Problem:

x follows $\int_0^T udW_u$ and y follows $\int_0^T W_u du$

what is the value of $cov(x, y)$

Answer: $cov(x, y) = E(xy) - E(x)E(y)$

$E(x) = E(\int_0^T udW_u)$

according to the martingality, $\int_0^T udW_u$ is a martingale, so E($\int_0^T udW_u$) equals to zero.

Then cov(x,y)= E(xy)

$x + y = \int_0^T (udW_u + W_u du) = \int_0^T d(W_u u) = TW_T$

so $(x + y)^2 = (TW_T)^2 = T^2 W_T^2$

$E(x^2 + 2xy + y^2) = E(T^2 W_T^2) = T^2 E(W_T^2) = T^3$

$E(x^2) = E((\int_0^T udW_u)^2) = E(\int_0^T udW_u * \int_0^T udW_u) = \int_0^T E(u^2)du = \frac{T^3}{3}$(According to the ito isometric)

$E(y^2) = E(\int_0^T W_u du)^2 = E(\int_0^T W_u du) * (\int_0^T W_s ds)) = \int_0^T \int_0^T E(W_s W_u) dsdu$

we know that $E(W_s W_u) = min(s, u)$, then

$\int_0^T \int_0^T E(W_s W_u) dsdu = \int_0^T \int_0^T min(s, u) dsdu = \int_0^T (\int_0^u s + \int_u^T u) dsdu$

$= \int_0^T (\frac{u^2}{2} + u(T - u))du = -\frac{T^3}{6} + \frac{T^3}{2} = \frac{T^3}{3}$

Then $2E(xy) = T^3 - T^3/3 - T^3/3 = T^3/3$

so $E(xy) = T^3/6$

So $cov(x, y) = T^3/6$
$x \sim N(0, 1)$
$\underbrace{u' - P(x)u^2 - Q(x)u - R(x)} = 0$, since u is a particular solution