G(0) + \sigma\big(z(t) - z(0)\big) + \left(u - \frac{\sigma^2}{2} \right) t \qquad s = e^t$$

$$s(t) = s(0)\, e^{\sigma[z(t) - z(0)] + \left(u - \frac{\sigma^2}{2} \right) t} \qquad z(t) - z(0) \sim \sqrt{t}\,\phi$$

3.1.1 The Black-Scholes Equation

Assumption:

(1) the stock price follows a geometric Brownian Motion

(2) risk free rate of return r = constant

(3) always risk free portfolios must earn the risk free rate.
No arbitrage

Let $V(S,t)$ = option price

let P be a portfolio with 1 option (V) and 1 stock that we borrow money for. Let α = amount of stock

$$P = V - \alpha S$$

Assume α = constant

Since α is a constant,

$$dP = dV - \alpha dS$$

$$dS = u\,dt \cdot S + \sigma S\,dZ$$

$$dV(S,t) = \left(uS\frac{dV}{dS} + \frac{\sigma^2 S^2}{2}\frac{\partial V^2}{\partial^2 S} + \frac{dV}{dt}\right)dt + \sigma S\frac{dV}{dS}dZ$$

So, $$dP = \left(uS\frac{dV}{dS} - \alpha u S\right)dt + \underbrace{\left(\sigma S\frac{dV}{dS} - \alpha\sigma S\right)dZ}_{\text{take } \alpha = \frac{dV}{dS}\text{, so no risk}} + \left(\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{dV}{dt}\right)dt$$

then $$dP = \left(\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{dV}{dt}\right)dt$$

the risk free

$$\frac{dP}{P} = r dt$$

$$dp = rp\,dt$$

$$rp\,dt = \left(\frac{dV}{dt} + \frac{\sigma^2 S^2}{2} \cdot \frac{\partial^2 V}{\partial S^2}\right) dt$$

$$P = V - \frac{dV}{dS} \cdot S$$

$$rV - r \cdot \frac{dV}{dS} \cdot S = \frac{dV}{dt} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}$$

$$\frac{dV}{dt} + \frac{\sigma^2 S^2}{2} \cdot \frac{\partial^2 V}{\partial S^2} + r \cdot \frac{dV}{dS} \cdot S - r \cdot V = 0 \rightarrow \text{B-S Equation}$$

3.29.

$$V_{SS} = \frac{\partial^2 V}{\partial S^2} \quad V_t = \frac{\partial V}{\partial t}$$

$$V_S = \frac{dV}{\partial S}$$

$$V_t + \frac{1}{2}\sigma^2 V_{SS} + rS V_S - rV = 0$$

1 derive in time

2 derives in S

$\Rightarrow$ 1 condition in time

2 conditions in S (boundary conditions)

Condition in time

$V(t, S) = \text{payoff} = \max(K-S, 0)$ put

or $V(t, S) = \text{payoff} = \max(S-K, 0)$ call

Conditions in S we use the put-call parity.

create portfolio P with 1 stock, 1 call and 1 put.

$$P = S + V_P - V_C$$

S: value of stock

V_P: value of put

V_C: value of call

payoff at T.
$$P(T,S) = \begin{cases} S + 0 - (S-K) = K & S > K \\ S + (K-S) - 0 = K & S < K \end{cases}$$

$P_{T,S}$ is risk-free

then $P(0) = S(0) + V_P(0) - V_C(0) = e^{-rT} K$

also for all time

$$P(t) = S(t) + V_P(t) - V_C(t) = e^{-r(T-t)} \cdot K$$

Boundary Conditions. for $S=0$, $S=\infty$.

