

# "PDE methods for option pricing" study notes

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## Chapter 1

20150107 Monday What is an option?

**Problem 1:** Stock today \$100, in one year \$110  
 $\implies$  10% profit and at risk \$100, which is 100% loss.

**Problem 2:** Leverage \$10 borrow \$90 = \$100  

in a year	\$110
<hr/>	
\$20 Profit	

 $\implies$  100%(20-10/10), at risk \$100(10+investment)

**Problem 3:** Call option: right to buy at an agreed on price in one year.

$$S_{today} = \$100$$

$$S_{1year} = \$110$$

Want 20% profit

Buy option to purchase the stock at price \$92

In 1 year: we can buy the stock at price \$92 (according to the right of call option), and can sell the stock to the market at price \$110. the profit is \$18.  
cost is the value of the option.  $\rightarrow$  How do we compute?  
at risk  $\mathcal{V}$  = option price

A two state example

Assume  $S(0) = \$100$

In one year

$$S(1) = \begin{cases} \$110 \\ \$90 \end{cases}$$

buying price \$92

Today	T = 1	
s=100	s=\$110	s=\$90
$\mathcal{V}=?$	\$18	0

Here 0 is payoff

Set up portfolio :

Buy n share stock and sell one call option:

$$P(T) = n \cdot 110 - 18 \qquad P(T) = n \cdot 90 - 0$$

risk free:

$$n \cdot 110 - 18 = 90 \cdot n$$

$$\text{solve : } \boxed{n=\$0.9}$$

Value of portfolio,  $P(T) = \$81$

Today it is worth, need present value of \$81

If we put into bank

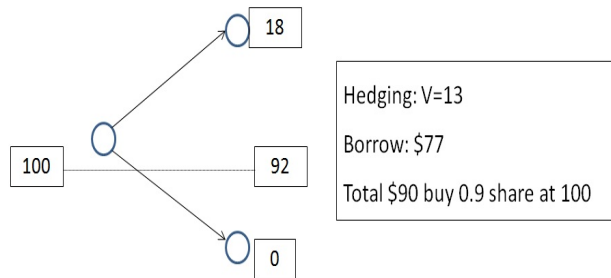
$$\begin{aligned} \frac{dp}{dt} &= rp \\ P(T) &= p(0)e^{rt} \\ P(0) &= e^{-rt}P(T) \end{aligned}$$

Here r is the risk free rate and the  $e^{-rt}$  is the discount factor., if  $r = 0.05, year 1, p(0) = e^{-0.05} \times \$81 = \$77$ (portfolio value today).

Today:

$$\begin{cases} 100 \times 0.9 - \mathcal{V} = \$77 \\ \mathcal{V} = \$13 \end{cases}$$

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hedging  $V=\$13$

Borrow \$ 47

Total \$90 buy 0.9 share at \$ 100

In a year:

so owe  $(0.9) \times 110 = \$99$   
 (1)  $s = \$110$  owe 18 to "BOB" lending \$81  
 owe  $77 \times \exp^{0.05} = \$81$

(2)  $s = \$90$  we have  $(0.9) \times 90 = \$81$  pay bank

Stocks don't work like this, they have random component, If there wasn't we expect.

$\frac{dS}{S} = \mu dt$ , here  $\mu$  is the drift rate.

know at  $t$ ,  $S = S_0 e^{\mu t}$

Otherwise add random component

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

$dZ = \phi \sqrt{dt}$ :  $\phi$  : normal distribution with mean 0 and variance 1

What is  $S(t)$ ?

Def expectation:

$E(Y) = E[f(\phi)]$ : Here  $f(\phi)$  is a random variable

$= \int_{\Omega} f(\phi) P(\phi) d\phi$  : Here  $P(\phi)$  is the probability density function

e.g:

$$P(\phi) = \frac{1}{\sqrt{s\pi}} e^{-1/2\phi^2}$$

$$E[1] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\phi^2} d\phi = 1$$

$$E[\phi] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi e^{-1/2\phi^2} d\phi = 0$$

$$E[\phi^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi^2 e^{-1/2\phi^2} d\phi = 1$$

$$E\left[\frac{dS}{S}\right] = E[\mu dt + \sigma dZ]$$

$$= \mu E[dt] + \sigma E[dZ]$$

$$= \mu dt E[1] (= 1, \text{ coz no variation}) + \sigma \sqrt{dt} E[d\phi] \quad (= 0, \text{ coz Gaussian disstribution})$$

