南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Algorithm Design and Analysis (H)

CS216

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(slides edited from Prof. Shiqi Yu)

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Randomized Algorithms

Randomization

- **Algorithm design patterns.**
  - Greedy
  - Divide and conquer
  - Dynamic programming
  - Duality (e.g., network flow)
  - Randomization

in practice, access to a pseudo-random number generator

- **Randomization.** Allow fair coin flip in unit time.
- **Why randomize?** Can lead to simplest, fastest, or only known algorithm for a particular problem.
- **Ex.** Symmetry breaking protocols, graph algorithms, quicksort, hashing, load balancing, Monte Carlo integration, cryptography, etc.

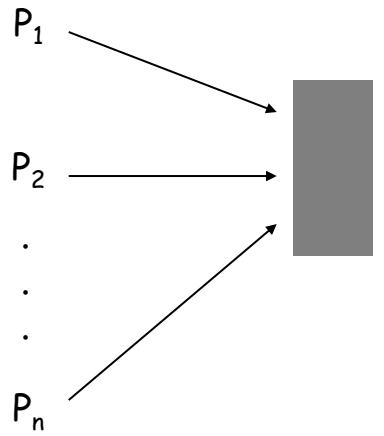
1. Content Resolution

Contention Resolution in a Distributed System

- **Contention resolution.** Given n processes P_1, …, P_n, each competing for access to a shared database. If 2 or more processes access the database simultaneously, all processes are locked out. Devise protocol to ensure all processes get through on a regular basis.

- **Restriction.** Processes can't communicate.

- **Challenge.** Need symmetry-breaking paradigm.



Contention Resolution: Randomized Protocol

- **Protocol.** Each process requests access to the database at time t with probability $p = 1/n$.
- **Useful facts from calculus.** As n increases from 2, the function:
  - $(1 - 1/n)^n$ converges monotonically from 1/4 up to $1/e$
  - $(1 - 1/n)^{n-1}$ converges monotonically from 1/2 down to $1/e$.
- **Lemma 1.** Let $S[i, t]$ = event that process i succeeds in accessing the database at time t. Then $1/(e \cdot n) \le \Pr[S(i, t)] \le 1/(2n)$.
- **Pf.** By independence, $\Pr[S(i, t)] = p(1 - p)^{n-1}$.

process i requests access, none of remaining processes request access

  - Setting $p = 1/n$, we have $\Pr[S(i, t)] = 1/n\,(1 - 1/n)^{n-1}$. ∎

value that maximizes Pr[S(i,t)]

between 1/e and 1/2

Contention Resolution: Randomized Protocol

- **Protocol.** Each process requests access to the database at time t with probability $p = 1/n$.
- **Lemma 2.** The probability that process i fails to access the database in $e \cdot n$ rounds is at most $1/e$. After $e \cdot n$ $(c \ln n)$ rounds, the probability $\leq n^{-c}$.
- **Pf.** Let $F[i, t]$ = event that process i fails to access database in rounds 1 through t. By independence and Lemma 1, $\Pr[F(i, t)] \leq (1 - 1/(en))^t$.
  - Choose $t = \lceil e \cdot n \rceil$: $$\Pr[F(i,t)] \leq \left(1 - \frac{1}{en}\right)^{\lceil en \rceil} \leq \left(1 - \frac{1}{en}\right)^{en} \leq \frac{1}{e}$$
  - Choose $t = \lceil e \cdot n \rceil \lceil c \cdot \ln n \rceil$: $$\Pr[F(i,t)] \leq \left(\frac{1}{e}\right)^{c \ln n} = n^{-c}$$

Contention Resolution: Randomized Protocol

- **Claim.** The probability that all processes succeed within $2en \ln n$ rounds is $\geq 1 - 1/n$.
- **Pf.** Let $F[t]$ = event that at least one of the n processes fails to access database in any of the rounds 1 through t.

$$\Pr[F[t]] = \Pr\left[\bigcup_{i=1}^{n} F[i,t]\right] \leq \sum_{i=1}^{n} \Pr[F[i,t]]$$

union bound

Lemma 2

➢ Choosing $t = \lceil e \cdot n \rceil \lceil 2 \ln n \rceil$ yields $\Pr[F[t]] \leq n \cdot n^{-2} = 1/n$. ▪

