Derivatives Pricing with Market Impact and Limit Order Book *

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Abstract

This paper investigates derivatives pricing under existence of liquidity costs and market impacts for the underlying asset in continuous time. Firstly, we formulate the charge for the liquidity cost and the market impact on the derivatives prices through a stochastic control problem that aims to maximize the mark-to-market value of the portfolio less the quadratic hedging error during the hedging period and the liquidation cost at maturity. Then, we obtain the derivatives price by reduction of this charge from the premium in the Bachelier model. Next, we solve a second order semilinear PDE of parabolic type reduced from the HJB equation for the control problem, which is analytically solved or approximated by an asymptotic expansion around a solution to an explicitly solvable nonlinear PDE. We also present numerical examples of the pricing for a quadratic payoff and a European call payoff in different settlement types, and show comparative static analyses.

1 Introduction

In this paper, we consider derivatives pricing under existence of liquidity costs and market impacts for the underlying asset, which are caused by the trader's hedging transactions. After formulating the charge for the liquidity cost and the effect on the derivatives prices by market impacts through a stochastic control problem, we provide a scheme to compute its value function which is solved analytically or approximated by an asymptotic expansion of a second order semilinear PDE of parabolic type. This asymptotic expansion is novel in that the expansions are made around an explicitly solvable semilinear PDE. This is different from previous works on asymptotic expansions in option pricing such as [8], [10], [13], [14], [15], [16], which typically make expansion around linear PDEs.

Estimation of the total liquidity costs during the hedging period is the most essential factor in pricing in practice, since banks may make losses by the price spreads which they pay in every hedging transaction. Also, prediction of the effect of market impacts on the

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hedging costs is important, especially when banks trade derivatives on illiquid underlying assets such as low liquidity stocks and illiquid currency pairs. Particularly, these matter in pricing exotic derivatives that require hedging with illiquid instruments. When banks quote derivatives prices in biddings, the estimation of these costs is occasionally the only factor that differentiates their prices from their competitors'. Despite these importance, they are usually done by traders' rules of thumb. This study provides a quantitative method to estimate these quantities.

Related literatures such as Guéant and Pu [5] and Li and Almgren [9] also deal with option pricing under existence of liquidity costs and market impacts. Guéant and Pu [5] adopts an expected exponential utility for the objective function to be maximized. After deriving the corresponding HJB partial differential equation for the optimization problem, Guéant and Pu [5] solves the HJB equation numerically with finding a maximum point at every grid in the discretized equation. On the other hand, by assuming the mark-to-market value of the hedging portfolio at maturity less the quadratic hedging error and the liquidation cost at maturity for the objective function to be maximized, we solve the optimization part in the HJB equation explicitly and reduce the equation to a second order semilinear PDE of parabolic type. Then, depending on the payoff type, we analytically solve the PDE or asymptotically expand the solution of the semilinear PDE up to the first order. In detail, the zeroth order part corresponds to a solution of a solvable semilinear PDE, which has a quadratic expression with coefficients satisfying an ODE or a linear PDE, and the first order part is a solution of a second order linear PDE. We solve the system of the ODE and PDEs through stochastic representations of the solutions by Feynman-Kac formula.

Our study extends the setting of Li and Almgren [9] which is originally based on Rogers and Sing [11], and provides an explicit calculation method and numerical examples. Our setting is different from the one in Li and Almgren [9] in the following points. First, while Li and Almgren [9] considers an intraday hedging problem of an option, we deal with the problem for the whole trading period from the starting date to the maturity. Second, while Li and Almgren [9] puts a strong assumption that the derivatives Gamma is a constant and does not take into account the liquidation cost of the underlying asset at the end of the day, we do not impose any assumptions on the derivatives Gamma and take into account the liquidation cost at maturity in the stochastic control problem. We also take into consideration a small execution error in every hedging transaction, which makes the semilinear parabolic PDE nondegenerate, in contrast to the PDE of degenerate type in Li and Almgren [9]. In this setting, with some suitable assumptions, the existence of a unique classical solution of the semiliear PDE is guaranteed. Third, in Li and Almgren [9], although an explicit expression of the solution of a semilinear PDE under their setting is obtained, it is not even solved numerically. Li and Almgren [9] only shows a direction to solve the PDE in some special cases such as no market impact case and a constant gamma case in intraday trading. Our study provides the first order asymptotic expansion of the solution to the semilinear PDE together with its stochastic representations computable by Monte Carlo simulation, even when the PDE is not analytically solvable. With this solution, we obtain the derivatives prices when the market impacts and the liquidity costs for the underlying asset exist.

In concrete examples, we provide derivatives prices for a quadratic payoff in physical settlement, which are computed analytically, as well as those for a European call payoff in physical and cash settlement, which are obtained through the asymptotic expansion. The derivatives with a quadratic payoff is an important example since the payoff corresponds to a variance contract that pays realized variance of the underlying asset at maturity. Compared to the European option payoff, the effect of the market impact on the derivatives price for the quadratic payoff is larger, since there is a constant derivatives Gamma in every underlying asset level for the quadratic payoff as opposed to for the European call payoff. In the quadratic payoff, traders have to rebalance their delta positions more often than those in the European call payoff as the underlying asset price is moved by the market impact due to the delta hedging.

This paper is organized as follows. After Section 2 introduces the setup and the stochastic control problem, Section 3 investigates the existence of a unique classical solution for the corresponding HJB equation. Section 4 provides the first order asymptotic expansion of the classical solution as a combination of solutions of a solvable semilinear PDE and a second order linear parabolic PDE. Section 5 shows stochastic representations of the approximate solution by Feynman-Kac formula and Section 6 provides concrete examples on the derivatives prices for the quadratic payoff and the European call payoff in physical settlement. Finally, Section 6 concludes. Appendix A shows that the conditions for the existence of a unique classical solution for the HJB equation in Section 3 are satisfied. Appendix B presents the asymptotic expansion for the European call payoff in cash settlement. Appendix C provides the poof of a proposition in Section 5.3 which indicates that the option price in this study agrees with the well-known expression in the perfect market for the case of the Bachelier model.

2 Setting

In this section, with some extensions of the one in Li and Almgren [9] and Rogers and Singh [11], we introduce the setting of the optimal hedging problem. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbf{P})$ be the filtered probability space satisfying the usual conditions. Let $[0, T]$ be the trading period, where 0 is the initial time of the trading and T is the maturity of a derivatives. Let θ_t be a $\{\mathcal{F}_t\}$-adapted process which satisfies $\mathbf{E}\left[\int_0^T \theta_s^2 ds\right] < \infty$. We assume that the trader buys the underlying asset at the speed of θ_t for hedging at time t. Let $(W_t, W_t^{\perp})$ be a two dimensional $\{\mathcal{F}_t\}$-Brownian Motion. Let $\lambda, \sigma, \delta, \epsilon$, and γ be positive constants, and let $0 \leq \epsilon \leq 1$.

We investigate a hedging problem of a derivatives for a trader, who is a sole large investor whose hedging activity impacts on the market price of the underlying asset. We consider an economy that consists of a money market account and the underlying asset, where we assume that the price of the money market account is always 1. We suppose that the trader buys a derivatives at time 0 by paying m_0 as the derivatives premium, and holds the long position of the derivatives until T. Considering the case of selling the derivatives, we take the opposite sign for the derivatives payoff and premium. We also suppose that the trader begins to hedge with the underlying asset position X_0, which is considered to be the initial delta exchange with the seller for the derivatives. Let P_t be the mid price process of the underlying asset, which is a $\{\mathcal{F}_t\}$-adapted process satisfying

$$P_t = P_0 + \sigma W_t + \epsilon \int_0^t \theta_s ds,\ 0 \leq t \leq T. \tag{1}$$

This indicates that there is an accumulated market impact of $\epsilon \int_0^t \theta_s ds$ on the price process. We can interpret that $\int_0^t \theta_s ds$ is the total amount of hedging order submitted by the trader and the market impact is proportional to this amount with the proportional constant ϵ. Note that if $\epsilon = 0$, P_t follows a Gaussian process.

Let $\tilde{P}(\theta_t)$ be the execution price when the trader submits a buying order of $\theta_t dt$ for the underlying asset during the period from t to $t + dt$, which satisfies

$$\tilde{P}(\theta_t) = P_t + \eta\theta_t. \tag{2}$$

This implies that η corresponds to thinness of the order book: if η is high, the spread between the execution price and the mid price is large. Note also that this corresponds to a limit order book where the order density increases uniformly in prices by $\frac{1}{2\eta}$ per unit of time. When the trader buys θ_t amount of the underlying asset per unit of time, the trader takes the offer orders from the mid price P_t to the maximum price $P_t + 2\eta\theta_t$, which results in buying at the average price $P_t + \eta\theta_t$. (For the relationship between the limit order book density and the price spread, see Saito [12] for instance.) Note that (1) and (2) correspond to equations (1) and (2) in Li and Almgren [9].

Let $h : \mathbf{R} \to \mathbf{R}$ be the derivatives payoff function, which satisfies an exponential growth condition and represents the mid value of the derivatives payoff at T as a function of the mid underlying price x_1. Let $g : [0, T] \times \mathbf{R} \to \mathbf{R}$ be the unique solution of the PDE

$$g_t + \frac{1}{2}\sigma^2 g_{x_1x_1} = 0, \tag{3}$$

$$g(T, x_1) = h(x_1), \tag{4}$$

satisfying an exponential growth condition: there exist positive constants C and k such that

$$|g(t, x_1)| \leq C \exp(k|x_1|^2), \ 0 \leq \forall t \leq T, \ \forall x_1 \in \mathbf{R}. \tag{5}$$

Here we have denoted $\frac{\partial}{\partial t} g$, $\frac{\partial^2}{\partial x_1 x_1} g$ by g_t, g_{x_1,x_1}. Hereafter, we use this notation with subscripts for partial derivatives. Note that g is given by

$$g(t, x_1) = \int_{\mathbf{R}} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left(-\frac{(x_1 - \xi)^2}{2\sigma^2(T-t)}\right) h(\xi)d\xi, \ 0 \leq \forall t < T, \ \forall x_1 \in \mathbf{R}. \tag{6}$$

(For instance, see Theorem 1.12 and 1.16 in Friedman [2].)

Let $\Gamma(t, x_1) = g_{x_1x_1}(t, x_1)$. We further assume the following.

Assumption 1. *There exist positive constants c_1 and c_2 such that*

$$c_1 \leq |\Gamma(t, x_1)| \leq c_2, \ \forall t \in [0, T], \ \forall x_1 \in \mathbf{R}. \tag{7}$$

Moreover, $\Gamma(t, x_1)$ is of class $C^{1,2}([0, T] \times \mathbf{R})$ and $\Gamma_{x_1}(t, x_1)$ is bounded on $[0, T] \times \mathbf{R}$.

Example 1. For $K > 0$, $h(x_1) = \frac{1}{4}(x_1 - K)^2 + \frac{1}{2}(x_1 - K) + \frac{1}{4}$ satisfies Assumption 1. In fact, by Feynman-Kac Theorem (e.g. Theorem 4.4.2 in Karatzas and Shreve [7]),

$$g(t, x_1) = \mathbf{E}[h(x_1 + \sigma W_{T-t})], \tag{8}$$

$$\Gamma(t, x_1) = g_{x_1x_1}(t, x_1) = \mathbf{E}[h''(x_1 + \sigma W_{T-t})] = \frac{1}{2}, \tag{9}$$

$$\Gamma_{x_1}(t, x_1) = g_{x_1x_1x_1}(t, x_1) = \mathbf{E}[h'''(x_1 + \sigma W_{T-t})] = 0. \tag{10}$$

Example 2. For $l, \delta > 0$ and $0 < c_1 < \frac{1}{2l}$, let

$$\begin{cases} h(x_1) = \frac{l}{4} + \int_K^{x_1} (\frac{1}{2} + \int_K^v h''(s)ds)dv \ (x_1 \leq K) \\ h(x_1) = \frac{l}{4} + \int_K^{x_1} \int_K^v h''(s)dsdv \ (x_1 > K), \end{cases} \tag{11}$$

where

$$h''(x_1) = c_1 + (\frac{1}{2l} - c_1) g_{l,\delta}(x_1 - K). \tag{12}$$

Here, $g_{l,\delta}(x_1)$ is a function of class $\mathcal{C}^\infty(\mathbf{R})$ satisfying

$$g_{l,\delta}(x_1) = \begin{cases} 1 \ (-l \leq x_1 \leq l), \\ 0 \ (x_1 \leq -l-\delta, \ l+\delta \leq x_1), \end{cases} \tag{13}$$

$$0 \leq g_{l,\delta}(x_1) \leq 1 \ (-l-\delta \leq x_1 \leq -l, \ l \leq x_1 \leq l+\delta). \tag{14}$$

In detail, we can take such $g_{l,\delta}(x_1)$ as

$$g_{l,\delta}(x_1) = \begin{cases} \frac{\phi(x_1-(-l-\delta))}{\phi(x_1-(-l-\delta))+\phi(-l-x_1)} \ (-l-\delta \leq x_1 \leq -l) \\ \frac{\phi(l+\delta-x_1)}{\phi(l+\delta-x_1)+\phi(x_1-l)} \ (\ l \leq x_1 \leq l+\delta), \end{cases} \tag{15}$$

where $\phi(x_1) = \exp(-\frac{1}{x}) \ (x_1 \geq 0), \ 0 \ (x_1 < 0)$. Then, $g(t, x_1)$ satisfies Assumption 1. Note that $g(t, x_1)$ is interpreted as an approximation of the call option payoff $(x_1 - K)^+$, since taking limit as $l, \delta, c_1 \downarrow 0$, $g(t, x_1)$ converges to $(x_1 - K)^+$.

