"PDE methods for option pricing" study notes

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Chapter 1

20150107 Monday What is an option?

Problem 1: Stock today \$100, in one year \$110
$\Longrightarrow$ 10% profit and at risk \$100, which is 100% loss.

Problem 2: Leverage \$10 borrow \$90 = \$100
in a year \$110
\$20 Profit
$\Longrightarrow$ 100%(20-10/10),at risk \$100(10+investment)

Problem 3: Call option: right to buy at an agreed on price in one year.

$S_{today} = \$100$

$S_{1year} = \$110$

Want 20% profit

Buy option to purchase the stock at price \$92

In 1 year: we can buy the stock at price \$92(according to the right of call option), and can sell the stock to the market at price \$110. the profit is \$18.
cost is the value of the option.$\rightarrow$ How do we compute?
at risk $\mathcal{V}$=option price

A two state example

Assume S(0)=\$100

In one year

$$S(1) = \begin{cases} \$110 \\ \$90 \end{cases}$$

buying price \$92

Today	T	=1
s=100	s=\$110	s=\$90
$\mathcal{V}$=?	\$18	0

Here 0 is payoff
Set up portfolio :
Buy n share stock and sell one call option:

$$P(T) = n \cdot 110 - 18 \qquad\qquad P(T) = n \cdot 90 - 0$$

risk free:

$$n \cdot 110 - 18 = 90 \cdot n$$
$$solve: \quad \boxed{\text{n=\$0.9}}$$

Value of portfolio, $P(T) = \$81$
Today it is worth, need present value of \$81
If we put into bank

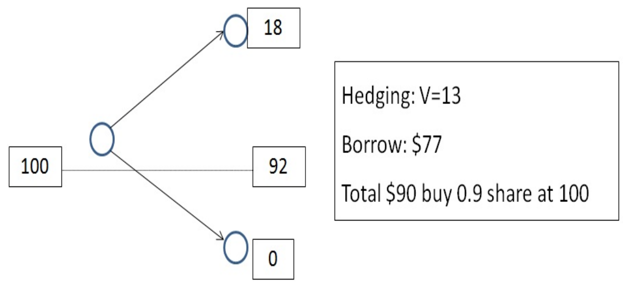
$$\frac{dp}{dt} = rp$$
$$P(T) = p(0)e^{rt}$$
$$P(0) = e^{-rt}P(T)$$

Here r is the risk free rate and the $e{-}rt$ is the discount factor., if $r = 0.05, year1, p(0) = e^{-0.05} \times \$81 = \$77$(portfolio value today).

Today:

$$\begin{cases} 100 \times 0.9 - \mathcal{V} = \$77 \\ \mathcal{V} = \$13 \end{cases}$$

20150109 Wednesday option pricing



hedging $\mathcal{V}$=$13

Borrow $ 47

Total $90 buy 0.9 share at $ 100

In a year:

	so owe $(0.9)\times$ 110=$99
(1) s=$110	owe 18 to "BOB" lending $81
	owe $77\times exp^{0.05}$ = $81
(2)s=$90	we have $(0.9)\times 90$=$81 pay bank

Stocks don't work like this, they have random component, If there wasn't we expect.

$\frac{dS}{S} = \mu dt$, here μ is the drift rate.

know at t, $S = S_0 e^{\mu t}$

Otherwise add random component

$\frac{dS}{S} = \mu dt + \sigma dZ$

$dZ = \phi\sqrt{dt}$: $\phi : normaldistribution$ with mean 0 and variance 1

What is S(t)?

