NPDE for Option Pricing : Assignment #1

Due on 2015/02/27

Kopriva 12:20am

Jian Wang

Contents

Problem 1

[Summary]

1) For the vanilla put option, our methods showed the second order convergence rate on both the put option price and all the three Greeks.

2)For the binary put option, our methods showed around first order convergence rate on both the put option price and all the three Greeks.

3) From my computation, the results of three methods are very similar, the Crank-Nicolson method performance a little bit better than the other two.

[Statement of the problem]

1.Compute the values of the European vanilla put for $E^* = \$10$, $r^* = 0.05/yr$, $\sigma^* = 0.20/yr$ and with a six month expiry with and without Rannacher smoothing. Report the error as a function of δS and $\delta\tau$. Compute the greeks, δ and Γ and their errors. Propose and implement a technique to compute the $v = \partial V/\partial\sigma$ and report on its performance. Test the effects of the outer boundary on the solution in the range $[0, K]$.

2. Redo the previous task for the European binary put. In particular, examine the solution for large $\delta\tau$ and no smoothing

[Description of The Mathematics]

The BSM and boundary condition for European put option is:

$$\begin{cases} V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_s - rV = 0 \\ V(0,t) = Ke^{-r(T-t)} \\ V(\infty, t) = 0 \\ V(S,T) = max(K-S, 0) \end{cases}$$

combine : $\delta_t^+ V_j^n + L_h[\alpha V_j^{n+1} + (1-\alpha)V_j^n] = 0$

$\alpha = 1$: Back Euler
$\alpha = \frac{1}{2}$: Crank-Nicolson method

Backward Euler method:
The scheme for the Backward Euler method is given by:

$$\frac{V_{i,j}-V_{i,j-1}}{\delta t}+\frac{1}{2}\sigma^2(i\delta S)^2\frac{V_{i+1,j-1}-2V_{i,j-1}+V_{i-1,j-1}}{\delta S^2}+r(i\delta S)\frac{V_{i+1,j-1}-V_{i-1}j-1}{2\delta S)}-rV_{i,j-1}=0$$

we can rewrite it as:

$$V_{i,j}=A_iV_{i-1,j-1}+B_iV_{i,j-1}+C_iV_{i+1,j-1}$$

where:

$$A_i=\frac{1}{2}\delta t(r_i-\sigma^2i^2),B_i=1+(\sigma^2i^2+r)\delta t,C_i=-\frac{1}{2}\delta t(ri+\sigma^2i^2)$$

Crank-Nicolson method:

$$\frac{V_{ij}-V_{i,j-1}}{\delta t}+\frac{ri\delta S}{2}+(\frac{V_{i+1,j-1}-V_{i-1,j-1}}{2\delta S}+\frac{ri\delta S}{2}(\frac{V_{i+1,j}-V_{i-1,j}}{2\delta S})+$$
$$\frac{\sigma^2i^2(\delta S)^2}{4}(\frac{V_{i+1,j-1}-2V_{i,j-1}+V_{i-1,j-1}}{(\delta S)^2})+$$
$$\frac{\sigma^2i^2(\delta S)^2}{4}(\frac{V_{i+1,j}-2V_{i,j}+V_{i-1,j}}{(\delta S)^2})=\frac{r}{2}V_{i,j-1}+\frac{r}{2}V_{ij}$$

We can rewrite the above equation as:

$$-\alpha_iV_{i-1,j-1}+(1-\beta_i)V_{i,j-1}-\gamma_iV_{i+1,j-1}=\alpha_iV_{i-1,j}+(1+\beta_i)V_{i,j}+\gamma_iV_{i+1,j}$$

Where:

$$\alpha_i=\frac{\Delta t}{4}(\sigma^2i^2-ri)$$
$$\beta_i=-\frac{\Delta t}{2}(\sigma^2i^2+r)$$
$$\gamma_i=\frac{\Delta t}{4}(\sigma^2i^2+ri)$$

Ranacher Smooth method

(1)We use backward Euler for a few n $\geq$ 2 time steps
(2)Use Crank-Nicolson after that: given second order accuracy

Close form Black Scholes formula

To test the result for the SDE model of the option pricing, we also need to know the close form solution of the Black -Scholes assumptions, which is the famous Black- Scholes formula.
For the European put options:

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$$

where:

$$d_1 = \frac{log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{log\frac{S}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty} xe^{-\frac{1}{2}S^2}ds$$

Delta for Vanilla put option:

$$-e^{-q\tau}\Phi(-d_1)$$

Gamma for Vanilla put option:

$$-e^{-q\tau}\frac{\Phi(d_1)}{S\sigma\sqrt{\tau}}$$

Vega for Vanilla put option:

$$Se^{-q\tau}\phi(d_1)\sqrt{\tau} = Ke^{-r\tau}\phi(d_2)\sqrt{\tau}$$

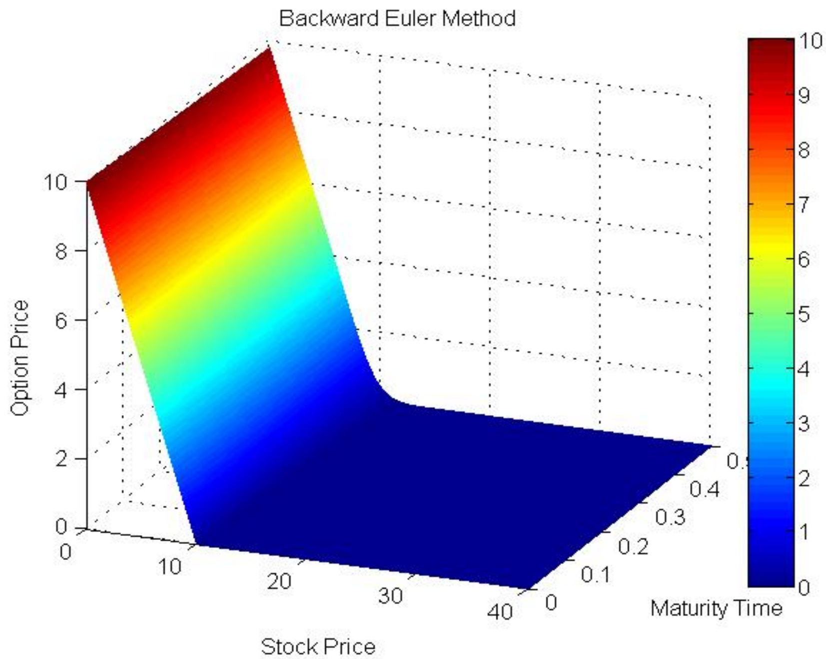
where q here is the dividend rate which is equal to 0 in our problem.

[Results]

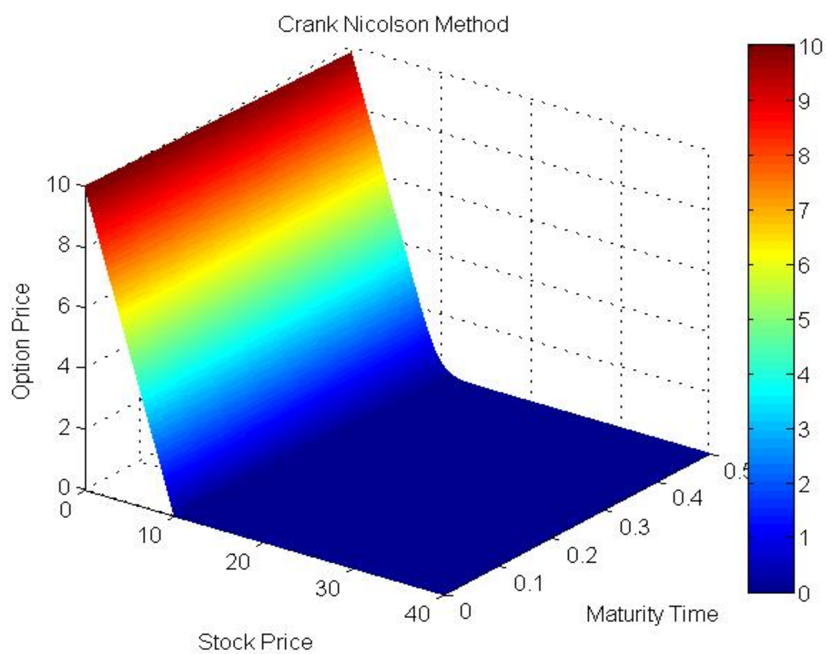
We used $E^* = \$10$, $r^* = 0.05/yr$, $\sigma^* = 0.20/yr$ T=0.5 as the example to build our model.
First, for the regular vanilla put option:

We run the models with different time and stock price steps, and the following is are results when we choose the number of time steps and the number of stock price steps both equal to 640.
Option price:

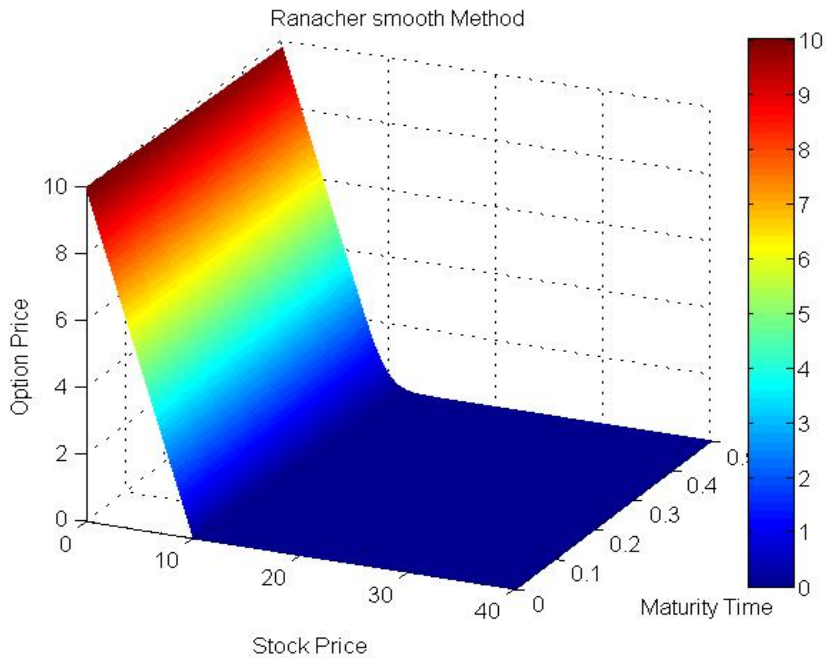
The surface of option price under Backward Euler Methods:



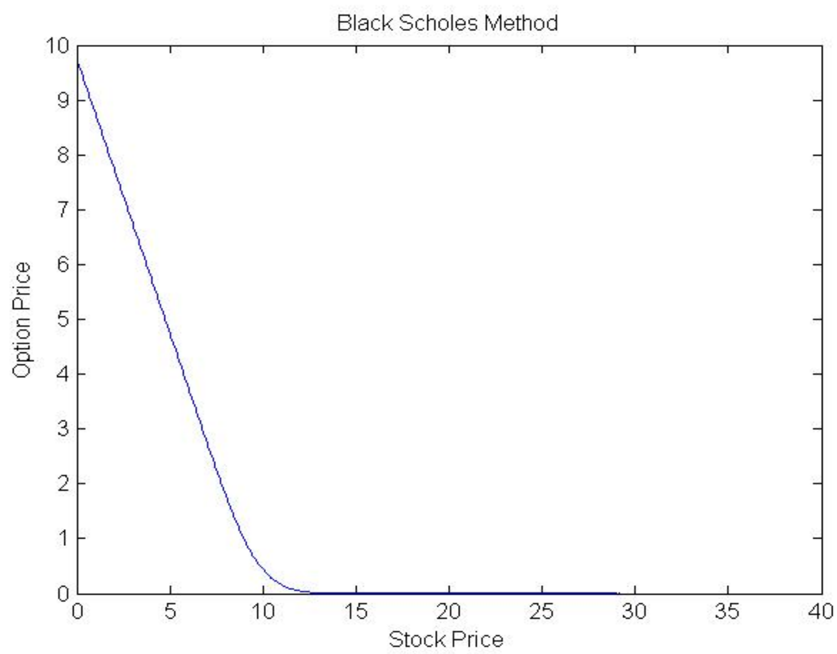
The surface of option price under Crank Nicolson Methods:



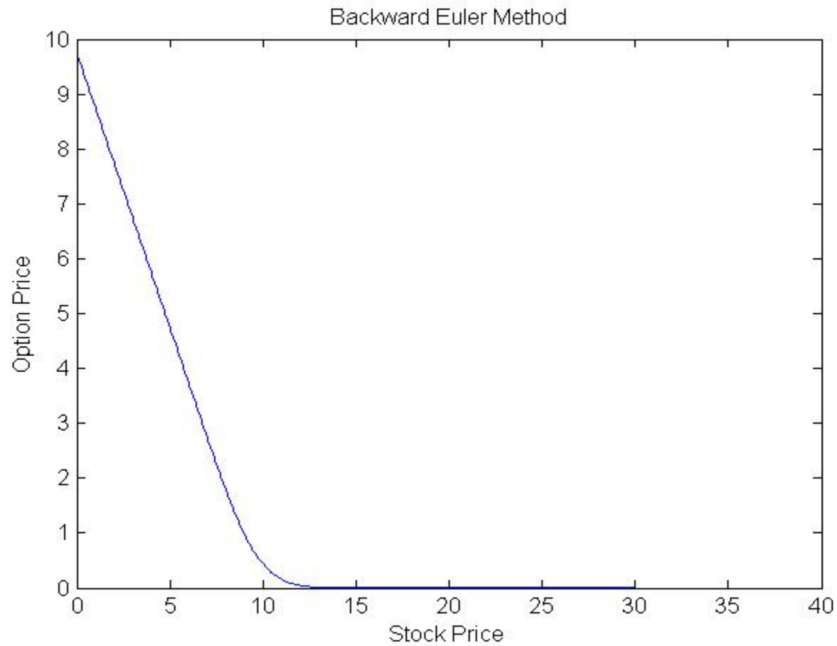
The surface of option price under Ranacher Smooth Methods:



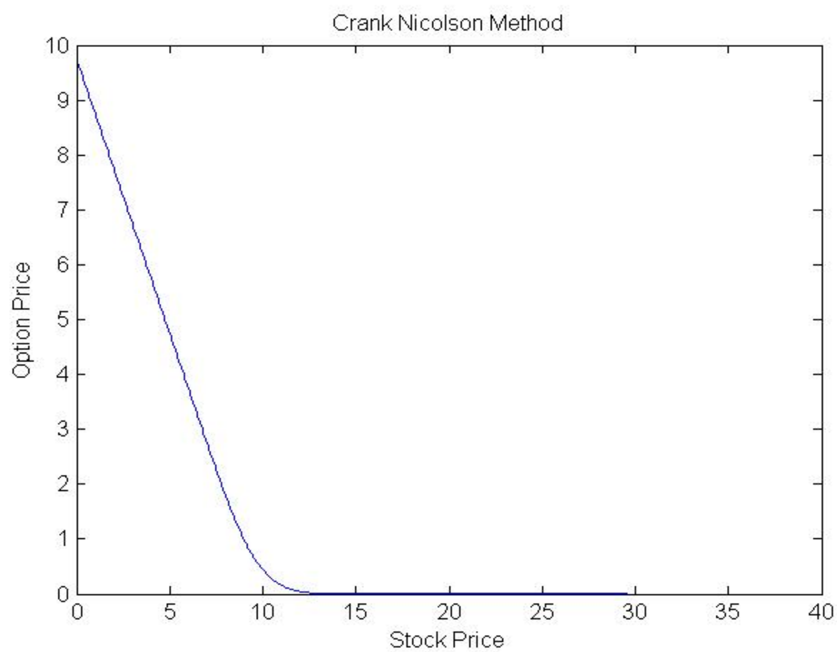
The option price under different stock prices of Black Scholes method:



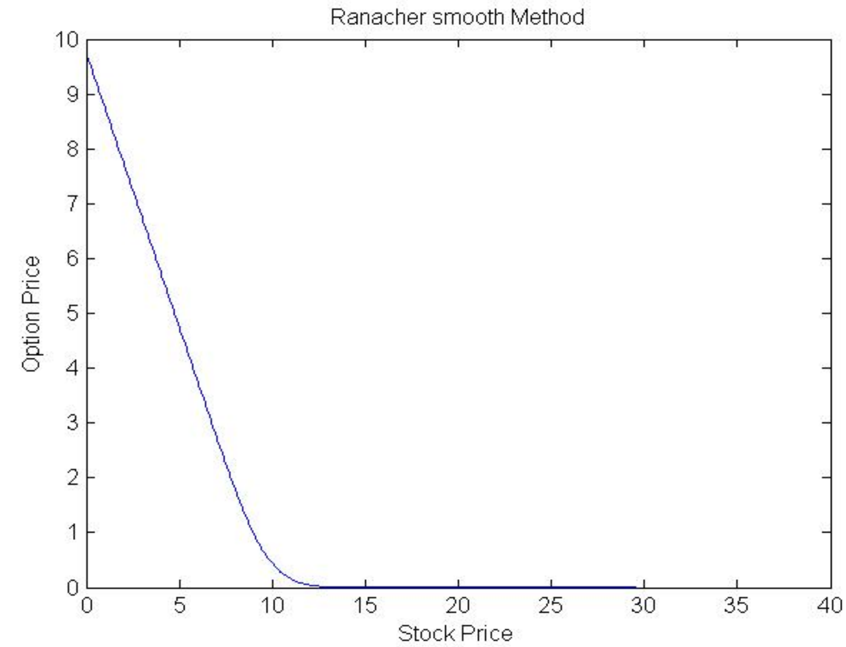
The option price under different stock prices of Backward Euler method:



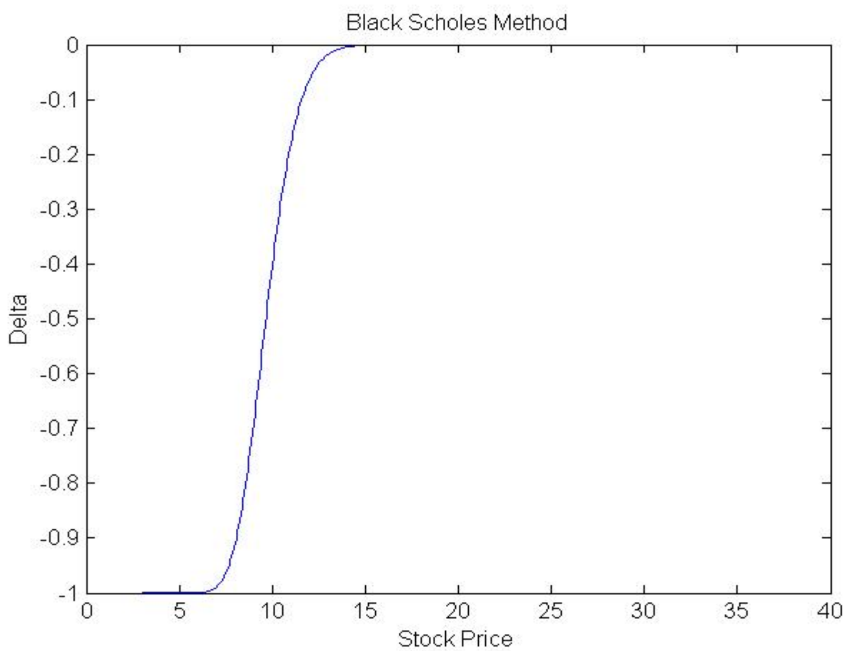
The option price under different stock prices of Crank Nicolson method:



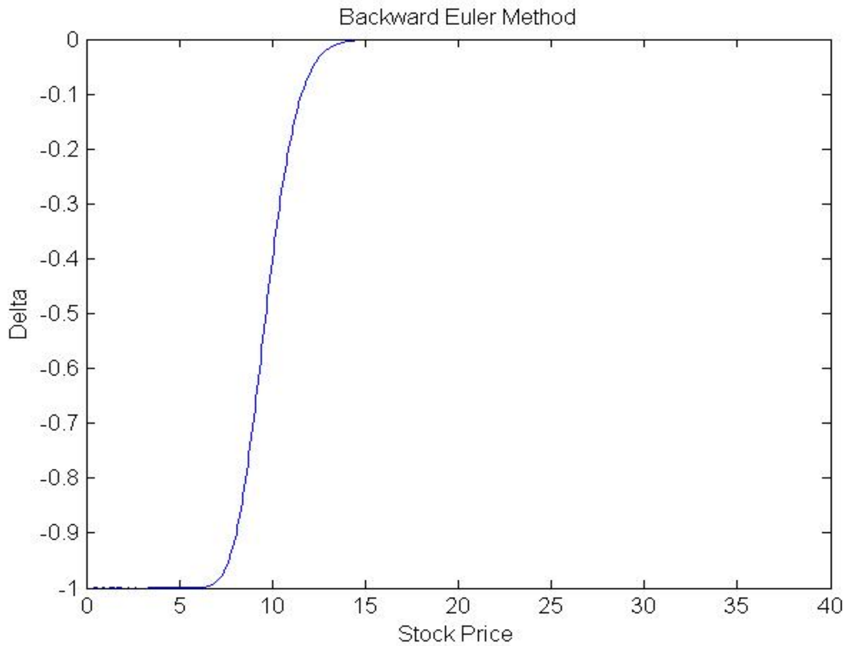
The option price under different stock prices of Ranacher Smooth method:



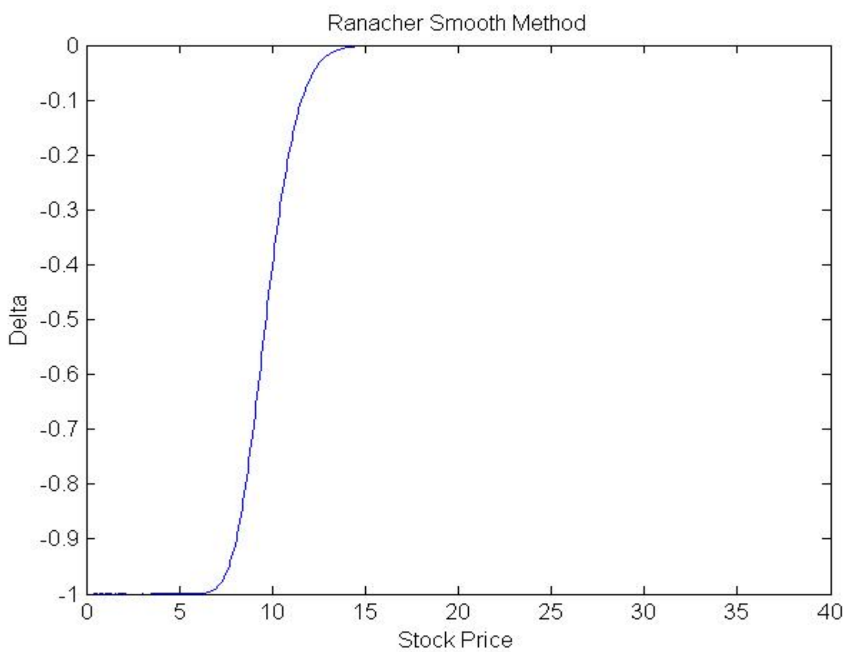
Delta:
The Delta of option price under Black Scholes Methods:



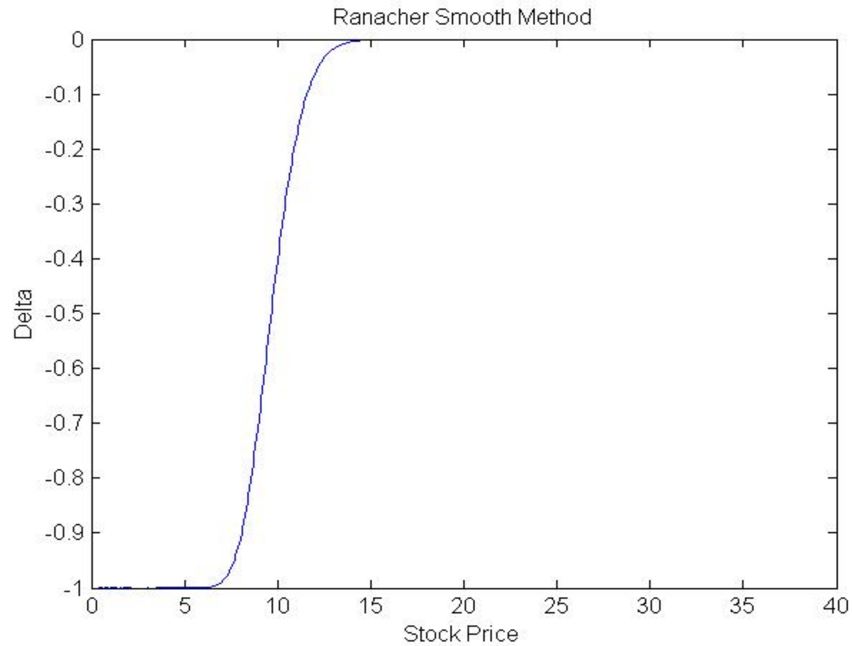
The Delta of option price under Backward Euler Methods:



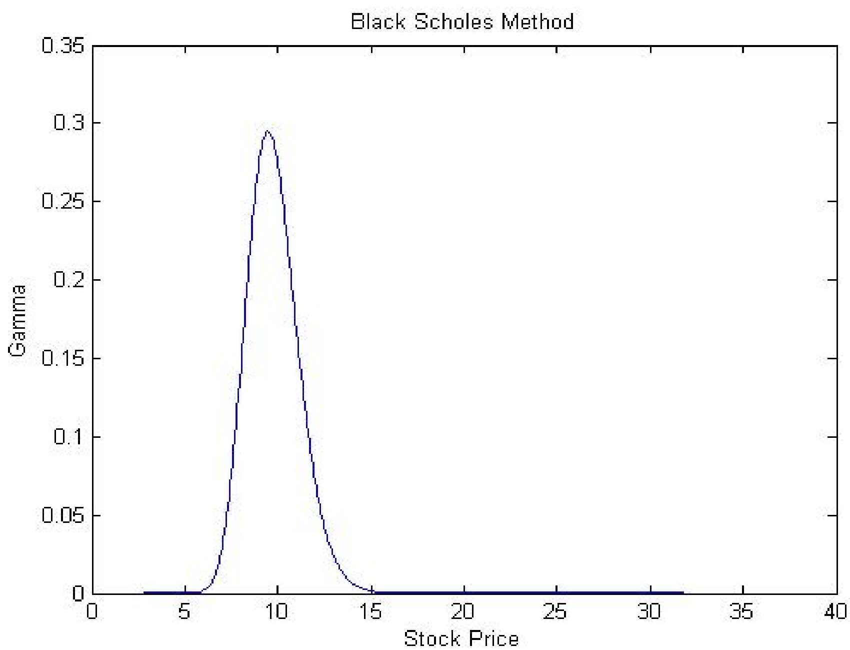
The Delta of option price under Crank Nicolson Methods:



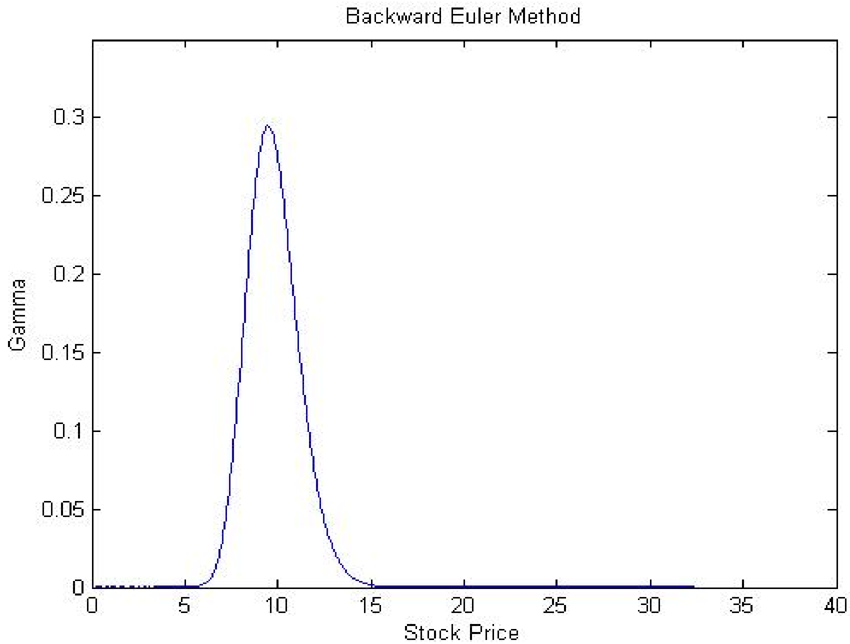
The Delta of option price under Ranacher Smooth Methods:



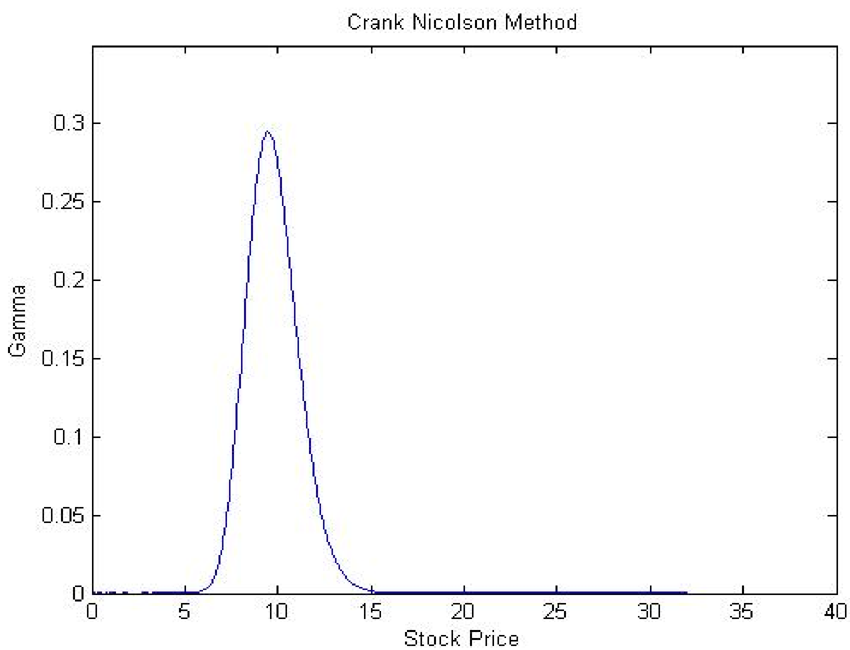
Gamma:
The Gamma of option price under Black Scholes Methods:



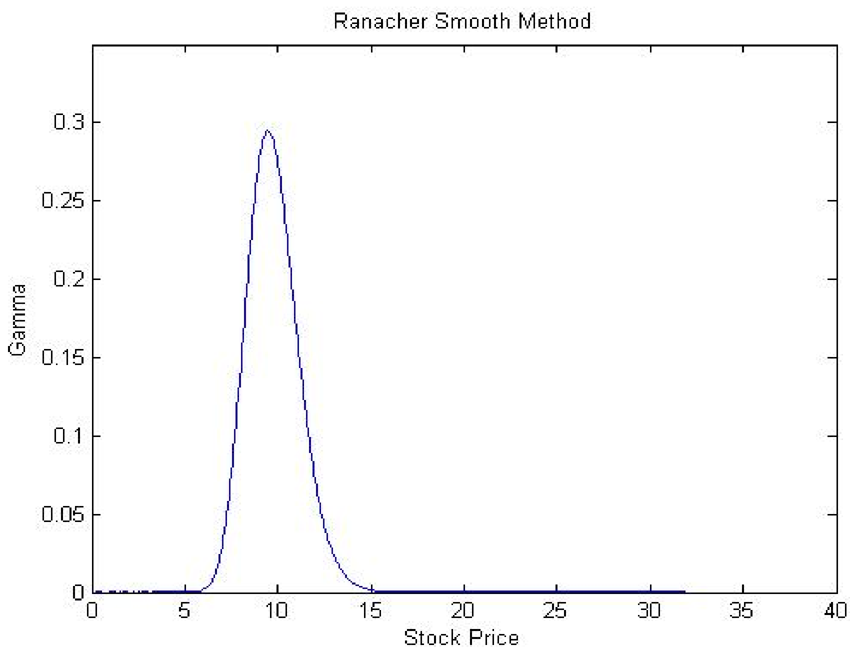
The Gamma of option price under Backward Euler Methods:



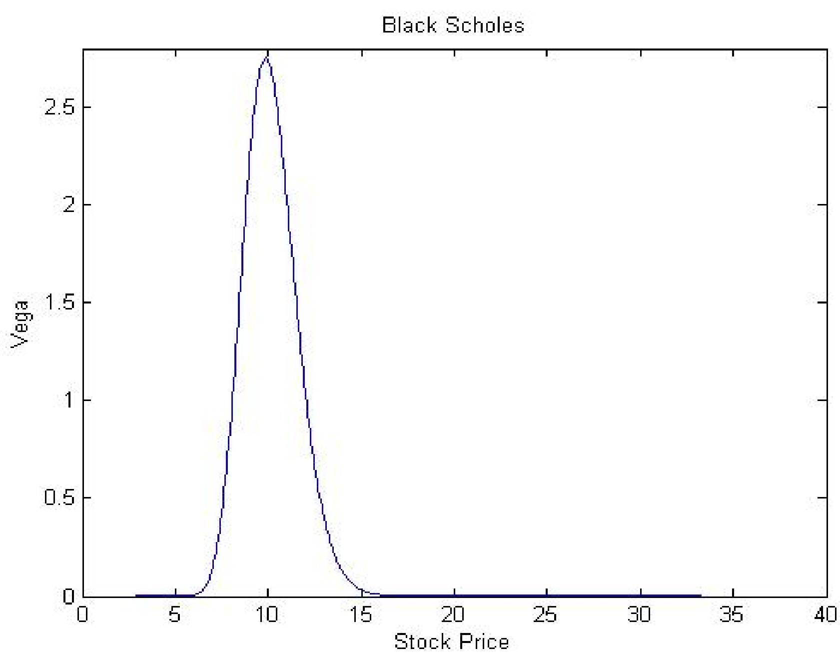
The Gamma of option price under Crank Nicolson Methods:



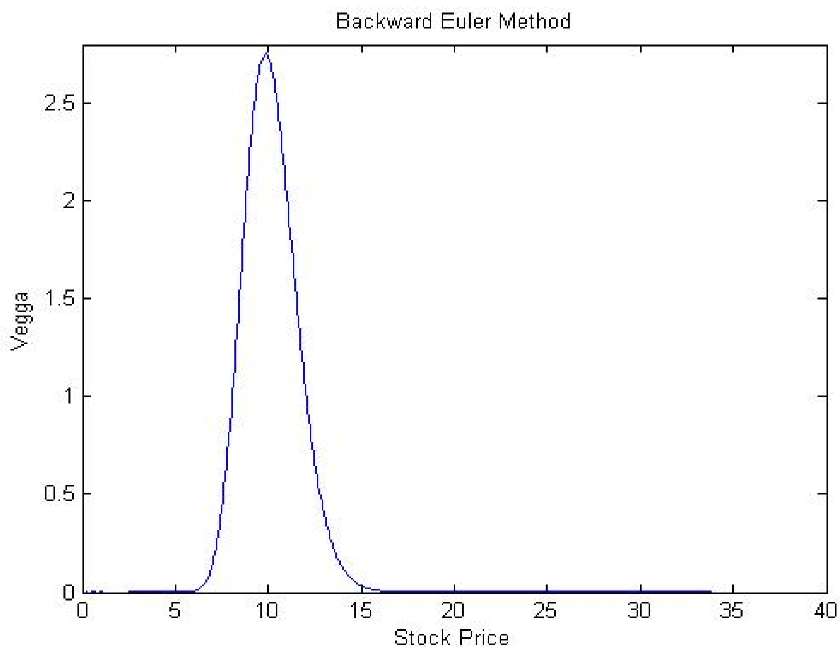
The Gamma of option price under Ranacher Smooth Methods:



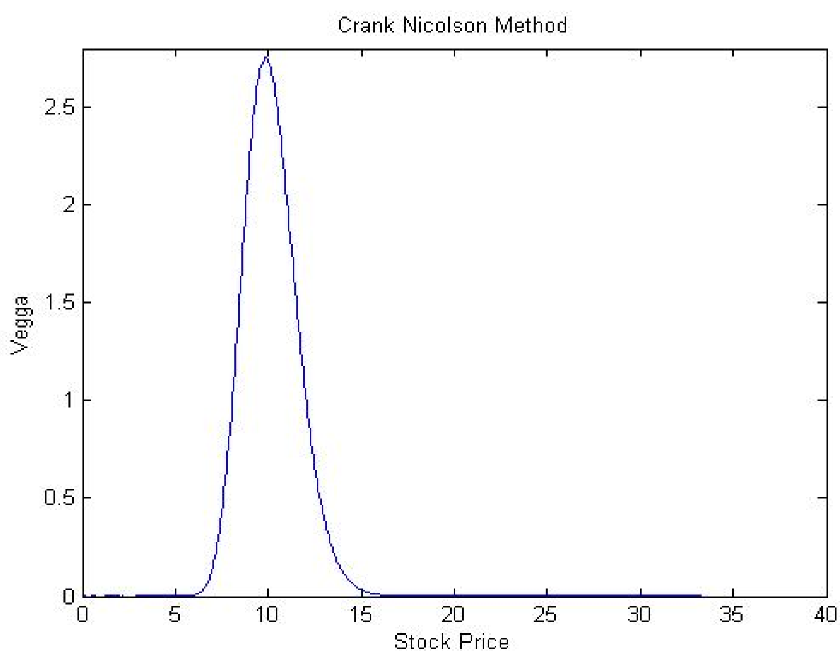
Vega: The Vega of option price under Black Scholes Methods:



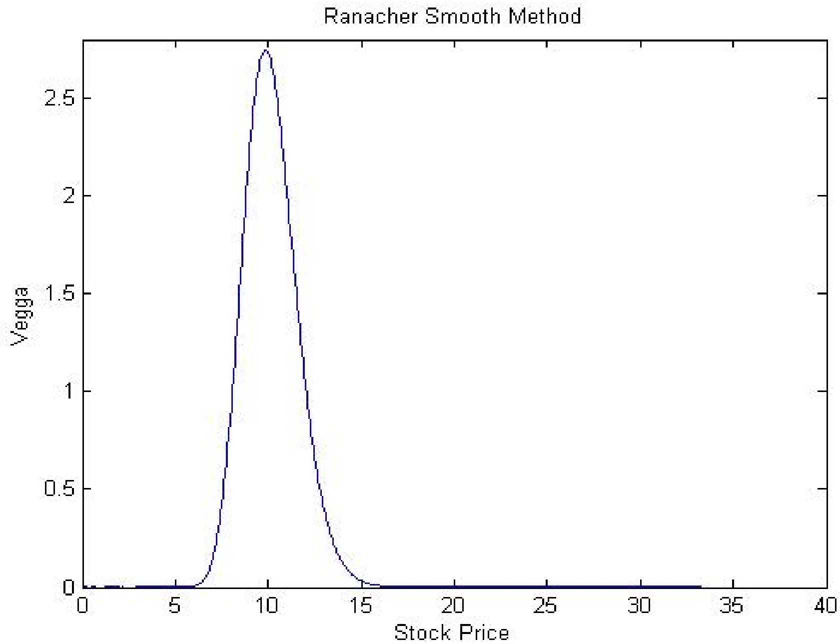
The Vega of option price under Backward Euler Methods:



The Vega of option price under Crank Nicolson Methods:



The Vega of option price under Ranacher Smooth Methods:



Error Test

First set the N_x equal to 640 and see the error change based on the change of N_t

N_t	Backward	Rate	Crank Nicolson	Rate	Ranacher Smooth	Rate
20	0.003594387		0.001567487		0.012230367	
40	0.001867687	1.92	0.000132867	11.80	0.006112979	2.00
80	0.00100197	1.86	0.000135258	0.98	0.003055939	2.00
160	0.000568873	1.76	0.000137234	0.99	0.001527832	2.00
320	0.000352555	1.61	0.000137727	1.00	0.000763881	2.00
640	0.000244465	1.44	0.000137851	1.00	0.000396422	1.93

Next set the N_t equal to 640 and see the error change based on the change of N_x

N_x	Backward	Rate	Crank Nicolson	Rate	Ranacher Smooth	Rate
20	0.163436618		0.163344756		0.163662174	
40	0.040302554	4.06	0.040159448	4.07	0.040454358	4.05
80	0.009109848	4.42	0.008996653	4.46	0.009257477	4.37
160	0.00231537	3.93	0.002206775	4.08	0.00246187	3.76
320	0.00065681	3.53	0.000552129	4.00	0.000802989	3.07
640	0.000244465	2.69	0.000137851	4.01	0.000396422	2.03

Finally change both N_t and N_s

N_x	N_t	Backward	Rate	Crank Nicolson	Rate	Ranacher Smooth	Rate
20	20	0.166238921		0.163327854		0.173703389	
40	40	0.042442619	3.92	0.040143068	4.07	0.044943594	3.86
80	80	0.009901673	4.29	0.008993691	4.46	0.011095262	4.05
160	160	0.002640911	3.75	0.002206118	4.08	0.003229567	3.44
320	320	0.000764333	3.46	0.000552005	4.00	0.001062514	3.04
640	640	0.000244465	3.13	0.000137851	4.00	0.000396422	2.68

For the Greeks, We only test the convergence rate that both N_x and N_t are changed.