$S=0$

$V_C(0,t) = 0$

then $V_P(0,t) = e^{-r(T-t)} \cdot K$

$S \to \infty$

$V_P(\infty, t) = 0$

$V_C(\infty, t) = S(t) - Ke^{-r(T-t)}$

PDE: $V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rS V_S - rV = 0$

put $\begin{cases} V(0,t) = Ke^{-r(T-t)} \\ V(-\infty, t) = 0 \\ V(S,T) = \cancel{K}\max(K-S, 0) \end{cases}$

call $\begin{cases} V(0,t) = 0 \\ V(\infty, t) = S - Ke^{-r(T-t)} \\ V(S,T) = \max(S-K, 0) \end{cases}$

3.2 Advection-Diffusion Equs.

$$u_t + a u_x = \nu u_{xx} \qquad a, \nu, \text{ const.}$$

The fundamental soln is

$$u = \hat{u}(w,t) e^{iwx} = u(x,t) \qquad i^2 = -1$$

provided that

$$\frac{d\hat{u}}{dt} e^{iwx} + aiw \hat{u} e^{iwx} = -\nu w^2 \hat{u} e^{iwx}$$

$$\frac{d\hat{u}}{dt} = \cancel{i(\nu w^2 + aw)\hat{u}} -(\nu w^2 + aiw)\hat{u}$$

soln:

$$\hat{u}(w,t) = \hat{u}(w,0) e^{-(\nu w^2 + aiw)t}$$

so.

$$u(x,t) = \hat{u}(w,0) e^{-(aiw + \nu w^2)t} e^{iwx}$$

$$= \hat{u}(w,0) \cdot e^{iw(x-a)} e^{-\nu w^2 t}$$

use ~~fourter~~ transform

(1) Given $f(x)$ is fourier transform $\hat{F}(w) = \mathcal{F}[f](w)$

$$\hat{F}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

(i) $\hat{F}$ is linely complex

(ii) $\mathcal{F}$ is linear

12) Integration by parts given derivative formulas

$$\mathcal{F}(f') = i\omega \mathcal{F}(f)$$

in general

$$\mathcal{F}[f^{(m)}] = (i\omega)^m \mathcal{F}(f)$$

13) The F.T is inveribl.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(\omega) e^{i\omega x} d\omega$$

$$= \mathcal{F}^{-1}[\hat{F}]$$

14) The F.T transform of a gaussian is a gaussian

$$\mathcal{F}(e^{-px^2}) = \frac{1}{\sqrt{2p}} e^{-\omega^2/4p}$$

$$\mathcal{F}(e^{-\omega^2/4p}) = e^{-px^2}$$

3.31

$$u_t + a u_x = \nu u_{xx}$$

$$\mathcal{F}[u_t + a u_x] = \mathcal{F}[\nu u_{xx}]$$

$$\mathcal{F}[u_t] + a\,\mathcal{F}[u_x] = \nu\,\mathcal{F}[u_{xx}]$$

$$\frac{d}{dt}\underbrace{\mathcal{F}[u_t]}_{\hat{u}} + a i w \hat{u} = -\nu w^2 \hat{u}$$

$$\frac{d\hat{u}}{dt} = -(\nu w^2 + a i w)\,\hat{u}$$

$$\hat{u}(w,t) = \hat{u}(w,0)\,e^{-(aiw + \nu w^2)t}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(w,0)\,e^{-(aiw+\nu w^2)t} \cdot e^{iwx}\,dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(w,0)\,\underbrace{e^{iw(x-at)}}_{1^{st}} \cdot \underbrace{e^{-\nu w^2 t}}_{2^{nd}}\,dw$$

Advection

$e^{iw(x-at)}$ of the form $u = f(x-at)$

which is a solution of $u_t + a u_x = 0$

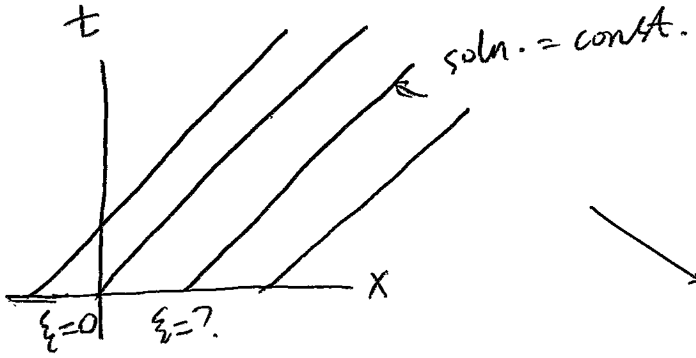
since $\frac{\partial u}{\partial t} = -af'$ $\quad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -af' + af' = 0$ ✓

