MAP 6437-4 - Monte Carlo Methods in Financial Mathematics - Spring 2010
Homework 2: Statistical Tests
Due Date: Feb 5, 2010

1. Compute $D_N = \max(D^+, D^-)$ for the data set $x_1 = 0.2, x_2 = 0.6, x_3 = 0.7$. Take F to be the c.d.f. of $U(0,1)$; the uniform distribution on $(0,1)$.(Do these computations by hand - no computer code.) What do you think D^+, D^-, D_N measure, intuitively?

2. Prove

$$\max\left[F(X_1), \max_{k=1,\ldots,N-1}\left(F(x_{k+1}) - \frac{k}{N}, \frac{k}{N} - F(x_k)\right), 1 - F(X_N)\right]$$
$$= \max\left[\max_{k=1,\ldots,N}\left(\frac{k}{N} - F(x_k)\right), \max_{k=1,\ldots,N}\left(F(x_k) - \frac{k-1}{N}\right)\right]$$

which was used in the derivation of D_N; see your lecture notes.

3. Consider the MCG with parameters: $a = 23, M = 10^8+1$, and let the seed be 47594118. (This is the original MCG proposed by Lehmer in 1948.) Apply the Kolmogorov-Smirnov test to the first 1000 random numbers (including the seed) from this generator. Compute the KS-statistic and find its p-value. What is your conclusion for the generator?

4. Implement an obviously "bad" random number generator of your choice - you should explain why it is bad. Then apply the χ^2-test together with the KS-statistic as explained in Remark 2, part 3. Take $M = 10, N = 1000$, and $k = 10$. What are your conclusions?

5. Write a computer program for the gap test and apply it to the Fibonacci generator. See the previous lecture notes and examples for the parameters of the generator. I want you to skip the first 100 terms of the generator, and start with the 101st term. Use $J = (0.3, 0.8)$ as the subinterval. All the other parameters of the test should be the same as the parameters discussed in Gap Test Examples. What is your conclusion?

6. Do you think we can apply the χ^2- test directly to the run-up counts, assuming that we know the probability of run-up of length i? (Hint: think about the basic assumptions required to apply the χ^2- test, and investigate whether these assumptions are satisfied by "run-up events".

7. Implement the collision test and apply it to the classical MCG by Lehmer that was discussed in the previous lecture examples. Use $M = 10,000$ and $N = 2000$. Let the seed of the MCG be 47594118. You should apply the test to the first 2000 numbers, including the number that corresponds to the seed (i.e., 0.475941). What is your conclusion?

8. Design a statistical test for random number generators, based on the following result. Then apply the test to any generator you want and explain the results.
FACT: A coin is flipped consecutively until the number of heads obtained equals the number of tails. The output of a flip is heads with probability p. Define the random

variable X as: X = the first time the total number of heads is equal to the total number of tails. Observe that X takes values 2,4,6,.... For example, if the outcomes of one experiment are: H,H,T,H,T,T then the value of X for this outcome is 6. Here is the probability density function of X :

$$P\{X = 2n\} = \frac{1}{2n-1}\binom{2n}{n}p^n(1-p)^n.$$

(For the interested student, a proof of this statement can be found in "Introduction to Probability Models", Sheldon Ross, 8th edition, page 128.)