Union bound. Given events $E_1, \ldots, E_n$,

$$\Pr\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_{i=1}^{n} \Pr[E_i]$$

2. Global Min Cut

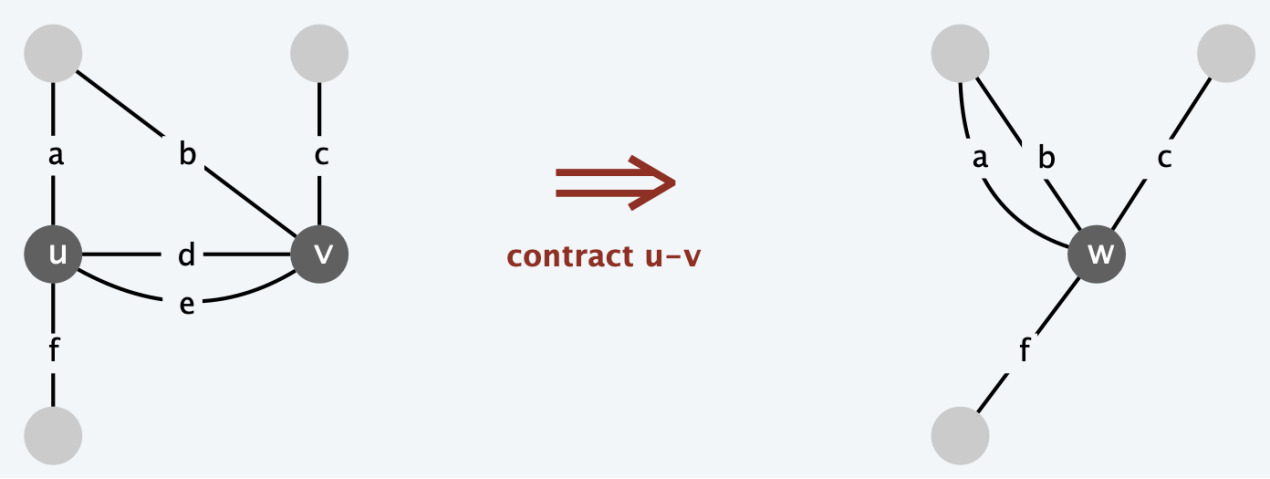
Global Minimum Cut

- **Global min cut.** Given a connected, undirected graph $G = (V, E)$, find a cut (A, B) of minimum cardinality.
- **Applications.** Partitioning items in a database, identify clusters of related documents, network reliability, circuit design, TSP solvers, etc.

- **Network flow solution.**
  - Replace every edge (u, v) with two antiparallel edges (u, v) and (v, u).
  - Pick some vertex $s \in V$ and compute min s-v cut separating s from each other node $v \in V$.
- **False intuition.** Global min-cut is harder than min s-t cut.

Global Min Cut: Contraction Algorithm

- **Contraction algorithm.** [Karger 1995]
  - Pick an edge $e = (u, v)$ uniformly at random. Contract edge e.
    - ✓replace u and v by single new super-node w
    - ✓preserve edges, updating endpoints of u and v to w
    - ✓keep parallel edges, but delete self-loops
  - Repeat until graph has just two super-nodes v_1 and v_2 .
  - Return the cut $(S(v_1), S(v_2))$ ($S(v_i)$: all nodes that were contracted to v_i).