Let X_t be the position of the underlying asset for the trader, which is a $\{\mathcal{F}_t\}$-adapted process satisfying

$$X_t = X_0 + \int_0^t \theta_s ds + \delta \int_0^t \sigma\Gamma(t, P_t) dW_t^\perp. \tag{16}$$

We interpret this as follows. The trader starts hedging with the initial position X_0 for the underlying asset, and changes the position absolutely continuously with respect to t at the speed of θ_t. Moreover, there exists an small execution error $\delta \int_0^t \sigma\Gamma(t, P_t)dW_t^\perp$, which accumulates proportionally to $\sigma\Gamma(t, P_t)dW_t^\perp$ during the period from t to $t + dt$ with the proportional constant δ. This also indicates that the size of the execution error is subject to a noise $dW_t^\perp$ independent of dW_t in the price process (1), and when the market is volatile and the derivatives' gamma is high, the execution error is large.

Let R_t be the mid mark to market value process of the trader's portfolio satisfying

$$\begin{aligned} R_t &= g(t, P_t) + X_t P_t - \left(m_0 + X_0 P_0 + \int_0^t \tilde{P}(\theta_s)\theta_s ds \right) \\ &= g(t, P_t) + X_t P_t - m_0 - X_0 P_0 - \int_0^t P_s \theta_s ds - \eta \int_0^t \theta_s^2 ds, \end{aligned} \tag{17}$$

where the first, second and third terms in the first line represent the mid mark to market values of the option, the underlying asset and the money market account positions, respectively.

Let Y_t be the difference between X_t, the underlying asset position, and $(-g_{x_1}(t, P_t))$, the delta amount to hold against the long option position if there is no market impact:

$$Y_t = X_t - (-g_{x_1}(t, P_t)). \tag{18}$$

First note that by (1),

$$dP_t = \epsilon\theta_t dt + \sigma dW_t. \tag{19}$$

Applying Ito's formula to (17),(18), we have

$$\begin{aligned}
dR_t = g_t(t, P_t)dt + g_{x_1}(t, P_t)dP_t + \frac{1}{2}g_{x_1x_1}(t, P_t)d\langle P\rangle_t + X_t dP_t + P_t\theta_t dt + \delta P_t\sigma\Gamma(t, P_t)dW_t^{\perp} \\
- P_t\theta_t dt - \eta\theta_t^2 dt \\
= (g_{x_1}(t, P_t) + X_t)dP_t + \delta P_t\sigma\Gamma(t, P_t)dW_t^{\perp} - \eta\theta_t^2 dt \\
= Y_t dP_t + \delta P_t\sigma\Gamma(t, P_t)dW_t^{\perp} - \eta\theta_t^2 dt,
\end{aligned} \tag{20}$$

$$\begin{aligned}
dY_t &= g_{tx_1}(t, P_t)dt + g_{x_1x_1}(t, P_t)dP_t + \frac{1}{2}g_{x_1x_1x_1}(t, P_t)d\langle P\rangle_t + \theta_t dt + \delta\sigma\Gamma(t, P_t)dW_t^{\perp} \\
&= \Gamma(t, P_t)dP_t + \theta_t dt + \delta\sigma\Gamma(t, P_t)dW_t^{\perp} \\
&= \theta_t(1 + \epsilon\Gamma(t, P_t))dt + \sigma\Gamma(t, P_t)dW_t + \delta\sigma\Gamma(t, P_t)dW_t^{\perp} \\
&= \theta_t(1 + \epsilon\Gamma(t, P_t))dt + \sqrt{1+\delta^2}\sigma\Gamma(t, P_t)d\tilde{W}_t \\
&= \theta_t(1 + \epsilon\Gamma(t, P_t))dt + \tilde{\sigma}\Gamma(t, P_t)d\tilde{W}_t,
\end{aligned} \tag{21}$$

where

$$\tilde{\sigma} = \sqrt{1+\delta^2}\sigma, \tag{22}$$

$$\tilde{W}_t = \frac{1}{\sqrt{1+\delta^2}}(W_t + \delta W_t^{\perp}). \tag{23}$$

In the second lines of (20),(21), we used (3) and Assumption 1.

Let $\psi : \mathbf{R}^2 \to \mathbf{R}$ be a penalty function, which stands for the cost to liquidate the mismatch of the underlying asset position at the maturity T. We assume that ψ is of class $C^3(\mathbf{R}^2)$ and ψ and ψ_x satisfy a polynomial growth condition. For example, if the option is cash-settled and ends up in-the-money at the maturity, a trader has to liquidate the underlying asset position holding for delta hedging. Similarly, if the option is physically settled, a trader has to clear the net underlying asset position, which is the sum of the hedging position and the delivered amount, at the maturity.

We suppose that there exists a counterparty who undertakes the liquidation amount from the trader instantly at T with the price spread proportional to the liquidation amount with the proportional constant k_1.

In cash settlement where the underlying asset position is liquidated at the delivery,

$$\begin{aligned}
\psi(P_T, Y_T) &= k_1(Y_T - g_{x_1}(T, P_T))^2 \\
&= k_1(Y_T^2 - 2g_{x_1}(T, P_T)Y_T + g_{x_1}(T, P_T)^2),
\end{aligned} \tag{24}$$

since $X_T = Y_T - g_{x_1}(T, P_T)$ is the amount to be liquidated.

In physical settlement where the delta amount of the underlying asset is delivered at the maturity,

$$\psi(P_T, Y_T) = k_1 Y_T^2, \tag{25}$$

since Y_T defined by (18), the difference between the underlying asset position and the delta amount, is the amount to be cleared.

Next, let us define:

$$f(t, x_1, \theta) = \begin{pmatrix} \epsilon\theta \\ (1 + \epsilon\Gamma(t, x_1))\theta \end{pmatrix}, \tag{26}$$

$$\sigma(t, x_1) = \begin{pmatrix} \sigma & 0 \\ \sigma\Gamma(t, x_1) & \delta\sigma\Gamma(t, x_1) \end{pmatrix}, \tag{27}$$

$$a(t, x_1) = \sigma\sigma', \tag{28}$$

$$L(t, x_1, x_2, \theta) = \eta\theta^2 - \epsilon x_2\theta + \lambda\sigma^2 x_2^2 + \delta^2\lambda\sigma^2 x_1^2 \Gamma(t, x_1)^2. \tag{29}$$

Then, we assume that we aim to minimize the following amount in this study by choosing a hedging strategy $\{\theta_s\}_{0 \le s \le T}$ among all admissible feedback controls.

$$\mathbf{E}\left[\int_0^T L(s, P_s, Y_s, \theta_s) ds + \psi(P_T, Y_T)\right], \tag{30}$$

subject to

$$\begin{pmatrix} dP_s \\ dY_s \end{pmatrix} = f(s, P_s, \theta_s) ds + \sigma(s, P_s) \begin{pmatrix} dW_s \\ dW_s^{\perp} \end{pmatrix}. \tag{31}$$

Here, for given (t, x_1, x_2), we define an admissible feedback control as a map $\theta : [0, T] \times \mathbf{R} \to U$ satisfying the following.

1. There exists a two dimensional Brownian Motion $(W, W^{\perp})$ such that a solution $(Y_s, P_s),\ t \le s \le T$ of the following SDE exists uniquely in law. Namely, a unique weak solution exists for

$$\begin{pmatrix} dP_s \\ dY_s \end{pmatrix} = f\left(s, P_s, \theta(s, P_s, Y_s)\right) ds + \sigma(s, P_s) \begin{pmatrix} dW_s \\ dW_s^{\perp} \end{pmatrix},\ t \le s \le T, \tag{32}$$

$$(P_t, Y_t) = (x_1, x_2). \tag{33}$$

2. For any $k > 0$, $\mathbf{E}^{(t,x_1,x_2)}\left[|(P_s, Y_s)|^k\right]$ is bounded for $t \le s \le T$, and

$$\mathbf{E}^{(t,x_1,x_2)}\left[\int_t^T |\theta(s, P_s, Y_s)|^k ds\right] < \infty. \tag{34}$$

Let $\mathcal{V}$ be the set of all admissible controls. Since

$$\mathbf{E}[R_T] = g(0, P_0) - m_0 + \mathbf{E}\left[\int_0^T (-\eta\theta_s^2 + \epsilon Y_s \theta_s) ds\right], \tag{35}$$

$$\mathbf{E}\left[\lambda \langle R \rangle_T\right] = \mathbf{E}\left[\lambda \int_0^T \left(\sigma^2 Y_s^2 + \delta^2 \sigma^2 P_s^2 \Gamma(s, P_s)^2\right) ds\right], \tag{36}$$

we have

$$\begin{aligned}
&\mathbf{E}\left[R_T - \lambda\langle R\rangle_T - \psi(P_T, Y_T)\right] \\
&= g(0, P_0) - m_0 - \mathbf{E}\left[\lambda \int_0^T \left(\sigma^2 Y_s^2 + \delta^2\sigma^2 P_s^2 \Gamma(s, P_s)^2\right) ds + \int_0^T (\eta\theta_s^2 - \epsilon Y_s\theta_s) ds + \psi(P_T, Y_T)\right] \\
&= g(0, P_0) - m_0 - \mathbf{E}\left[\int_0^T L(s, P_s, Y_s, \theta_s) ds + \psi(P_T, Y_T)\right]. \qquad (37)
\end{aligned}$$

This equation explains that for given P_0, X_0 and m_0, we maximize the expectation of the mid mark to market value of the portfolio at the maturity in (35), less the quadratic variation of the mid portfolio value rescaled by a constant λ in (36) and the mismatch cost of the underlying asset in (24) or (25). Equivalently, we minimize the expectation of the rescaled quadratic variation, the liquidity costs together with the small market impact, and the liquidation cost of the underlying asset at the maturity:

$$\inf_{\theta \in \mathcal{V}} \mathbf{E}\left[\int_0^T L(s, P_s, Y_s, \theta_s) ds + \psi(P_T, Y_T)\right], \tag{38}$$

subject to

$$\begin{pmatrix} dP_s \\ dY_s \end{pmatrix} = f(s, P_s, \theta_s) ds + \sigma(s, P_s) \begin{pmatrix} dW_s \\ dW_s^\perp \end{pmatrix}. \tag{39}$$

Note that if Y_s and θ_s have the same sign, when the trader's underlying asset position X_s is more short or long position than $-g_x(s, P_s)$ and the trader's optimal hedging is to accelerate the deviation, the market impact effect is a gain and offsets with the liquidity cost. Otherwise, if the trader's optimal hedging is to narrow the position gap between X_s and $-g_x(s, P_s)$, the market impact effect is also a cost.

3 HJB Equation and its Classical Solution

Let U be a compact subset of $\mathbf{R}$ and $u : [0, T] \times \mathbf{R}^2 \to \mathbf{R}$. In this section we consider existence of a classical solution for the following HJB equation corresponding to the minimization problem (38), (39):

$$\begin{aligned}
0 = u_t &+ \frac{1}{2}\sigma^2 u_{x_1 x_1} + \sigma^2 \Gamma(t, x_1) u_{x_1 x_2} + \frac{1}{2}\tilde{\sigma}^2 \Gamma(t, x_1)^2 u_{x_2 x_2} + \lambda\sigma^2 (x_2^2 + \delta^2 x_1^2 \Gamma(t, x_1)^2) \\
&+ \inf_{\theta \in U}\left[\left\{(1 + \epsilon\Gamma(t, x_1)) u_{x_2} + \epsilon u_{x_1} - \epsilon x_2\right\}\theta + \eta\theta^2\right], \\
&u(T, x_1, x_2) = \psi(x_1, x_2). \qquad (40)
\end{aligned}$$

Let $\Theta(t, x_1, \theta) = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}$ with

$$\Theta_1 = \frac{\epsilon}{\sigma}\theta, \tag{41}$$

$$\Theta_2 = \frac{\theta}{\delta\sigma\Gamma(t, x_1)}. \tag{42}$$

Note that

$$f(t, x_1, \theta) = \sigma(t, x_1)\Theta(t, x_1, \theta), \tag{43}$$

$$a(t, x_1) = \begin{pmatrix} \sigma^2 & \sigma^2\Gamma(t, x_1) \\ \sigma^2\Gamma(t, x_1) & (1 + \delta^2)\sigma^2\Gamma^2(t, x_1) \end{pmatrix}. \tag{44}$$

Let $Q_0 = (0, T) \times \mathbf{R}^2$ and $\bar{Q}_0$ be the closure of Q_0. For $Q \subseteq \bar{Q}_0$, let $C^{1,2}(Q)$ be the space of functions $\phi(t, x)$ continuous on Q together with the partial derivatives ϕ_t, ϕ_{x_i} and ϕ_{x_i,x_j}, $i, j = 1, 2$ continuous on Q. Let $C_p^{1,2}(Q)$ be the class of $\phi \in C^{1,2}(Q)$ which satisfy a polynomial growth condition on Q. We say that ϕ satisfies a polynomial growth condition on Q if, for some constants D and k, $|\psi(t, x)| \leq D(1 + |x|^k)$ when $(t, x) \in Q$. In this setting, the following conditions for Theorem VI.6.2 in Fleming and Rishel [1] are satisfied and a unique classical solution $u \in C_p^{1,2}(\bar{Q}_0)$ exists as a result. See Appendix A for the details.

- Uniform parabolicity for the matrices $a(t, x_1)$. That is, $a(t, x_1)$ are symmetric non-negative definite and their characteristic values are bounded below by some $c > 0$.

$$\frac{1}{2}a_{1,1}(t, x_1)\xi_1^2 + \frac{1}{2}a_{2,2}(t, x_1)\xi_2^2 + a_{1,2}(t, x_1)\xi_1\xi_2 \geq c|\xi|^2, \ \forall t \in [0, T], \forall x_1, x_2 \in \mathbf{R}, \forall \xi \in \mathbf{R}^2. \tag{45}$$

- $\sigma(t, x_1) \in C^{1,2}(\bar{Q}_0)$.
- σ, σ^{-1} and σ_x are bounded on $\bar{Q}_0$.
- $\Theta \in C^1(\bar{Q}_0 \times U)$ with bounded Θ and Θ_x.
- $L \in C^1(\bar{Q}_0 \times U)$ with L and L_x satisfying a polynomial growth condition.
- $\psi \in C^2(\mathbf{R}^2)$ with ψ and ψ_x satisfying a polynomial growth condition.

