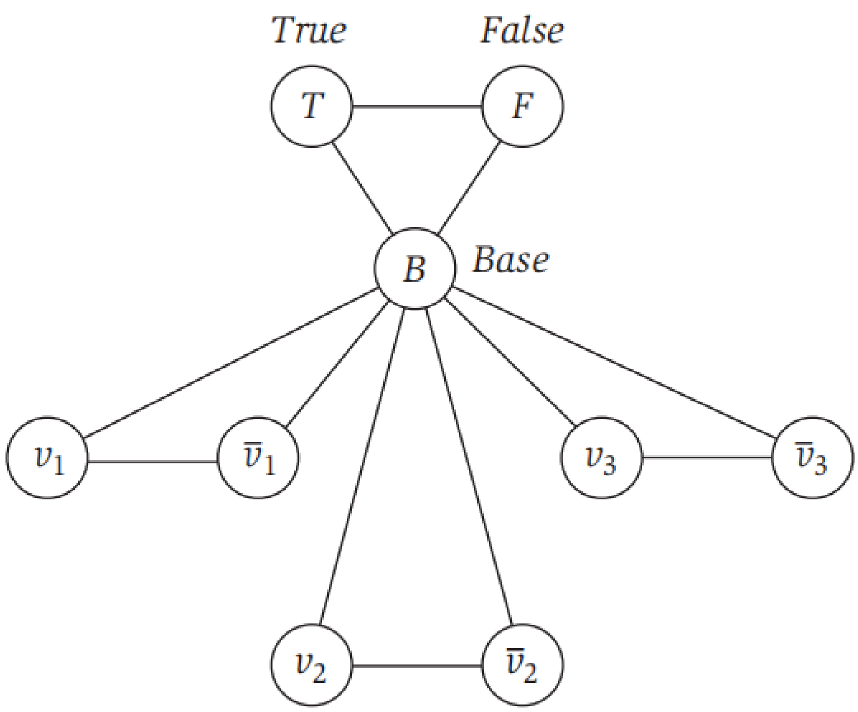
Assignment 5

3-Color Problem

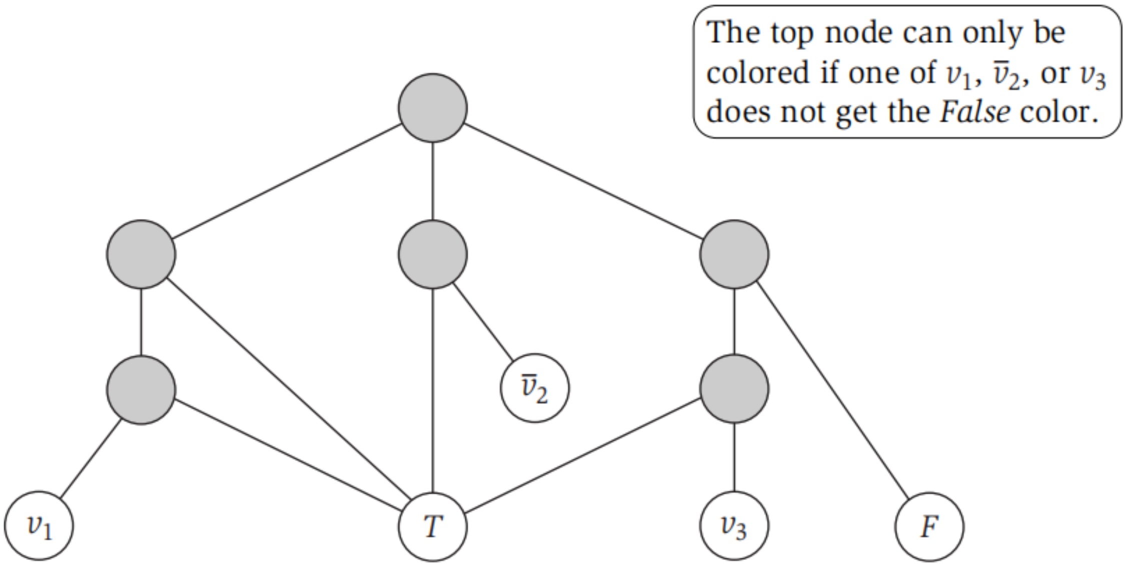
Description: A gragh G we call it has a 3-coloring if there exists a function $f: V \rightarrow \{1,2,3\}$, so that for every edge (u,v), we have $f(u) \neq f(v)$.

Theorem: 3-Color is NP-complete.

Proof: We have learned that 3-SAT is NP-complete in class, so we only need to prove 3-SAT $\leq_p$ 3-Color. In preparation, we should define something to convert the input of 3-SAT to the input of 3-Color. We define variable of 3-SAT as a vertex in G and if there exists an edge between two vertices it means they are adjacent. Then we define there are three kinds of color in G. They are True, False and Base. And we set the value of a variable as True/1 which is equivalent to coloring the node of this variable with color True. Similarly, we can map the input value of each variable in 3-SAT problem to the input color of each node in 3-Color problem. After preparation we need to construct to transfer 3-SAT to 3-Color completely. Here, we construct the gragh like below:



So, according to the goal of 3-Color three nodes of a single triangle should have different colors, which means v_1 and $\overline{v_1}$ should have different colors picked in {True, False}. Then our construction indicates that for any G, an 3-Color on it is equivalent to set True or False value in 3-SAT. We pick a clause in 3-SAT in the form of $x_1 \vee \overline{x_2} \vee x_3$. Then what we only need to do is to convert the output of 3-Color to that of 3-SAT. Here, we construct each clause of 3-SAT as a subgraph like below:



We notice that suppose all of the value of v_1 ,$\overline{v_2}$,v_3 are False, then the lowest level of shadow nodes must be Base. Then the shadow nodes on second level must be False, Base and True, which uses up all of the colors. So, 3-color on above gragh indicates there is at least a True value among v_1 ,$\overline{v_2}$,v_3. Based on these conclusions, we have completed the whole construction. Given 3-SAT, we construct each clause like above and change it to a 3-Color problem and if the graph can be colored by only 3 colors then each clause must be True according to analysis above. So, 3-SAT is satisfied. Conversely, given a 3-SAT, we construct a graph like above, if 3-SAT is satisfied then, each clause can be colored only by three colors, and we can extend these subgraph to the whole graph and consequently the whole gragh can be colored only by three colors.

Therefore, we give a convertion between 3-SAT can 3-Color within polynomial time. And 3-SAT $\leq_p$ 3-Color is valid. Thus, we prove the 3-Color problem is a NP-complete problem.