NPDE for Option Pricing : Assignment #2

Due on Today

Instructor:Dr.Kopriva 12:20am

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Problem 1

[Summary]

1) The iterative number of PSOR algorithm was first decreasing and then increasing according to the increasing of the overrelaxation parameters.

2)We may need 4096 points and 160 time steps to ensure the 4 digit numbers of both American and European option.

3) The value of American put option was higher than the value of the European put option.

4) When stock price was less than the optimal exercise boundary, the American option price was same as the payoff function, the delta was -1 and gamma remain 0. Gamma will increase dramatically when $S > S_f$

[Statement of the problem]

1. Repeat the computation of the price of the European vanilla put in the last assignment and compare the CPU time for computing the American VS the European option. For the PSOR, test the sensitivity of the CPU time based on the optimal relaxation parameter and the iteration tolerance.

2.Compute and plot the values of the European and American vanilla put for K =\$10, r= 0.1/yr, σ =0.4/$\sqrt{yr}$ amd wotj a six month expiry with ranacher smoothing for S $\in$ [0,2K]. Compute the greeks, Δ and Γ for both. Compare and discuss the solutions. Test the sufficient number of points and time steps so that the European option has four significant digits. Test if the American option also converged with this grid. If not, how many points and time steps are required?

3. Calculate the optimal exercise boundary as a function of time. Draw the picture.

[Description of The Mathematics]

The BSM and boundary condition for American put option is:

$$\begin{cases} V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_s - rV = 0 \\ V(0,t) = Ke^{-r(T-t)} \\ V(\infty,t) = 0 \\ V(S,T) = max(K-S,0) \\ V(s,t) = Max(V(s,t), K-s) \text{ for s<}S_{max} \text{ and t < T} \end{cases}$$

Problem 1 continued on next page...

combine : $\delta_t^+ V_j^n + L_h[\alpha V_j^{n+1} + (1-\alpha)V_j^n] = 0$

$\alpha = 1$: Back Euler
$\alpha = \frac{1}{2}$: Crank-Nicolson method

Backward Euler method:
The scheme for the Backward Euler method is given by:

$$\frac{V_{i,j} - V_{i,j-1}}{\delta t} + \frac{1}{2}\sigma^2(i\delta S)^2 \frac{V_{i+1,j-1} - 2V_{i,j-1} + V_{i-1,j-1}}{\delta S^2} + r(i\delta S)\frac{V_{i+1,j-1} - V_{i-1}j - 1}{2\delta S)} - rV_{i,j-1} = 0$$

we can rewrite it as:

$$V_{i,j} = A_i V_{i-1,j-1} + B_i V_{i,j-1} + C_i V_{i+1,j-1}$$

where:

$$A_i = \frac{1}{2}\delta t(r_i - \sigma^2 i^2), B_i = 1 + (\sigma^2 i^2 + r)\delta t, C_i = -\frac{1}{2}\delta t(ri + \sigma^2 i^2)$$

Crank-Nicolson method:

$$\frac{V_{ij} - V_{i,j-1}}{\delta t} + \frac{ri\delta S}{2} + (\frac{V_{i+1,j-1} - V_{i-1,j-1}}{2\delta S} + \frac{ri\delta S}{2}(\frac{V_{i+1,j} - V_{i-1,j}}{2\delta S}) +$$
$$\frac{\sigma^2 i^2(\delta S)^2}{4}(\frac{V_{i+1,j-1} - 2V_{i,j-1} + V_{i-1,j-1}}{(\delta S)^2}) +$$
$$\frac{\sigma^2 i^2(\delta S)^2}{4}(\frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\delta S)^2}) = \frac{r}{2}V_{i,j-1} + \frac{r}{2}V_{ij}$$

We can rewrite the above equation as:

$$-\alpha_i V_{i-1,j-1} + (1-\beta_i)V_{i,j-1} - \gamma_i V_{i+1,j-1} = \alpha_i V_{i-1,j} + (1+\beta_i)V_{i,j} + \gamma_i V_{i+1,j}$$

Where:

$$\alpha_i = \frac{\Delta t}{4}(\sigma^2 i^2 - ri)$$
$$\beta_i = -\frac{\Delta t}{2}(\sigma^2 i^2 + r)$$
$$\gamma_i = \frac{\Delta t}{4}(\sigma^2 i^2 + ri)$$

Problem 1 continued on next page. . .