We further assume the following.

Assumption 2. *There exists a compact set $U \subset \mathbf{R}$ such that for the unique classical solution u of (40), $-\frac{1}{2\eta}((1 + \epsilon\Gamma(t, x_1))u_{x_2} + \epsilon u_{x_1} - \epsilon x_2) \in U$, $0 \leq \forall t \leq T$, $\forall x_1, x_2 \in \mathbf{R}$, meaning that $-\frac{1}{2\eta}((1 + \epsilon\Gamma(t, x_1))u_{x_2} + \epsilon u_{x_1} - \epsilon x_2)$ is an inner point in U and* inf *in (40) is attained at this point.*

As a result, the uniqueness and existence of the classical solution of the below semilinear PDE follows.

$$
\begin{aligned}
0 =& u_t + \frac{1}{2}\sigma^2 u_{x_1,x_1} + \sigma^2\Gamma(t,x_1)u_{x_1,x_2} + \frac{1}{2}\tilde{\sigma}^2\Gamma(t_1,x_1)^2 u_{x_2,x_2} \\
&+ \lambda\sigma^2(x_2^2 + \delta^2 x_1^2\Gamma(t,x_1)^2) - \frac{1}{4\eta}[(1+\epsilon\Gamma(t,x_1))u_{x_2} + \epsilon u_{x_1} - \epsilon x_2]^2,
\end{aligned}
$$

$$
u(T,x_1,x_2) = \psi(x_1,x_2). \tag{46}
$$

Then, we note that the value function of the minimization problem and its optimal hedging strategy are obtained by a verification theorem. (e.g. Theorem VI.4.1 in Fleming and Rishel [1].) Let

$$
J(t,x_1,x_2;\theta) = \mathbf{E}^{(t,x_1,x_2)}\left[\int_t^T L(s,P_s,Y_s,\theta_s)ds + \psi(P_T,Y_T)\right], \tag{47}
$$

where $\theta : [0,T]\times\mathbf{R} \to U$ is an admissible feedback control.

Then, for the classical solution u of (46) and any admissible feedback control θ,

$$
u(t,x_1,x_2) \le J(t,x_1,x_2;\theta). \tag{48}
$$

Note that by Assumption 1, the sufficient condition (iii) for the admissibility for a feedback control in p.156 in Fleming and Rishel [1] is satisfied, and

$$
\theta^*(t,x_1,x_2) = -\frac{1}{2\eta}\left((1+\epsilon\Gamma(t,x_1))u_{x_2}(t,x_1,x_2) + \epsilon u_{x_1}(t,x_1,x_2) - \epsilon x_2\right) \tag{49}
$$

is an admissible feedback control. By the verification theorem, θ satisfies

$$
u(t,x_1,x_2) = J(t,x_1,x_2;\theta^*). \tag{50}
$$

4 Asymptotic Expansion of the Semilinear PDE

In this section, we consider the first order expansion of the classical solution of (46), the Cauchy problem of the second order semilinear PDE of parabolic type, by asymptotic expansion. Let η be a positive constant and $b_1, b_2 : \mathbf{R} \to \mathbf{R}$ be functions of class $\mathcal{C}^2$ satisfying a polynomial growth condition together with their first and second order derivatives.

For $0 \le \epsilon \le 1$, let $u^{(\epsilon)} : [0,T]\times\mathbf{R}^2 \to \mathbf{R}$ be a unique classical solution of the following semilinear parabolic PDE,

$$
\begin{aligned}
0 =& u_t^{(\epsilon)} + \frac{1}{2}\sigma^2 u_{x_1,x_1}^{(\epsilon)} + \sigma^2\Gamma(t,x_1)u_{x_1,x_2}^{(\epsilon)} + \frac{1}{2}\tilde{\sigma}^2\Gamma(t_1,x_1)^2 u_{x_2,x_2}^{(\epsilon)} \\
&+ \lambda\sigma^2(x_2^2 + \delta^2 x_1^2\Gamma(t,x_1)^2) - \frac{1}{4\eta}[(1+\epsilon\Gamma(t,x_1))u_{x_2}^{(\epsilon)} + \epsilon u_{x_1}^{(\epsilon)} - \epsilon x_2]^2,
\end{aligned} \tag{51}
$$

$$
u^{(\epsilon)}(T,x_1,x_2) = b_2 x_2^2 + b_1(x_1)x_2 + b_0(x_1). \tag{52}
$$

We assume that for all $(t, x_1, x_2) \in [0,T] \times \mathbf{R}^2$, $u^{(\epsilon)}(t, x_1, x_2)$ is twice differentiable with respect to ϵ and and the second derivative is continuous on $\epsilon \in [0,1]$. Applying Taylor's theorem with respect to ϵ around $\epsilon = 0$ to the classical solution $u^{(\epsilon)}(t, x_1, x_2)$, we have

$$u^{(\epsilon)} = u^{(0,0)} + \epsilon u^{(1,0)} + \epsilon^2 \int_0^1 (1-r) u^{(2,r\epsilon)} dr. \tag{53}$$

Let $u^{(k,r\epsilon)} = \frac{\partial^k}{\partial \epsilon^k} u^{(\epsilon)}|_{\epsilon = r\epsilon}$, $\forall k \in \mathbf{N} \cup \{0\}$, $0 \leq \forall r \leq 1$. Then asymptotic expansions with respect to ϵ on (51), (52) yield the following PDEs that $u^{(0,0)}, u^{(1,0)}, u^{(1,\epsilon)}$ and $u^{(2,\epsilon)}$ satisfy.

- 0th order ($\epsilon = 0$).

$$\begin{aligned} 0 = u_t^{(0,0)} &+ \frac{1}{2}\sigma^2 u_{x_1,x_1}^{(0,0)} + \sigma^2 \Gamma(t,x_1) u_{x_1,x_2}^{(0,0)} + \frac{1}{2}\tilde{\sigma}^2 \Gamma(t_1,x_1)^2 u_{x_2,x_2}^{(0,0)} \\ &+\lambda\sigma^2(x_2^2 + \delta^2 x_1^2 \Gamma(t,x_1)^2) - \frac{1}{4\eta} u_{x_2}^{(0,0)2}, \\ &u^{(0,0)}(T, x_1, x_2) = \psi(x_1, x_2). \end{aligned} \tag{54}$$

- 1st order ($\epsilon = 0$).

$$\begin{aligned} 0 = u_t^{(1,0)} &+ \frac{1}{2}\sigma^2 u_{x_1,x_1}^{(1,0)} + \sigma^2 \Gamma(t,x_1) u_{x_1,x_2}^{(1,0)} + \frac{1}{2}\tilde{\sigma}^2 \Gamma(t_1,x_1)^2 u_{x_2,x_2}^{(1,0)} \\ &-\frac{1}{2\eta} u_{x_2}^{(0,0)} u_{x_2}^{(1,0)} - \frac{1}{2\eta} u_{x_2}^{(0,0)} [\Gamma(t,x_1) u_{x_2}^{(0,0)} + u_{x_1}^{(0,0)} - x_2], \\ &u^{(1,0)}(T, x_1, x_2) = 0. \end{aligned} \tag{55}$$

- 2nd order ($0 \leq \epsilon \leq 1$).

$$\begin{aligned} 0 = u_t^{(2,\epsilon)} &+ \frac{1}{2}\sigma^2 u_{x_1,x_1}^{(2,\epsilon)} + \sigma^2 \Gamma(t,x_1) u_{x_1,x_2}^{(2,\epsilon)} + \frac{1}{2}\tilde{\sigma}^2 \Gamma(t_1,x_1)^2 u_{x_2,x_2}^{(2,\epsilon)} \\ -\frac{1}{2\eta}&[(1+\epsilon\Gamma(t,x_1)) u_{x_2}^{(\epsilon)} + \epsilon u_{x_1}^{(\epsilon)} - \epsilon x_2] \cdot (\epsilon u_{x_1}^{(2,\epsilon)} + (1+\epsilon\Gamma(t,x_1)) u_{x_2}^{(2,\epsilon)}) \\ &-\frac{1}{\eta}[(1+\epsilon\Gamma(t,x_1)) u_{x_2}^{(\epsilon)} + \epsilon u_{x_1}^{(\epsilon)} - \epsilon x_2] \cdot (u_{x_1}^{(1,\epsilon)} + \Gamma(t,x_1) u_{x_2}^{(1,\epsilon)}) \\ &-\frac{1}{2\eta}[\Gamma(t,x_1) u_{x_2}^{(\epsilon)} + (1+\epsilon\Gamma(t,x_1)) u_{x_2}^{(1,\epsilon)} + u_{x_1}^{(\epsilon)} + \epsilon u_{x_1}^{(1,\epsilon)} - x_2]^2, \\ &u^{(2,\epsilon)}(T, x_1, x_2) = 0. \end{aligned} \tag{56}$$

Note that $u^{(0,0)}$ satisfies a polynomial growth condition since $u^{(\epsilon)}$ $(0 \leq \epsilon \leq 1)$ is a unique classical solution in $C_p^{1,2}(\bar{Q}_0)$, and $u^{(0,0)} = u^{(\epsilon)}|_{\epsilon=0}$. We further assume the following.

Assumption 3. *$u_{x_1}^{(0,0)}, u_{x_2}^{(0,0)}, u^{(1,0)}, u^{(1,\epsilon)}$ and $u^{(2,\epsilon)}$ satisfy a polynomial growth condition.*

Let

$$\alpha_1^{(\epsilon)}(t, x_1, x_2) = -\frac{1}{2\eta}[(1+\epsilon\Gamma(t,x_1)) u_{x_2}^{(\epsilon)} + \epsilon u_{x_1}^{(\epsilon)} - \epsilon x_2] \cdot \begin{pmatrix} \epsilon \\ 1 + \epsilon\Gamma(t,x_1) \end{pmatrix}, \tag{57}$$

$$\alpha_2^{(\epsilon)}(t, x_1, x_2) = \begin{pmatrix} 0 \\ -\frac{1}{2\eta} u_{x_2}^{(0,0)} \end{pmatrix}. \tag{58}$$

In order to conduct an error estimate in (53), we consider the Feynman-Kac representation of $u^{(2,\epsilon)}$.

Note that $\sigma(t,x_1)$ satisfies the Lipschitz and linear growth condition by (27) and Assumption 1. We further assume the following.

Assumption 4. *For $i=1,2$, there exist constants $L_T^{(i)}>0$ such that for all $\epsilon\in[0,1]$*

$$\|\alpha_i^{(\epsilon)}(t,x_1,x_2)-\alpha_i^{(\epsilon)}(t,x_1',x_2')\|^2 \le L_T^{(i)}\|(x_1,x_2)-(x_1',x_2')\|, \tag{59}$$

$$\|\alpha_i^{(\epsilon)}(t,x_1,x_2)\|^2 \le L_T^{(i)}(\|(x_1,x_2)\|^2+1). \tag{60}$$

While the conditions for $i=1$ is used in Theorem 1, those for $i=2$ will be used in Section 6.

Let

$$\begin{aligned} h^{(\epsilon)}(t,x_1,x_2) = &-\frac{1}{\eta}[(1+\epsilon\Gamma(t,x_1))u^{(\epsilon)}_{x_2}+\epsilon u^{(\epsilon)}_{x_1}-\epsilon x_2]\cdot(u^{(1,\epsilon)}_{x_1}+\Gamma(t,x_1)u^{(1,\epsilon)}_{x_2}) \\ &-\frac{1}{2\eta}[\Gamma(t,x_1)u^{(\epsilon)}_{x_2}+(1+\epsilon\Gamma(t,x_1))u^{(1,\epsilon)}_{x_2}+u^{(\epsilon)}_{x_1}+\epsilon u^{(1,\epsilon)}_{x_1}-x_2]^2. \end{aligned} \tag{61}$$

Assumption 5. *The following term satisfies a polynomial growth condition uniformly on $\epsilon\in[0,1]$. That is, there exists $D>0$ and $k\in\mathbf{N}$ such that for all $0\le\epsilon\le 1$*

$$|h^{(\epsilon)}(t,x_1,x_2)| \le D(1+|x|^k). \tag{62}$$

From Assumption 3, 5, $i=1$ in Assumption 4 and Theorem 5.7.6 in Karatzas and Shreve [7], $u^{(2,\epsilon)}$ has a Feynman-Kac representation,

$$u^{(2,\epsilon)}(t,x_1,x_2) = \mathbf{E}^{(t,x_1,x_2)}\left[\int_t^T h^{(\epsilon)}(s,X_{1,x},X_{2,s})ds\right], \tag{63}$$

where

$$\begin{pmatrix} dX_{1,t} \\ dX_{2,t}\end{pmatrix} = \alpha_1(t,X_{1,t},X_{2,t})dt + \sigma(t,X_{1,t})\begin{pmatrix} dW_{1,t} \\ dW_{2,t}\end{pmatrix}, \tag{64}$$

and the following estimation holds.