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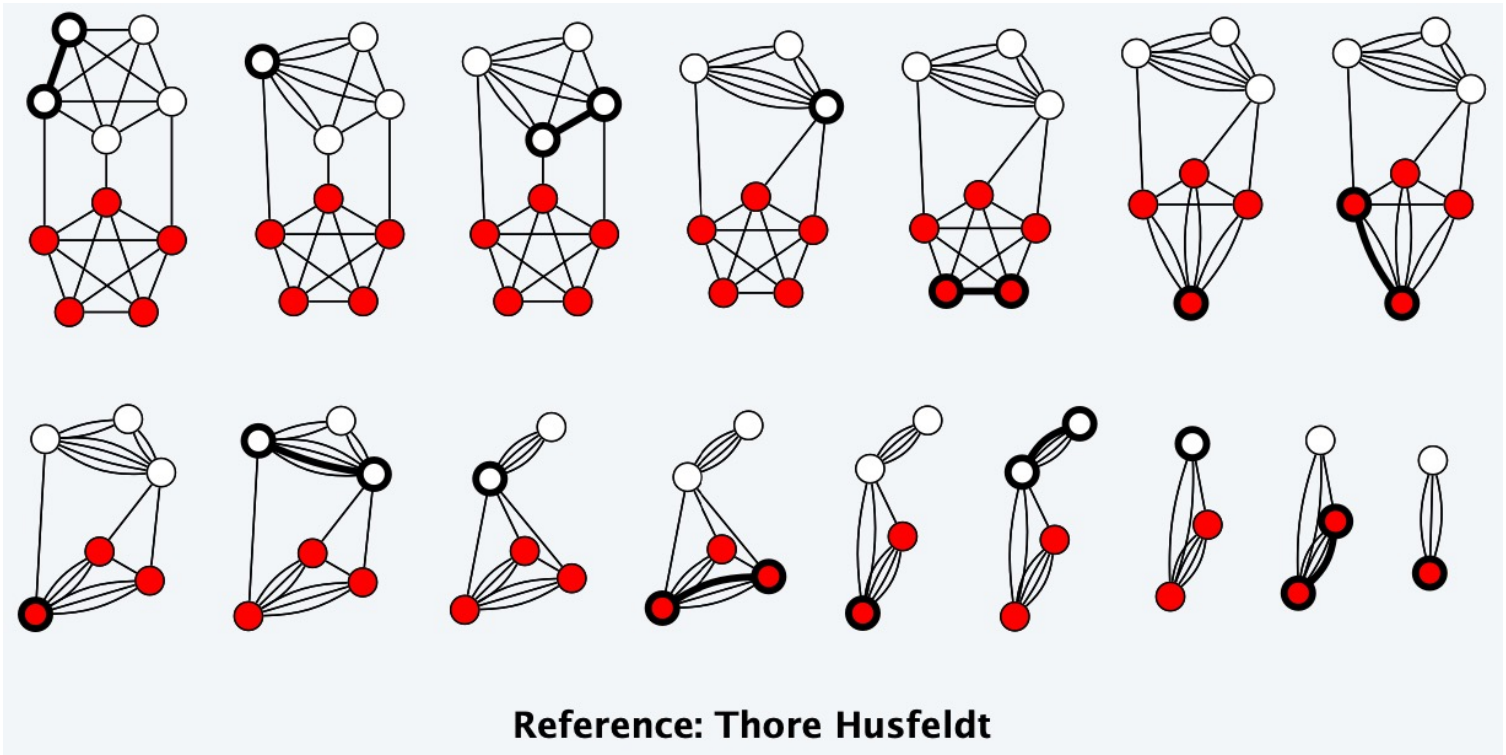

