# Algorithm Design and Analysis (H)

## CS216

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(slides edited from Prof. Shiqi Yu)

# Greedy Algorithms

# 8. Huffman Codes

# Encoding

- **Q.** Why do we need encoding?
- **A.** Encoding transforms data of human language to numbers such that they can be processed by digital computers.
- Ex. Postcode, character codes (Unicode), etc.

- **Q.** Given a text that uses 32 symbols (26 different letters, space, and some punctuation characters), how can we encode this text in bits?
- **A.** We can encode $2^5$ different symbols using a fixed length of 5 bits per symbol. This is called fixed length encoding.

# Fixed Length Encoding

- 64 samples (1 poison)
- How many guinea pigs do we need to find the poison?

| | 🐁 | 🐁 | 🐁 | 🐁 | 🐁 | 🐁 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 1 |
| 4 | 0 | 0 | 0 | 1 | 0 | 0 |
| … | | | | | | |
| 63 | 1 | 1 | 1 | 1 | 1 | 1 |

# Data Compression

- **Q.** Some symbols (e, t, a, o, i, n) are used far more often than others. How can we use this to reduce our encoding?
- **A.** Encode such characters with fewer bits and the others with more bits.

- **Q.** How do we know when the next symbol begins?
- **A.** Use a separation symbol (like the pause in Morse), or make sure that there is no ambiguity by ensuring that no code is a prefix of another one.
- **Ex.** c(a) = 01, c(b) = 010, c(e) = 1.
- Q. What is 0101?

# Prefix Codes

- **Def.** A prefix code for a set S is a function $\gamma$ that maps each x ∈ S to a bit string such that for x, y ∈ S, x ≠ y, $\gamma(x)$ is not a prefix of $\gamma(y)$.

- **Ex.** c(a) = 11, c(e) = 01, c(k) = 001, c(l) = 10, c(u) = 000.
- Q. What is the meaning of 1001000001?
- A. "leuk"
- **Ex.** Frequencies in 1G text: $f_a = 0.4$, $f_e = 0.2$, $f_k = 0.2$, $f_l = 0.1$, $f_u = 0.1$.
- Q. What is the size of the encoded text?
- A. $2f_a + 2f_e + 3f_k + 2f_l + 3f_u = 2.3G$

# Optimal Prefix Codes

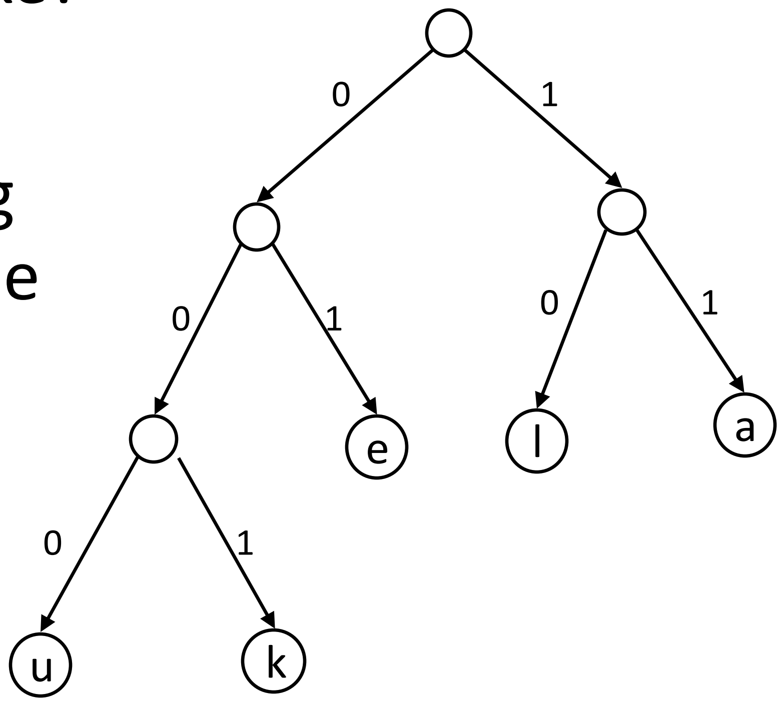
- **Def.** The average bits per letter (ABL) of a prefix code $\gamma$ is the sum, over all symbols $x \in S$, of its frequency $f_x$ times the number of bits of its encoding $\gamma$(x):