Delta:

N_x	N_t	Backward	Rate	Crank Nicolson	Rate	Ranacher Smooth	Rate
20	20	0.040638928		0.040343351		0.042947343	
40	40	0.012125597	3.35	0.011656525	3.46	0.012815246	3.35
80	80	0.004392592	2.76	0.004071524	2.86	0.004890106	2.62
160	160	0.001321169	3.32	0.001036892	3.93	0.003048254	1.60
320	320	0.000460403	2.87	0.000262897	3.94	0.003048076	1.00
640	640	0.000182361	2.52	6.58197E-05	3.99	0.003047988	1.00

Gamma:

N_x	N_t	Backward	Rate	Crank Nicolson	Rate	Ranacher Smooth	Rate
20	20	0.07304392		0.073453933		0.071442482	
40	40	0.035599504	2.05	0.035458092	2.07	0.0345424	2.07
80	80	0.009692153	3.67	0.009562398	3.71	0.00911713	3.79
160	160	0.002839872	3.41	0.002495421	3.83	0.006096616	1.50
320	320	0.000856993	3.31	0.000625908	3.99	0.012192524	0.50
640	640	0.00030478	2.81	0.000156592	4.00	0.02438434	0.50

Vega:

N_x	N_t	Backward	Rate	Crank Nicolson	Rate	Ranacher Smooth	Rate
20	20	0.39866406		0.440434356		0.341570382	
40	40	0.264507211	1.51	0.272835462	1.61	0.245209753	1.39
80	80	0.04499004	5.88	0.049463898	5.52	0.057953377	4.23
160	160	0.01112004	4.05	0.011662777	4.24	0.020572162	2.82
320	320	0.003324887	3.34	0.002888742	4.04	0.007843445	2.62
640	640	0.001170097	2.84	0.000721944	4.00	0.00334992	2.34

Problem 2

Re do the problem 1 for the Binary option. **Binary put option:**

The binary put option is that the payoff will be 1 if the maturity time stock price is lower than the strike price,otherwise it will be 0;
All the other assumptions are same with the vanilla put option.So we can rewrite the formula as follows:

Note that the boundary conditions changed.

$$\begin{cases} V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_s - rV = 0 \\ V(0,t) = e^{-r(T-t)} \\ V(\infty,t) = 0 \\ V(S,T) = I_{\{S \leq K\}} \end{cases}$$

Close form solution of the binary put: The Black-Scholes formula for binary put option is:

$$-e^{-r(T-t)}N(-d_2)$$

where the $N(x)$ here is the cumulative function of standard normal distribution:

Delta:

$$-\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{(T-t)}}$$

Gamma:

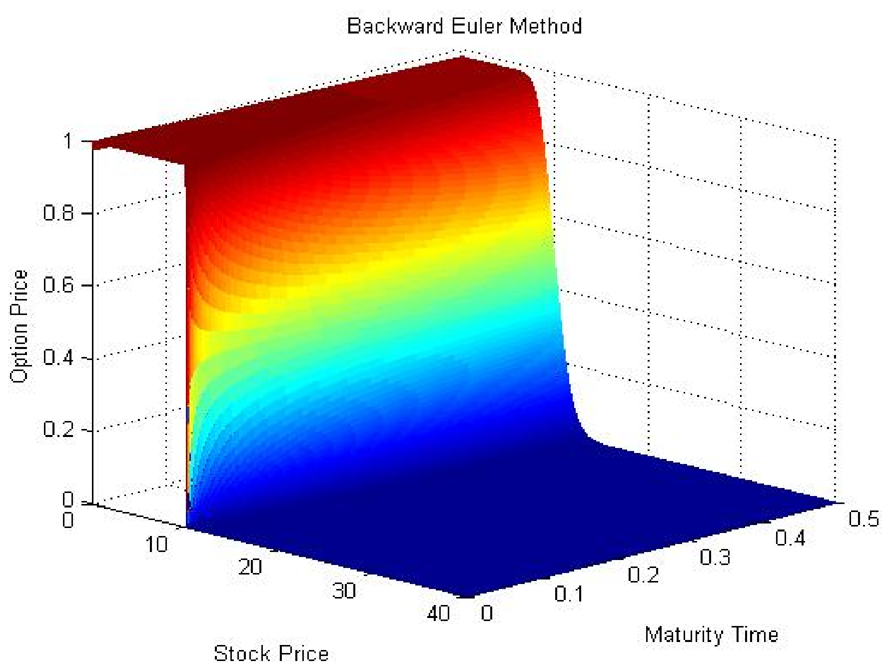
$$-\frac{e^{-r(T-t)}d_1 N'(d_2)}{\sigma^2 S^2 (T-t)}$$

Vega:

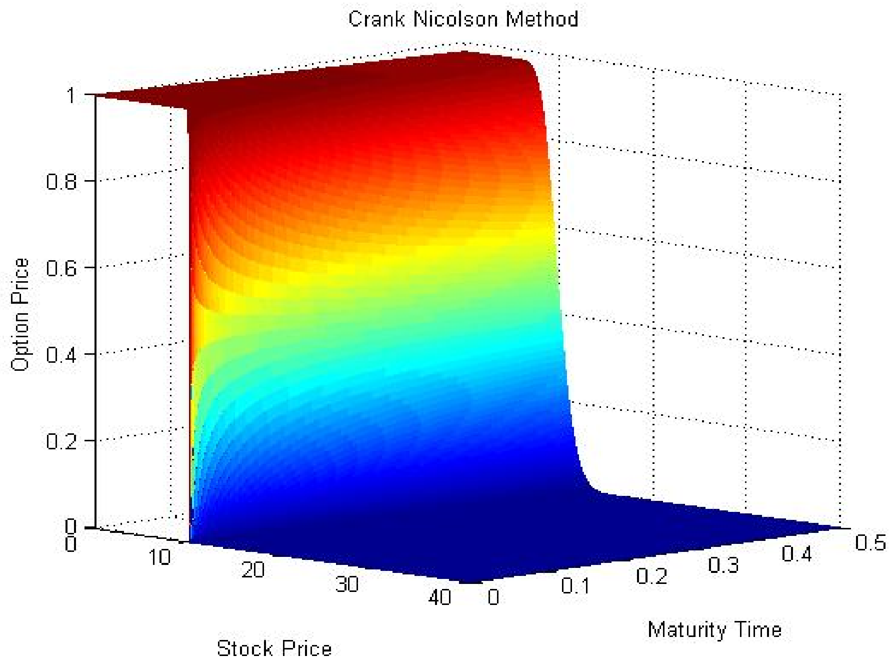
$$-e^{-r(T-t)}N'(d_2)\frac{d_1}{\sigma}$$

Option price:
We still used $E^* = \$10$, $r^* = 0.05/yr$, $\sigma^* = 0.20/yr$ T=0.5 as the example to build our model. The number of time steps and the number of stock price steps are both equal to 640

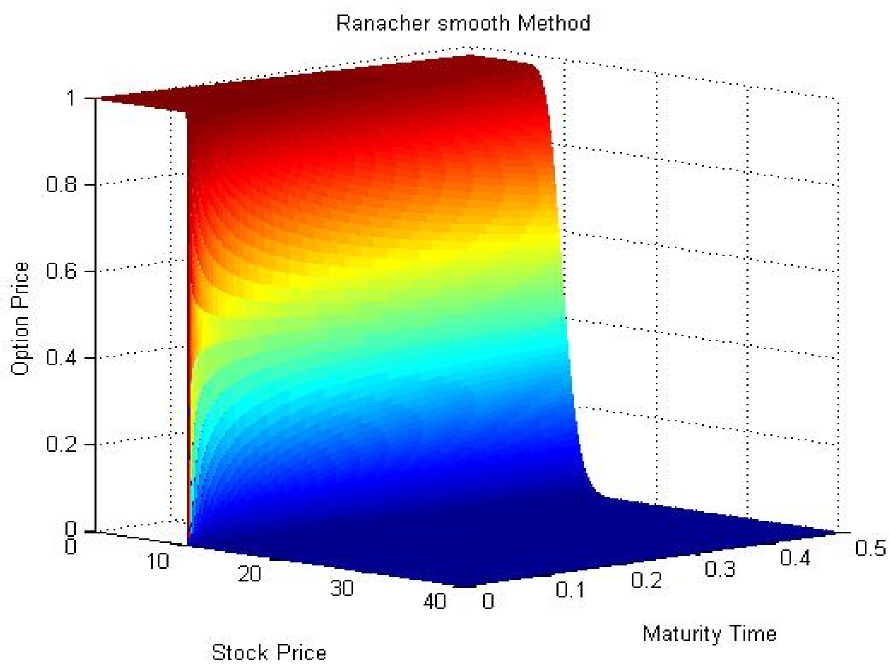
The surface of option price under Backward Euler Methods:



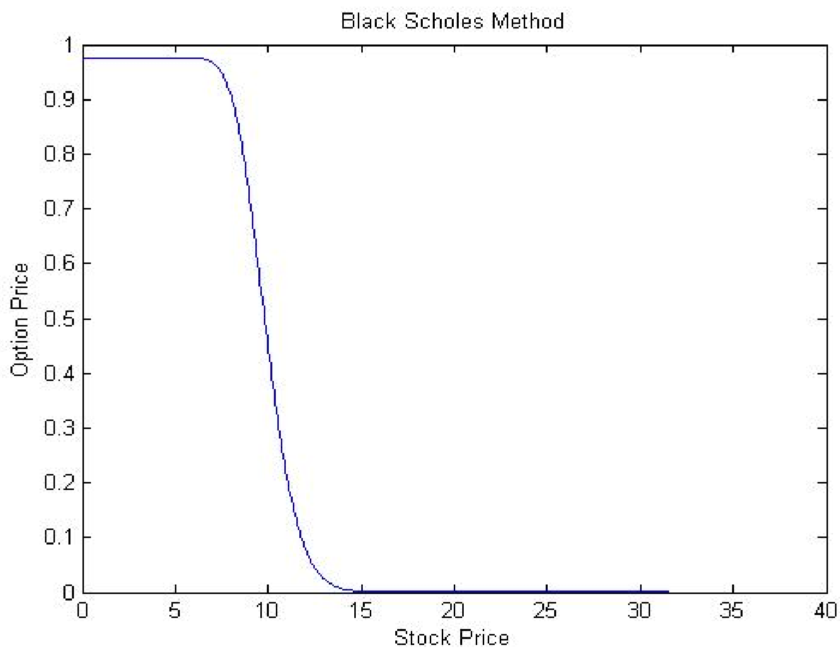
The surface of option price under Crank Nicolson Methods:



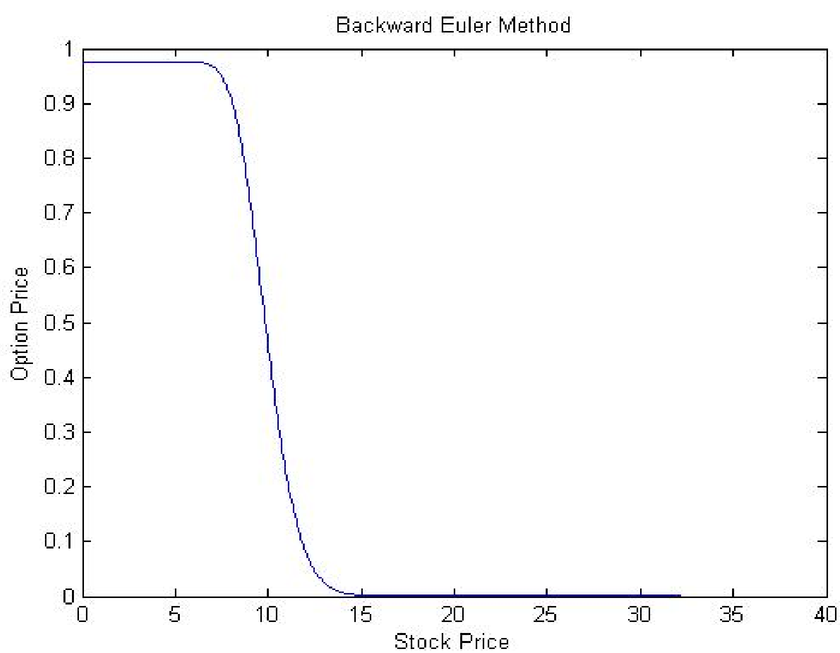
The surface of option price under Ranacher Smooth Methods:



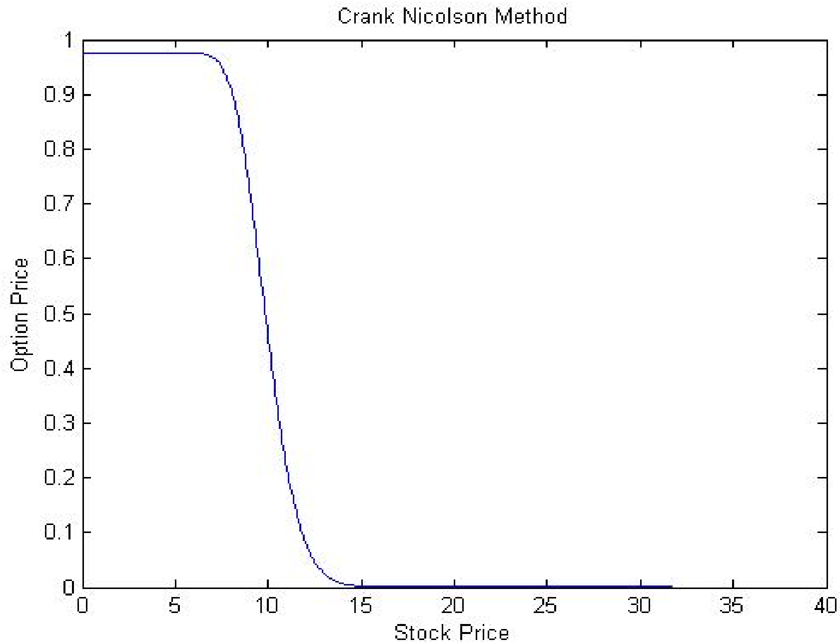
The option price under different stock prices of Black Scholes method:



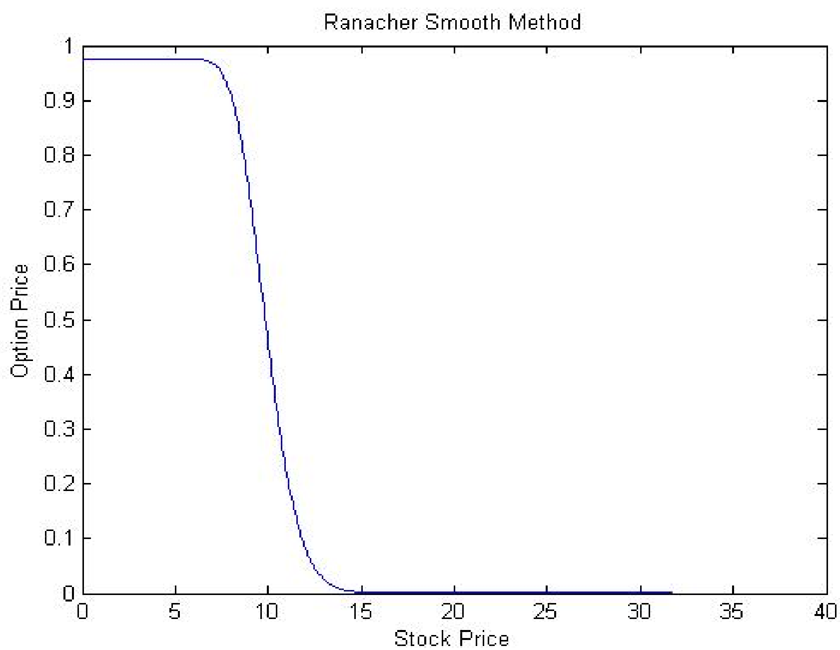
The option price under different stock prices of Backward Euler method:



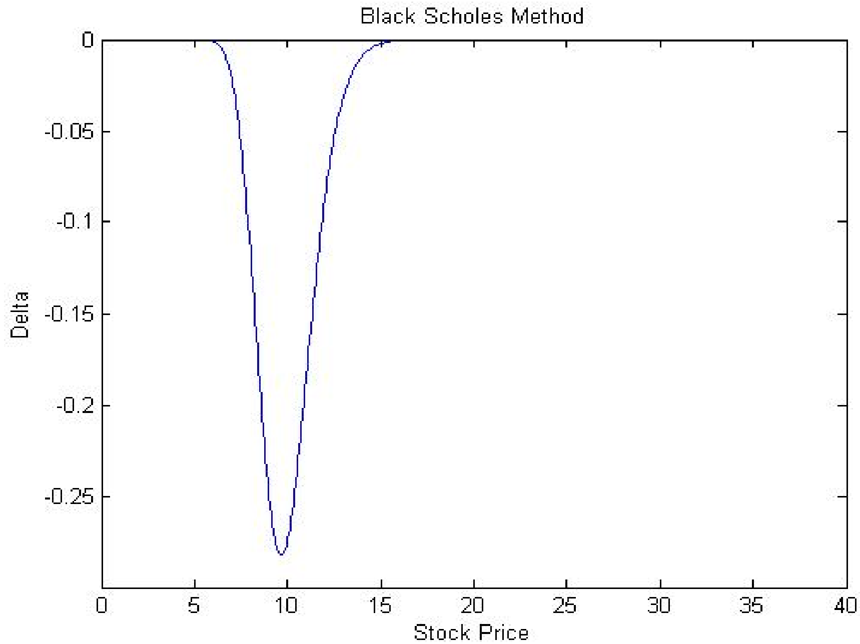
The option price under different stock prices of Crank Nicolson method:



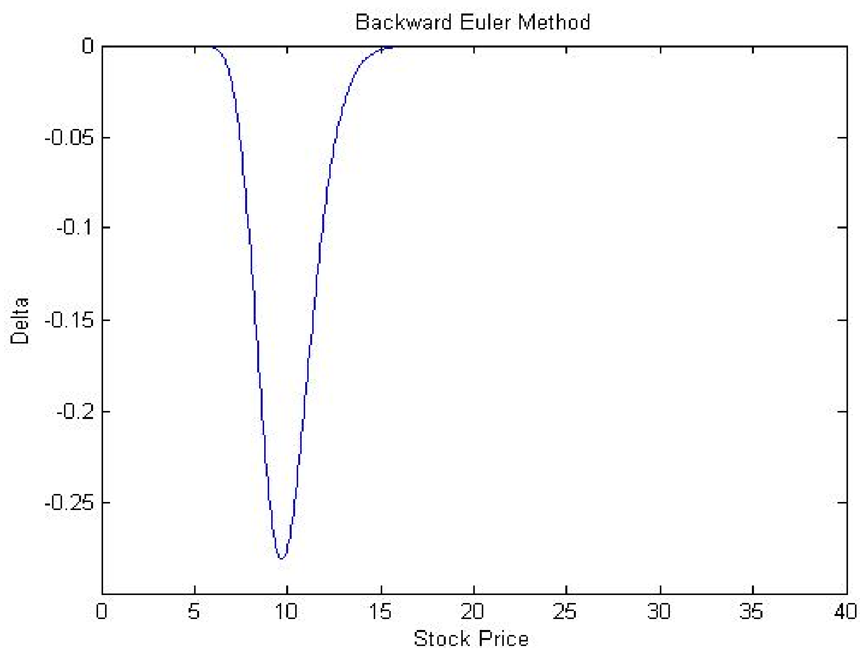
The option price under different stock prices of Ranacher Smooth method:



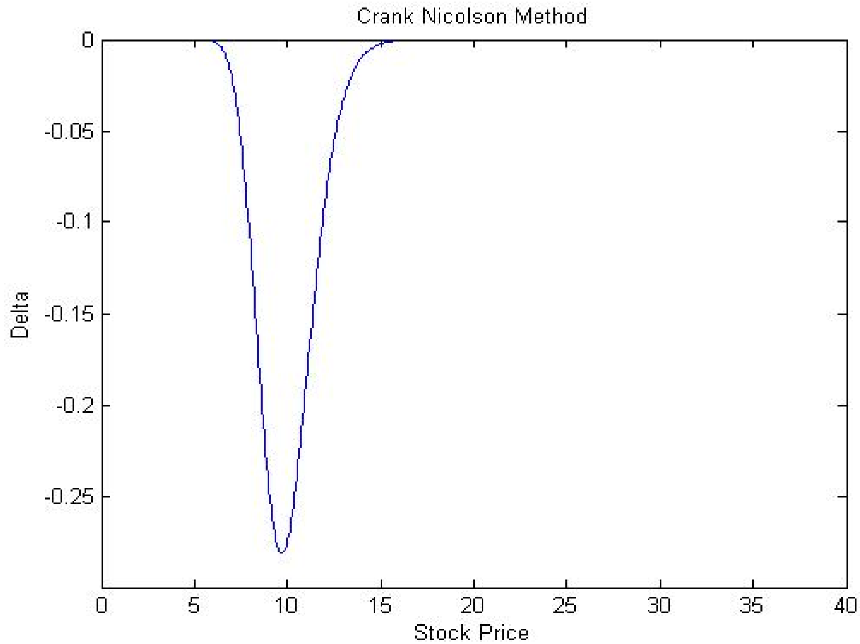
Delta:
The Delta of option price under Black Scholes Methods:



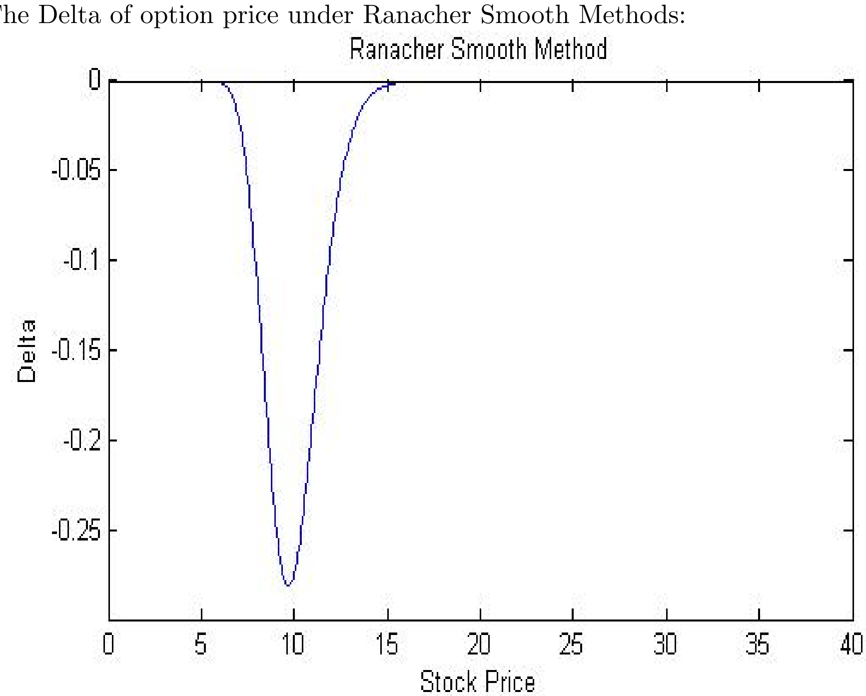
The Delta of option price under Backward Euler Methods:



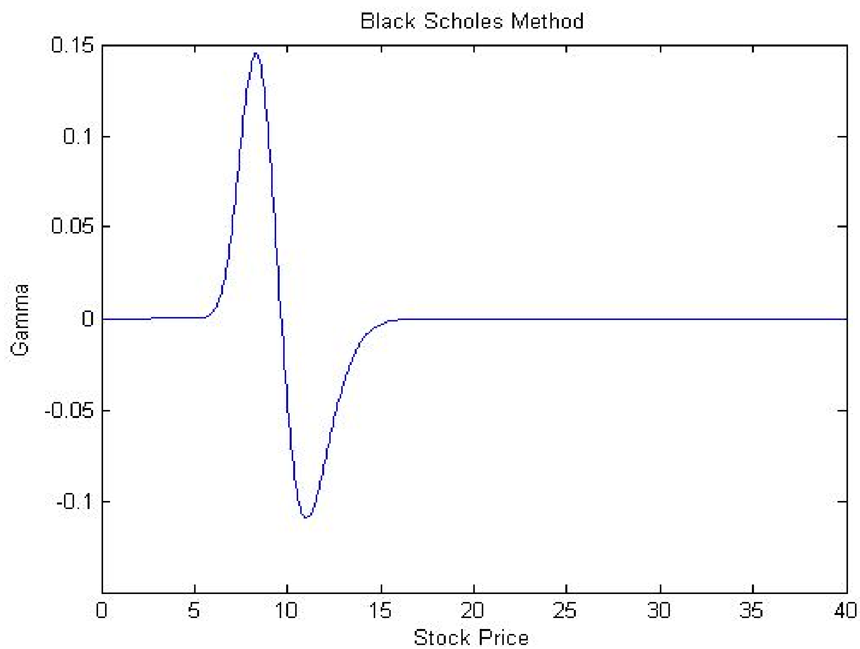
The Delta of option price under Crank Nicolson Methods:



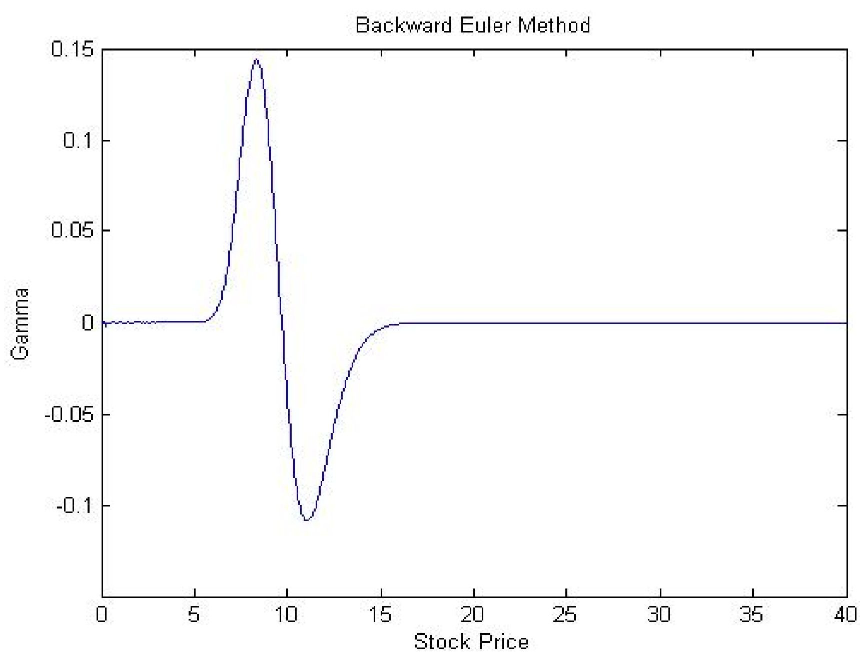
The Delta of option price under Ranacher Smooth Methods:



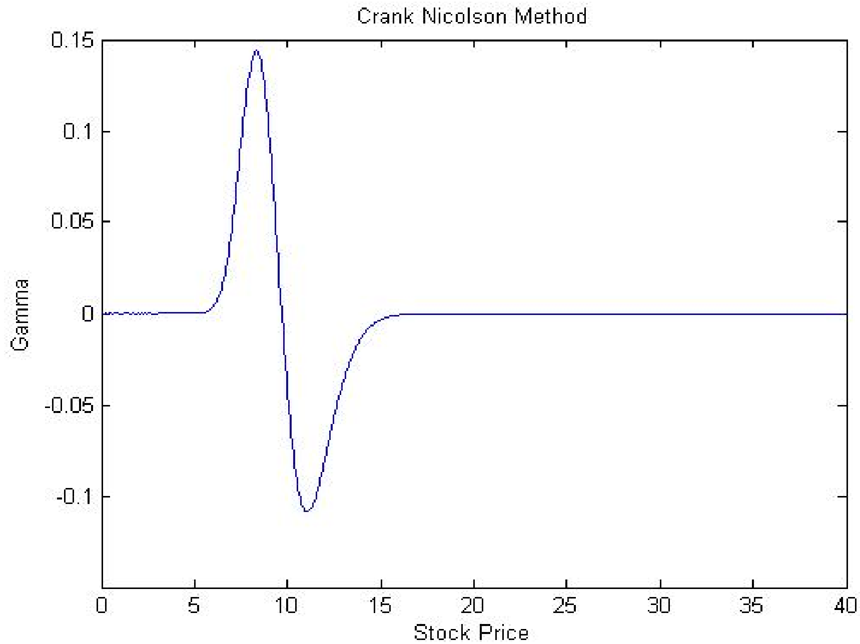
Gamma:
The Gamma of option price under Black Scholes Methods:



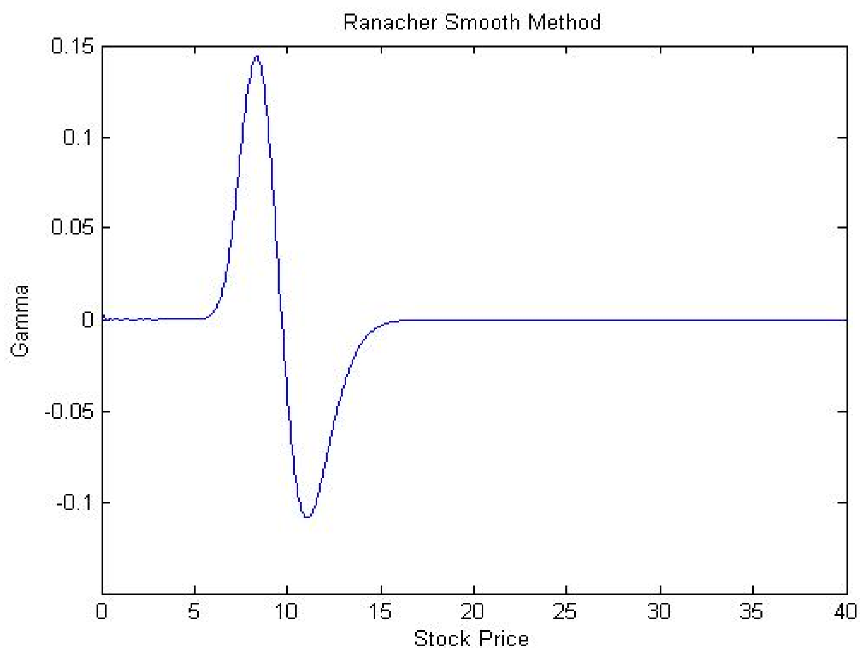
The Gamma of option price under Backward Euler Methods:



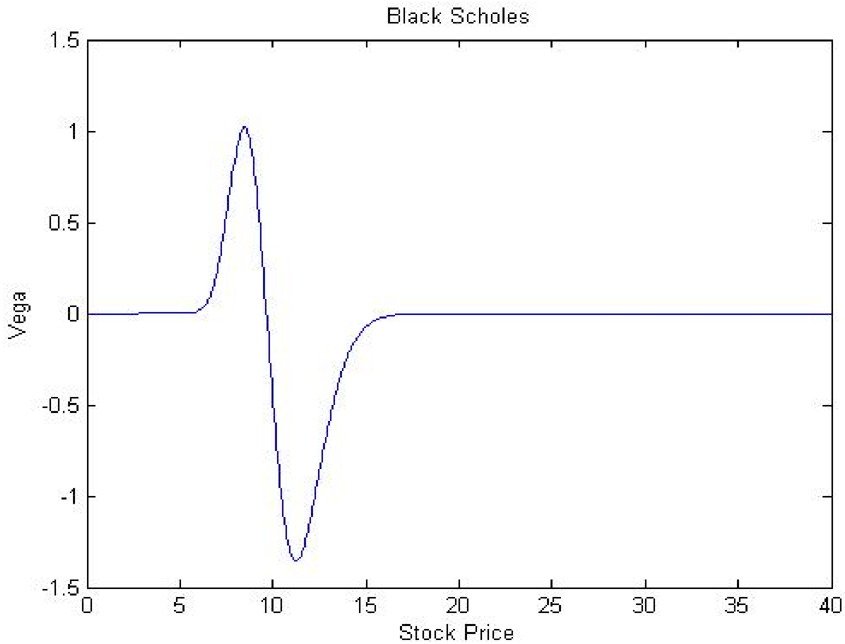
The Gamma of option price under Crank Nicolson Methods:



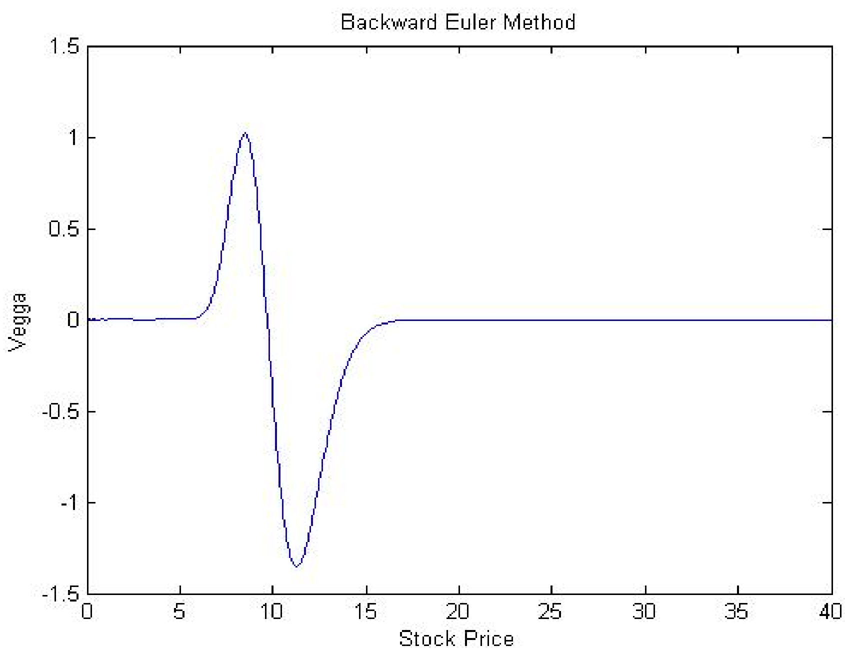
The Gamma of option price under Ranacher Smooth Methods:



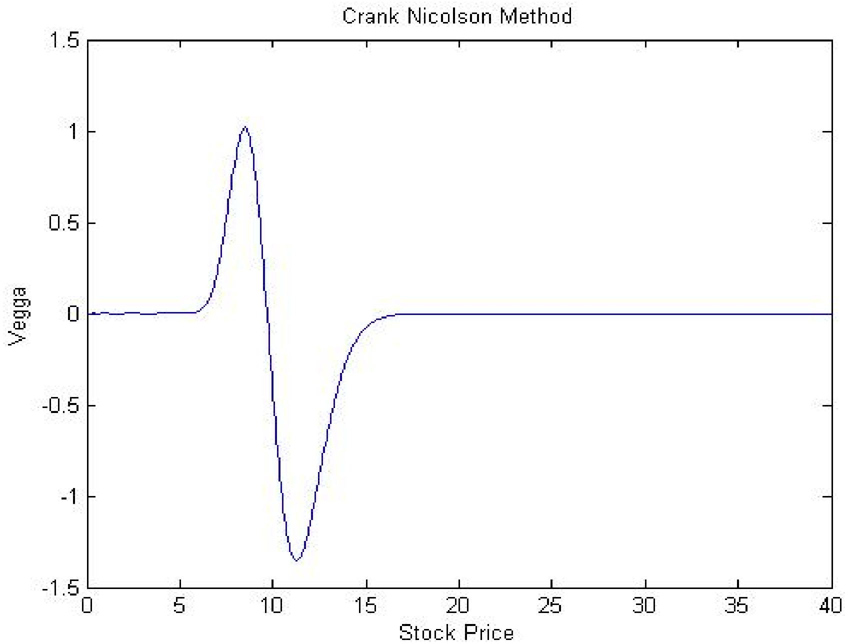
Vega: The Vega of option price under Black Scholes Methods:



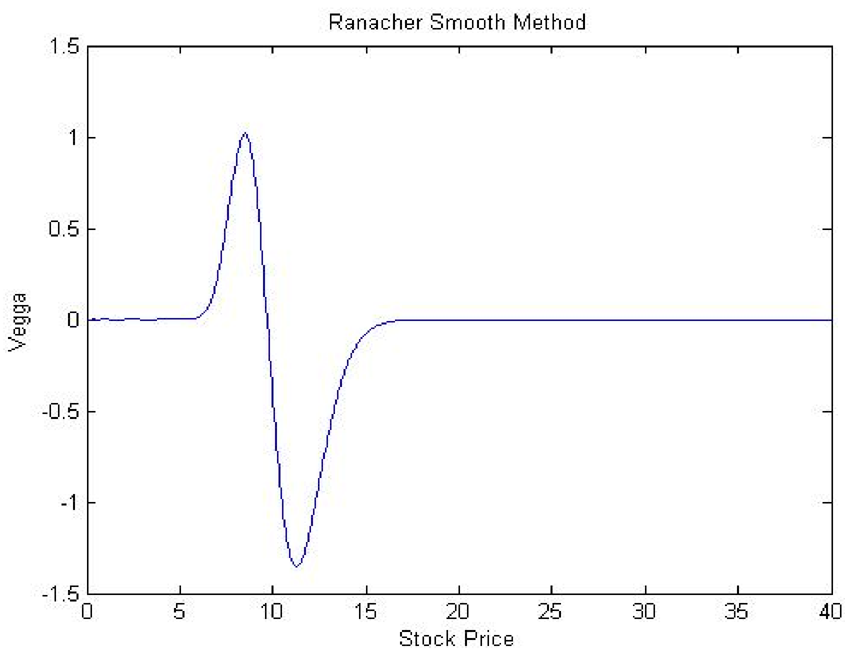
The Vega of option price under Backward Euler Methods:



The Vega of option price under Crank Nicolson Methods:



The Vega of option price under Ranacher Smooth Methods:



Error Test:
We increased both the N_x and N_t from 20 to 640, to see the convergence rate of different methods for the binary put option.

1)For the put option price:

N_x	N_t	Backward	Rate	Crank Nicolson	Rate	Ranacher Smooth	Rate
20	20	0.317074957		0.314437672		0.322944148	
40	40	0.150341433	2.11	0.148175535	2.12	0.151402278	2.13
80	80	0.069756168	2.16	0.0693145	2.14	0.070194038	2.16
160	160	0.034760674	2.01	0.034475478	2.01	0.035016767	2.00
320	320	0.017298542	2.01	0.017207956	2.00	0.017418478	2.01
640	640	0.008643274	2.00	0.008598123	2.00	0.00869713	2.00

2) For the Delta:

N_x	N_t	Backward	Rate	Crank Nicolson	Rate	Ranacher Smooth	Rate
20	20	0.096672268		0.096032301		0.098693105	
40	40	0.077343851	1.25	0.076095613	1.26	0.076609725	1.29
80	80	0.038565233	2.01	0.03756055	2.03	0.038250679	2.00
160	160	0.018763329	2.06	0.018244999	2.06	0.018504687	2.07
320	320	0.009159206	2.05	0.008909075	2.05	0.009012556	2.05
640	640	0.004519495	2.03	0.004394301	2.03	0.004449629	2.03

3) For the vega:

N_x	N_t	Backward	Rate	Crank Nicolson	Rate	Ranacher Smooth	Rate
20	20	1.419941566		1.444646893		1.440652837	
40	40	0.943121636	1.51	0.931003917	1.55	0.954989352	1.51
80	80	0.364156019	2.59	0.36099817	2.58	0.366642911	2.60
160	160	0.179932815	2.02	0.178287731	2.02	0.18146443	2.02
320	320	0.088114546	2.04	0.087399967	2.04	0.088831093	2.04
640	640	0.043696289	2.02	0.043363191	2.02	0.04404213	2.02

We found that the value and greeks of binary put options are around first order convergence rate under all three methods.

Algorithm 1: PSOR Algoithm

begin
- $x \longleftarrow x_0$
- **for** k = *1 to maxiter* **do**
  - $rmax \longleftarrow 0$
  - **for** j = *1 to N-1* **do**
    - $r = y_j - (l_j x_{j-1} + d_j x_j + u_j x_{j+1})$
    - $x_{test} = x_j + \omega r / d_j$
    - **if** $x_{test} > payoff_j$ **then**
      - $x_j = x_{test}$
      - $rmax = max(rmax, |r|)$
    - **else**
      - $x_j = payoff_j$
    - **if** $r_{max} < Tol$ **then**
      - exit