Reference: Thore Husfeldt

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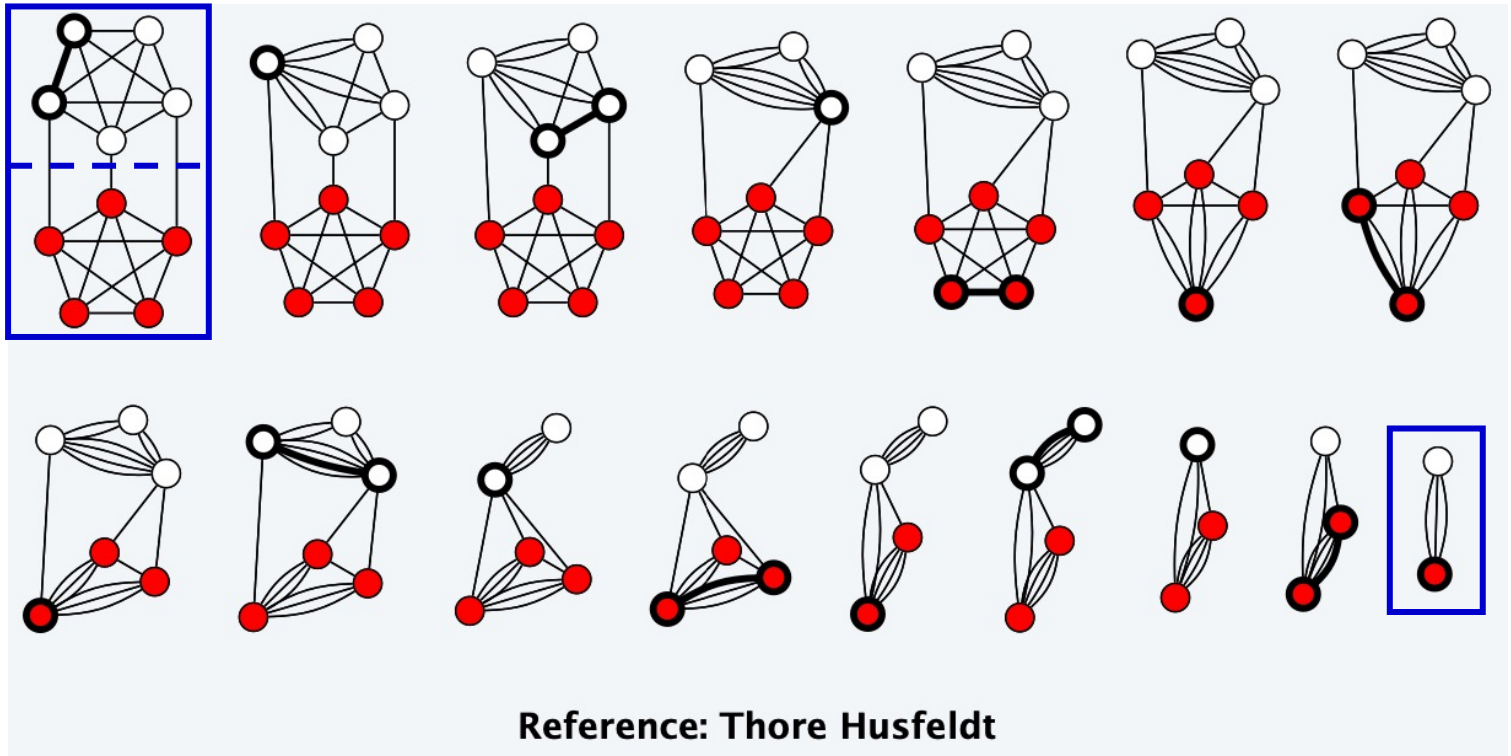

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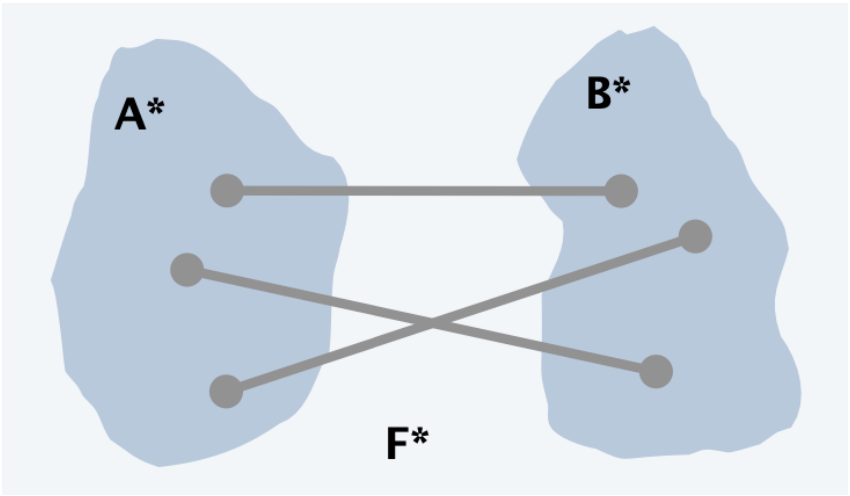
Contraction Algorithm: Analysis