Theorem 1. *Under Assumption 3,4 and 5, there exists $C>0$ and $k\in\mathbf{N}$ such that*

$$\left|\int_0^1(1-r)u^{(2,r\epsilon)}(t,x_1,x_2)dr\right| < C(1+|x_1|+|x_2|)^{2k},\ \forall t\in[0,T],\ \forall x_1,x_2\in\mathbf{R},\ 0\le\forall\epsilon\le 1. \tag{65}$$

Proof.

$$\begin{aligned} |\int_0^1(1-r)u^{(2,r\epsilon)}(t,x_1,x_2)dr| &\le \sup_{0\le r\le 1}|u^{(2,r\epsilon)}(t,x_1,x_2)| \\ &= \sup_{0\le r\le 1}\left|\mathbf{E}^{(t,x_1,x_2)}\left[\int_t^T h^{(r\epsilon)}(s,X^{(r\epsilon)}_{1,x},X^{(r\epsilon)}_{2,s})ds\right]\right| \\ &\le \sup_{0\le r\le 1}\mathbf{E}^{(t,x_1,x_2)}\left[\int_t^T D(1+|X^{(r\epsilon)}_s|^{2k})ds\right] \\ &\le DC_{2k,T}(1+|x|^{2k})(T-t) \\ &\le DC_{2k,T}(1+|x_1|+|x_2|)^{2k}(T-t). \end{aligned} \tag{66}$$

In the second last inequality, we have used the moment estimation result on a solution of SDEs. (e.g. Theorem V.4.2 in Fleming and Rishel [1].) □

5 Computation of Coefficients in the Expansion

In this section, we calculate the 0th order and the first order expansion in (54) and (55).

5.1 $u^{(0,0)}(t, x_1, x_2)$

We assume that $u^{(0,0)}$, a solution of the semilinear PDE (54), has the following expression,

$$u^{(0,0)}(t, x_1, x_2) = A_2(T-t)x_2^2 + A_1(T-t, x_1)x_2 + A_0(T-t, x_1), \tag{67}$$

$$u^{(0,0)}(T, x_1, x_2) = b_2 x_2^2 + b_1(x_1)x_2 + b_0(x_1), \tag{68}$$

where A_2, A_1 and A_0 are unique solutions of the following system of ODE and PDEs

$$\begin{aligned} 0 &= -A_2'(t') - \frac{1}{\eta}A_2^2(t') + \lambda\sigma^2, \\ A_2(0) &= b_2, \end{aligned} \tag{69}$$

$$\begin{aligned} 0 &= -A_{1,t'}(t', x_1) + \frac{1}{2}\sigma^2 A_{1,x_1x_1}(t', x_1) - \frac{1}{\eta}A_2(t')A_1(t', x_1), \\ A_1(0) &= b_1(x_1), \end{aligned} \tag{70}$$

$$\begin{aligned} 0 &= -A_{0,t'}(t', x_1) + \frac{1}{2}\sigma^2 A_{0,x_1x_1}(t', x_1) \\ &+\sigma^2\Gamma(T-t', x_1)A_{1,x_1}(t', x_1) + \tilde{\sigma}^2\Gamma(T-t', x_1)^2 A_2(t') - \frac{1}{4\eta}A_1(t', x_1)^2 + \delta^2\lambda\sigma^2 x_1^2\Gamma(T-t', x_1)^2, \\ A_0(0) &= b_0(x_1). \end{aligned} \tag{71}$$

Note that the ODE (69) is a Riccati type and has the unique solution

$$A_2(t') = \begin{cases} \sqrt{\lambda\eta\sigma^2}\tanh\left(-\frac{1}{2}\log\frac{1-h_0}{1+h_0} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right) & (0 < b_2 < \sqrt{\lambda\eta\sigma^2}), \\ \sqrt{\lambda\eta\sigma^2} & (b_2 = \sqrt{\lambda\eta\sigma^2}), \\ \sqrt{\lambda\eta\sigma^2}\coth\left(-\frac{1}{2}\log\frac{h_0-1}{h_0+1} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right) & (\sqrt{\lambda\eta\sigma^2} < b_2), \end{cases} \tag{72}$$

where

$$h_0 = \frac{b_2}{\sqrt{\lambda\eta\sigma^2}}. \tag{73}$$

Note that the PDEs (70) and (71) have a unique solution with Feynman-Kac representation as follows.

Since $0 \le A_2(t') \le 1$ and b_1 satisfies a polynomial growth condition, by Theorem 1.12 in Friedman [2], the PDE (70) has a solution of class $C^{1,2}$ with exponential growth. Then by Theorem 4.4.2 in Karatzas and Shreve [7], A_1 has a Feynman-Kac representation. Furthermore, for the PDE (71), the nonhomogeneous term

$$\sigma^2\Gamma(T-t', x_1)A_{1,x_1}(t', x_1) + \tilde{\sigma}^2\Gamma(T-t', x_1)^2 A_2(t') - \frac{1}{4\eta}A_1(t', x_1)^2 \tag{74}$$

is locally Hölder continuous uniformly in t'. This follows from the fact that the first order derivative with respect to x_1 is bounded uniformly in t' on any compact subset of $\mathbf{R}$. This can be checked by (77), (78), (79) below, Assumption 1 and the polynomial growth condition on b_1, b_1' and b_1''. Then by Theorem 1.12 in Friedman [2], the PDE (71) has a solution of class $C^{1,2}$ with exponential growth and A_0 also has a Feynman-Kac representation.

Setting $B_2(t) = A_2(T-t)$, $B_1(t,x_1) = A_1(T-t,x_1)$ and $B_0(t,x_1) = A_0(T-t,x_1)$, we have

$$\begin{aligned} -B_{1,t}(t,x_1) + \frac{1}{\eta}B_2(t)B_1(t,x_1) &= +\frac{1}{2}\sigma^2 B_{1,x_1x_1}(t,x_1), \\ B_1(T,x_1) &= b_1(x_1), \end{aligned} \tag{75}$$

$$\begin{aligned} -B_{0,t}(t,x_1) = &+\frac{1}{2}\sigma^2 B_{0,x_1x_1}(t,x_1) \\ &+\sigma^2\Gamma(t,x_1)B_{1,x_1}(t,x_1) + \tilde{\sigma}^2\Gamma(t,x_1)^2 B_2(t) - \frac{1}{4\eta}B_1(t,x_1)^2 + \delta^2\lambda\sigma^2 x_1^2\Gamma(T-t',x_1)^2, \\ B_0(T,x_1) = &\ b_0(x_1). \end{aligned} \tag{76}$$

First, for B_1, B_{1,x_1} and B_{1,x_1x_1},

$$B_1(t,x_1) = \exp\left(-\int_t^T \frac{1}{\eta}B_2(s)ds\right)\mathbf{E}\left[b_1(x_1+\sigma W_{T-t})\right], \tag{77}$$

$$B_{1,x_1}(t,x_1) = \exp\left(-\int_t^T \frac{1}{\eta}B_2(s)ds\right)\mathbf{E}\left[b_1'(x_1+\sigma W_{T-t})\right], \tag{78}$$

$$B_{1,x_1x_1}(t,x_1) = \exp\left(-\int_t^T \frac{1}{\eta}B_2(s)ds\right)\mathbf{E}\left[b_1''(x_1+\sigma W_{T-t})\right]. \tag{79}$$

By (72), it follows that

$$\exp\left(-\int_t^T \frac{1}{\eta}B_2(s)ds\right) = \begin{cases} \dfrac{\cosh\left(-\frac{1}{2}\log\frac{1-h_0}{1+h_0}\right)}{\cosh\left(-\frac{1}{2}\log\frac{1-h_0}{1+h_0}+\sqrt{\frac{\lambda\sigma^2}{\eta}}(T-t)\right)} \ (0 < b_2 < \sqrt{\lambda\eta\sigma^2}) \\ \exp(-\sqrt{\frac{\lambda\sigma^2}{\eta}}(T-t)) \ (b_2 = \sqrt{\lambda\eta\sigma^2}) \\ \dfrac{\sinh\left(-\frac{1}{2}\log\frac{h_0-1}{h_0+1}\right)}{\sinh\left(-\frac{1}{2}\log\frac{h_0-1}{h_0+1}+\sqrt{\frac{\lambda\sigma^2}{\eta}}(T-t)\right)} \ (\sqrt{\lambda\eta\sigma^2} < b_2) \end{cases}, \tag{80}$$

where

$$h_0 = \frac{b_2}{\sqrt{\lambda\eta\sigma^2}}. \tag{81}$$

Next, as for B_0,

$$\begin{aligned} B_0(t,x_1) = \mathbf{E}^{t,x_1}\Bigg[b_0(X_T) + \int_t^T &\sigma^2\Gamma(s,X_s)B_{1,x_1}(s,X_s) + \tilde{\sigma}^2\Gamma(s,X_s)^2B_2(s) \\ &-\frac{1}{4\eta}B_1(s,X_s)^2 + \delta^2\lambda\sigma^2X_s^2\Gamma(s,X_s)^2ds\Bigg], \end{aligned} \tag{82}$$

$$dX_s = \sigma dW_s, \ X_t = x_1. \tag{83}$$

This can be rewritten as

$$\begin{aligned}B_0(t,x_1) =& \mathbf{E}\Bigg[b_0(x_1+\sigma W_{T-t}) + \int_t^T \sigma^2\Gamma(s,x_1+\sigma W_{s-t})B_{1,x_1}(s,x_1+\sigma W_{s-t}) \\ &+\tilde{\sigma}^2\Gamma(s,x_1+\sigma W_{s-t})^2 B_2(s) - \frac{1}{4\eta}B_1(s,x_1+\sigma W_{s-t})^2 \\ &+\delta^2\lambda\sigma^2(x_1+\sigma W_{s-t})^2\Gamma(s,x_1+\sigma W_{s-t})^2 ds\Bigg].\end{aligned} \tag{84}$$

Then B_{0,x_1} is given by

$$\begin{aligned}&B_{0,x_1}(t,x_1) = \mathbf{E}\Bigg[b_0'(x_1+\sigma W_{T-t}) \\ &+\int_t^T \sigma^2\left\{\Gamma_{x_1}(s,x_1+\sigma W_{s-t})B_{1,x_1}(s,x_1+\sigma W_{s-t}) + \Gamma(s,x_1+\sigma W_{s-t})B_{1,x_1x_1}(s,x_1+\sigma W_{s-t})\right\} \\ &+2\tilde{\sigma}^2\Gamma(s,x_1+\sigma W_{s-t})\Gamma_{x_1}(s,x_1+\sigma W_{s-t})B_2(s) - \frac{1}{2\eta}B_1(s,x_1+\sigma W_{s-t})B_{1,x_1}(s,x_1+\sigma W_{s-t}) \\ &+2\delta^2\lambda^2\sigma^2(x_1+\sigma W_{s-t})\Gamma^2(s,x_1+\sigma W_{s-t}) \\ &+2\delta^2\lambda\sigma^2(x_1+\sigma W_{s-t})^2\Gamma(s,x_1+\sigma W_{s-t})\Gamma_{x_1}(s,x_1+\sigma W_{s-t})ds\Bigg].\end{aligned} \tag{85}$$

5.2 $u^{(1,0)}(t,x_1,x_2)$

Finally, by Theorem 5.7.6 in Karatzas and Shreve [7] together with Assumption 3 and $i=2$ in Assumption 4, $u^{(1,0)}$ has the following Feynman-Kac representation.

$$\begin{aligned}&u^{(1,0)}(t,x_1,x_2) = \\ &\mathbf{E}^{t,x_1,x_2}\left[\int_t^T -\frac{1}{2\eta}u^{(0,0)}_{x_2}(s,X_{1,s},X_{2,s})\{\Gamma(s,X_{1,s})u^{(0,0)}_{x_2}(s,X_{1,s},X_{2,s}) + u^{(0,0)}_{x_1}(s,X_{1,s},X_{2,s}) - X_{2,s}\}ds\right],\end{aligned} \tag{86}$$

where

$$dX_{1,s} = \sigma dW_{1,s}, \tag{87}$$

$$dX_{2,s} = \sigma\Gamma(s,X_{1,s})(dW_{1,s}+\delta dW_{2,s}) - \frac{1}{2\eta}u^{(0,0)}_{x_2}(s,X_{1,s},X_{2,s})ds. \tag{88}$$

Note that by (67) and (68),

$$u^{(0,0)}(t,x_1,x_2) = B_2(t)x_2^2 + B_1(t,x_1)x_2 + B_0(t,x_1), \tag{89}$$

$$u^{(0,0)}_{x_2}(t,x_1,x_2) = 2B_2(t)x_2 + B_1(t,x_1), \tag{90}$$

$$u^{(0,0)}_{x_1}(t,x_1,x_2) = B_{1,x_1}(t,x_1)x_2 + B_{0,x_1}(t,x_1), \tag{91}$$

and B_2, B_1, B_0 and B_{0,x_1} are calculated in the previous subsection.

5.3 Derivatives Pricing and Optimal Hedging Strategy

Taking the objective function for maximization (37) into consideration, we define an option price as the largest m satisfying the following inequality for some threshold $K \geq 0$,

$$g(0, P_0) - m - u(0, P_0, Y_0) \geq K, \tag{92}$$

that is,

$$g(0, P_0) - u(0, P_0, Y_0) - K. \tag{93}$$

In other words, the option price for the buyer is the highest premium which the trader is willing to pay in order to make profit K. Conversely, the option price for the seller is the lowest premium which the trader is willing to receive to gain K. Note that the threshold $K = 0$ corresponds to the state where the trader does not trade anything, since (37) is 0 in such a case. For the same reason, K is considered to be the profit which the trader aims to make by the whole transactions. We note that $X_0 = -g_{x_1}(t, P_0)$ when there is an initial delta exchange with the counter party, and $X_0 = 0$ when there is no initial exchange.

Example 3. In Bachelier model, where $\epsilon = 0, \eta = 0, \delta = 0, \psi(x_1, x_2) = 0$, the European option price defined in this section with $K = 0$ agrees with the option price by the standard definition. In fact, in (38), the minimizing target amount becomes

$$\mathbf{E}\left[\lambda \int_0^T \sigma^2 Y_s^2 ds\right]. \tag{94}$$

Considering an approximating strategy of the negative of the delta amount in the Bachelier model $-g_x(t, P_t)$, we can make the target amount (94) as small as possible. In detail, the following proposition, whose proof is given in Appendix C, holds. Hence, the European option price becomes $g(0, P_0)$, which agrees with the option price by the standard definition.