Ranacher Smooth method

(1)We use backward Euler for a few n $\geq$ 2 time steps
(2)Use Crank-Nicolson after that: given second order accuracy

Close form Black Scholes formula

To test the result for the SDE model of the option pricing, we also need to know the close form solution of the Black -Scholes assumptions, which is the famous Black- Scholes formula.
For the European put options:

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$$

where:

$$d_1 = \frac{log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{log\frac{S}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty} xe^{-\frac{1}{2}S^2}ds$$

Delta for Vanilla put option:

$$-e^{-q\tau}\Phi(-d_1)$$

Gamma for Vanilla put option:

$$-e^{-q\tau}\frac{\Phi(d_1)}{S\sigma\sqrt{\tau}}$$

Vega for Vanilla put option:

$$Se^{-q\tau}\phi(d_1)\sqrt{\tau} = Ke^{-r\tau}\phi(d_2)\sqrt{\tau}$$

where q here is the dividend rate which is equal to 0 in our problem.

[Description of Algorithm]

Algorithm 1: PSOR Algoithm

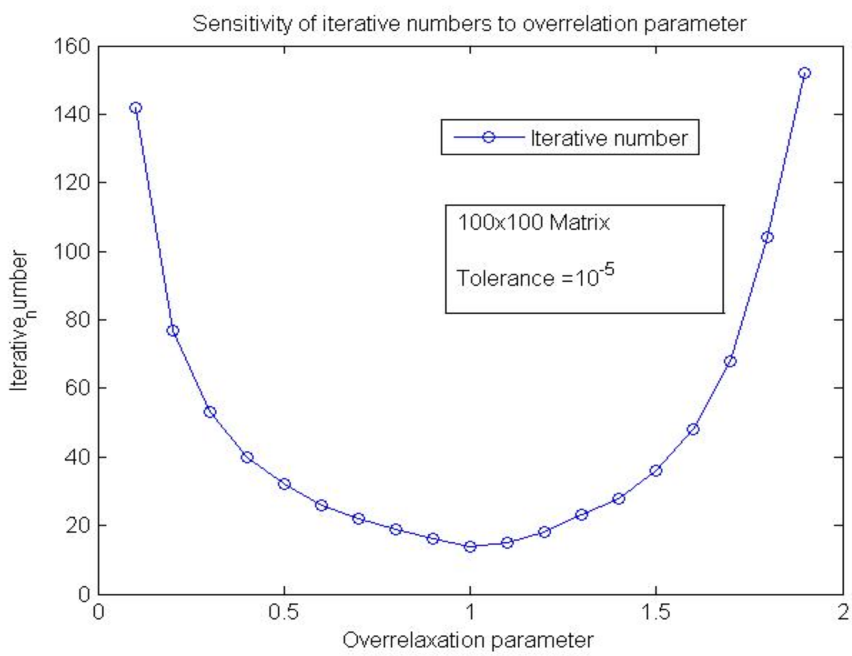
begin
    $x \longleftarrow x_0$
    for *k = 1 to maxiter* **do**
        $rmax \longleftarrow 0$
        for *j = 1 to N-1* **do**
            $r = y_j - (l_j x_{j-1} + d_j x_j + u_j x_{j+1})$
            $x_{test} = x_j + \omega r/d_j$
            if $x_{test} > payoff_j$ **then**
                $x_j = x_{test}$
                $rmax = max(rmax, |r|)$
            else
                $x_j = payoff_j$
            if $r_{max} < Tol$ **then**
                exit

