A New Structural Model for Multidimensional Default Risk

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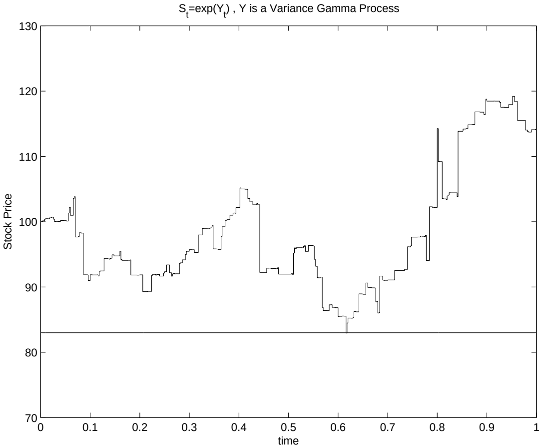
Overview

Structural versus Reduced Form Models

- Structural
  - Black and Scholes in 1973, Merton 1974, Black and Cox(1976),etc
- Reduced Form
  - Jarrow and Turnbull in 1995



Structural versus Reduced Form Models

- Structural or Reduced Form?
- default time: predictable vs inaccessible
- Robert A.Jarrow and Philip Protter: complete information vs incomplete information

Structural versus Reduced Form Models

Structural Model with Partial Information

Let X_t be the cash balances of the firm, $g(t) := sup\{s \leq t : X_s = 0\}$ denote the last time (before t) that cash balances hit zero.Let

$$\tau_\alpha := inf\{t > 0 : t - g(t) \geq \frac{\alpha^2}{2}, where \quad X_s < 0 \quad for \quad s \in (g(t-), t)\}$$

τ_α is the first time that the firm's cash balances have continued to be negative for at least $\frac{\alpha^2}{2}$ units of time. The default time is defined as

$$\tau := inf\{t > \tau_\alpha : X_t = 2X_{\tau_\alpha}\}$$

Reduced Form Model

Model Setup

Suppose τ is the default time, $H_t = I_{\tau \leq t}$, and $\mathcal{H}_t = \sigma(H_s : s \leq t)$ a filtration denotes the default time information. Denote $F(t)$ the survival probability,i.e. $F(t) = P(\tau > t)$.

Definition

Hazard Function The function Γ: $R^+ \to R^+$ given by

$$\Gamma(t) = -log(F(t))$$

is called the hazard function. If F is absolutely continuous, define the intensity function

$$\lambda(t) = \Gamma'(t)$$

$$F(t) = e^{-\int_0^t \lambda(s)ds}$$

Reduced Form Model

- advantage: explicit formulas for survival probabilities

$$P(t, T) = 1_{\tau > t} E^Q[e^{-\int_t^T \lambda(X_s)ds} | \mathcal{F}_t]$$

- interest rate model
- disadvantage: lack economic interpretation

A Hybrid Model

Definition

(New structural Model: one dimension) Let S be the stock price of a firm which is described by an exponential Lévy process . We define the time of default as the first time the log-return of S_t jumps below a level $a(t) < 0$.

$$\tau = inf\{t > 0 : log S_t / S_{t^-} \leq a(t)\}$$

We call $a(t)$ the default level of the firm, $a(t)$ could be stochastic.

- stock price proxy for firm's value
- exponential lévy process to describe stock price
- jump of log return

Lévy Process

Definition

(Lévy Process) A process $X = X_t : t \geq 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it possesses the following properties:
(i) The paths of X are $\mathbb{P}$-almost surely right continuous with left limits.
(ii)P($X_0 = 0$)=1.
(iii)For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.
(iv) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $X_u : u \leq s$.

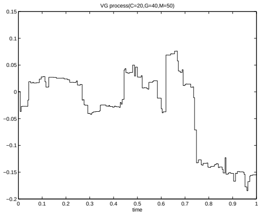


Figure: A sample path of a Variance Gamma process.