Def expectation:
$E(Y) = E[f(\phi)]$: Here f(ϕ) is a random variable
$= \int_\Omega f(\phi)P(\phi)d\phi$: Here $P(\phi)$ is the probability density function

e.g:

$$P(\phi) = \frac{1}{\sqrt{s\pi}} e^{-1/2\phi^2}$$
$$E[1] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\phi^2} d\phi = 1$$
$$E[\phi] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi e^{-1/2\phi^2} d\phi = 0$$
$$E[\phi^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi^2 e^{-1/2\phi^2} d\phi = 1$$
$$E[\frac{dS}{S}] = E[\mu dt + \sigma dZ]$$
$$= \mu E[dt] + \sigma E[dZ]$$
$$= \mu dt E[1] (= 1, coz \quad no \quad variation) + \sigma \sqrt{dt} E[d\phi] \quad (= 0, coz \quad Gaussian \quad disstribution)$$
$$= \mu dt$$
$$Var[Y] = E[Y^2] - E^2[Y]$$
$$Var[\phi] = E[\phi^2] = E[\phi^2] - E^2[\phi] = 1$$
$$Var[\frac{dS}{S}] = E[(\frac{dS}{S})^2] - E^2[\frac{ds}{s}]$$
$$since: \frac{dS}{S} = \mu dt + \sigma\phi\sqrt{dt}$$
$$E[(\frac{dS}{S})^2] = E[(\mu dt + \sigma\sqrt{dt}\phi)^2]$$
$$= (\mu dt)^2 + 2\mu\sigma dt E[dZ] + \sigma^2 E[(dZ)^2] - (\mu dt)^2$$
$$= \sigma^2 dt$$

$\sigma \rightarrow$ volatility standard deviation

Lemma($it\hat{o}$) suppose $G = G(s, t)$ where $\frac{dS}{S} = \mu dt + \sigma dZ$

then: $dG = (\mu S \frac{dG}{dS} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t}) dt + \sigma S \frac{\partial G}{\partial S} d\mathcal{Z}$

To find $S(t)$, let $G(s) = log(s), G_s = \frac{1}{S}, G_t = 0, G_{SS} = -\frac{1}{S^2}$

$dG = G_S S \sigma d\mathcal{Z} + (\mu S G_S + \frac{\sigma^2 S^2}{2} G_{SS} + G_t) dt$
Substitution for partial derivatives,
$= \sigma d\mathcal{Z} + \mu dt - \frac{\sigma^2}{2} dt$

$dG = \sigma d\mathcal{Z} + (\mu - \frac{\sigma^2}{2}) dt$

$\int_0^t dG = \sigma \int_0^t d\mathcal{Z} + (c) \int_0^t dt$

$$G(t) - G(0) = \sigma(\mathcal{Z}(t) - \mathcal{Z}(0)) + (\mu - \tfrac{\sigma^2}{2})t$$

$$G = log(S)$$

$$log(S) - log(S_0) = \sigma(\mathcal{Z}(t) - \mathcal{Z}(0)) + (\mu - \tfrac{\sigma^2}{2})t$$

$$\Longrightarrow \boxed{S = s_0 e^{\sigma(\mathcal{Z}(t) - \mathcal{Z}(0)) + (\mu - \frac{\sigma^2}{2})t}}$$

Black-Scholes Equation
Assume (1)the stock price follows a geometric brownian motion
(2)Risk free rate is r = constant and also σ = constant
(3) No arbitrage ("All risk-free portfolio grow at risk free rate")
Set up portfolio with one option and a share of stocks (borrowed)

$$P = V - \alpha S$$
$$dP = dV - \alpha ds$$
$$dp = dV - \alpha ds = dV - \alpha(\mu dt + \sigma d\mathcal{Z})S$$
$$dV(s) = (\mu s V_s + \frac{\sigma^2 s^2}{2} V_{SS} + V_t)dt + \sigma S V_s d\mathcal{Z}$$

$\Rightarrow dP = \sigma S(V_s - \alpha)d\mathcal{Z} + (\mu S V_s + \frac{\sigma^2 S^2}{2} V_{SS} + V_t - \alpha\mu_s)dt$
If we choose $\alpha = V_s$, dP doesn't depend on $d\mathcal{Z}$
So: $dP = (\mu s V_s - \alpha(= V_s)\mu_s + \frac{\sigma^2 S^2}{2} V_{SS} + V_t)dt$
$dP = (\frac{\sigma^2 S^2}{2} V_{SS} + V_t)dt = rPdt$
with: $P = V - V_S S \Rightarrow V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_s - rV = 0(*)$

