Analysis on Global Economic Data

K. David Yao
Florida State University, Department of Mathematics

Contact Information:
Email: `kdy10@my.fsu.edu`

Motivation

•More attention is being given to the spatial distribution of financial systems. Some researchers such as Martin et. al. (2005)[R.Martin] have been working on showing that the spatial structure of the financial system is far from neutral in its effects, but rather influences the allocation of funds, capital and credit across regions and localities.
•We will direct our focus on global financial data. We will start with financial data on World's currencies and especially the US dollar exchange rates. To investigate these data on the Earth, we will approximate the Earth surface with a unit Sphere.

Some Background on Spherical Data

Spherical means and spherical sample means

Let X be a $\mathbb{S}^2$-measurable random variable defined on the probability space $(\Omega, \mathcal{A},\ Pr)$ with $Q = P_X$ on $\mathcal{B}_{\mathbb{S}^2}$

The *spherical mean set* is the set of minimizers of the *Fréchet function* defined by

$$\mathcal{F}(p) = E\left[\|X-p\|^2\right] = \int_{\mathbb{S}^2} \|x-p\|^2\, Q(dx)$$

It turns out that if the mean vector $\mu = E[X]$ is not the zero vector then the spherical mean set has one point only, the spherical mean $\mu_E = \frac{\mu}{\|\mu\|}$, and if $X_1, X_2, \ldots, X_n$ are independent random vectors with a common distribution Q on $\mathbb{S}^2$, then their spherical sample mean is $\overline{X}_E$ given by

$$\overline{X}_E = \frac{\bar{X}}{\|\bar{X}\|}$$

Assigning Meaning to Location using

We now turn our focus back to currency exhange rates. We will first work on establishing what we would refer to as **mean locations**. It will be done using the **Global Financial Centres Index 14** abrieviated GFCI14. It is a list of rankings of financial centers using online surveys and over 102 indices.

Mean Locations

Since our focus in on US exchange rates, we will create weighted means using the scores and locations for cities sharing the same currencies. We will use the Euro currency to illustrate these means.
And for cities located in countries that share the Euro as a currency we have the following pairs $\{(X_1, g_1), (X_2, g_2) \ldots, (X_{17}, g_{17})\}$ used to give the weighted means. The mean location are computed as follow;

$$U = g_1 X_1 + \ldots + g_n X_n. \text{ and } \overline{U} = \frac{1}{\|U\|} U$$

Mean Location for the EURO currency

Weighted mean location= $U = (0.667028094, 0.108890018, 0.717598711)^T$ with $\|U\| = 0.985764386$
Mean Location=$\overline{U} = (0.676660776, 0.11046252, 0.727961692)^T$ The mean location was found around Pitasch, Switzerland . Since Switzerland is not in the Eurozone, we select as mean location, the closest city in the Eurozone to Pitasch, Switzerland.

Function Estimation on a Sphere

•Laplace-Beltrami Operator
Let $C^\infty(\mathcal{M})$ be the space of real valued infinitely differentiable continuous functions on $\mathcal{M}$. We denote the Laplace-Beltrami operator on $\mathcal{M}$ by Δ and it is locally given by:

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{j,k} \partial_j \left(g^{p,j,k} \sqrt{g}\, \partial_k \right)$$

Here $g^{p,i,j}$ is the inverse of $g_{p,ij}$ the Riemannian metric tensor, and g is the determinant of the matrix $(g_{p,ij})$.

•Real Basis
Let ϕ_k be an eigenfunction of Δ. For $f \in L^2(\mathcal{M})$, the eigenfunction expansion will be defined by

$$f = \sum_{k=0}^{\infty} \sum_{\mathcal{E}_k} \hat{f}_k \phi_k, \text{ where } \hat{f}_k = \int_{\mathcal{M}} f\ \phi_k, \text{ for } k \in \mathbb{N}$$

The summation over $\mathcal{E}_k$ means over all eigenfunctions ϕ_k in the eigenspace $\mathcal{E}_k$.

Statistical Inverse Estimation

Function Estimation in Finance

Let's consider a regression function f, of the response variable Y, on the measurement variable$X \in \mathcal{M}$, so that $\mathbb{E}(Y|X) = f(X)$, where X is uniformly distributed on $\mathcal{M}$ and $\mathbb{E}$ denotes the expectation. We will consider the signal plus noise model,

$$Y = f(X) + \varepsilon$$
$$\text{with } \mathbb{E}(\varepsilon) = 0 \text{ and } \mathbb{E}(\varepsilon^2) = \sigma^2 > 0$$

- We will attempt to estimate the function $f(X)$ defined on the Sphere $\mathbb{S}^2$ at the various values of the mean locations.
- At multiple values of mean locations $f(X)$ may represent the percentage of increase or decrease of Yearly Average Exchange Rates which represent the value of one US dollar in the countries currency. The graph below illustrate such values of f for over 30 currencies.