Lévy Process: random measure

Definition

(Random measure) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(S, \mathcal{S})$ a measurable space. The function M: $\Omega \times \mathcal{S} \to [0, \infty]$ is called a random measure if
(i) $\forall A \in \mathcal{S}$,M(.,A) is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.
(ii) $\forall \omega \in \Omega$, $M(\omega, .)$ is a measure on $(S, \mathcal{S})$

Definition

(Poisson random measure) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(S, \mathcal{S}, \eta)$ a measurable space. A random measure X is a Poisson random measure on $\mathcal{S}$ with intensity η if:
(i) $\forall A \in \mathcal{S}$, X(A) is a Poisson random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with parameter $\eta(A)$.
(ii)$\forall \{A_i\}_{i=1}^{n} \in \mathcal{S}$ such that $A_i \bigcap A_j = \emptyset$ if $i \neq j$, the r.v. $X(A_i)$ and $X(A_j)$ are independent.

Definition

(Temporal Poisson random measure-TPRM) Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space, $(E, \mathcal{E}, \pi)$ a measure space, and Leb denote Lebesgue measure on $[0, \infty)$. A temporal Poisson random measure (TPRM) X on E with intensity π is an $\mathcal{F}$-adapted Poisson random measure on $([0, \infty) \times E, \mathcal{B}[0, \infty) \otimes \mathcal{E})$ with intensity $Leb \times \pi$. For $t \geq 0, A \in \mathcal{E}$ we write $X_t(A) = X([0, t] \times A)$.

With this TPRM, we can define the integral with this measure. For $t \geq 0, A \in \mathcal{E}$, we can write the measure in a integral form

$$X(\omega, [0, t] \times A) = \int_0^t \int_A X(\omega, ds \times dx) \tag{1}$$

Further, we consider integral with measurable function x,

$$\int_0^t \int_A xX(\omega, ds \times dx) \tag{2}$$

Tail Process

Definition

(Tail Process) Let Y be a Lévy process with Levy measure λ. The tail process Y^a of Y at the level a is the counting process defined by

$$Y_t^a = \sharp\{0 \leq s \leq t : \triangle Y_s(\omega) \leq a\}, \omega \in \Omega, t \geq 0$$

Important Fact

The tail process Y^a of Y is a Poisson process with mean $\Lambda(a)$ and the survival time is an exponential random variable with parameter $\Lambda(a)$. where $\Lambda(x) = \int_{-\infty}^{x} \lambda(d\omega)$ is the tail integral of the process Y.

survival probability

Then the survival probability of the firm up to time $t > 0$ is given by

$$P(\tau > t) = E[e^{-\int_0^t \Lambda(a_u)du}]$$

Further more, for almost every $t \geq 0$, the local default rate LDR_t exists and is given by

$$LDR_t \equiv lim_{h \downarrow 0} \frac{\mathbb{P}(\tau \leq t + h | \tau > t)}{h} = E(\Lambda(a_t) | \tau > t)$$

Multidimensional Poisson Processes

Theorem

Let N and M be two Poisson processes on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ with parameters λ and μ. Suppose (N, M) is a two dimensional Poisson process. Then there exists three independent adapted Poisson processes L_1,L_2,L_{12}, with respective parameters $\lambda - \rho$, $\mu - \rho$, ρ, such that,

$$N_t^1 = L_1 + L_{12}$$
$$M_t^2 = L_2 + L_{12}$$

Multidimensional Lévy process

$Y = (Y_t^1, Y_t^2, ..., Y_t^d)$ is a d dimensional Lévy process, then there exists a vector $\vec{b} \in \mathbb{R}^d$, a unique Σ, and a Temporal Poisson random measure(TPRM) X on $E = \mathbb{R}^d \setminus \{0\}$ with intensity π such that

$$Y_t(\omega) = -\vec{b}t + \Sigma W_t + \int_0^t \int_{|x|\geq 1} xX(ds \times dx) + lim_{\epsilon\downarrow 0} \int_0^t \int_{\epsilon<|x|<1} x\{X(dx \times dx) - ds\pi(dx)\} \quad (3)$$