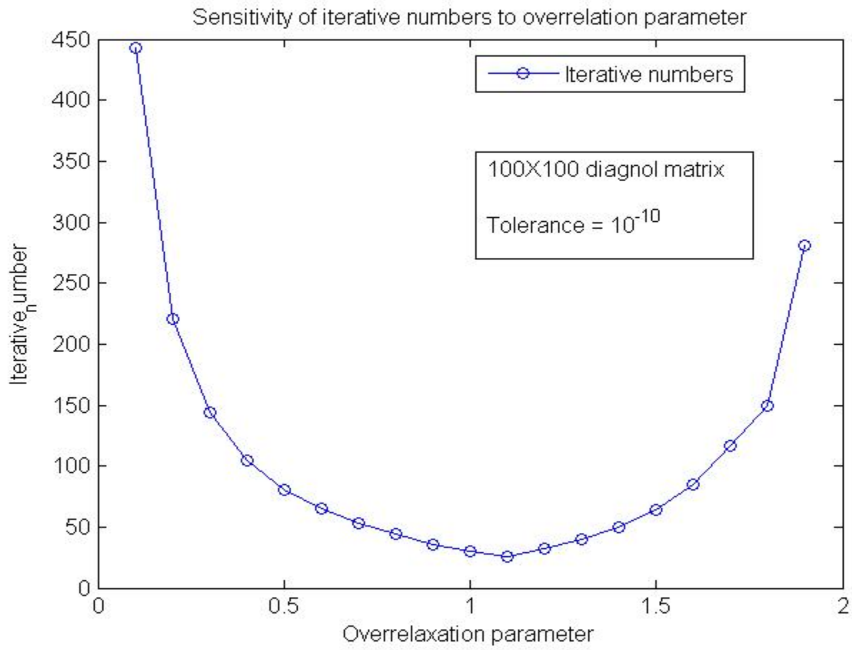
Convergence theorem:
For the PSOR solver, if the next equation holds, the PSOR will converge to the real solutions.

$$\rho = max \frac{\{(\frac{1}{\omega} - 1)\,|d_i| + |u_i|\}}{\{\frac{|d_i|}{\omega} - l_i\}} < 1; \tag{1}$$

[Results]

First we need to test if our PSOR method works. we choose a 100x100 diagnal matrix, $B = DIAG(1, 3, 1)$. We set the tolerance equal to 10^{-5} and 10^{-10} to see the number of iteration based on the over-relaxation parameter. The chart is as follows:





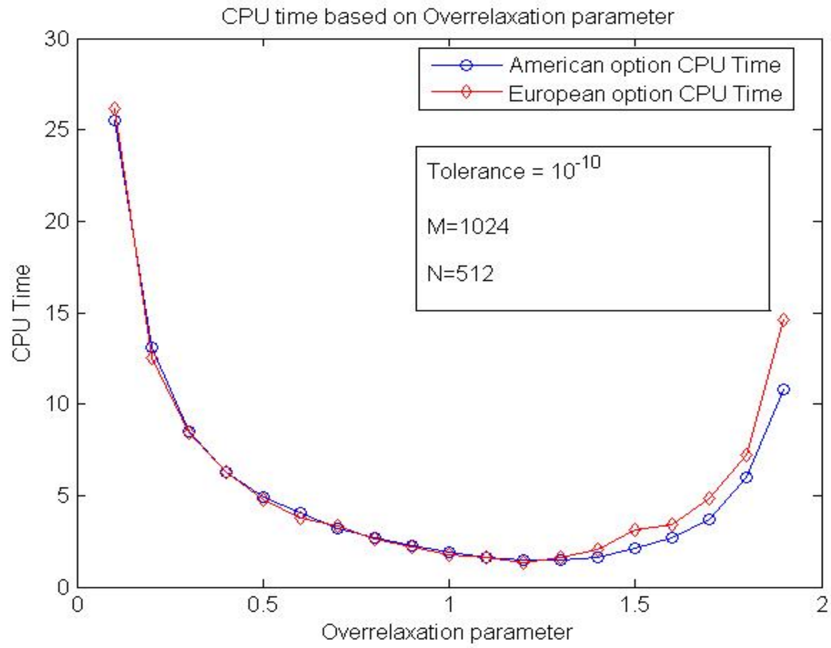
The above charts showed that the iteration number was first decreasing with the increasing of the overrelation parameter, when the overrelaxation parameter was around 1.1, the iterative number started to increase. The situation for tolerance 10^{-10} and tolerance 10^{-10} was similar.
We can also found that:

$$\rho = max\frac{\{(\frac{1}{\omega}-1)\,|d_i| + |u_i|\}}{\{\frac{|d_i|}{\omega} - l_i\}} = \frac{\{(\frac{1}{\omega}-1)\,|3| + |1|\}}{\{\frac{|3|}{\omega} - l_i\}} < 1; \tag{2}$$

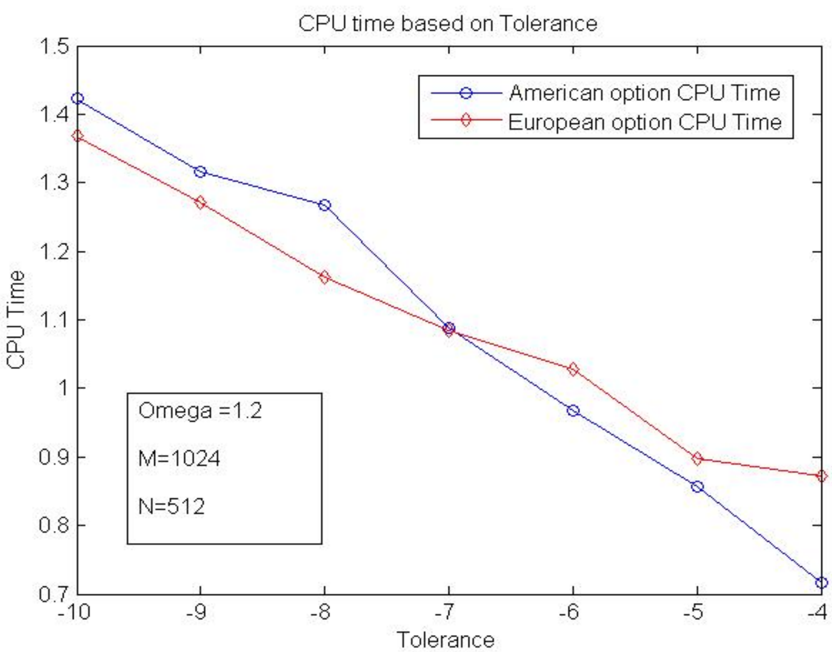
So the convergence theorem holds and our PSOR solver works.

Our first problem is to redo the assignment 1 problem based on the POSR method and compute the CPU time.

So we set $E^* = \$10$, $r^* = 0.05/yr$, $\sigma^* = 0.20/yr$ T=0.5 as the example to build our model.
We watch the cpu time in the following two situation:
First under different overrelaxation parameter:



The above figure showed that the cpu time for the European option was a little bit higher than American option on the small and big overrelaxation paramater, and a little bit lower around 1.The main reason may be that the American option need less iterative numbers to converge, when the ω was very high and low. Besides, both European and American Option reached the minimal CPU time around parameter 1.2.



The second situation is the tolerance, both American and European option CPU time decreased with the increasing of the tolerance. The CPU time for American option is higher than the European when the tolerance was less than 10^{-7} and lower when the tolerance increased after that.

Next we try to see the influence of the mesh grid to CPU time. The tolerance was set as 10^{-10} and overrelaxation parameter was set as 1.2.

N_x	N_t	European	Increasing ratio	American	Increasing ratio
16	8	0.001486		0.001150	
32	16	0.002443	1.64	0.002138	1.86
64	32	0.005997	2.45	0.005571	2.61
128	64	0.019255	3.21	0.015861	2.85
256	128	0.063979	3.32	0.052949	3.34
512	256	0.253021	3.95	0.202332	3.82
1024	512	1.348758	5.33	1.467072	7.25

With the increasing of the mesh grids, the CPU time is increasing, and the ratio of time change was also increasing, which meant that the computer consumed more as the grids increased.