Proposition 1. *Let*

$$U = [-A, A], \;\; A > 0, \tag{95}$$

and

$$Y_t^{(\epsilon,A)} = \int_0^t \theta_s^{(\epsilon,A)} ds - (-g_x(t, P_t)), \;\; (0 \leq t \leq T). \tag{96}$$

Then, for all $\epsilon > 0$, *there exist* $A > 0$ *and* $\theta_s^{(\epsilon,A)}$ *satisfying* $\mathbf{E}\left[\int_0^T \theta_s^{(\epsilon,A)2} ds\right] < \infty$ *such that*

$$\mathbf{E}\left[\int_0^T Y_s^{(\epsilon,A)2} ds\right] < \epsilon. \tag{97}$$

6 Examples

In this section, we first present an example of a quadratic payoff in physical settlement where $u^{(\epsilon)}$ in (51) is obtained explicitly together with numerical results on accuracy of the first order approximation. We also show examples of the computation of the first order expansion for a European call option in both physical settlement and cash settlement.

Example 4. For the payoff $h(x_1) = \frac{1}{4}(x_1 - K)^2 + \frac{1}{2}(x_1 - K) + \frac{1}{4}$ in Example 1 in the case of physical settlement where $b_1 = b_0 = 0$, with $\delta = 0$, $u(t, x_1, x_2)$ in (46) is solved explicitly as follows. Note that this payoff can be regarded as the variance contract. We assume that $\frac{\epsilon}{2+\epsilon} < b_2$.

$$u^{(\epsilon)}(t, x_1, x_2) = A_2^{(\epsilon)}(T-t)x_2^2 + A_0^{(\epsilon)}(T-t), \tag{98}$$

$$A_2^{(\epsilon)}(t') = \begin{cases} \frac{1}{2+\epsilon}\left\{2\sqrt{\lambda\eta\sigma^2}\tanh\left(-\frac{1}{2}\log\frac{1-h_0}{1+h_0} + \sqrt{\frac{(2+\epsilon)^2\lambda\sigma^2}{4\eta}}t'\right) + \epsilon\right\} & \left(b_2 < \frac{2\sqrt{\lambda\eta\sigma^2}+\epsilon}{2+\epsilon}\right) \\ \frac{1}{2+\epsilon}(2\sqrt{\lambda\eta\sigma^2} + \epsilon) \quad \left(b_2 = \frac{2\sqrt{\lambda\eta\sigma^2}+\epsilon}{2+\epsilon}\right) & \\ \frac{1}{2+\epsilon}\left\{2\sqrt{\lambda\eta\sigma^2}\coth\left(-\frac{1}{2}\log\frac{h_0-1}{h_0+1} + \sqrt{\frac{(2+\epsilon)^2\lambda\sigma^2}{4\eta}}t'\right) + \epsilon\right\} & \left(\frac{2\sqrt{\lambda\eta\sigma^2}+\epsilon}{2+\epsilon} < b_2\right) \end{cases}, \tag{99}$$

$$A_0^{(\epsilon)}(t') = \begin{cases} \frac{\sigma^2\eta}{(2+\epsilon)^2}\log\frac{\cosh\left(-\frac{1}{2}\log\frac{1-h_0}{1+h_0} + \sqrt{\frac{(2+\epsilon)^2\lambda\sigma^2}{4\eta}}t'\right)}{\cosh\left(-\frac{1}{2}\log\frac{1-h_0}{1+h_0}\right)} + \frac{\sigma^2\epsilon}{4(2+\epsilon)}t' & \left(b_2 < \frac{2\sqrt{\lambda\eta\sigma^2}+\epsilon}{2+\epsilon}\right) \\ \frac{\sigma^2\sqrt{\lambda\eta\sigma^2}}{2(2+\epsilon)} \quad \left(b_2 = \frac{2\sqrt{\lambda\eta\sigma^2}+\epsilon}{2+\epsilon}\right) & \\ \frac{\sigma^2\eta}{(2+\epsilon)^2}\log\frac{\sinh\left(-\frac{1}{2}\log\frac{h_0-1}{h_0+1} + \sqrt{\frac{(2+\epsilon)^2\lambda\sigma^2}{4\eta}}t'\right)}{\sinh\left(-\frac{1}{2}\log\frac{h_0-1}{h_0+1}\right)} + \frac{\sigma^2\epsilon}{4(2+\epsilon)}t' & \left(\frac{2\sqrt{\lambda\eta\sigma^2}+\epsilon}{2+\epsilon} < b_2\right) \end{cases}, \tag{100}$$

where

$$h_0 = \frac{(2+\epsilon)b_2 - \epsilon}{2\sqrt{\lambda\eta\sigma^2}}. \tag{101}$$

The first order approximation is as follows.

$$u^{(0,0)}(t, x_1, x_2) + \epsilon u^{(1,0)}(t, x_1, x_2), \tag{102}$$

where

$$u^{(0,0)}(t, x_1, x_2) = A_2^{(0,0)}(T-t)x_2^2 + A_0^{(0,0)}(T-t), \tag{103}$$

$$u^{(1,0)}(t, x_1, x_2) = A_2^{(1,0)}(T-t)x_2^2 + A_0^{(1,0)}(T-t), \tag{104}$$

$$A_i^{(0,0)}(t') = A_i^{(\epsilon)}(t')\Big|_{\epsilon=0}, \quad (i = 0, 2) \tag{105}$$

$$A_i^{(1,0)}(t') = \frac{\partial}{\partial\epsilon}A_i^{(\epsilon)}(t')\Big|_{\epsilon=0}, \quad (i = 0, 2). \tag{106}$$

$A_2^{(1,0)}$ and $A_0^{(1,0)}$ are calculated as follows. We set $h_0^{(0)} = \frac{b_2}{\sqrt{\lambda\eta\sigma^2}}$.

The case $b_2 < \sqrt{\lambda\eta\sigma^2}$.

$$
\begin{aligned}
&A_2^{(1,0)} \\
&= \frac{1}{2} - \frac{1}{2}\sqrt{\lambda\eta\sigma^2}\tanh\left(-\frac{1}{2}\log\frac{1-h_0^{(0)}}{1+h_0^{(0)}} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right) \\
&+ \sqrt{\lambda\eta\sigma^2}\left\{1-\tanh^2\left(-\frac{1}{2}\log\frac{1-h_0^{(0)}}{1+h_0^{(0)}} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right)\right\}\left(\frac{1}{(1+h_0^{(0)})(1-h_0^{(0)})}\frac{(b_2-1)}{2\sqrt{\lambda\eta\sigma^2}} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right),
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
&A_0^{(1,0)} \\
&= \frac{1}{8}\sigma^2 t' - \frac{\sigma^2\eta}{4}\log\frac{\cosh\left(-\frac{1}{2}\log\frac{1-h_0^{(0)}}{1+h_0^{(0)}} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right)}{\cosh\left(-\frac{1}{2}\log\frac{1-h_0^{(0)}}{1+h_0^{(0)}}\right)} \\
&\quad + \frac{\sigma^2\eta}{4}\left\{\tanh\left(-\frac{1}{2}\log\frac{1-h_0^{(0)}}{1+h_0^{(0)}} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right)\left(\frac{1}{(1+h_0^{(0)})(1-h_0^{(0)})}\frac{(b_2-1)}{2\sqrt{\lambda\eta\sigma^2}} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right)\right. \\
&\quad \left. - \tanh\left(-\frac{1}{2}\log\frac{1-h_0^{(0)}}{1+h_0^{(0)}}\right)\left(\frac{1}{(1+h_0^{(0)})(1-h_0^{(0)})}\frac{(b_2-1)}{2\sqrt{\lambda\eta\sigma^2}}\right)\right\}.
\end{aligned}
\tag{108}
$$

The case $\sqrt{\lambda\eta\sigma^2} < b_2$.

$$
\begin{aligned}
&A_2^{(1,0)} \\
&= \frac{1}{2} - \frac{1}{2}\sqrt{\lambda\eta\sigma^2}\coth\left(-\frac{1}{2}\log\frac{h_0^{(0)}-1}{h_0^{(0)}+1} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right) \\
&+ \sqrt{\lambda\eta\sigma^2}\left\{1-\coth^2\left(-\frac{1}{2}\log\frac{h_0^{(0)}-1}{h_0^{(0)}+1} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right)\right\}\left(\frac{-1}{(h_0^{(0)}+1)(h_0^{(0)}-1)}\frac{(b_2-1)}{2\sqrt{\lambda\eta\sigma^2}} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right),
\end{aligned}
\tag{109}
$$

$$
\begin{aligned}
&A_0^{(1,0)} \\
&= \frac{1}{8}\sigma^2 t' - \frac{\sigma^2\eta}{4}\log\frac{\sinh\left(-\frac{1}{2}\log\frac{h_0^{(0)}-1}{h_0^{(0)}+1} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right)}{\sinh\left(-\frac{1}{2}\log\frac{h_0^{(0)}-1}{h_0^{(0)}+1}\right)} \\
&\quad + \frac{\sigma^2\eta}{4}\left\{\coth\left(-\frac{1}{2}\log\frac{h_0^{(0)}-1}{h_0^{(0)}+1} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right)\left(\frac{-1}{(h_0^{(0)}+1)(h_0^{(0)}-1)}\frac{(b_2-1)}{2\sqrt{\lambda\eta\sigma^2}} + \sqrt{\frac{\lambda\sigma^2}{\eta}}t'\right)\right. \\
&\quad \left. - \coth\left(-\frac{1}{2}\log\frac{h_0^{(0)}-1}{h_0^{(0)}+1}\right)\left(\frac{-1}{(h_0^{(0)}+1)(h_0^{(0)}-1)}\frac{(b_2-1)}{2\sqrt{\lambda\eta\sigma^2}}\right)\right\}.
\end{aligned}
\tag{110}
$$

Table 1 and 2 show the error of the first order approximation from the exact value for this quadratic payoff where $K = 100.0$ when $\epsilon = 0.10$ and $\epsilon = 0.01$, respectively. We set the other parameters as $b_2 = 0.20$, $\lambda = 0.50$, $\eta = 0.000000267$, $\sigma = 10.0$, $x_1 = 100.0$, $x_2 = 0.5$, $t = 0$ and $T = 0.50$.

$u^{(\epsilon)}(t, x_1, x_2)$	0.651529
$u^{(0,0)}(t, x_1, x_2)$	0.0466077
$u^{(1,0)}(t, x_1, x_2)$	6.35167
$u^{(0,0)}(t, x_1, x_2) + \epsilon u^{(1,0)}(t, x_1, x_2)$	0.656190
Bachelier price $g(0, x_1)$	12.75

Table 1: Comparison between the exact value and the first order approximation, $\epsilon = 0.1$

$u^{(\epsilon)}(t, x_1, x_2)$	0.109808
$u^{(0,0)}(t, x_1, x_2)$	0.0466077
$u^{(1,0)}(t, x_1, x_2)$	6.35167
$u^{(0,0)}(t, x_1, x_2) + \epsilon u^{(1,0)}(t, x_1, x_2)$	0.110274
Bachelier price $g(0, x_1)$	12.75

Table 2: Comparison between the exact value and the first order approximation, $\epsilon = 0.01$

We interpret the parameters as follows. η is the cost from the mid in terms of the price when the trader buys the underlying asset at the speed of θ. For example, let us consider the case where the trader buys 100 million notional of the USDJPY option. Here we take 100 million notional as one unit, 1 year as the unit of time and assume that the USDJPY exchange rate is 100. If the liquidity of the exchange rate is such that when the trader buys this one unit of USDJPY in one minute and the trader pays 0.10 of the exchange rate as the price spread, η is as follows. Considering that there are five trading days in a week, one minute is $1/(365 * 5/7 * 24 * 60) = 2.67 * 10^{-6}$ years, $\theta = 1/(1/(365 * 5/7 * 24 * 60)) = 374{,}400$ and $\eta = 0.10/\theta = 2.67 * 10^{-7}$ by (2). If, as for the other illiquid currency, when the trader buys one unit of the underlying asset in one day and pays 1.0 of the exchange rate as the spread, $\theta = 1/(1/(365 * 5/7)) = 260$ and $\eta = 1.0/\theta = 3.85 * 10^{-3}$.

As we have observed in Section 2, $b_2 x_2^2$ corresponds to the liquidity cost when the third party undertakes x_2 unit of the underling asset from the trader immediately, while $b_2 x_2$ is the spread from the mid in terms of the underlying asset price. If a third party buys this one unit of USD from the trader instantly and requires 0.20 of the exchange rate as a cost, b_2 will be 0.20. As for ϵ, if the price impact when the trader buys this one unit of USDJPY is 0.10, $\epsilon = 0.10$. λ corresponds to the risk aversion parameter in the mean-variance optimization and we take $\lambda = 1$ here. σ corresponds to the volatility in the Bachelier model. For instance, $\sigma = 10.0$ indicates that the volatility of P_1 when $\epsilon = 0.0$ is 10.0.

Finally, δ is a proportional constant for the execution error amount. In illiquid markets, the execution amount often differs from the order amount due to mismatch of the quantities which the buyer and the seller aim to trade. For example, when a trader intends to buy some quantity of the underlying asset for hedging, a seller may willing to sell only some slightly different amount. In such a case, the trader buys the amount which is slightly different from what the trader first intended to buy. Here the execution error during the period $[t, t+dt]$ is $\delta\sigma\Gamma(t, P_t)dW_t^{\perp}$. This indicates that the quantity is proportional to $\sigma\Gamma(t, P_t)$, which represents the rehedging amount in the Bachelier model.