Apply to Tail Processes

$$N_t := (X_1^{a_1(t)}, X_2^{a_2(t)}, X_3^{a_3(t)})$$

where $X_i^{a_i(t)}$ is the tail process of X_i at level $a_i(t), i = 1, 2, 3$. there exists 7 independent adapted Poisson processes L_I

$$N_t^1 = L_1 + L_{12} + L_{13} + L_{123}$$
$$N_t^2 = L_2 + L_{12} + L_{23} + L_{123}$$
$$N_t^3 = L_3 + L_{13} + L_{23} + L_{123}$$

$$\begin{aligned}
&\mathbb{P}(\tau_u^1 > x_1, \tau_u^2 > x_2, \tau_u^3 > x_3) \\
&= E[e^{-\int_u^{u+x_1} \Lambda_1(s)ds} e^{-\int_u^{u+x_2} \Lambda_2(s)ds} e^{-\int_u^{u+x_3} \Lambda_3(s)ds} \\
&e^{\int_u^{u+min(x_1,x_2)} \Lambda_{12}(s)ds} e^{\int_u^{u+min(x_1,x_3)} \Lambda_{13}(s)ds} e^{\int_u^{u+min(x_2,x_3)} \Lambda_{23}(s)ds} \\
&e^{-\int_u^{u+min(x_1,x_2,x_3)} \Lambda_{123}(s)ds} | \mathcal{F}_u]
\end{aligned}$$

Idea of poof

$$N_t^1 = \int_0^t \int_{A_1} X(ds \times d\vec{x})$$

$$A_1 = B_1 \bigcup B_{12} \bigcup B_{13} \bigcup B_{123}$$

$$\begin{aligned} N_t^1 &= \int_0^t \int_{B_1} X(ds \times d\vec{x}) + \int_0^t \int_{B_{12}} X(ds \times d\vec{x}) + \int_0^t \int_{B_{13}} X(ds \times d\vec{x}) \\ &+ \int_0^t \int_{B_{123}} X(ds \times d\vec{x}) \\ &= L_1 + L_{12} + L_{13} + L_{123} \end{aligned}$$

In summary,

$$N_t^1, N_t^2, N_t^3$$

represented by 7 independent Poisson processes.

Multidimensional Model

Suppose there is a portfolio consisted of d obligors. Consider a d-dimensional stock price vector $\overrightarrow{S_t} = (S_t^1, S_t^2, ..., S_t^d)$ where S_t^i denotes i^{th} firm's stock price. Let τ_i be the default time of name i, τ_i is defined as the first time log-return of S_t^i jumps below a level $a^i(t) < 0$.

$$\tau^i = inf\{t > 0 : logS_t^i / S_{t^-}^i \leq a^i(t)\}$$

$a^i(t)$ is called default level, $a^i(t)$ could be stochastic.
Assume the price vector $\overrightarrow{S_t}$ follows an exponential Lévy process, i.e.

$$S_t^j = S_0^j exp\{rt + Y_t^j + t\psi^j(-i)\}$$

where S_0^j is the initial stock price, r is interest rate, $Y = (Y_t^1, Y_t^2, ..., Y_t^d)$ is a d dimensional Lévy process defined on $\mathbb{R}^n$, ψ^j is the characteristic exponent of Y_1^j, $j = 1, 2, ..., n$.

If the default level $\vec{a} = (a_1(t), a_2(t), a_3(t))$ is stochastic, then

$$\mathbb{P}(\tau_u^1 > x_1, \tau_u^2 > x_2, \tau_u^3 > x_3) = E[e^{-\int_u^{u+max(x_1,x_2,x_3)} \vec{1}^T \Lambda^* \vec{1} ds} | \mathcal{F}_u]$$

$\Lambda^* =$

$$\begin{pmatrix} \Lambda_1 1_{s<min(x_1)} & -\Lambda_{12} 1_{s<min(x_1,x_2)} & -\Lambda_{13} 1_{s<min(x_1,x_3)} \\ 0 & \Lambda_2 1_{s<min(x_2)} & -\Lambda_{23} 1_{s<min(x_2,x_3)} \\ \Lambda_{123} 1_{s<min(x_1,x_2,x_3)} & 0 & \Lambda_3 1_{s<min(x_3)} \end{pmatrix}$$

Multidimensional Risk

- default dependency
- dependency measure
  - correlation coefficient, rank correlation, coefficients of tail dependence.
  - copula function