$$ABL(\gamma) = \sum_{x \in S} f_x |\gamma(x)|$$

- **Q.** Can we construct a prefix code that has the lowest ABL?
- **Observation.** A prefix code can be modeled in a binary tree.
  - ➢ Any binary code can be modeled in a binary tree.

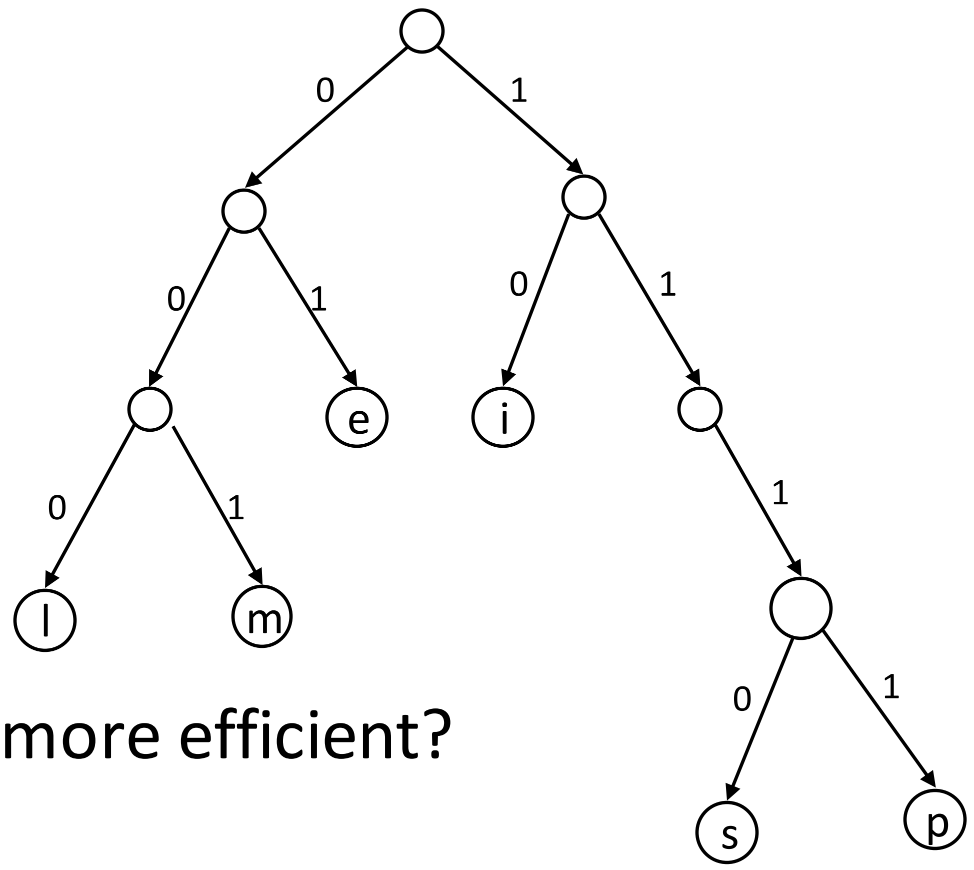
# Representing Prefix Codes with Binary Trees

- **Ex.** c(a) = 11, c(e) = 01, c(k) = 001, c(l) = 10, c(u) = 000.
- **Q.** How does the tree of a prefix code look like?
- **A.** Only the leaves have a label.
- Pf. An encoding of x is a prefix of an encoding of y if and only if the path of x is a prefix of the path of y.