$\frac{\partial u}{\partial x} = f'$

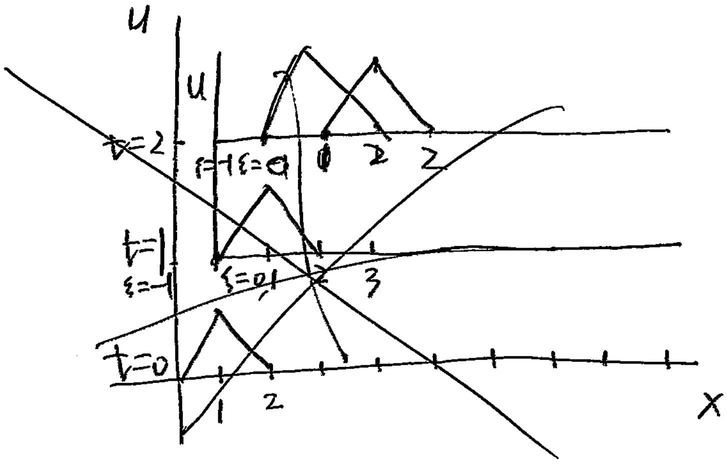
then if the phase is $\xi = x - at$

the soln. at constant phase is a constant. The line

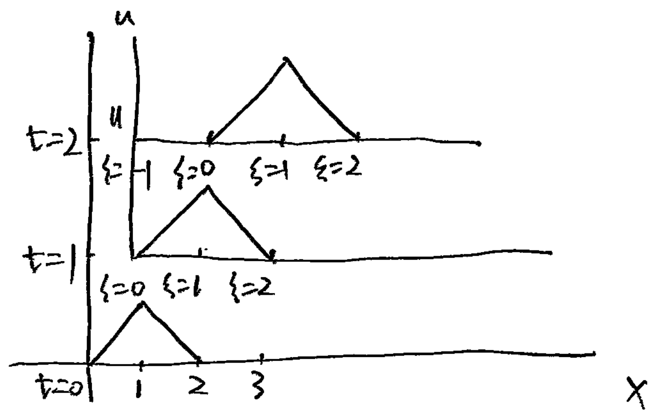
$x - at = \xi$ is a curve in space-time, called "characteristic curves."

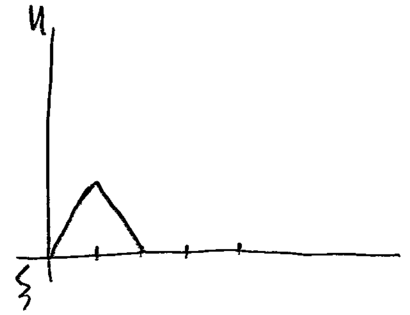


$a=1 \quad \xi = x-at$



shape doesn't change & moves to right with speed a, called advection





moving wave $\xi = x-at$

Advection + Diffusion

take $u(x,0) = \sqrt{2\pi}\,\delta(x-x_0)$

$\uparrow$ dirac delta function.

δ is the function such that $\int_{-\infty}^{\infty} \delta(x-x_0)F(x)dx \equiv F(x_0)$

so $\int_{-\infty}^{\infty} \delta(x-x_0)dx = 1$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(w,t) \cdot e^{iwx} \, dx$$

$$\hat{u}(w,t) = \hat{u}(w,0) e^{iw(x-at)} e^{-vw^2 t}$$

$$\hat{u}(w,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{u(x,\delta)}_{\sqrt{2\pi}\,\delta(x-x_0)} e^{-iwx} \, dx = e^{-iwx_0}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx_0} e^{iw(x-at)} \cdot e^{-vw^2 t} \, dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iw(x-at-x_0)} e^{-vw^2 t} \, dw$$

let $\xi = x - at - x_0$

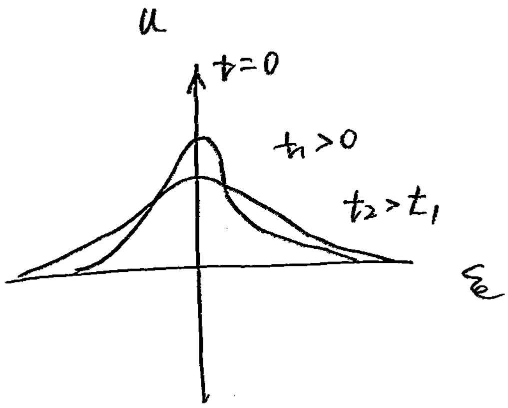
$$u(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iw\xi} (e^{-vw^2 t}) \, dt$$