Problem 2

Now we try to see how many points we need to ensure that the numerical result would have 4 significant digits.
We set the S_{max} equal to 8*S, $\omega = 1.2$, and $Tol = 10^{-10}$, In order to ensure 4 digit numbers, we need the relative error less than 0.00005.

First we set N equal to 160 and watch the relative error with the increasing of stock price steps.

Steps		Option Price		Delta		Gamma	
M	N	European	American	European	American	European	American
32	160	0.30031	0.33386	0.04397	0.04128	0.03837	0.05317
64	160	0.10214	0.09587	0.02373	0.03679	0.00997	0.01855
128	160	0.02094	0.02115	0.00455	0.00726	-0.00005	-0.00281
256	160	0.00504	0.00560	0.00106	0.00129	-0.00013	-0.00035
512	160	0.00125	0.00146	0.00026	0.00032	-0.00004	-0.00021
1024	160	0.00031	0.00036	0.00007	0.00008	-0.00001	-0.00005
2048	160	0.00008	0.00009	0.00002	0.00002	0.00000	-0.00001
4096	160	0.00002	0.00002	0.00000	0.00001	0.00002	0.00002

When M increased to 4096, both European and American option ensure the 4 digit numbers.

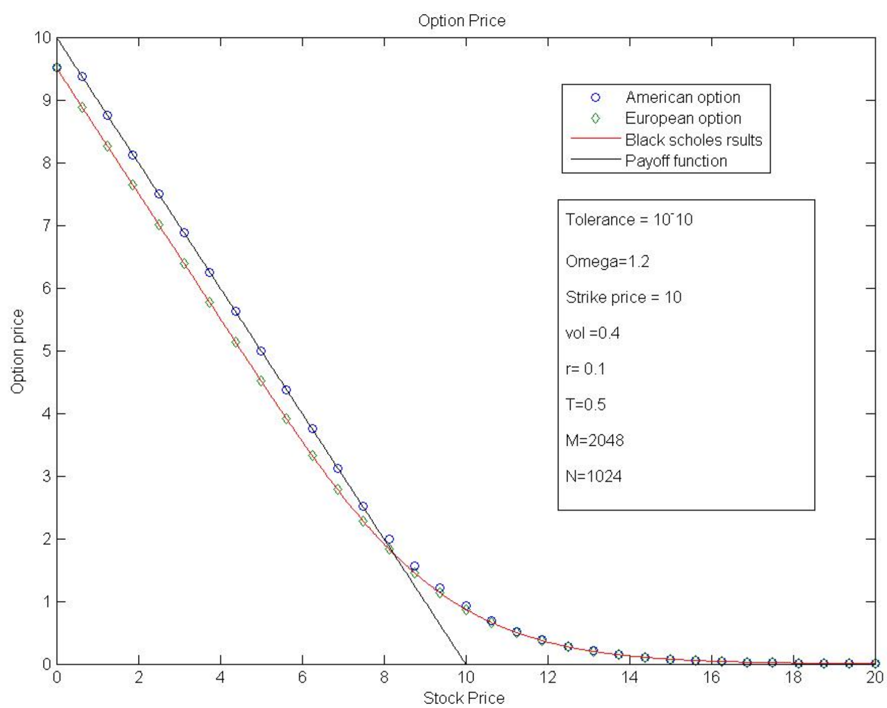
Now we set the N equal to 320, the relative error was listed in the following:

Steps		Option Price		Delta		Gamma	
M	N	European	American	European	American	European	American
32	320	0.30045	0.33409	0.04406	0.04144	0.03826	0.05309
64	320	0.10191	0.09563	0.02369	0.03675	0.00989	0.01841
128	320	0.02090	0.02113	0.00454	0.00724	-0.00007	-0.00280
256	320	0.00503	0.00559	0.00106	0.00128	-0.00013	-0.00035
512	320	0.00125	0.00146	0.00026	0.00033	-0.00004	-0.00022
1024	320	0.00031	0.00037	0.00007	0.00008	-0.00001	-0.00005
2048	320	0.00008	0.00009	0.00002	0.00002	0.00000	-0.00001
4096	320	0.00002	0.00002	0.00000	0.00001	0.00000	0.00000

The situation was similar, we still need 4096 ponts to ensure the 4 digit numbers.