σ	δ	t	T	x_1	K	x_2	λ	η	b_2	$g(t, x_1)$	u_0	u_1
5	0	0	0.0417	100	100	0.5	1	2.7E-07	0.2	0.407169	0.00130	0.25380
5	0	0	0.0417	100	100	0.5	1	2.7E-06	0.2	0.407169	0.00431	0.23566
5	0	0	0.0417	100	100	0.5	1	2.7E-05	0.2	0.407169	0.01430	0.22259
5	0	0	0.0417	100	100	0.5	1	2.7E-04	0.2	0.407169	0.04379	0.20077
5	0	0	0.0417	100	100	0.5	1	2.7E-03	0.2	0.407169	0.12384	0.15194
5	0	0	0.0417	100	100	0.5	1	2.7E-02	0.2	0.407169	0.29886	0.06724
5	0.002	0	0.0417	100	100	0.5	1	2.7E-07	0.2	0.407169	0.01109	0.25725
5	0.002	0	0.0417	100	100	0.5	1	2.7E-06	0.2	0.407169	0.01410	0.23878
5	0.002	0	0.0417	100	100	0.5	1	2.7E-05	0.2	0.407169	0.02410	0.22522
5	0.002	0	0.0417	100	100	0.5	1	2.7E-04	0.2	0.407169	0.05359	0.20296
5	0.002	0	0.0417	100	100	0.5	1	2.7E-03	0.2	0.407169	0.13363	0.15343
5	0.002	0	0.0417	100	100	0.5	1	2.7E-02	0.2	0.407169	0.30865	0.06776
5	0.004	0	0.0417	100	100	0.5	1	2.7E-07	0.2	0.407169	0.04048	0.26762
5	0.004	0	0.0417	100	100	0.5	1	2.7E-06	0.2	0.407169	0.04349	0.24816
5	0.004	0	0.0417	100	100	0.5	1	2.7E-05	0.2	0.407169	0.05348	0.23312
5	0.004	0	0.0417	100	100	0.5	1	2.7E-04	0.2	0.407169	0.08297	0.20952
5	0.004	0	0.0417	100	100	0.5	1	2.7E-03	0.2	0.407169	0.16302	0.15789
5	0.004	0	0.0417	100	100	0.5	1	2.7E-02	0.2	0.407169	0.33804	0.06930
5	0.006	0	0.0417	100	100	0.5	1	2.7E-07	0.2	0.407169	0.08946	0.28490
5	0.006	0	0.0417	100	100	0.5	1	2.7E-06	0.2	0.407169	0.09247	0.26379
5	0.006	0	0.0417	100	100	0.5	1	2.7E-05	0.2	0.407169	0.10246	0.24629
5	0.006	0	0.0417	100	100	0.5	1	2.7E-04	0.2	0.407169	0.13195	0.22046
5	0.006	0	0.0417	100	100	0.5	1	2.7E-03	0.2	0.407169	0.21200	0.16533
5	0.006	0	0.0417	100	100	0.5	1	2.7E-02	0.2	0.407169	0.38702	0.07187
5	0.008	0	0.0417	100	100	0.5	1	2.7E-07	0.2	0.407169	0.15802	0.30909
5	0.008	0	0.0417	100	100	0.5	1	2.7E-06	0.2	0.407169	0.16104	0.28568
5	0.008	0	0.0417	100	100	0.5	1	2.7E-05	0.2	0.407169	0.17103	0.26473
5	0.008	0	0.0417	100	100	0.5	1	2.7E-04	0.2	0.407169	0.20052	0.23578
5	0.008	0	0.0417	100	100	0.5	1	2.7E-03	0.2	0.407169	0.28057	0.17575
5	0.008	0	0.0417	100	100	0.5	1	2.7E-02	0.2	0.407169	0.45559	0.07547
5	0.01	0	0.0417	100	100	0.5	1	2.7E-07	0.2	0.407169	0.24618	0.34021
5	0.01	0	0.0417	100	100	0.5	1	2.7E-06	0.2	0.407169	0.24919	0.31383
5	0.01	0	0.0417	100	100	0.5	1	2.7E-05	0.2	0.407169	0.25919	0.28845
5	0.01	0	0.0417	100	100	0.5	1	2.7E-04	0.2	0.407169	0.28868	0.25546
5	0.01	0	0.0417	100	100	0.5	1	2.7E-03	0.2	0.407169	0.36873	0.18914
5	0.01	0	0.0417	100	100	0.5	1	2.7E-02	0.2	0.407169	0.54375	0.08009

Table 3: Bachilier price, u_0 and u_1 for different parameter sets, $\sigma = 5$, Variance contract.

σ	δ	t	T	x_1	K	x_2	λ	η	b_2	$g(t, x_1)$	u_0	u_1
10	0	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.00257	0.31606
10	0	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.00827	0.23903
10	0	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.02660	0.22804
10	0	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.08167	0.21116
10	0	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.23975	0.16509
10	0	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.65885	0.07549
10	0.002	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.01236	0.31794
10	0.002	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.01807	0.24061
10	0.002	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.03640	0.22939
10	0.002	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.09146	0.21237
10	0.002	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.24955	0.16603
10	0.002	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.66864	0.07597
10	0.004	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.04175	0.32357
10	0.004	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.04746	0.24533
10	0.004	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.06578	0.23344
10	0.004	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.12085	0.21598
10	0.004	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.27893	0.16885
10	0.004	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.69803	0.07743
10	0.006	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.09073	0.33295
10	0.006	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.09644	0.25321
10	0.006	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.11476	0.24018
10	0.006	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.16983	0.22199
10	0.006	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.32791	0.17353
10	0.006	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.74702	0.07985
10	0.008	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.15930	0.34608
10	0.008	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.16501	0.26424
10	0.008	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.18333	0.24962
10	0.008	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.23840	0.23041
10	0.008	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.39649	0.18009
10	0.008	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.81559	0.08323
10	0.01	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.24746	0.36298
10	0.01	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.25317	0.27842
10	0.01	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.27150	0.26176
10	0.01	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.32657	0.24124
10	0.01	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.48465	0.18853
10	0.01	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.90376	0.08759

Table 4: Bachilier price, u_0 and u_1 for different parameter sets, $\sigma = 10$, Variance contract.

Table 3 and 4 show $g(0, x_1)$, $u^{(0)}$ and $u^{(1)}$ for different sets of parameters. We observe the following. First of all, u_1 is positive in all the cases, which indicates that as ϵ increases, $u^{(0)} + \epsilon u^{(1)}$, which is an approximate value of u, increases. According to (93), this implies that when the price impact on the price process is large, the option premium which the buyer has to pay is less. Second, as η increases, u_0 increases. This agrees with the intuition that when the order book is thin, the liquidity cost for hedging for the option buyer becomes large. Third, as σ increases, u_0 in all the cases and u_1 in most of the cases increase. This indicates that as the volatility of the underlying asset rises, the liquidity cost and the market impact on the option price also increase. Fourth, as δ increases, u_0 increases. This implies that as the rate of execution errors rises, the liquidity cost becomes large.

Example 5. In physical settlement of a call option, [1]
where $b_2 = b_2, b_1 = b_0 = 0$ and $h(x_1) = (x_1 - K)^+$, for fixed $0 \leq t \leq T$, we have

[1]Strictly speaking, the call option payoff does not satisfy Assumption 1 and the smoothed payoff in Example 2 should be used instead. However, the European call payoff the most important example in practice, we consider this in both physical and cash settlement.

$B_1 = B_{1,x_1} = B_{1,x_1 x_1} = 0,$

$$g(0,x_1) = \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} \exp\left(-\left(\frac{x_1-K}{\sigma\sqrt{T-t}}\right)^2\right) + (x_1-K)N\left(\frac{x_1-K}{\sigma\sqrt{T-t}}\right), \tag{111}$$

$$\Gamma(s,x_1) = \frac{1}{\sqrt{2\pi\sigma^2(T-s)}} \exp\left(-\frac{(x_1-K)^2}{2\sigma^2(T-s)}\right),\ t \le \forall s < T. \tag{112}$$

For $X_s = x_1 + \sigma W_{s-t}$, it follows that

$$\mathbf{E}\left[\Gamma^2(s,x_1+\sigma W_{s-t})\right] = \frac{1}{2\pi\sigma^2\sqrt{T-s}\sqrt{T+s-2t}} \exp\left(-\frac{(x_1-K)^2}{\sigma^2(T+s-2t)}\right), \tag{113}$$

$$\begin{aligned}\mathbf{E}\left[(x_1+\sigma W_{s-t})^2\Gamma^2(s,x_1+\sigma W_{s-t})\right] &= \frac{1}{2\pi\sigma^2\sqrt{T-s}\sqrt{T+s-2t}} \exp\left(-\frac{(x_1-K)^2}{\sigma^2(T+s-2t)}\right) \\ &\quad \times \left\{\frac{\sigma^2(s-t)(T-s)}{T+s-2t} + \left(x_1 - \frac{2(x_1-K)(s-t)}{T+s-2t}\right)^2\right\}.\end{aligned} \tag{114}$$

By (84), we have

$$\begin{aligned}B_0(t,x_1) = \int_t^T &\frac{1}{2\pi\sigma^2\sqrt{T-s}\sqrt{T+s-2t}} \exp\left(-\frac{(x_1-K)^2}{\sigma^2(T+s-2t)}\right) \\ \times &\left[\tilde{\sigma}^2 B_2(s) + \delta^2\lambda\sigma^2\left\{\frac{\sigma^2(s-t)(T-s)}{T+s-2t} + \left(x_1 - \frac{2(x_1-K)(s-t)}{T+s-2t}\right)^2\right\}\right] ds.\end{aligned} \tag{115}$$

$$\begin{aligned}&B_{0,x_1}(t,x_1) \\ &= \int_t^T -\frac{(x_1-K)}{\pi\sigma^4\sqrt{T-s}(\sqrt{T+s-2t})^3} \exp\left(-\frac{(x_1-K)^2}{\sigma^2(T+s-2t)}\right) \\ &\quad \times \left[\tilde{\sigma}^2 B_2(s) + \delta^2\lambda\sigma^2\left\{\frac{\sigma^2(s-t)(T-s)}{T+s-2t} + \left(x_1 - \frac{2(x_1-K)(s-t)}{T+s-2t}\right)^2\right\}\right] ds \\ &\quad + \int_t^T \frac{\delta^2\lambda\sqrt{T-s}}{\pi(\sqrt{T+s-2t})^3}\left(x_1 - \frac{2(x_1-K)(s-t)}{T+s-2t}\right) \exp\left(-\frac{(x_1-K)^2}{\sigma^2(T+s-2t)}\right) ds.\end{aligned} \tag{116}$$

Note that

$$u^{(0,0)}(t,x_1,x_2) = B_2(t)x_2^2 + B_0(t,x_1), \tag{117}$$

$$u^{(0,0)}_{x_2}(t,x_1,x_2) = 2B_2(t)x_2, \tag{118}$$

$$u^{(0,0)}_{x_1}(t,x_1,x_2) = B_{0,x_1}(t,x_1). \tag{119}$$

Then, we have

$$
\begin{aligned}
u^{(1,0)}(t,x_1,x_2) &= \mathbf{E}^{(t,x_1,x_2)}\left[\int_t^T -\frac{1}{2\eta}u^{(0,0)}_{x_2}(s,X_{1,s},X_{2,s})\{\Gamma(s,X_{1,s})u^{(0,0)}_{x_2}(s,X_{1,s},X_{2,s})\right.\\
&\qquad \left. + u^{(0,0)}_{x_1}(s,X_{1,s},X_{2,s}) - X_{2,s}\}ds\right],\\
&= \mathbf{E}^{(t,x_1,x_2)}\left[\int_t^T -\frac{1}{\eta}B_2(s)X_{2,s}\{\Gamma(s,X_{1,s})2B_2(s)X_{2,s} + B_{0,x_1}(s,X_{1,s}) - X_{2,s}\}ds\right],
\end{aligned}
\tag{120}
$$

where

$$X_{1,s} = x_1 + \sigma\int_t^s dW_{1,v}, \tag{121}$$

$$X_{2,s} = x_2 + \int_t^s \sigma\Gamma(v,X_{1,v})(dW_{1,v}+\delta dW_{2,v}) - \int_t^s \frac{1}{\eta}B_2(v)X_{2,v}dv. \tag{122}$$

σ	δ	t	T	x_1	K	x_2	λ	η	b_2	$g(t,x_1)$	u_0	u_1
5	0	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.407169	0.00129751	0.248278
5	0	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.407169	0.00430841	0.235578
5	0	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.407169	0.0143023	0.232727
5	0	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.407169	0.0437907	0.210719
5	0	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.407169	0.123837	0.138302
5	0	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.407169	0.298857	0.0492516
5	0.002	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.407169	0.0110929	0.253246
5	0.002	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.407169	0.0141038	0.240013
5	0.002	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.407169	0.0240977	0.236563
5	0.002	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.407169	0.0535862	0.213771
5	0.002	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.407169	0.133633	0.139843
5	0.002	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.407169	0.308653	0.0495965
5	0.004	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.407169	0.0404791	0.268163
5	0.004	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.407169	0.04349	0.253278
5	0.004	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.407169	0.053484	0.247937
5	0.004	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.407169	0.0829726	0.222789
5	0.004	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.407169	0.16302	0.144351
5	0.004	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.407169	0.33804	0.0505595
5	0.006	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.407169	0.089456	0.293041
5	0.006	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.407169	0.0924669	0.27539
5	0.006	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.407169	0.102461	0.266869
5	0.006	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.407169	0.13195	0.237792
5	0.006	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.407169	0.211998	0.151842
5	0.006	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.407169	0.387019	0.0521441
5	0.008	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.407169	0.158024	0.327897
5	0.008	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.407169	0.161035	0.306367
5	0.008	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.407169	0.171029	0.293381
5	0.008	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.407169	0.200518	0.258799
5	0.008	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.407169	0.280567	0.162328
5	0.008	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.407169	0.45559	0.0543541
5	0.01	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.407169	0.246182	0.372744
5	0.01	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.407169	0.249193	0.346226
5	0.01	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.407169	0.259188	0.327492
5	0.01	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.407169	0.288678	0.285829
5	0.01	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.407169	0.368728	0.175823
5	0.01	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.407169	0.543753	0.0571931

Table 5: Bachilier price, u_0 and u_1 for different parameter sets, $\sigma = 5$, European call option.