# Representing Prefix Codes with Binary Trees

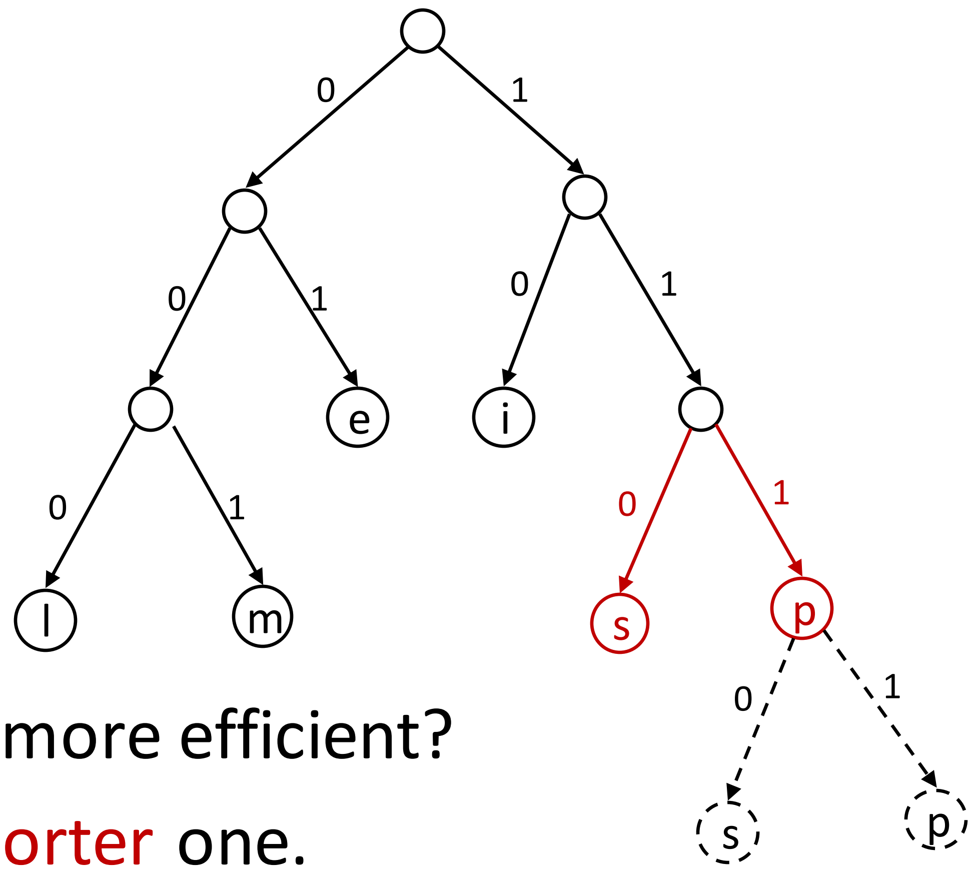
- **Q.** What is the meaning of 111010001111101000?
- **A.** "simpel"




- **Q.** How can this prefix code be made more efficient?

# Representing Prefix Codes with Binary Trees

- **Q.** What is the meaning of 111010001111101000?
- **A.** "simpel"



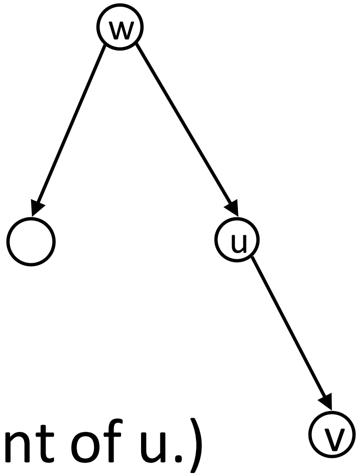

- **Q.** How can this prefix code be made more efficient?
- **A.** Change encoding of s and p to a shorter one.

# Representing Prefix Codes with Binary Trees

- **Def.** A tree is full if every node that is not a leaf has two children.
- **Claim.** The binary tree corresponding to an optimal prefix code is full.
- Pf. (by contradiction)
  - ➢ Suppose T is binary tree of optimal prefix code and is not full.
  - ➢ This means there is a node u with only one child v.
  - ➢ Case 1: u is the root.