Now we set M = 2048 and N = 1024 to see the results of the option price, delta and gamma under the PSOR methods

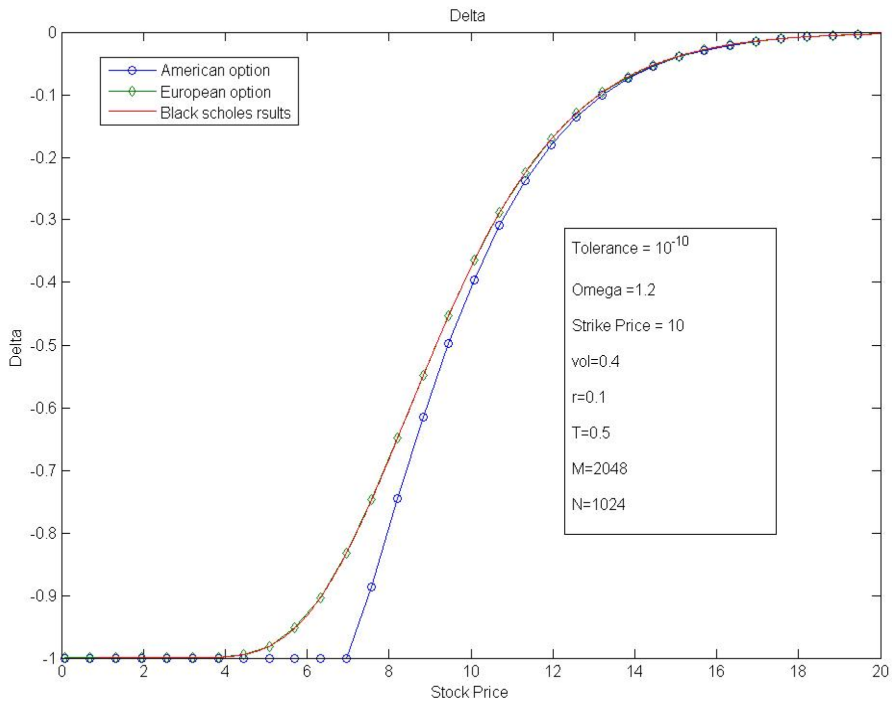
[option price]



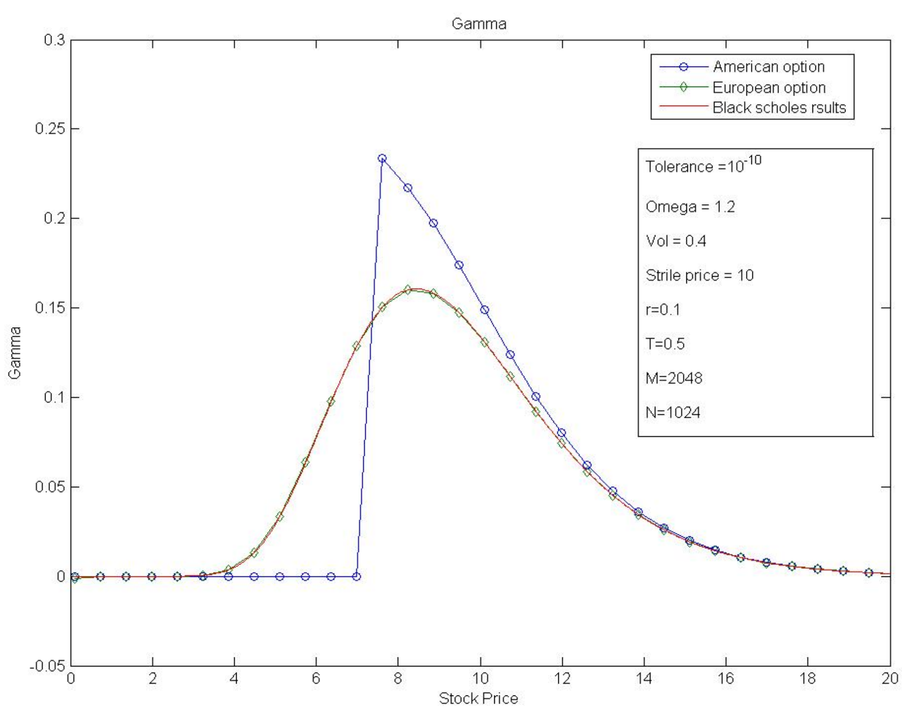
We can see that American option price was higher than the European option, which made sense, since American put option had the inside early exercise right, so the value was more valuable.
From this chart,we can also find the phenomenon of the Optimal Exercise Boundary.