20150112 Monday option pricing
Hedging:
(1)Sell option of value V computed by *

(2) Buy $SV_s - V$ from bank

(3) Buy V_s shares of stocks(SV_s)

always keep V_s shares $\Leftrightarrow$ Dynamic hedge

$\Delta = V_s$

Greeks:

Name	Symbol	Def
Delta	Δ	V_s
Gamma	Γ	V_{ss}
Vega	ν	V_σ
Rho	ρ	V_r
Theta	Θ	V_t

Knowing V we can compute
eg. Δ and Γ by finite differences $V_j{}_{j=0}^N$
$\Delta_j \approx \delta_x^0 V_j = \frac{V_{j+1}-V_{j-1}}{2\Delta S}$
$\Gamma_j \approx \delta_x^+ \delta_x^- V_j = \frac{V_{j+1}-2V_j+V_{j-1}}{\Delta S^2}$
To get ρ or ν, use two values $r \pm \Delta r$
Gives $\frac{V_j(r+\Delta r)-V_j(r-\Delta r)}{2\Delta r} \approx \rho_j$
But the Greeks also satisfy advection-diffusion equations:
eg:find equation for Δ

$$\frac{\partial}{\partial S}(V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV) = 0 (remark : BS.equation)$$
$$(V_s)_t + \sigma^2 S V_{SS} + \frac{\sigma^2 S^2}{2} V_{SSS} + rV_s + rSV_{SS} - rV_S = 0$$
$$\Delta_t + \sigma^2 S V_{SS} + \frac{\sigma^2 S^2}{2} V_{SSS} + r\Delta + rS\Delta_S - r\Delta = 0$$
$$\Delta_t + \frac{\sigma^2 S^2}{2} \Delta_{SS} + (\sigma^2 + r) S \Delta_S = 0$$

Similar for Γ

For ρ:

$$\frac{\partial}{\partial r}(V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV) = 0$$
$$(V_r)_t + \frac{\sigma^2 S^2}{2}(V_r)_{SS} + SV_S + SV_S + rs(V_r)_S - V - rV_r = 0$$

$$\begin{cases} \rho_t + \frac{\sigma^2 S^2}{2}(\rho)_{SS} + SV_S + rs(\rho)_S - r\rho + SV_S - V = 0 \\ V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV = 0 \end{cases}$$

$\begin{bmatrix} \rho \\ V \end{bmatrix}_t = \vec{q}_t$

$\vec{q}_{SS}|_j \approx \delta_S^+ \delta_S^- \vec{q}_j$
$V_S|_j \approx \delta_S^0 V|_j$

Need BC's and final conditions

(B-S equation is backward diffusion equation)

forward in $\tau = T - t$
so we change $\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\frac{\tau}{\partial t} = -\frac{\partial}{\partial \tau}$

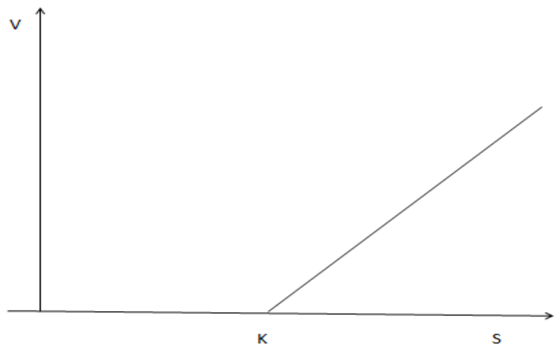
$V_\tau - \frac{\sigma^2 S^2}{2} V_{SS} - rSV_S + rV = 0$

The I.C. tell us the type of option +exercise rights

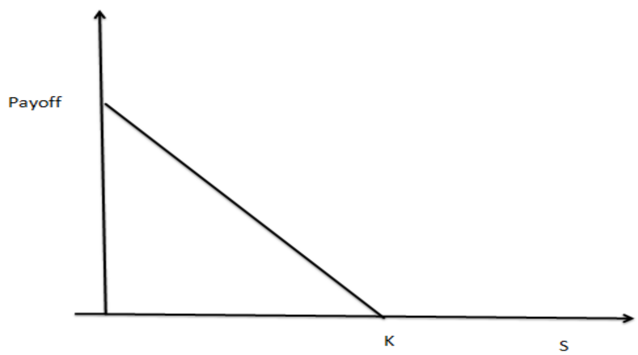
European options: Can only exercise at $t = T$