$$= \mu dt$$

$$\text{Var}[Y] = E[Y^2] - E^2[Y]$$

$$\text{Var}[\phi] = E[\phi^2] = E[\phi^2] - E^2[\phi] = 1$$

$$\text{Var}\left[\frac{dS}{S}\right] = E\left[\left(\frac{dS}{S}\right)^2\right] - E^2\left[\frac{dS}{S}\right]$$

$$\text{since : } \frac{dS}{S} = \mu dt + \sigma \phi \sqrt{dt}$$

$$E\left[\left(\frac{dS}{S}\right)^2\right] = E[(\mu dt + \sigma \sqrt{dt} \phi)^2]$$

$$= (\mu dt)^2 + 2\mu\sigma dt E[dZ] + \sigma^2 E[(dZ)^2] - (\mu dt)^2$$

$$= \sigma^2 dt$$

$\sigma \rightarrow$  volatility standard deviation

Lemma(itô) suppose  $G = G(s, t)$  where  $\frac{dS}{S} = \mu dt + \sigma dZ$

$$\text{then: } dG = \left(\mu S \frac{dG}{dS} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t}\right) dt + \sigma S \frac{\partial G}{\partial S} dZ$$

To find  $S(t)$ , let  $G(s) = \log(s)$ ,  $G_s = \frac{1}{s}$ ,  $G_t = 0$ ,  $G_{SS} = -\frac{1}{s^2}$

$$dG = G_S S \sigma dZ + \left(\mu S G_S + \frac{\sigma^2 S^2}{2} G_{SS} + G_t\right) dt$$

$$\text{Substitution for partial derivatives,}$$

$$= \sigma dZ + \mu dt - \frac{\sigma^2}{2} dt$$

$$dG = \sigma dZ + \left(\mu - \frac{\sigma^2}{2}\right) dt$$

$$\int_0^t dG = \sigma \int_0^t dZ + (c) \int_0^t dt$$

$$G(t) - G(0) = \sigma(\mathcal{Z}(t) - \mathcal{Z}(0)) + (\mu - \frac{\sigma^2}{2})t$$

$$G = \log(S)$$

$$\log(S) - \log(S_0) = \sigma(\mathcal{Z}(t) - \mathcal{Z}(0)) + (\mu - \frac{\sigma^2}{2})t$$

$$\implies \boxed{S = s_0 e^{\sigma(\mathcal{Z}(t) - \mathcal{Z}(0)) + (\mu - \frac{\sigma^2}{2})t}}$$

Black-Scholes Equation

Assume (1) the stock price follows a geometric brownian motion

(2) Risk free rate is  $r = \text{constant}$  and also  $\sigma = \text{constant}$

(3) No arbitrage ("All risk-free portfolio grow at risk free rate")

Set up portfolio with one option and a share of stocks (borrowed)

$$P = V - \alpha S$$

$$dP = dV - \alpha ds$$

$$dp = dV - \alpha ds = dV - \alpha(\mu dt + \sigma d\mathcal{Z})S$$

$$dV(s) = (\mu s V_s + \frac{\sigma^2 s^2}{2} V_{SS} + V_t)dt + \sigma S V_s d\mathcal{Z}$$

$$\Rightarrow dP = \sigma S(V_s - \alpha) d\mathcal{Z} + (\mu S V_s + \frac{\sigma^2 S^2}{2} V_{SS} + V_t - \alpha \mu_s)dt$$

If we choose  $\alpha = V_s$ ,  $dP$  doesn't depend on  $d\mathcal{Z}$

So:  $dP = (\mu s V_s - \alpha(= V_s)\mu_s + \frac{\sigma^2 S^2}{2} V_{SS} + V_t)dt$

$$dP = (\frac{\sigma^2 S^2}{2} V_{SS} + V_t)dt = rPdt$$

$$\text{with: } P = V - V_s S \Rightarrow V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rS V_s - rV = 0(*)$$

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Hedging:

(1) Sell option of value  $V$  computed by \*

(2) Buy  $SV_s - V$  from bank

(3) Buy  $V_s$  shares of stocks( $SV_s$ )

always keep  $V_s$  shares  $\Leftrightarrow$  Dynamic hedge

$$\Delta = V_s$$

Greeks:

Name	Symbol	Def
Delta	$\Delta$	$V_s$
Gamma	$\Gamma$	$V_{ss}$
Vega	$\nu$	$V_\sigma$
Rho	$\rho$	$V_r$
Theta	$\Theta$	$V_t$

Knowing  $V$  we can compute

eg.  $\Delta$  and  $\Gamma$  by finite differences  $V_{j,j=0}^N$

$$\Delta_j \approx \delta_x^0 V_j = \frac{V_{j+1} - V_{j-1}}{2\Delta S}$$

$$\Gamma_j \approx \delta_x^+ \delta_x^- V_j = \frac{V_{j+1} - 2V_j + V_{j-1}}{\Delta S^2}$$

To get  $\rho$  or  $\nu$ , use two values  $r \pm \Delta r$

$$\text{Gives } \frac{V_j(r+\Delta r) - V_j(r-\Delta r)}{2\Delta r} \approx \rho_j$$

But the Greeks also satisfy advection-diffusion equations:

eg: find equation for  $\Delta$

$$\frac{\partial}{\partial S}(V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_s - rV) = 0 (\text{remark : BS.equation})$$

$$(V_s)_t + \sigma^2 SV_{SS} + \frac{\sigma^2 S^2}{2} V_{SSS} + rV_s + rSV_{SS} - rV_s = 0$$

$$\Delta_t + \sigma^2 SV_{SS} + \frac{\sigma^2 S^2}{2} V_{SSS} + r\Delta + rS\Delta_S - r\Delta = 0$$

$$\Delta_t + \frac{\sigma^2 S^2}{2} \Delta_{SS} + (\sigma^2 + r)S\Delta_S = 0$$

Similar for  $\Gamma$

For  $\rho$ :

$$\begin{aligned}\frac{\partial}{\partial r}(V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV) &= 0 \\ (V_r)_t + \frac{\sigma^2 S^2}{2} (V_r)_{SS} + SV_S + SV_S + rs(V_r)_S - V - rV_r &= 0\end{aligned}$$

$$\left\{ \begin{array}{l} \rho_t + \frac{\sigma^2 S^2}{2} (\rho)_{SS} + SV_S + rs(\rho)_S - r\rho + SV_S - V = 0 \\ V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV = 0 \end{array} \right.$$

$$\begin{bmatrix} \rho \\ V \end{bmatrix}_t = \vec{q}_t$$

$$\vec{q}_{SS}|_j \approx \delta_S^+ \delta_S^- \vec{q}_j$$

$$V_S|_j \approx \delta_S^0 V|_j$$

Need BC's and final conditions

(B-S equation is backward diffusion equation)

forward in  $\tau = T - t$

so we change  $\frac{\partial}{\partial t} = \frac{\partial}{\partial t} \frac{\tau}{\partial t} = -\frac{\partial}{\partial \tau}$

$$V_\tau - \frac{\sigma^2 S^2}{2} V_{SS} - rSV_S + rV = 0$$

The I.C. tell us the type of option +exercise rights

European options: Can only exercise at  $t = T$

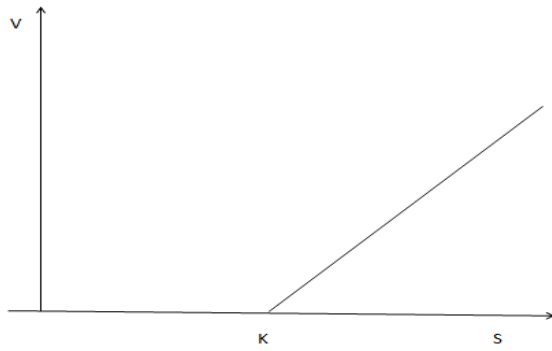
American options : Can exercise any time between  $t = [0, T]$