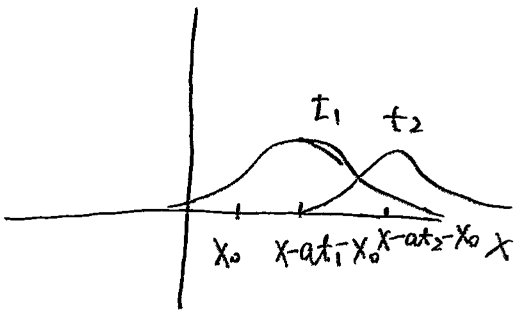
$$u(\xi, t) = \mathcal{F}^{-1}[e^{-vw^2 t}]$$

we had $\mathcal{F}[e^{-w^2/4p}] = \sqrt{2p}\, e^{-px^2}$

let $vt = \frac{1}{4p} \Rightarrow \sqrt{2p} = \frac{1}{\sqrt{2vt}}$

so $u(\xi, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2vt}} e^{-\xi^2/4vt}$





diffusion ~~smooth~~ smooths the soln.

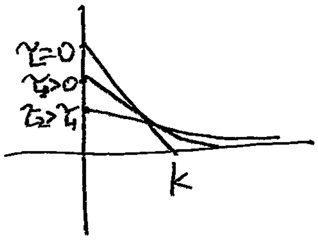
advection moves " "

B-S Equation

let $\tau = T - t$

$$\frac{\partial V}{\partial \tau} - \underbrace{r \cdot S}_{-a} \cdot \frac{\partial V}{\partial S} = \underbrace{\frac{\sigma^2 S^2}{2}}_{V} \frac{\partial^2 V}{\partial S^2} - r \cdot V$$

put



4.2

3.3 Scaling & non-dimensionalization

Any measurable quantity has unit dimensions.

$S \, [=]$ currency, \$, €
$\downarrow$
has dimension of.

$t \, [=]$
$\downarrow$
time

$r \, [=] \rightarrow \frac{1}{\text{time}}$

$V \, [=] \rightarrow$ currency

Equations must be dimensionally consistant.

$$-V_t + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + r\cdot S\cdot V_S - \underbrace{r\,V}_{\frac{1}{\text{time}}\ \text{currency}} = 0$$

$-V_t$ [=] $\frac{\text{currency}}{\text{time}}$

$$\sigma \,[=]\, \frac{1}{\sqrt{\text{time}}}$$

$$\frac{dS}{S} = u\,dt + \sigma\sqrt{dt}\,\phi$$

Aduection – Diffusion

$$u_t + a u_x = \nu u_{xx}$$

u [=] density or sth.

x [=] length

a [=] ? length/time

t [=] time

ν [=] $\frac{\text{length}^2}{\text{time}}$.

two time scales from a & ν

let $\mathcal{L}$ be a length scale.

then $\frac{\mathcal{L}}{a}$ [=] time $\quad \propto \tau_{ad}$

$\frac{\mathcal{L}^2}{\nu}$ [=] time $\quad \propto \tau_{diff}$

Ratio of the scales tell us which process is more important.

The behavior of the soln. depends on the ratio of the time scales

$$\frac{\tau_{diff}}{\tau_{ad}} = \frac{\mathcal{L}^2/\nu}{\mathcal{L}/a} = \frac{\mathcal{L}a}{\nu} = \frac{1}{R}$$

Scaling equations: traditionally

we represent - dimensional variables with α^*

x^* [=] length

t^* [=] time

a^* [=] length/time

And then define nondimensional variables using refererence quantities, eg.