- **Theorem.** The contraction algorithm returns a min cut with prob ≥ 2 / n^2.
- **Pf.** Consider a global min-cut (A^*, B^*) of G. Let F^* be edges with one endpoint in A^* and the other in B^*. Let $k = |F^*|$ = size of min cut.
  - In first step, algorithm contracts an edge in F^* with probability $k / |E|$.
  - Every node has degree ≥ k since otherwise (A^*, B^*) would not be a min-cut $\Rightarrow |E| \geq 1/2\ k\ n \Leftrightarrow k / |E| \leq 2 / n$.
  - Thus, algorithm contracts an edge in F^* with probability ≤ $2/n$.



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  - Let G' be graph after j iterations. There are $n' = n - j$ super-nodes.
  - Suppose no edge in F^* has been contracted. The min-cut in G' is still k.
  - Thus, algorithm contracts an edge in F^* with probability ≤ 2 / n'.
  - Let E_j = event that no edge in F^* is contracted in iteration j.

$$
\begin{aligned}
\Pr[E_1 \cap E_2 \cdots \cap E_{n-2}] &= \Pr[E_1] \times \Pr[E_2 \mid E_1] \times \cdots \times \Pr[E_{n-2} \mid E_1 \cap E_2 \cdots \cap E_{n-3}] \\
&\geq \left(1 - \tfrac{2}{n}\right)\left(1 - \tfrac{2}{n-1}\right) \cdots \left(1 - \tfrac{2}{4}\right)\left(1 - \tfrac{2}{3}\right) \\
&= \left(\tfrac{n-2}{n}\right)\left(\tfrac{n-3}{n-1}\right) \cdots \left(\tfrac{2}{4}\right)\left(\tfrac{1}{3}\right) \\
&= \frac{2}{n(n-1)} \\
&\geq \frac{2}{n^2}
\end{aligned}
$$

Contraction Algorithm: Amplification

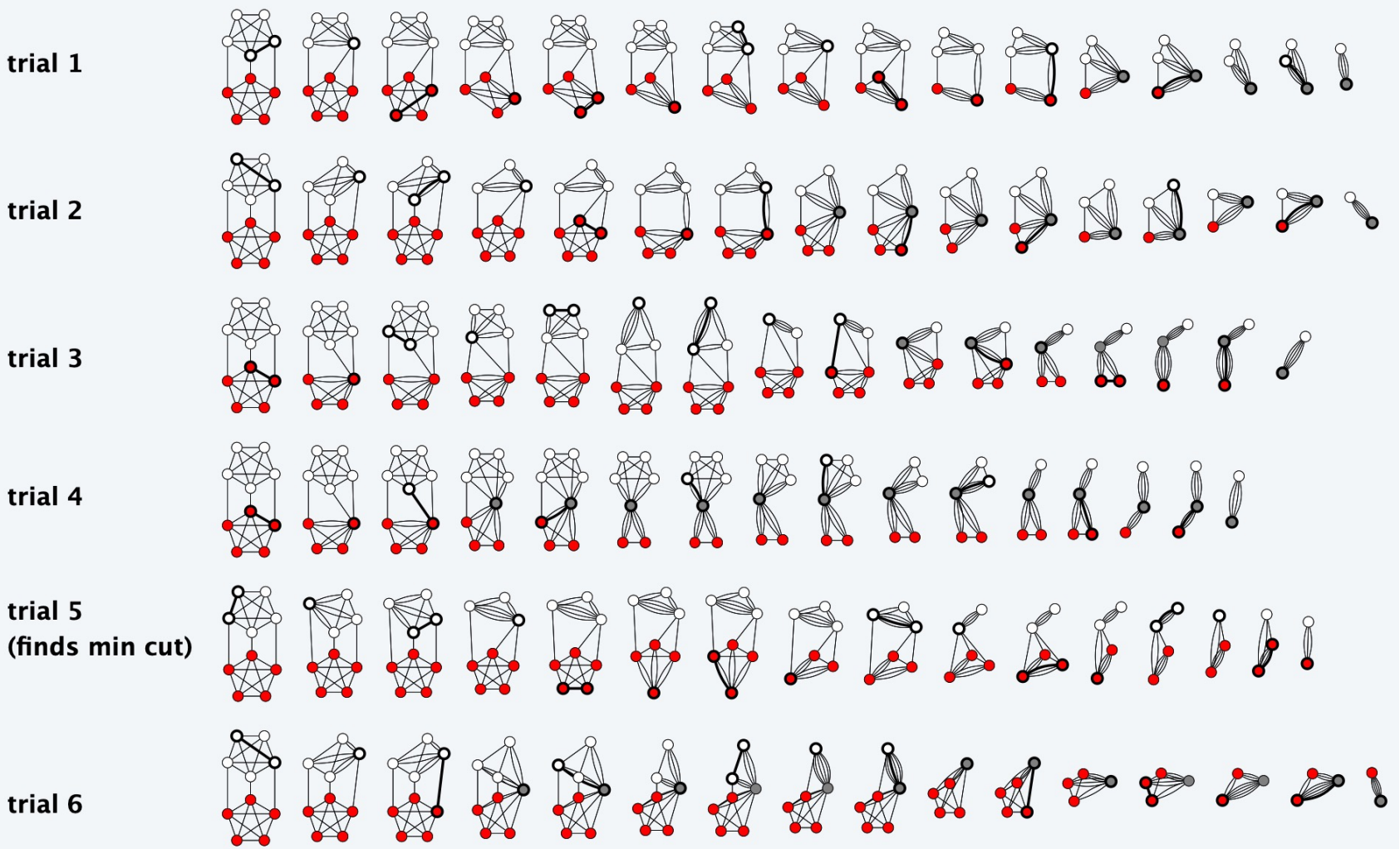
- **Amplification.** To amplify the probability of success, run the contraction algorithm many times with independent randomness.