American options : Can exercise any time between $t = [0, T]$

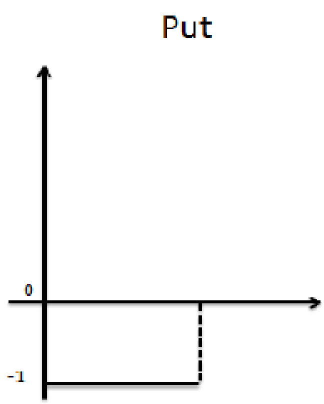
Two vanilla Europe options
(1)European call
Right to buy S at $t = T$ (expiry at T), but not obligation
$V(\tau = 0) = payoff = max(S - k, 0)$(Here S is underlying and K is the exercise price)

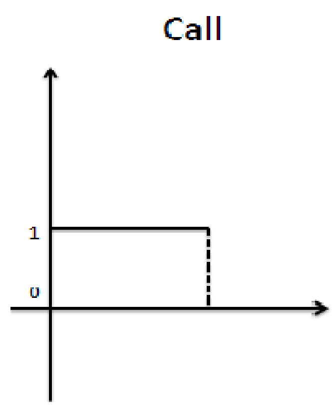


Put: right to sell S at price K at time t. payoff $=max(K-S,0)$,payoff $\notin C^2$



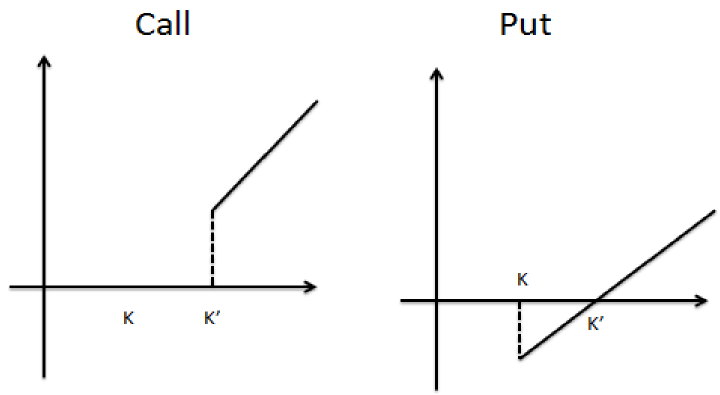
Note for Δ:



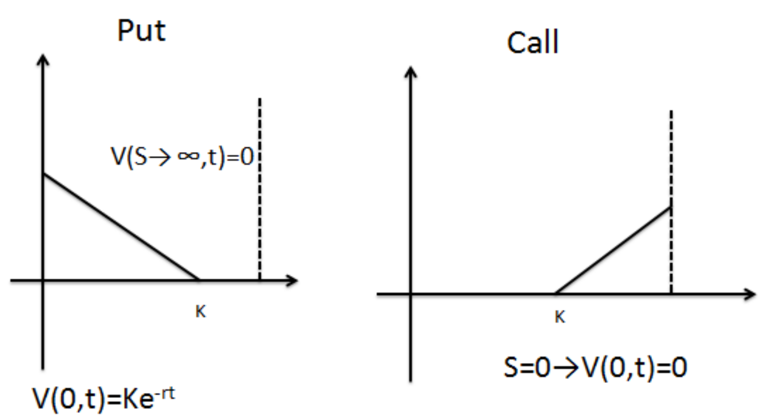


For Gap option:

$$V_{call}(t=T)=\begin{cases} S-K & K^{'}<S \\ 0 & \text{otherwise} \end{cases}$$



For vanilla options:



20150206 Friday option pricing

$w_t + \frac{\sigma^2}{2}W_{xx} + (r - \frac{\sigma^2}{2})$

Here W_t is forward time, given final condition.

$\delta_t^- W_j^{n+1} + \frac{\sigma^2}{2}\delta_x^+\delta_x^- W_j^{n+1} + (r - \frac{\sigma^2}{2})\delta_x^0 = 0$

$n = N_t, ..., 0$

write out and gather terms.