    ✓Delete u and use v as the root.
  - ➢ Case 2: u is not the root.

    ✓Delete u and make v be a child of w in place of u. (w is the parent of u.)
  - ➢ In both cases the number of bits needed to encode any leaf in the subtree of v is decreased. The rest of the tree is not affected.
  - ➢ Clearly this new tree T' has a smaller ABL than T. Contradiction!



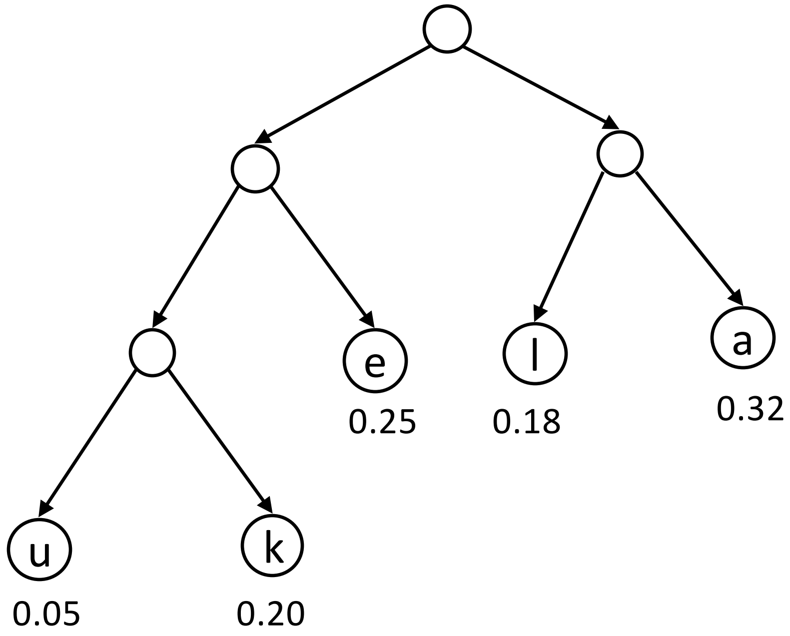
# Optimal Prefix Codes: False Start

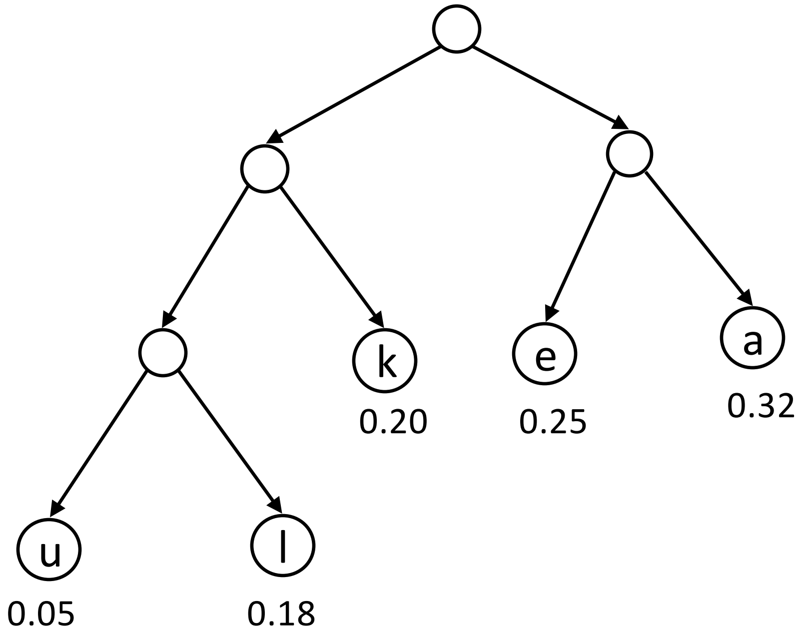
- **Q.** Where in the tree of an optimal prefix code should letters with a high frequency be placed?
- **A.** Near the top.