- **Claim.** If we repeat the contraction algorithm $n^2 \ln n$ times, then the probability of failing to find the global min-cut is $\leq 1 / n^2$.
- **Pf.** By independence, the probability of failure is at most

$$\left(1-\frac{2}{n^2}\right)^{n^2 \ln n} = \left[\left(1-\frac{2}{n^2}\right)^{\frac{1}{2}n^2}\right]^{2\ln n} \leq \left(e^{-1}\right)^{2\ln n} = \frac{1}{n^2}$$

$(1 - 1/x)^x \leq 1/e$

Contraction Algorithm: Demo



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More on Global Minimum Cut

- **Remark.** Overall running time $\Theta(n^2 m \log n)$ is slow since we perform $\Theta(n^2 \log n)$ iterations and each takes $\Omega(m)$ time.
- **Improvement.** [Karger-Stein 1996] $O(n^2 \log^3 n)$.
  - Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n / \sqrt{2}$ nodes remain.
  - Run contraction algorithm until $n / \sqrt{2}$ nodes remain.
  - Run contraction algorithm twice on resulting graph and return best of two cuts.

- **Extensions.** Naturally generalizes to handle positive weights.
- **Best known.** [Karger 2000] $O(m \log^3 n)$.

faster than best known max flow algorithm
or deterministic global min cut algorithm

3. MAX 3-SAT

Maximum 3-Satisfiability

- **MAX 3-SAT.** Given a 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$
\begin{aligned}
C_1 &= x_2 \vee \overline{x_3} \vee \overline{x_4} \\
C_2 &= x_2 \vee x_3 \vee \overline{x_4} \\
C_3 &= \overline{x_1} \vee x_2 \vee x_4 \\
C_4 &= \overline{x_1} \vee \overline{x_2} \vee x_3 \\
C_5 &= x_1 \vee \overline{x_2} \vee \overline{x_4}
\end{aligned}
$$

- **Remark.** **NP**-hard optimization problem.
- **Simple idea.** Flip a coin, and set each variable true with probability ½, independently for each variable.

Maximum 3-Satisfiability: Analysis

- **Claim.** Given a 3-SAT formula with k clauses, the expected number of clauses satisfied by a random assignment is $7k / 8$.