Statistical Estimator

•Statistical Inverse Problem
Let T be an operator on $L^2(\mathbb{S}^2)$ (T could be unbounded). A statistical inverse problem on $\mathbb{S}^2$, is an attempt to statistically estimate $T(f)$. Since $T(f)$ is unknown, we first estimate f by observing a random sample of size n, $\{(X_1, Y_1), \ldots (X_n, Y_n)\}$ and it is done as follow;

$$f^n_\Lambda(X) = \sum_{\lambda_k \le \Lambda} \hat{f}^n_k \phi_k(X) \sim f(X) = \sum_{k=0}^{\infty} \sum_{\mathcal{E}_k} \hat{f}_k(X) \phi_k(X) \quad X \in \mathbb{S}^2 \text{ and}$$
$$\hat{f}^n_k = \frac{1}{n} \sum_{j=1}^{n} Y_j \phi_k(X_j) \sim \hat{f}_k = \int_{\mathcal{M}} f\ \phi_k, \text{ for } k \in \mathbb{N}$$

•Statistical Estimator:
We get the statistical estimator $T(f^n_\Lambda)$ of $T(f)$ with;

$$T(f^n_\Lambda) = \sum_{\lambda_k \le \Lambda} t_k \hat{f}^n_k \phi_k \ \sim T(f) = \sum_{k=0}^{\infty} \sum_{\varepsilon_k} t_k \hat{f}_k \phi_k$$

•In general if If we have a bounded operator, then, as long as f^n_Λ is a consistent estimator of f, we have that $T(f^n_\Lambda)$ will consistently estimate $T(f)$. However, the unbounded case, also known as ill-posed problem, is of more practical relevance. It is with respect to ill-posed estimation that we will proceed.

Functional PCA

•For this part we think of a random vector $X \in L^2(\mathbb{S}^2)$ an infinite dimensional space of square integrable functions on the sphere. And $X = (X_1, X_2, .., X_p, \ldots)$ with respect to the basis of eigenfunctions of the Laplace-Beltrami operator.

•The principal components are those orthogonal affine combinations in the space square integrable functions $Y_1, Y_2, \ldots, Y_p, \ldots$ whose variances are as large as possible and;

$$X = [X_1\ X_2 \vdots X_p \vdots]^T = [X]_e \ \to [X]_a = Y = [Y_1\ Y_2 \vdots Y_p \vdots]^T$$

And this rotation is done when we have $a_i = \mathbf{f_i}$, for the eigenvalue-eigenvector pairs $(\lambda_1, \mathbf{f_1}), (\lambda_\mathbf{2}, \mathbf{f_2}), \ldots, (\lambda_\mathbf{p}, \mathbf{f_p}), \ldots$ And for $Y_i = \mathbf{f_i^T}(X - \mu)$ then the *total population variance* is;

$$\sum_{i=1}^{\infty} \text{Var}(X_i) = \sum_{i=1}^{\infty} \text{Var}(X_i) = \sum_{i=1}^{\infty} \lambda_i < \infty$$

And one can now, express the proportion of total variance explained by the kth principal component by

$$\frac{\lambda_k}{\lambda_1 + \lambda_2 + \ldots + \lambda_p + \ldots}$$

Future Work

- For functional PCA we consider each random function as a random element $X \in L^2(\mathcal{M}), X = (X_1, X_2, \ldots)$, whose random covariance operator can be diagonalized, with the nonzero diagonal entries, being the variances of the P,C.A.'s. To avoid a singular sample covariance matrix, here , one might consider to regularize the sample covariance matrix, by adding to it a small error σId. Next, for dimensionality reduction, one may follow, the multivariate case.
- Certainly the functional PCA is applicable to a small data set, that included a number of covariates, that might be all represented in terms of spherical harmonics. To this end an important goal is data collection of multiple financial indices for as many countries as possible, that are available in the public domain.
- The next step consists in a FPCA data dimensionality reduction, that may lead to new interpretations of variability of a changing global financial market.
- Instead of using the Laplace-Beltrami in our function estimation process, we may use spherical wavelets as well.

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