Definition

(Copula Functions) U is a uniform random variable if it has a uniform distribution on the interval [0,1]. For d uniform random variables $U_1, U_2, .., U_d$, the joint distribution function C, defined as

$$C(u_1, u_2, ..., u_d) = P(U_1 \leq u_1, U_2 \leq u_2, ..., U_d \leq u_d)$$

is called a copula function

Theorem

Sklar's theorem Let F be a joint distribution function with margins F_1,F_2,...,F_d, then there exists a copula C such that for all $(x_1, x_2, ..., x_d) \in R^d$,

$$F(x_1, x_2, ...x_d) = C(F_1(x_1), F_2(x_2), ..., F_d(x_d))$$

Conversely, if C is a copula function and $F_1, F_2, ...F_d$ are distributions, then the function F defined by above formula is a joint distribution function with margins $F_1, F_2, ..., F_d$.

Lévy Copulas

- Lévy copulas parallels the notion of a copula on the level of Lévy measures.
- Lévy copulas are used to characterize the dependence among components of multidimensional Lévy processes.
- A version of Sklar's theorem: the law of a general multivariate Lévy process is obtained by combining arbitrary univariate Lévy processes with an arbitrary Lévy copula.
- Copulas allow to separate the dependence structure of a random vector from its univariate margins.

Multidimensional Risk

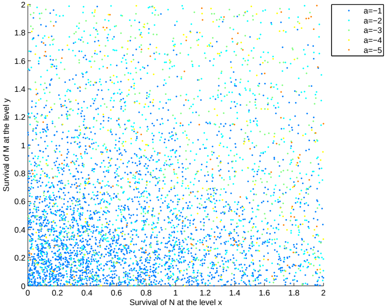
- Assume that the 2 stock prices $S = (S_t^1, S_t^2)$ of 2 firms respectively can be written in terms of a 2 dimensional Lévy process $Y = (Y_t^1, Y_t^2)$
- Default Levels:$\vec{a} = (a_1(t), a_2(t))$
- Joint Survival Probability

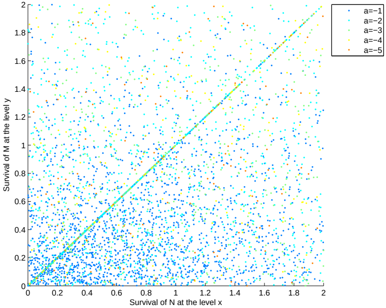
$$\mathbb{P}(\tau^1 > t_1, \tau^2 > t_2)$$

- tail integrals

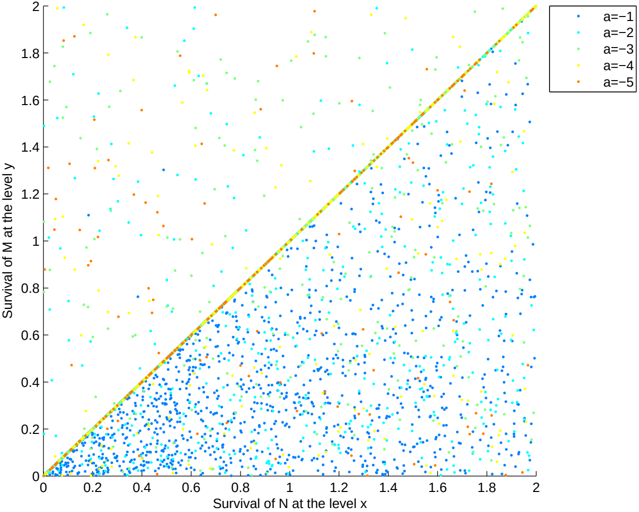
$$\Lambda_{12}(x_1, x_2) := \pi((-\infty, x_1) \times (-\infty, x_2))$$
$$\Lambda_1(x) := \mu_1((-\infty, x))$$
$$\Lambda_2(x) := \mu_2((-\infty, x))$$