- **Pf.** Consider random variable $Z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

  - Let Z = number of clauses satisfied by random assignment.

$$\begin{aligned} E[Z] &= \sum_{j=1}^{k} E[Z_j] \\ &= \sum_{j=1}^{k} \Pr[\text{clause } C_j \text{ is satisfied}] \\ &= \tfrac{7}{8} k \end{aligned}$$

linearity of expectation

disjunction of 3 literals
each literal corresponds to a different variable

The Probabilistic Method

- **Claim.** Given a 3-SAT formula with k clauses, the expected number of clauses satisfied by a random assignment is $7k / 8$.

- **Corollary.** For any instance of 3-SAT, there exists a truth assignment that satisfies at least a 7/8 fraction of all clauses.
- **Pf.** Random variable is at least its expectation some of the time. ▪

- **Probabilistic method.** [Paul Erdös] Prove the existence of a non-obvious property by showing that a random construction produces it with positive probability!

Maximum 3-Satisfiability: Analysis

- **Q.** Can we turn this idea into a 7/8-approximation algorithm?
- **A.** Yes (but a random variable can almost always be below its mean).
- **Lemma.** The probability that a random assignment satisfies $\geq 7k / 8$ clauses is at least $1/(8k)$.
- **Pf.** Let p_j be probability that exactly j clauses are satisfied; let p be probability that $\geq 7k / 8$ clauses are satisfied.

$$\frac{7}{8}k = E[Z] = \sum_{j \geq 0} j\, p_j = \sum_{j < 7k/8} j\, p_j + \sum_{j \geq 7k/8} j\, p_j$$

$$\leq \left(\frac{7k}{8} - \frac{1}{8}\right) \sum_{j < 7k/8} p_j + k \sum_{j \geq 7k/8} p_j \leq \left(\frac{7}{8}k - \frac{1}{8}\right) \cdot 1 + k\, p$$

- Rearranging terms yields $p \geq 1 / (8k)$. ▪

Maximum 3-Satisfiability: Analysis

- **Johnson's algorithm.** Repeatedly generate random truth assignments until one of them satisfies ≥ 7k/8 clauses.

- **Theorem.** Johnson's algorithm is a 7/8-approximation algorithm.
- **Pf.**
  - By previous Lemma, each iteration succeeds with probability at least p = 1/(8k).
  - The expected number of trials to find the satisfying assignment is $\sum_{j=0}^{\infty} j \Pr[j \text{ trials}] \leq \sum_{j=0}^{\infty} (1-p)^{j-1} p = \frac{1}{p} \geq 8k$ ▪

- **Takeaway.** **NP**-hard problems may have good approximation algorithms.

Maximum Satisfiability

- **Extensions.**
  - Allow one, two, or more literals per clause.
  - Find max weighted set of satisfied clauses.
- **Theorem.** [Asano–Williamson 2000] There exists a 0.784-approximation algorithm for MAX-SAT.
- **Theorem.** [Karloff–Zwick 1997, Zwick+computer 2002] There exists a 7/8-approximation algorithm for version of MAX 3-SAT in which each clause has at most 3 literals.
- **Theorem.** [Håstad 1997] Unless **P** = **NP**, no ρ-approximation algorithm for MAX 3-SAT (and hence MAX SAT) for any ρ > 7/8.

very unlikely to improve over
simple randomized algorithm for MAX 3-SAT

Monte Carlo vs Las Vegas Algorithms

- **Monte Carlo.** Guaranteed to run poly-time, likely to find correct answer.
- **Ex:** Contraction algorithm for global min cut.

- **Las Vegas.** Guaranteed to find correct answer, likely to run in poly-time.
- **Ex:** Randomized quicksort, Johnson's MAX 3-SAT algorithm.

stop algorithm after a certain point

- **Remark.** Can always convert a Las Vegas algorithm into Monte Carlo, but no known method (in general) to convert the other way.