σ	δ	t	T	x_1	K	x_2	λ	η	b_2	$g(t, x_1)$	u_0	u_1
10	0	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.00256696	0.306871
10	0	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.00827422	0.237448
10	0	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.0266017	0.23294
10	0	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.0816688	0.220228
10	0	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.239749	0.163846
10	0	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.658846	0.0561582
10	0.002	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.0123626	0.309293
10	0.002	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.0180699	0.239866
10	0.002	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.0363974	0.235026
10	0.002	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.0914646	0.222015
10	0.002	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.249545	0.165045
10	0.002	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.668643	0.0565496
10	0.004	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.0417496	0.316576
10	0.004	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.0474569	0.247114
10	0.004	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.0657845	0.241199
10	0.004	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.120852	0.227229
10	0.004	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.278934	0.16853
10	0.004	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.698033	0.0576462
10	0.006	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.090728	0.328727
10	0.006	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.0964354	0.259199
10	0.006	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.114763	0.251468
10	0.006	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.169831	0.235877
10	0.006	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.327914	0.174311
10	0.006	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.747016	0.0594519
10	0.008	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.159298	0.345754
10	0.008	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.165005	0.276129
10	0.008	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.183333	0.265843
10	0.008	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.238402	0.247971
10	0.008	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.396487	0.182396
10	0.008	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.815593	0.0619703
10	0.01	0	0.0417	100	100	0.5	1	2.67E-07	0.2	0.814338	0.247459	0.367666
10	0.01	0	0.0417	100	100	0.5	1	2.67E-06	0.2	0.814338	0.253166	0.297912
10	0.01	0	0.0417	100	100	0.5	1	2.67E-05	0.2	0.814338	0.271495	0.284332
10	0.01	0	0.0417	100	100	0.5	1	2.67E-04	0.2	0.814338	0.326565	0.263518
10	0.01	0	0.0417	100	100	0.5	1	2.67E-03	0.2	0.814338	0.484652	0.192794
10	0.01	0	0.0417	100	100	0.5	1	2.67E-02	0.2	0.814338	0.903763	0.065205

Table 6: Bachilier price, u_0 and u_1 for different parameter sets, $\sigma = 10$, European call option.

Table 5 and 6 also show $g(0, x_1), u_0, u_1$ for different sets of parameters, where we are able to observe the similar features as in Example 4.

7 Conclusion

This paper has presented a stochastic model in continuous time under the existence of market impact and liquidity cost for the underlying asset. This study also provides derivatives pricing with this model through a stochastic control problem, which is solved analytically or approximately by an asymptotic expansion. This method is useful since traders in financial institutions are able to estimate both the liquidity cost and the market impact charge when they quote derivatives prices. This is particularly important when financial institutions trade derivatives on an underlying asset with low liquidity and there exist market impacts on the underlying asset price which are caused by the hedging transactions.

First, we have formulated the charge for the liquidity cost and the market impact through a stochastic control problem that maximizes the mark-to-market value of the portfolio at maturity less the quadratic hedging error during the hedging period and the liquidation cost. The model and the control problem incorporate execution errors in the hedging as well as different settlement types, which are physical settlement and cash settlement. After we derived the HJB equation corresponding to the control problem and reduced it to a semilinear PDE, we have given an analytic solution or a numerical algorithm by an asymptotic expansion depending on the payoff type in order to solve the equation. In detail, as for the asymptotic expansion, we expand the semilinear PDE with respect to the market impact parameter up to the first order and reduce it to an explicitly solvable semilinear PDE for the zeroth part and a second order linear parabolic PDE for the first part. Then, we compute the solutions of the PDEs through their Feynman-Kac representations by Monte Carlo simulation. Furthermore, we have provided concrete examples of the charges on the liquidity cost and the market impact for both a quadratic payoff in physical settlement, where the charge is solved analytically, and a European call payoff in physical and cash settlement, where the charge is obtained by the asymptotic expansion. We have also presented comparative static analyses for the parameters' changes in the quadratic payoff and the call payoff in physical settlement.

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A Existence of a Unique Classical Solution of (40)

In this appendix, we show that the assumptions for existence of a classical solution for HJB equations in Fleming and Rishel [1] are satisfied.

1. Uniform Parabolicity of $a(t,x_1)$. For all $\xi \in \mathbf{R}^2, t \in [0,T], x_1 \in \mathbf{R}$,

$$\begin{aligned}
&\frac{1}{2}\sigma^2\xi_1^2 + \frac{1}{2}(1+\delta^2)\sigma^2\Gamma(t,x_1)^2\xi_2^2 + \sigma^2\Gamma(t,x_1)\xi_1\xi_2 \\
&= \frac{1}{2}\Bigg\{(\sigma\xi_1 + \sqrt{1+\delta^2}\sigma\Gamma(t,x_1)\xi_2)^2\frac{1}{\sqrt{1+\delta^2}} \\
&\quad +(1-\frac{1}{\sqrt{1+\delta^2}})\sigma^2\xi_1^2 + (1-\frac{1}{\sqrt{1+\delta^2}})(1+\delta^2)\sigma^2\Gamma(t,x_1)^2\xi_2^2\Bigg\} \\
&\geq \frac{1}{2}(1-\frac{1}{\sqrt{1+\delta^2}})\sigma^2\xi_1^2 + \frac{1}{2}(1-\frac{1}{\sqrt{1+\delta^2}})(1+\delta^2)\sigma^2\Gamma(t,x_1)^2\xi_2^2 \\
&\geq \min\Big(\frac{1}{2}(1-\frac{1}{\sqrt{1+\delta^2}})\sigma^2,\ \frac{1}{2}(1-\frac{1}{\sqrt{1+\delta^2}})(1+\delta^2)\sigma^2\Gamma(t,x_1)^2\Big)(\xi_1^2+\xi_2^2) \\
&\geq \min\Big(\frac{1}{2}(1-\frac{1}{\sqrt{1+\delta^2}})\sigma^2,\ \frac{1}{2}(1-\frac{1}{\sqrt{1+\delta^2}})(1+\delta^2)\sigma^2c_1^2\Big)|\xi|^2. \qquad (123)
\end{aligned}$$

   In the last inequality, we have used (7) in Assumption 1.

2. $\sigma(t,x_1) \in C^{1,2}(\bar{Q}_0)$ follows from Assumption 1.

3. σ, σ^{-1} and σ_x are bounded on $\bar{Q}_0$. This follows from (27), Assumption 1, and

$$\sigma^{-1}(t,x_1,x_2) = \begin{pmatrix} \frac{1}{\sigma} & 0 \\ -\frac{1}{\delta\sigma} & \frac{1}{\delta\sigma\Gamma(t,x_1)} \end{pmatrix}. \qquad (124)$$

4. $\Theta \in C^1(\bar{Q}_0 \times U)$, with Θ and Θ_x bounded. This follows from (41),(42) and (7) in Assumption 1.

5. $L \in C^1(\bar{Q}_0 \times U)$ with L and L_x satisfying a polynomial growth condition. This follows from (29) and Assumption 1.

B Asymptotic Expansion for European Call Payoff in Cash Settlement

In the case of cash settlement of European call option, where $b_2 = k_1, b_1 = -2k_1 g_{x_1}(T, x_1), b_0 = k_1 g_{x_1}^2(T, x_1)$ as in (24) and $h(x_1) = (x_1 - K)^+$, the first order approximation is calculated as follows. Noting that

$$b_2 = k_1, \tag{125}$$

$$b_1 = -2k_1 g_{x_1}(T, x_1) = -2k_1 1_{\{x_1 > K\}}, \tag{126}$$

$$b_0 = k_1 g_{x_1}^2(T, x_1) = k_1 1_{\{x_1 > K\}}, \tag{127}$$

by (77),(78) and (79), we have

$$B_1(t, x_1) = \exp\left(-\int_t^T \frac{1}{\eta} B_2(s) ds\right)\left(-2k_1 N(\frac{x_1 - K}{\sigma\sqrt{T-t}})\right), \tag{128}$$

$$B_{1,x_1}(t, x_1) = \exp\left(-\int_t^T \frac{1}{\eta} B_2(s) ds\right)\left(-2k_1 \frac{1}{\sigma\sqrt{T-t}} n(\frac{x_1 - K}{\sigma\sqrt{T-t}})\right), \tag{129}$$

$$B_{1,x_1x_1}(t, x_1) = \exp\left(-\int_t^T \frac{1}{\eta} B_2(s) ds\right)\left(2k_1 \frac{x_1 - K}{\sigma^3(\sqrt{T-t})^3} n(\frac{x_1 - K}{\sigma\sqrt{T-t}})\right), \tag{130}$$

where $\exp\left(-\int_t^T \frac{1}{\eta} B_2(s) ds\right)$ is the one obtained in (80).

Next, we note that

$$\mathbf{E}\left[b_0(x_1 + \sigma W_{T-t})\right] = -k_1 N\left(\frac{x_1 - K}{\sigma\sqrt{T-t}}\right), \tag{131}$$

$$\begin{aligned}
&\mathbf{E}\left[\sigma^2 \Gamma(s, x_1 + \sigma W_{s-t}) B_{1,x_1}(s, x_1 + \sigma W_{s-t})\right] \\
&= -2k_1\sigma^2 \exp\left(-\int_s^T \frac{1}{\eta} B_2(v) dv\right) \mathbf{E}\left[\Gamma^2(s, x_1 + \sigma W_{s-t})\right], \\
&= -2k_1\sigma^2 \exp\left(-\int_s^T \frac{1}{\eta} B_2(v) dv\right) \frac{1}{2\pi\sigma^2\sqrt{T-s}\sqrt{T+s-2t}} \exp\left(-\frac{(x_1 - K)^2}{\sigma^2(T+s-2t)}\right).
\end{aligned} \tag{132}$$

$$\mathbf{E}\left[-\frac{1}{4\eta}B_1^2(s,x_1+\sigma W_{s-t})\right]$$
$$= -\frac{1}{\eta}k_1^2\exp\left(-2\int_s^T\frac{1}{\eta}B_2(v)dv\right)\mathbf{E}\left[N^2\left(\frac{x_1+\sigma W_{s-t}-K}{\sigma\sqrt{T-s}}\right)\right]$$
$$= -\frac{1}{\eta}k_1^2\exp\left(-2\int_s^T\frac{1}{\eta}B_2(v)dv\right)\left[1-\sqrt{\frac{2(s-t)}{\pi(T-t)}}\exp\left(\frac{1}{2}(x_1-K)^2\left\{\frac{s-t}{T-t}-\frac{1}{\sigma^2(T-s)}\right\}\right)\right.$$
$$\left. N\left(\sqrt{\frac{T-t}{T+s-2t}}\left\{\frac{1}{\sigma\sqrt{T-s}}-\frac{s-t}{T-t}\right\}(x_1-K)\right)\right]. \tag{133}$$

Then, in a similar manner as in Example 5, we have

$$B_0(t,x_1)$$
$$= -k_1N\left(\frac{x_1-K}{\sigma\sqrt{T-t}}\right)$$
$$+\int_t^T\frac{1}{2\pi\sigma^2\sqrt{T-s}\sqrt{T+s-2t}}\exp\left(-\frac{(x_1-K)^2}{\sigma^2(T+s-2t)}\right)$$
$$\times\left[\tilde{\sigma}^2B_2(s)+\delta^2\lambda\sigma^2\left\{\frac{\sigma^2(s-t)(T-s)}{T+s-2t}+\left(x_1-\frac{2(x_1-K)(s-t)}{T+s-2t}\right)^2\right\}\right]ds$$
$$+\int_t^T -2k_1\sigma^2\exp\left(-\int_s^T\frac{1}{\eta}B_2(v)dv\right)\frac{1}{2\pi\sigma^2\sqrt{T-s}\sqrt{T+s-2t}}\exp\left(-\frac{(x_1-K)^2}{\sigma^2(T+s-2t)}\right)ds$$
$$+\int_t^T -\frac{1}{\eta}k_1^2\exp\left(-2\int_s^T\frac{1}{\eta}B_2(v)dv\right)\left[1-\sqrt{\frac{2(s-t)}{\pi(T-t)}}\exp\left(\frac{1}{2}(x_1-K)^2\left\{\frac{s-t}{T-t}-\frac{1}{\sigma^2(T-s)}\right\}\right)\right.$$
$$\left.\times N\left(\sqrt{\frac{T-t}{T+s-2t}}\left\{\frac{1}{\sigma\sqrt{T-s}}-\frac{s-t}{T-t}\right\}(x_1-K)\right)\right]ds. \tag{134}$$

B_{0,x_1} is obtained by taking partial derivative of this with respect to x_1; $u^{(0,0)}$ and $u^{(1,0)}$ are calculated by (90) and (86).

C Proof of Proposition 1

In this appendix, we give the proof of Proposition 1.

Set $D_t := g_x(t,P_t),\ D_t^M := D_t 1_{[-M,M]},\ M>0$.