Simulation of Two Dimensional Default





Simulation of Two Dimensional Default



survival probability by Lévy copula

$$\mathbb{P}(\tau_u^1 > x_1, \tau_u^2 > x_2, \tau_u^3 > x_3) = E[e^{-\int_u^{u+max(x_1,x_2,x_3)} \vec{1}^T \Lambda^* \vec{1} ds} | \mathcal{F}_u]$$

where $\Lambda^* =$

$$\Lambda_1 1_{s<min(x_1)} \quad -F^{12}(-\Lambda_1, -\Lambda_2) 1_{s<min(x_1,x_2)} \quad -F^{13}(-\Lambda_1, -\Lambda_3) 1_{s<min(x_1,x_3)}$$

$$0 \quad \Lambda_2 1_{s<min(x_2)} \quad -F^{23}(-\Lambda_2, -\Lambda_3) 1_{s<min(x_2,x_3)}$$

$$-F^{123}(-\Lambda_1, -\Lambda_2, -\Lambda_3) 1_{s<min(x_1,x_2,x_3)} \quad 0 \quad \Lambda_3 1_{s<min(x_3)}$$

Conclusion: Two threshold

reduced form model and structural model

Suppose there are d firms. For each obligor $1 \leq i \leq d$, we define

1. The default intensity $\lambda^i(t)$: a deterministic function. We usually assume it to be a step function.
2. The survival function $S^i(t)$:

$$S^i(t) := exp(-\int_0^t \lambda^i(u)du).$$

3. The default trigger variables U_i: uniform random variables on [0,1]. The d-dimensional vector $U = (U_1, U_2, ..., U_d)$ is distributed according to the d-dimensional copula C.
4. The time of default τ_i of obligor i, where $i = 1, ..., d$,

$$\tau_i := inf\{t : S^i(t) \leq 1 - U_i\}.$$

Closed Form

Basket CDS Pricing formula

$$m_t = (1-R)\frac{\int_t^T E^Q(e^{-\int_t^u r_s+\Lambda(\vec{a}(s))ds}\cdot(\Lambda(\vec{a}(u)))|\mathcal{H}_t)du}{\int_t^T E^Q(e^{-\int_t^u r_s+\Lambda(\vec{a}_s)ds}|\mathcal{H}_t)du}$$

$$\begin{aligned}\Lambda(\vec{a}(s)) = {} & \Lambda_1(a_1)+\Lambda_2(a_2)+\Lambda_3(a_3)\\ & -F_{12}(\bar{\Lambda}(a_1),\bar{\Lambda}(a_2))-F_{13}(\bar{\Lambda}(a_1),\bar{\Lambda}(a_3))-F_{23}(\bar{\Lambda}(a_2),\bar{\Lambda}(a_3))\\ & +F_{123}(\bar{\Lambda}(a_1),\bar{\Lambda}(a_2),\bar{\Lambda}(a_3))\end{aligned}$$

$$\bar{\Lambda}(x_1,x_2,x_3) = \prod sgn(x_1,x_2,x_3)\lambda(\prod_{i=1}^{3}\mathcal{I}(x_i))$$

Closed Form

$$E(exp(-\int_0^T (r_t + \lambda_t)dt))$$

and

$$E(\lambda_t exp(-\int_0^t (r_u + \lambda_u)du))$$

Closed Form

Interest rate and Default rate

$$\lambda_t = k_t r_t + b_t,$$

where λ_t is the default rate and r_t is interest rate, k_t and b_t are deterministic functions.

One Factor Model

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t), r(0) = r_0$$
$$P(t, T) = A(t, T)e^{-B(t,T)r(t)},$$

where

$$A(t, T) = exp\{(\theta - \frac{\sigma^2}{2k^2})[B(t, T) - T + t] - \frac{\sigma^2}{4k}B(t, T)^2\}$$
$$B(t, T) = \frac{1}{k}[1 - e^{-k(T-t)}]$$

Closed Form: One dimensional case

$$m_t = (1 - R) \cdot \frac{f_1}{f_2} \tag{4}$$

$$\begin{aligned} f_1 = \int_t^T exp\{-r_t \int_t^u (a_s + 1)e^{k(t-s)} ds + \int_t^u (-k\theta_s \int_s^u (a_s + 1)e^{k(s-m)} dm \\ + \frac{1}{2}\sigma^2 (\int_s^u (a_s + 1)e^{k(s-m)} dm)^2 - b_s) ds\} du \end{aligned} \tag{5}$$

$$\begin{aligned} f_2 = \int_t^T exp\{-r_t \int_t^u (a_s + 1)e^{k(t-s)} ds + \int_t^u (-k\theta_s \int_s^u (a_s + 1)e^{k(s-m)} dm \\ + \frac{1}{2}\sigma^2 (\int_s^u (a_s + 1)e^{k(s-m)} dm)^2 - b_s) ds - ln(a_u r_u + b_u)\} du \end{aligned} \tag{6}$$

Closed Form: Model from Structural Perspective

Interest rate and Default level

$$a_t = k_t r_t + b_t,$$

where a_t is the default level and r_t is interest rate, k_t and b_t are deterministic functions.