$W_j^n = (\frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} - (r - \frac{\sigma^2}{2})\frac{\Delta t}{2\Delta x})W_{j-1}^{n+1} + (1 - \frac{\sigma^2 \Delta t}{\Delta x^2})W_j^{n+1} + (\frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} + (r - \frac{\sigma^2}{2})\frac{\Delta t}{2\Delta x})W_{j+1}^{n+1}$

Binomial: $V_j^n = e^{-r\Delta t}(PV_{j+1}^{n+1} + (1-p)V_j^{n+1})$

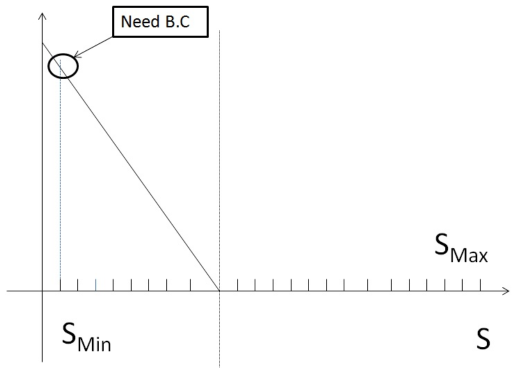
we choose Δx so that :

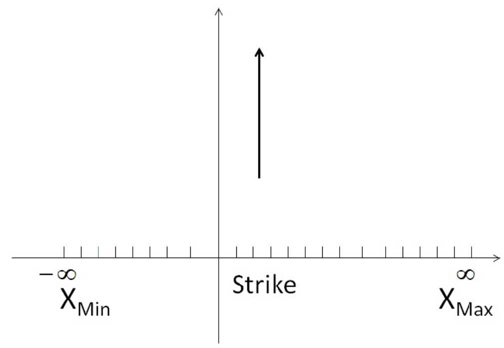
$$\frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} - (r - \frac{\sigma^2}{2})\frac{\Delta t}{2\Delta x} = 0$$

$$\boxed{\Delta x = \frac{\sigma^2}{r - \frac{\sigma^2}{2}}} \quad \frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

Then: $W_j^n = PW_{j+1}^{n+1} + (1-p)W_j^{n+1}$
$P = \frac{\Delta t(r - \frac{\sigma^2}{2})}{\sigma^2}$
$V_j^{n+1} = e^{rt_{n+1}}W_j^{n+1}$
$V_j^n = e^{rt_n}W_j^n$
so: $V_j^n = e^{-r\Delta t}\{PV_{j+1}^{n+1} + (1-p)V_j^{n+1}\}$
$\rightarrow$ For Binomial tree:
(1) 1^{st} order in time (not accurate, however, manager like it since easy)
(2)stability limit for Δt
$\frac{2^P}{2^p - 1}, \frac{2}{1}, ..., 0$
$R_{Exact} \approx R_{\Delta x} + C\Delta x^P$
$R_{Exact} \approx R_{\frac{\Delta x}{2}} + C(\frac{\Delta x^P}{2})$
$\Rightarrow 0 \approx (R_{\Delta x} - R_{\frac{\Delta x}{2}}) + C\Delta x^P(1 - \frac{1}{2^P})$(remark:$C\Delta x^P$ is Error, $C\Delta x^P = E\Delta_x$)
$\Rightarrow E_{\Delta x} \approx \frac{2^P}{2^p - 1}(R_{\frac{\Delta x}{2}} - R_{\Delta x})$
$w_t + \frac{\sigma^2}{2}W_{xx} + (r - \frac{\sigma^2}{2})$
$X = log(S)$
$S \in (0, \infty)$
$x \in (-\infty, \infty)$

Next Time European options with Jump Diffusion Diffusion:





$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} + (r - \kappa\lambda) S V_s - (r + \lambda) V + \int_0^\infty V(JS, t) g(J) dJ) = 0$$

20150213 Friday option pricing

$\widetilde{I}_j = \sum_{l=-N/2}^{N/2-1} V_{j-l} \tilde{\tilde{g}}_l$