RP and ZPP

- **RP.** [Monte Carlo] Decision problems solvable with one-sided error in poly-time.
- **One-sided error.**
  - If the correct answer is *no*, always return *no*.
  - If the correct answer is *yes*, return *yes* with probability ≥ 1/2.

can decrease probability of false negative to 2^{-100} by 100 independent repetitions

- **ZPP.** [Las Vegas] Decision problems solvable in expected poly-time.

running time can be unbounded, but fast on average

- **Theorem.** **P** ⊆ **ZPP** ⊆ **RP** ⊆ **NP**.
- **Fundamental open questions.** To what extent does randomization help?
  - Does **P** = **ZPP**? Does **ZPP** = **RP**? Does **RP** = **NP**?

4. Load Balancing

Load Balancing

- **Load balancing.** System in which *m* jobs arrive in a stream and need to be processed immediately on *n* identical processors. Find an assignment that balances the workload across processors.

- **Centralized controller.** Assign jobs in round-robin manner. Each processor receives at most $\lceil m / n \rceil$ jobs.

- **Decentralized controller.** Assign jobs to processors uniformly at random. How likely is it that some processor is assigned "too many" jobs?

Chernoff Bounds

- **Setting.**
  - $X_1, \ldots, X_n$: independent 0-1 random variables
  - $X = X_1 + \ldots + X_n$
  - $\mathbf{E}(X) = \mathbf{E}(X_1) + \ldots + \mathbf{E}(X_n)$

- **Theorem.** (above mean) For any $\delta > 0$ and $\mu \geq \mathbf{E}(X)$, we have

$$\Pr[X > (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$$

usually we choose μ = E(X)

- **Theorem.** (below mean) For any $\delta > 0$ and $\mu \leq \mathbf{E}(X)$, , we have

$$\Pr[X < (1-\delta)\mu] < e^{-\delta^2\mu/2}$$

Load Balancing: # Jobs = # Processors

- **Analysis.** (number of jobs m = number of processors n)
  - Let X_i = number of jobs assigned to processor i.
  - Let Y_{ij} = 1 if job j is assigned to processor i, and 0 otherwise.
  - Thus, $X_i = \sum_j Y_{ij}$. We have $\mathbf{E}[Y_{ij}] = 1/n$ and $\mathbf{E}[X_i] = 1$.
  - Chernoff bounds with $\delta = c - \mu > 0$ and $\mu = \mathbf{E}[X_i] = 1 \Rightarrow \Pr[X_i > c] < e^{c-1} / c^c$.
  - Let $\gamma(n)$ be number x such that $x^x = n$, and choose $c = e\ \gamma(n)$.

$$\Pr[X_i > c] < \frac{e^{c-1}}{c^c} < \left(\frac{e}{c}\right)^c = \left(\frac{1}{\gamma(n)}\right)^{e\gamma(n)} \leq \left(\frac{1}{\gamma(n)}\right)^{2\gamma(n)} = \frac{1}{n^2}$$

  - Union bound $\Rightarrow$ with probability $\geq 1 - 1/n$ no processor receives more than $c = e\ \gamma(n) = \Theta(\log n / \log \log n)$ jobs.

    ✓ $x \log x = \log n$ & $\log x + \log \log x = \log \log n \Rightarrow \gamma(n) / 2 \leq \log n / \log \log n \leq \gamma(n)$

Load Balancing: # Jobs > # Processors

- **Theorem.** Suppose the number of jobs $m = 16\, n \ln n$. Then on average, each of the n processors handles $16 \ln n$ jobs. With high probability, every processor will have between half and twice the average load.
- **Pf.**
  - Let X_i = number of jobs assigned to processor i .
  - Applying Chernoff bounds with $\delta = 1$ and $\mu = \mathbf{E}(X_i) = 16 \ln n$ yields

$$\Pr[X_i > 2\mu] < \left(\frac{e}{4}\right)^{16 \ln n} < \left(\frac{1}{e}\right)^{2 \ln n} = \frac{1}{n^2}$$

$$\Pr\left[X_i < \frac{1}{2}\mu\right] < e^{-\frac{1}{2}\left(\frac{1}{2}\right)^2 16 \ln n} = \frac{1}{n^2}$$

  - Union bound ⇒ every processor has load between half and twice the average with probability ≥ $1 - 2/n$. ▪