Lévy measure

Assume the stock price follows an exponential compound poisson process. We are only interested in jump component

$$\sum_{i=1}^{N_t} U_i$$

where N_t is a Poisson process with intensity λ, and U_i is uniform random on $[-\lambda, 0)$, thus the Lévy measure, which is product of jump arrival rate λ and $\mu(F)$, where F is some jump size set. In this case, the Lévy measure is $\lambda \frac{1}{\lambda} 1_{[-\lambda,0)}$. So the tail integral would be

$$\Lambda(a_t) = \begin{cases} 0 & a_t < -\lambda \\ a_t + \lambda & -\lambda \leq a_t < 0 \end{cases} \tag{7}$$

Definition

(Clayton Levy Copula,two dimensional case). A Clayton Levy copula is a function F(u,v) with parameter θ, η defined as

$$(|u|^{-\theta} + |v|^{-\theta})^{-\frac{1}{\theta}}(\eta 1_{uv \geq 0} - (1-\eta)1_{uv<0}) \tag{8}$$

where $\theta > 0$ controls the dependence of absolute value of jumps for two levy processes, and $0 \leq \eta \leq 1$ controls measures the correlation between two levy processes. The closer η to 1, the more two levy processes are related.

To get explicit formula, I select $\theta = 1$. So the Clayton copula becomes

$$\frac{|uv|}{|u| + |v|}\eta \tag{9}$$

We assume that

$$|u| = p_k|v| \tag{10}$$

$$|u|\eta \tag{11}$$

$$m_t/(1-R) = \frac{\int_t^T E^Q(e^{-\int_t^u r_s+2(a_s+\lambda)-(a_s+\lambda)\eta ds}\cdot(2(a_u+\lambda)-(a_u+\lambda)\eta)|\mathcal{H}_t)}{\int_t^T E^Q(e^{-\int_t^u r_s+2(a_s+\lambda)-(a_s+\lambda)\eta ds}|\mathcal{H}_t)du}$$

$$\frac{E^Q(e^{-\int_t^u r_s+(2-\eta)a_s+2\lambda-\eta\lambda ds}\cdot((2-\eta)a_u+2\lambda-\eta\lambda)|\mathcal{H}_t)}{E^Q(e^{-\int_t^u r_s+(2-\eta)a_s+2\lambda-\eta\lambda ds}|\mathcal{H}_t)}$$

$$\frac{E^Q(e^{-\int_t^u r_s+(2-\eta)(k_t r_s+b_s)+2\lambda-\eta\lambda ds}\cdot((2-\eta)(k_u r_u+b_u)+2\lambda-\eta\lambda)|\mathcal{H}_t)}{E^Q(e^{-\int_t^u r_s+(2-\eta)(k_t r_s+b_s)+2\lambda-\eta\lambda ds}|\mathcal{H}_t)}$$

$$\frac{E^Q(e^{-\int_t^u (1+(2-\eta)k_t)r_s+(2-\eta)b_s+2\lambda-\eta\lambda ds}\cdot((2-\eta)k_u r_u+(2-\eta)b_u+2\lambda-\eta\lambda)}{E^Q(e^{-\int_t^u (1+(2-\eta)k_t)r_s+(2-\eta)b_s+2\lambda-\eta\lambda)ds}|\mathcal{H}_t)} \tag{12}$$

Closed Form: Interest rate Model

Two Factor Model

$$r(t) = x(t) + y(t) + \varphi(t), r(0) = r_0$$

$$dx(t) = -ax(t)dt + \sigma dW_1(t), x(0) = 0,$$
$$dy(t) = -by(t)dt + \eta dW_2(t), y(0) = 0,$$

where (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ as from

$$dW_1(t)dW_2(t) = \rho dt,$$

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Thank You!