Basics of Trapezoidal and Simpson Rules

Let f be a continuous function on $[a,b]$. We subdivide the interval into n pieces and let $x_0 = a$, $x_1 = a + (b-a)/n$, $x_2 = a + 2(b-a)/n$, $\ldots$, $x_n = a + n(b-a)/n = b$. The Trapezoidal Rule approximation to

$$\int_a^b f(x)\,dx$$

is

$$\frac{b-a}{2n}\Big(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\Big).$$

Note that we are taking a kind of weighted average of values of f at $n+1$ points, $n-1$ of them weighted by 2 and 2 of them weighted by 1. The sum of the weights is thus $2(n-1)+2 = 2n$, which is precisely the denominator. The error bound for this approximation is

$$|\text{error}| \le \frac{\max_{[a,b]}|f''(x)|}{12n^2}(b-a)^3.$$

The Simpson's Rule approximation to the integral (assuming n even) is

$$\frac{b-a}{3n}\Big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big).$$

Again, the sum of the weights (all 1, 2, or 4) in the numerator is the denominator, $3n$. The error bound for this approximation is

$$|\text{error}| \le \frac{\max_{[a,b]}|f''''(x)|}{180n^4}(b-a)^5.$$

Example. Say we want to approximate $\int_0^1 x^4\,dx = 1/5$. Take $a=0$, $b=1$, $x_j = j/n$, $f(x) = x^4$. Here $f''(x) = 12x^2$, with maximum value of 12. So the error bound in the Trapezoidal Rule is $1/n^2$. Since $f''''(x) = 24$, the error bound in Simpson's Rule is

$$\frac{24}{180n^4} = \frac{2}{15n^4}.$$

So suppose we want accuracy to 4 decimal places, that is, an error no bigger than 10^{-4}. To guarantee this with the Trapezoidal Rule, we could take n big enough so that $1/n^2 \leq 10^{-4}$, or $n^2 \geq 10^4$. So $n = 100$ would work. But to guarantee this with Simpson's Rule, it would suffice to choose n so that $15n^4 \geq 20000$, or $n^4 \geq 1334$. For this, $n = 6$ almost suffices, and we certainly could get the desired accuracy with $n = 8$.

Indeed, we find that the trapezoidal rule with $n = 100$ gives the approximation 0.200033333 to the integral, good to 4 but not to 5 decimal places, while Simpson's rule with $n = 6$ gives 0.200102881 and Simpson's rule with $n = 8$ gives 0.200032552 (very slightly better than the trapezoidal rule with $n = 100$). So certainly with smooth integrands like x^4, Simpson's rule is much more efficient.