We can see that the American put prices were same with the European put prices when the stock prices was bigger than Optimal Exercise Boundary S_f. When stock prices was less than S_f, American put option price was equal to the payoff function.

[Delta]



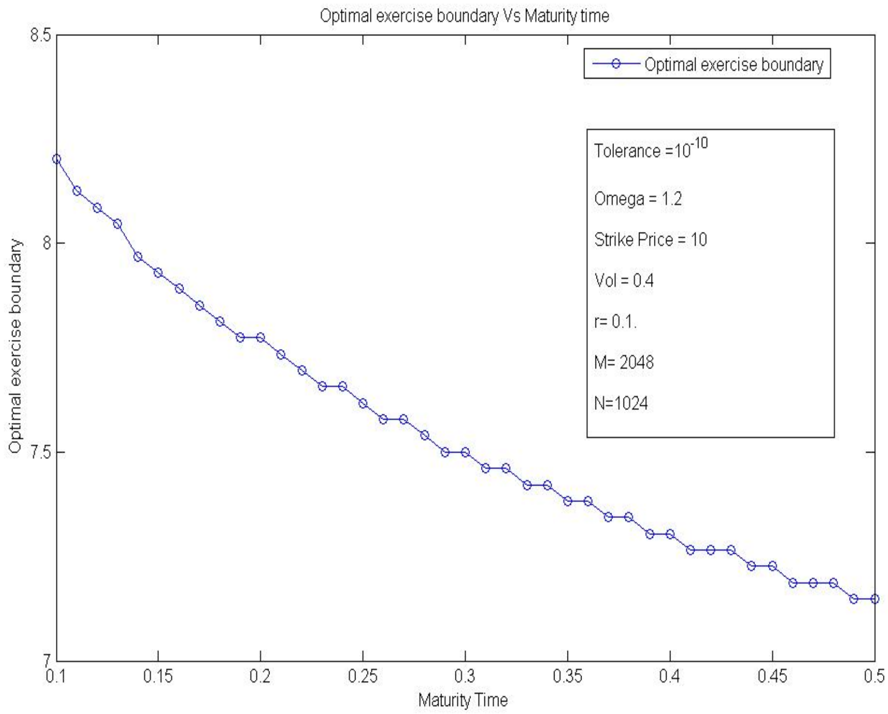
Delta for the American put option was equal to -1, when stock prices was less than S_f
[Gamma]



The gamma of American put option increased from 0 dramatically, when stock price was around S_f, it is because that the delta was increased from -1 to some higher value, when the stock price touched the S_f.

Problem 3

We set tolerance equal to 10^{-10}, ω equal to 1.2 ,M=2048 and N=1024, To see the optimal exercise boundary VS maturity time



From the above chart, we found when the maturity time increased, the optimal exercise boundary decreased, which meant that when time was close to the maturity time, the optimal exercise boundary would aproach to the strike price K which was 10 in this problem.