Two vanilla Europe options

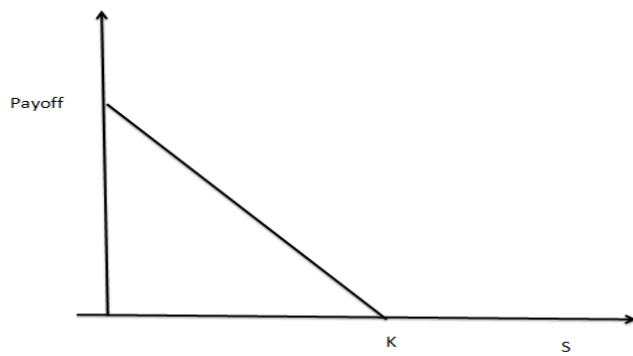
(1) European call

Right to buy S at  $t = T$  (expiry at T ), but not obligation

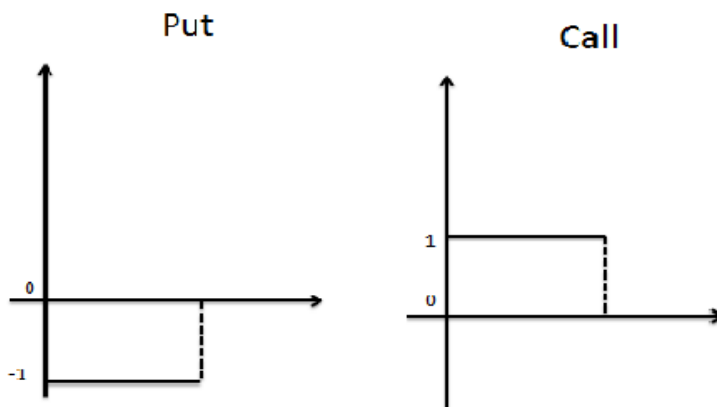
$V(\tau = 0) = payoff = \max(S - k, 0)$  (Here S is underlying and K is the exercise price)



Put: right to sell S at price K at time t.  $\text{payoff} = \max(K - S, 0)$ ,  $\text{payoff} \notin C^2$



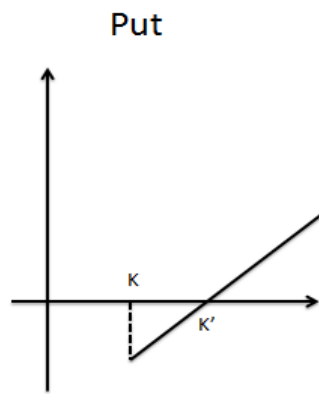
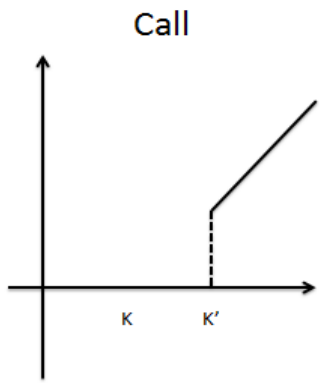
Note for  $\Delta$ :



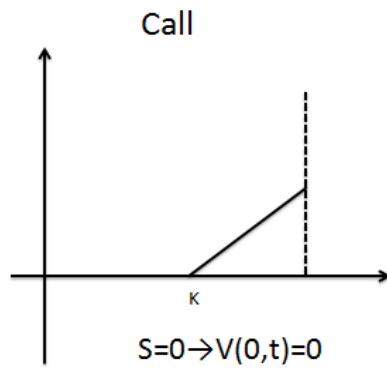
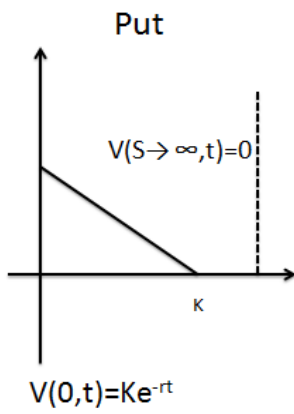
For Gap option:

$$V_{call}(t = T) = \begin{cases} S - K & K' < S \\ 0 & \text{otherwise} \end{cases}$$





For vanilla options:



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$$w_t + \frac{\sigma^2}{2} W_{xx} + (r - \frac{\sigma^2}{2})$$

Here  $W_t$  is forward time, given final condition.

$$\delta_t^- W_j^{n+1} + \frac{\sigma^2}{2} \delta_x^+ \delta_x^- W_j^{n+1} + (r - \frac{\sigma^2}{2}) \delta_x^0 = 0$$

$$n = N_t, \dots, 0$$

write out and gather terms.