- **Greedy template.** [Shannon-Fano, 1949] Create tree top-down. Split S into two sets $S_1$ and $S_2$ with (almost) equal frequencies. Recursively build tree for $S_1$ and $S_2$.
- **Q.** Does this approach output an optimal prefix code?

# Optimal Prefix Codes: False Start

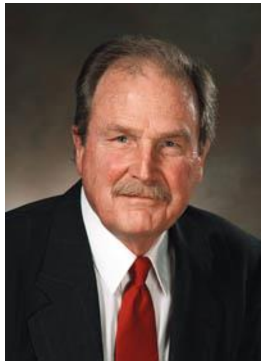
- **Greedy template.** [Shannon-Fano, 1949] Create tree top-down. Split S into two sets $S_1$ and $S_2$ with (almost) equal frequencies. Recursively build tree for $S_1$ and $S_2$.
- **Counterexample:** $f_a = 0.32$, $f_e = 0.25$, $f_k = 0.20$, $f_l = 0.18$, $f_u = 0.05$



# Optimal Prefix Codes: Huffman Encoding

- **Observation.** Lowest frequency items should be at the lowest level in the tree of an optimal prefix code.
- **Observation.** For n > 1, the lowest level always contains at least two leaves.
- **Observation.** The order in which items appear in a level does not matter.
- **Claim.** There is an optimal prefix code with tree T* where the two lowest-frequency letters are assigned to leaves that are siblings in T*.
- **Greedy template.** [Huffman, 1952] Create tree bottom-up. Make two leaves for two lowest-frequency letters y and z. Recursively build tree for the rest using a meta-letter for y, z.



# Optimal Prefix Codes: Huffman Encoding

```
Huffman(S) {
   if |S| == 2 {
      return tree with root and 2 leaves
   } else {
      let y and z be lowest-frequency letters in S
      let S′ be S with y and z removed
      insert new letter ω in S′ with fω = fy + fz
      T′= Huffman(S′)
      T = add two children y and z to leaf ω from T′
      return T
   }
}
```

- **Q.** What is the time complexity?
- **A.** $T(n) = T(n - 1) + O(\log n)$, therefore $T(n) = O(n \log n)$.

ExtractMin from priority queue S

# Huffman Encoding: Greedy Analysis

- **Claim.** ABL(T) = ABL(T') + $f_\omega$.
- Pf. Recall that T = T' with two children x, y added to leaf ω.

$$
\begin{aligned}
\text{ABL}(T) &= \sum_{x \in S} f_x \cdot \text{depth}_T(x) \\
&= f_{y^*} \cdot \text{depth}_T(y^*) + f_{z^*} \cdot \text{depth}_T(z^*) + \sum_{x \neq y^*, z^*} f_x \cdot \text{depth}_T(x) \\
&= (f_{y^*} + f_{z^*}) \cdot (1 + \text{depth}_{T'}(\omega)) + \sum_{x \neq y^*, z^*} f_x \cdot \text{depth}_{T'}(x) \\
&= f_\omega \cdot (1 + \text{depth}_{T'}(\omega)) + \sum_{x \neq y^*, z^*} f_x \cdot \text{depth}_{T'}(x) \\
&= f_\omega + f_\omega \cdot \text{depth}_{T'}(\omega) + \sum_{x \neq y^*, z^*} f_x \cdot \text{depth}_{T'}(x) \\
&= f_\omega + \sum_{x \in S'} f_x \cdot \text{depth}_{T'}(x) \\
&= f_\omega + \text{ABL}(T'). \quad \blacksquare
\end{aligned}
$$

# Huffman Encoding: Greedy Analysis

- **Claim.** Huffman code for S achieves the minimum ABL of any prefix code.
- Pf. (by induction over n = |S|)
  - **Basis Step:** For n = 2 there is no shorter code than root and two leaves.
  - **Inductive Hypothesis (IH):** Suppose Huffman tree T' for S' of size n – 1, with leaf ω instead of the lowest-frequency nodes y and z, is optimal.
  - **Inductive Step:** (by contradiction) [proof idea]
    - ✓Suppose some other tree Z of size n is better.
    - ✓Delete lowest frequency items y and z from Z creating Z'.
    - ✓Z' cannot be better than T' by IH.