$\mathcal{L}^*$ reference length

Θ^* reference time

$\mathcal{U}^*$ reference value

then $x = x^*/\mathcal{L}^*$ is dimensions

$t = t^*/\Theta^*$ " "

$u = u^*/\mathcal{U}^*$ " "

then we rewrite in terms of new variables

eg. $$\frac{\partial u^*}{\partial t^*} = \frac{\partial (u\, u^*)}{\partial (t\, \Theta^*)} = \frac{u^*}{\Theta^*} \cdot \frac{\partial u}{\partial t}$$

then $$\frac{u^*}{\Theta^*} \cdot \frac{\partial u}{\partial t} + \frac{a^* u^*}{\ell^*} \cdot \frac{\partial u}{\partial x} = \frac{\nu^* u^*}{\ell^{*2}} \cdot \frac{\partial^2 u}{\partial x^2}$$

so. $$\frac{\partial u}{\partial t} + \underbrace{\left(\frac{a^* \Theta^*}{\ell^*}\right)}_{a} \frac{\partial u}{\partial x} = \underbrace{\left(\frac{\nu^* \Theta^*}{\ell^{*2}}\right)}_{\nu} \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} + a u_x = \nu u_{xx}$$ dimensionaless

$$\left(\frac{\nu^* \ell^*}{a^* \cdot \ell^{*2}}\right)$$

But suppose we choose

$$\Theta^* = \left(\frac{a^*}{\ell^*}\right)^{-1} = \frac{\ell^*}{a^*}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\nu^*}{a^* \ell^*} \cdot \frac{\partial^2 u}{\partial x^2} = \frac{1}{R} \frac{\partial^2 u}{\partial x^2}$$

$R \gg 1$ soln. is adecfection dominated

$R \ll 1$ soln. is diffusion dominated.

(4.5)

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \left(\frac{1}{R}\right)\frac{\partial^2 u}{\partial x^2}$$

↑

reynolds #

peclet #

B. S. Eqn. is:

$$\frac{\partial V^*}{\partial t^*} + \frac{1}{2}(\sigma^*)^2(S^*)^2 \cdot \frac{\partial^2 V^*}{\partial (S^*)^2} + r^* S^* \frac{\partial V^*}{\partial S^*} - r^* \cdot V^* = 0$$

$$x = S^*/K^* \qquad V = \frac{V^*}{K^*}$$

$$\tau = (T^* - t^*)/\Theta^*$$

Then.

$$-\frac{K^*}{\Theta^*}\frac{\partial V}{\partial \tau} + \frac{1}{2}(\sigma^*)^2 (K^*)^2 x^2 \frac{K^*}{(K^*)^2}\frac{\partial^2 V}{\partial x^2} + r^* x \cdot \frac{K^* \cdot K^*}{K^*}\frac{\partial V}{\partial x} - r^* K^* V$$

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}\underbrace{(\sigma^*)^2(\Theta^*)}_{\sigma^2} x^2 \cdot \frac{\partial^2 V}{\partial x^2} + \underbrace{(r^*\Theta^*)}_{r} x \cdot \frac{\partial V}{\partial x} - r^*\Theta^* V = 0$$

$$\frac{\partial V}{\partial \tau} - r x \frac{\partial V}{\partial x} - rV = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}$$

if $\Theta^* = T^*$, nothing changes.

if $\Theta^* = \frac{1}{r_0^*}$,

↑

reference interest rate

$$(\sigma^*)^2 \textcircled{H}^* = \frac{(\sigma \cdot \sigma_0^*)^2}{r_0^*} = \sigma^2\left(\frac{\sigma_0^{*2}}{r_0^*}\right) = \frac{\sigma^2 \cdot (\sigma_0^*)^2}{r_0^*}$$

$$r^* \textcircled{H}^* = \frac{r^*}{r_0^*} = r$$

$$\frac{\partial V}{\partial \tau} - rX \cdot \frac{\partial V}{\partial X} - rV = \frac{1}{2} \frac{(\sigma_0^*)^2}{r_0^*} \cdot \sigma^2 X^2 \frac{\partial^2 V}{\partial X^2} \quad \equiv \frac{1}{R}$$

$$\frac{\partial V}{\partial \tau} - rX\frac{\partial V}{\partial X} - rV = \frac{1}{R} \frac{\sigma^2 X^2}{2} \cdot \frac{\partial^2 V}{\partial X^2}$$

we must also scale B.C.S & I.C. for a call

$$\begin{cases} V(X,0) = \max(X-1, 0) \\ V(0,\tau) = 0 \\ V(X,\tau) = X - e^{-r\tau} \end{cases}$$

why scale

(1) option price is independent of the currency

but $V^* = V \cdot K^*$

(2) price doesn't depend on two parameters.