$$W_j^n = (\frac{\sigma^2}{2} \frac{\Delta t}{\Delta x^2} - (r - \frac{\sigma^2}{2}) \frac{\Delta t}{2\Delta x}) W_{j-1}^{n+1} + (1 - \frac{\sigma^2 \Delta t}{\Delta x^2}) W_j^{n+1} + (\frac{\sigma^2}{2} \frac{\Delta t}{\Delta x^2} + (r - \frac{\sigma^2}{2}) \frac{\Delta t}{2\Delta x}) W_{j+1}^{n+1}$$

$$\text{Binomial: } V_j^n = e^{-r\Delta t} (P V_{j+1}^{n+1} + (1-p) V_j^{n+1})$$

we choose  $\Delta x$  so that :

$$\frac{\sigma^2}{2} \frac{\Delta t}{\Delta x^2} - (r - \frac{\sigma^2}{2}) \frac{\Delta t}{2\Delta x} = 0$$

$$\boxed{\Delta x = \frac{\sigma^2}{r - \frac{\sigma^2}{2}}} \quad \frac{\sigma^2}{2} \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

$$\text{Then: } W_j^n = P W_{j+1}^{n+1} + (1-p) W_j^{n+1}$$

$$P = \frac{\Delta t(r - \frac{\sigma^2}{2})}{\sigma^2}$$

$$V_j^{n+1} = e^{rt_{n+1}} W_j^{n+1}$$

$$V_j^n = e^{rt_n} W_j^n$$

$$\text{so: } V_j^n = e^{-r\Delta t} \{P V_{j+1}^{n+1} + (1-p) V_j^{n+1}\}$$

→ For Binomial tree:

(1) 1<sup>st</sup> order in time (not accurate, however, manager like it since easy)

(2) stability limit for  $\Delta t$

$$\frac{2^P}{2^P - 1}, \frac{2}{1}, \dots, 0$$

$$R_{Exact} \approx R_{\Delta x} + C \Delta x^P$$

$$R_{Exact} \approx R_{\frac{\Delta x}{2}} + C (\frac{\Delta x}{2})^P$$

$$\Rightarrow 0 \approx (R_{\Delta x} - R_{\frac{\Delta x}{2}}) + C \Delta x^P (1 - \frac{1}{2^P}) \text{ (remark: } C \Delta x^P \text{ is Error, } C \Delta x^P = E \Delta x)$$

$$\Rightarrow E_{\Delta x} \approx \frac{2^P}{2^P - 1} (R_{\frac{\Delta x}{2}} - R_{\Delta x})$$

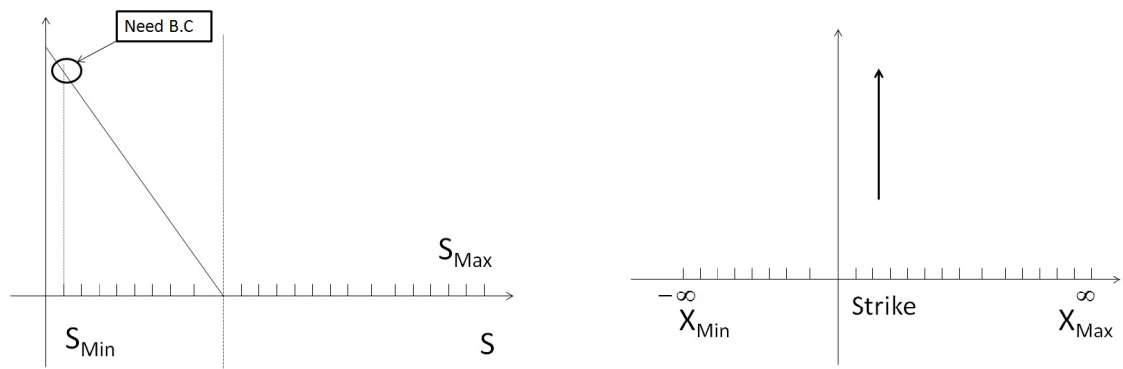
$$w_t + \frac{\sigma^2}{2} W_{xx} + (r - \frac{\sigma^2}{2})$$

$$X = \log(S)$$

$$S \in (0, \infty)$$

$$x \in (-\infty, \infty)$$

Next Time European options with Jump Diffusion Diffusion:



$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} + (r - \kappa \lambda) S V_s - (r + \lambda) V + \int_0^\infty V(JS, t) g(J) dJ = 0$$

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$$\tilde{I}_j = \sum_{l=-N/2}^{N/2-1} V_{j-l} \tilde{g}_l$$