First, note that

$$\mathbf{E}\left[\int_0^T D_s^2ds\right] = D_0^2T+\mathbf{E}\left[\sigma^2\int_0^T\int_0^s\Gamma(u,P_u)^2duds\right] \tag{135}$$
$$\leq D_0^2T+\sigma^2c_2^2T^2<\infty, \tag{136}$$

and hence, there exists $M>0$ such that

$$\mathbf{E}\left[\int_0^T(D_s-D_s^M)^2ds\right]<\frac{\epsilon}{9}. \tag{137}$$

Then, by the continuity of D_t^M and the bounded convergence theorem, there exists $N > 0$ such that for all $n \geq N$,

$$\mathbf{E}\left[\int_0^T (D_s^M - D_s^{(n)})^2 ds\right] < \frac{\epsilon}{9}, \tag{138}$$

where

$$D_s^{(n)} := \begin{cases} D^M_{\frac{(i-1)T}{2^n}}, \ s \in (\frac{(i-1)T}{2^n}, \frac{iT}{2^n}], \\ D_0^M, \ s = 0. \end{cases} \tag{139}$$

Hereafter, we formally set $D^M_{\frac{-T}{2^n}} := 0$ for a notational purpose.

Next, we define

$$\theta_s^{(\epsilon)} = \begin{cases} -\max\left(\frac{2^N}{T}|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|, |D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^3\left(\frac{3\cdot 2^N}{\epsilon}\right)\right)\mathrm{sgn}(D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}) \\ \times 1_{(\frac{(i-1)T}{2^N}, \frac{(i-1)T}{2^N} + \min(\frac{T}{2^N}, \frac{\epsilon}{|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^2 3\cdot 2^N})]}(s), s \in (\frac{(i-1)T}{2^N}, \frac{iT}{2^N}], \ 2 \leq i \leq 2^N, \\ -\max\left(\frac{2^N}{T}|D_0^M|, |D_0^M|^3\left(\frac{3\cdot 2^N}{\epsilon}\right)\right)\mathrm{sgn}(D_0^M) \\ \times 1_{(\frac{(i-1)T}{2^N}, \frac{(i-1)T}{2^N} + \min(\frac{T}{2^N}, \frac{\epsilon}{|D_0^M|^2 3\cdot 2^N})]}(s), \ s \in [0, \frac{T}{2^N}]. \end{cases}$$

First, we note that for $1 \leq i \leq 2^N$,

$$\begin{aligned} D^{(N)}_{\frac{iT}{2^N}} - D^{(N)}_{\frac{(i-1)T}{2^N}} + \int_{\frac{(i-1)T}{2^N}}^{\frac{iT}{2^N}} \theta_u^{(\epsilon)} du = D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}} + \int_{\frac{(i-1)T}{2^N}}^{\frac{iT}{2^N}} \theta_u^{(\epsilon)} du \\ = 0. \end{aligned} \tag{140}$$

Then,

$$\begin{aligned} \int_0^T (D_s^{(N)} + \int_0^s \theta_u^{(\epsilon)} du)^2 ds &= \sum_{i=1}^{2^N} \int_{\frac{(i-1)T}{2^N}}^{\frac{iT}{2^N}} (D_s^{(N)} + \int_0^s \theta_u^{(\epsilon)} du)^2 ds \\ &= \sum_{i=1}^{2^N} \int_{\frac{(i-1)T}{2^N}}^{\frac{iT}{2^N}} ((D_s^{(N)} - D^{(N)}_{\frac{(i-1)T}{2^N}}) + \int_{\frac{(i-1)T}{2^N}}^s \theta_u^{(\epsilon)} du)^2 ds \\ &= \sum_{i=1}^{2^N} \int_{\frac{(i-1)T}{2^N}}^{\frac{iT}{2^N}} ((D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}) + \int_{\frac{(i-1)T}{2^N}}^s \theta_u^{(\epsilon)} du)^2 ds. \end{aligned} \tag{141}$$

Note that when

$$\frac{T}{2^N} \geq \frac{\epsilon}{|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^2 3\cdot 2^N}, \tag{142}$$

$$
\begin{aligned}
&\int_{\frac{(i-1)T}{2^N}}^{\frac{iT}{2^N}} ((D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}) + \int_{\frac{(i-1)T}{2^N}}^{s} \theta^{(\epsilon)}_u du)^2 ds \\
&= \int_{\frac{(i-1)T}{2^N}}^{\frac{(i-1)T}{2^N} + \frac{\epsilon}{|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^2 3\cdot 2^N}} \\
&\quad \left(|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}| - |D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^3 \left(\frac{3\cdot 2^N}{\epsilon}\right)(s - \frac{(i-1)T}{2^N}) \right)^2 ds \\
&= \int_0^{\frac{\epsilon}{|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^2 3\cdot 2^N}} \left(|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^3 \left(\frac{3\cdot 2^N}{\epsilon}\right) v \right)^2 dv \\
&= \frac{1}{3} |D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^6 \left(\frac{3\cdot 2^N}{\epsilon}\right)^2 \left(\frac{\epsilon}{|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^2 3\cdot 2^N} \right)^3 \\
&= \frac{\epsilon}{9\cdot 2^N}.
\end{aligned} \tag{143}
$$

In the third line of (143), we have changed the variable as

$$
v = \frac{(i-1)T}{2^N} + \frac{\epsilon}{|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^2 3\cdot 2^N} - s. \tag{144}
$$

Note also that when

$$
\frac{T}{2^N} < \frac{\epsilon}{|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^2 3\cdot 2^N}, \tag{145}
$$

$$
\begin{aligned}
&\int_{\frac{(i-1)T}{2^N}}^{\frac{iT}{2^N}} ((D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}) + \int_{\frac{(i-1)T}{2^N}}^{s} \theta^{(\epsilon)}_u du)^2 ds \\
&= \int_{\frac{(i-1)T}{2^N}}^{\frac{iT}{2^N}} \left(|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}| - \frac{2^N}{T} |D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}| (s - \frac{(i-1)T}{2^N}) \right)^2 ds \\
&\leq \int_{\frac{(i-1)T}{2^N}}^{\frac{iT}{2^N}} \left(|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}| - |D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^3 \left(\frac{3\cdot 2^N}{\epsilon}\right)(s - \frac{(i-1)T}{2^N}) \right)^2 ds \\
&\leq \int_{\frac{(i-1)T}{2^N}}^{\frac{(i-1)T}{2^N} + \frac{\epsilon}{|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^2 3\cdot 2^N}} \\
&\quad \left(|D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}| - |D^M_{\frac{(i-1)T}{2^N}} - D^M_{\frac{(i-2)T}{2^N}}|^3 \left(\frac{3\cdot 2^N}{\epsilon}\right)(s - \frac{(i-1)T}{2^N}) \right)^2 ds \\
&= \frac{\epsilon}{9\cdot 2^N}.
\end{aligned} \tag{146}
$$

Hence, we have

$$
\mathbf{E}\left[\int_0^T (D^{(N)}_s + \int_0^s \theta^{(\epsilon)}_u du)^2 ds \right] \leq \frac{\epsilon}{9}. \tag{147}
$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T (D_s+\int_0^s \theta_u^{(\epsilon)}du)^2 ds\right]\\
&=\mathbf{E}\left[\int_0^T (D_s-D_s^M+D_s^M-D_s^{(N)}+D_s^{(N)}+\int_0^s \theta_u^{(\epsilon)}du)^2 ds\right]\\
&\leq 3\mathbf{E}\left[\int_0^T (D_s-D_s^M)^2 ds\right]+3\mathbf{E}\left[\int_0^T (D_s^M-D_s^{(N)})^2 ds\right]+3\mathbf{E}\left[\int_0^T (D_s^{(N)}+\int_0^s \theta_u^{(\epsilon)}du)^2 ds\right]\\
&<\epsilon. \qquad (148)
\end{aligned}$$

Next, we consider $\theta_s^{(\epsilon,A)}$ taking values in $U=[-A,A],\ A>0$.
Set

$$\theta_s^{(\epsilon,A)}=\begin{cases}
-\min\left(A,\ \max\left(\frac{2^N}{T}|D^M_{\frac{(i-1)T}{2^N}}-D^M_{\frac{(i-2)T}{2^N}}|,|D^M_{\frac{(i-1)T}{2^N}}-D^M_{\frac{(i-2)T}{2^N}}|^3\left(\frac{3\cdot 2^N}{\epsilon}\right)\right)\right)\\
\times\mathrm{sgn}(D^M_{\frac{(i-1)T}{2^N}}-D^M_{\frac{(i-2)T}{2^N}})\\
\times 1_{(\frac{(i-1)T}{2^N},\frac{(i-1)T}{2^N}+\min(\frac{T}{2^N},\frac{\epsilon}{|D^M_{\frac{(i-1)T}{2^N}}-D^M_{\frac{(i-2)T}{2^N}}|^2 3\cdot 2^N})]}(s),\ \ s\in(\frac{(i-1)T}{2^N},\frac{iT}{2^N}],\ 2\leq i\leq 2^N,\\
-\min\left(A,\ \max\left(\frac{2^N}{T}|D_0^M|,|D_0^M|^3\left(\frac{3\cdot 2^N}{\epsilon}\right)\right)\right)\\
\times\mathrm{sgn}(D_0^M)1_{(\frac{(i-1)T}{2^N},\frac{(i-1)T}{2^N}+\min(\frac{T}{2^N},\frac{\epsilon}{|D_0^M|^2 3\cdot 2^N})]}(s),\ \ s\in[0,\frac{T}{2^N}].
\end{cases}$$

Note that for $s\in(\frac{(i-1)T}{2^N},\frac{iT}{2^N}]$, $1\leq i\leq 2^N$,

$$\begin{aligned}
(D_s+\int_0^s \theta_u^{(\epsilon,A)}du)^2 &\leq 3(D_s-D_s^M)^2+3(D_s^M-D_s^{(N)})^2+3(D_s^{(N)}+\int_0^s \theta_u^{(\epsilon,A)}du)^2\\
&\leq 3(D_s-D_s^M)^2+3(D_s^M-D_s^{(N)})^2+3\cdot 2^N\sum_{j=1}^{2^N}(D^M_{\frac{(j-1)T}{2^N}}-D^M_{\frac{(j-2)T}{2^N}})^2\\
&\leq 3(D_s-D_s^M)^2+3(D_s^M-D_s^{(N)})^2+3\cdot 2^N\sum_{j=1}^{2^N}(D_{\frac{(j-1)T}{2^N}}-D_{\frac{(j-2)T}{2^N}})^2,
\end{aligned} \qquad (149)$$

and

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T 3(D_s-D_s^M)^2+3(D_s^M-D_s^{(N)})^2+3\cdot 2^N\sum_{j=1}^{2^N}(D_{\frac{(j-1)T}{2^N}}-D_{\frac{(j-2)T}{2^N}})^2 ds\right]\\
&<\frac{2}{3}\epsilon+3\cdot 2^N T^2\sigma^2 c_2^2.
\end{aligned} \qquad (150)$$

In the second line of (149), we have used

$$
\begin{aligned}
&(D_s^{(N)} + \int_0^s \theta_u^{(\epsilon,A)} du)^2 \\
&= \left[(D_s^{(N)} - D_{\frac{(i-1)T}{2^N}}^{(N)} + \int_{\frac{(i-1)T}{2^N}}^{s} \theta_u^{(\epsilon,A)} du) + \sum_{j=1}^{i-1} (D_{\frac{jT}{2^N}}^{(N)} - D_{\frac{(j-1)T}{2^N}}^{(N)} + \int_{\frac{(j-1)T}{2^N}}^{\frac{jT}{2^N}} \theta_u^{(\epsilon,A)} du) \right]^2 \\
&= \left[(D_{\frac{(i-1)T}{2^N}}^{M} - D_{\frac{(i-2)T}{2^N}}^{M} + \int_{\frac{(i-1)T}{2^N}}^{s} \theta_u^{(\epsilon,A)} du) + \sum_{j=1}^{i-1} (D_{\frac{(j-1)T}{2^N}}^{M} - D_{\frac{(j-2)T}{2^N}}^{M} + \int_{\frac{(j-1)T}{2^N}}^{\frac{jT}{2^N}} \theta_u^{(\epsilon,A)} du) \right]^2 \\
&\le i \left[(D_{\frac{(i-1)T}{2^N}}^{M} - D_{\frac{(i-2)T}{2^N}}^{M} + \int_{\frac{(i-1)T}{2^N}}^{s} \theta_u^{(\epsilon,A)} du)^2 + \sum_{j=1}^{i-1} (D_{\frac{(j-1)T}{2^N}}^{M} - D_{\frac{(j-2)T}{2^N}}^{M} + \int_{\frac{(j-1)T}{2^N}}^{\frac{jT}{2^N}} \theta_u^{(\epsilon,A)} du)^2 \right] \\
&\le i \left[(D_{\frac{(i-1)T}{2^N}}^{M} - D_{\frac{(i-2)T}{2^N}}^{M})^2 + \sum_{j=1}^{i-1} (D_{\frac{(j-1)T}{2^N}}^{M} - D_{\frac{(j-2)T}{2^N}}^{M})^2 \right] \\
&\le 2^N \sum_{j=1}^{2^N} (D_{\frac{(j-1)T}{2^N}}^{M} - D_{\frac{(j-2)T}{2^N}}^{M})^2. \qquad (151)
\end{aligned}
$$

Since

$$\lim_{A\to\infty} (D_s + \int_0^s \theta_u^{(\epsilon,A)} du)^2 = (D_s + \int_0^s \theta_u^{(\epsilon)} du)^2, \qquad (152)$$

by the dominated convergence theorem, we obtain

$$\lim_{A\to\infty} \mathbf{E}\left[\int_0^T (D_s + \int_0^s \theta_u^{(\epsilon,A)} du)^2 ds \right] = \mathbf{E}\left[\int_0^T (D_s + \int_0^s \theta_u^{(\epsilon)} du)^2 ds \right] < \epsilon. \qquad (153)$$

□