(σ^*, r^*), but only on 1 paratleter: R

(3) for fixed R, price is the same.

eg. double σ^* & quadrature r^*

$\Rightarrow$ R doesn't change $\Rightarrow$ r doesn't change.

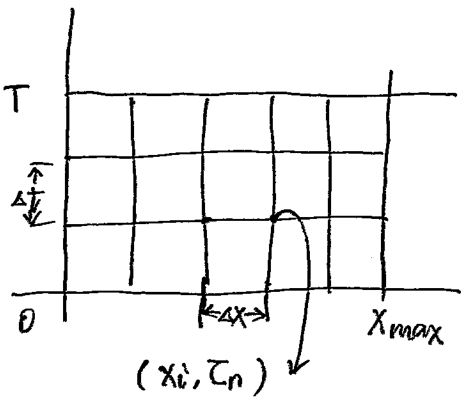
3.4 A finite difference approx. to the B.S Equ.

3.4.1 The big picture.

Initial Boundary Value Problem. IBVP.

$$\begin{cases} \text{IBVP}: \quad V_\tau - rX\,V_X - rV = \dfrac{\sigma^2 X^2}{R} \cdot \dfrac{\partial^2 V}{\partial X^2} \\ \dfrac{1}{R} = \dfrac{(\sigma_0^*)^2}{2r_0^*} \\ V(X,0) = \max(1-X, 0) \\ V(0,\tau) = e^{-r\tau} \\ V(X,\tau) = 0 \quad \text{as} \quad X \to \infty \end{cases}$$

Subdivide (X,τ) into a grid



approx.

$$V_i^n \approx V(X_i, \tau_n)$$

$\tau_n = n\Delta\tau$

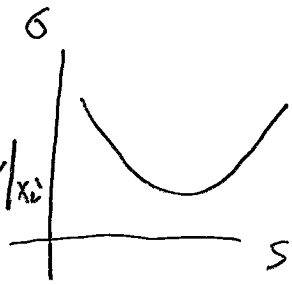
$X_i = i \cdot \Delta X$

we know

$$V_X\Big|_{X_i} \approx \delta_X^0 V(X_i,\tau) + O(\Delta X^2)$$

$$V_{XX}\Big|_{X_i} \approx \delta_X^+ \delta_X^- V(X_i,\tau) + O(\Delta X^2)$$

$$\frac{\partial V(\tau)}{\partial \tau}\Big|_{X_i} - r_i X_i\, \delta_X^0 V\Big|_{X_i} - r_i \cdot V\Big|_{X_i} = \frac{1}{R}\sigma_i^2 X_i^2\, \delta_X^+ \delta_X^- V\Big|_{X_i} + O(\Delta X^2)$$



or $\dfrac{\partial V}{\partial \tau}\Big|_{X_i} = L_h + O(\Delta X^2)$

σ → tends not to be const

$\begin{cases} h \text{ for } \Delta X \\ k \text{ for } \Delta\tau \end{cases}$

+ BC's

$$\begin{cases} v|_{x_0} = \beta^L(\tau) = e^{-r\tau} \\ v|_{x_{N_x}} = \beta^R(\tau) = 0 \end{cases}$$

then create a vector

$\cancel{\vec{u} = \{u_i(\tau)\}_{i=1}^{N_x-1}}$ $\quad\quad$ $\vec{V} = \{V_i(\tau)\}_{i=1}^{N_x-1}$

so, $\dfrac{d}{d\tau}\vec{V} = \hat{L}_h \vec{V}$ $\quad$ sytem of ODEs

$\uparrow$ includes BCs.

A common approximation is trapezoidal rule

→ Crank-Nicholson.

$$\frac{V^{n+1} - V^n}{\Delta t} = \frac{1}{2}\hat{L}_h(\vec{V}^{n+1} + \vec{V}^n)$$

$$\vec{V}^{n+1} = \vec{V}^n + \frac{\Delta t}{2}\hat{L}_h(\vec{V}^{n+1} + \vec{V}^n)$$

$$(I - \frac{\Delta t}{2}\hat{L}_h)\vec{V}^{n+1} = (I + \frac{\Delta t}{2}\hat{L}_h)\vec{V}^n$$

diagnol.