# Huffman Encoding: Greedy Analysis

- **Claim.** Huffman code for S achieves the minimum ABL of any prefix code.
- Pf. (by induction over n = |S|)
  - **Basis Step:** For n = 2 there is no shorter code than root and two leaves.
  - **Inductive Hypothesis (IH):** Suppose Huffman tree T' for S' of size n – 1, with leaf ω instead of the lowest-frequency nodes y and z, is optimal.
  - **Inductive Step:** (by contradiction)
    - ✓Let T be the Huffman tree. Suppose there is tree Z such that ABL(Z) < ABL(T).
    - ✓Then w.l.o.g. we assume Z has leaves y and z as siblings (see Observation).
    - ✓Let Z' be Z with y and z deleted and their former parent labeled ω'.
    - ✓We know that ABL(Z) = ABL(Z') + $f_{\omega'}$, as well as ABL(T) = ABL(T') + $f_{\omega}$.
    - ✓Recall ABL(Z) < ABL(T) and $f_{\omega'} = f_{\omega}$, so ABL(Z') < ABL(T'). Contradiction with IH!

# Huffman Encoding: Summary

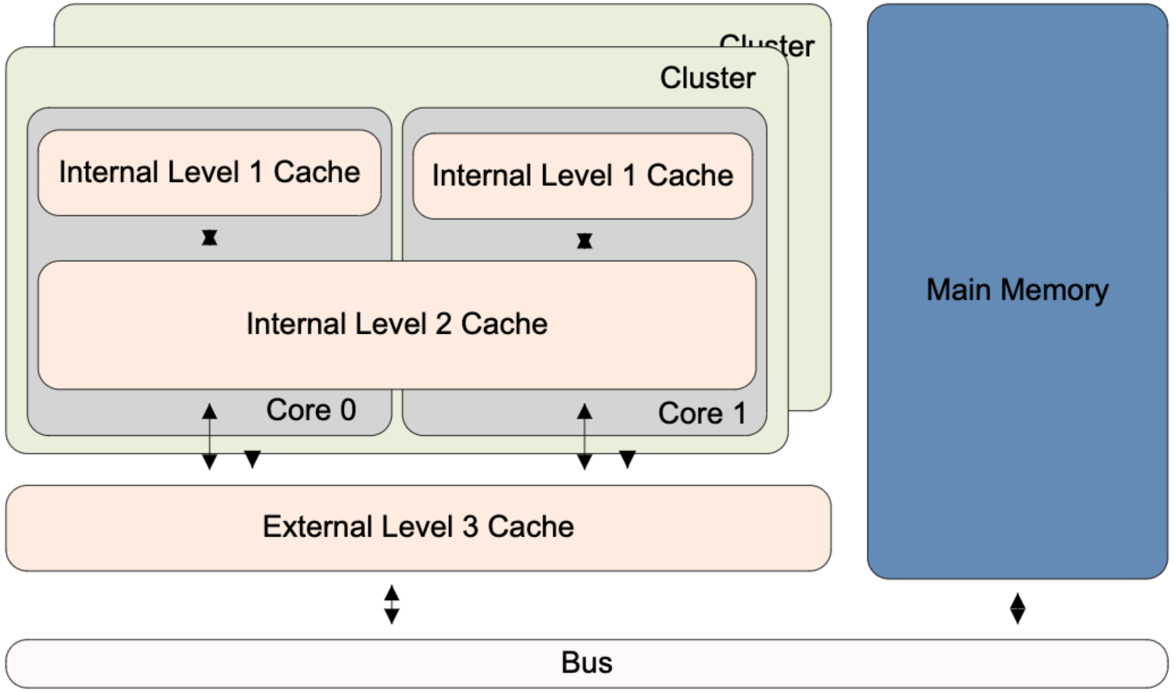
- **Greedy approach:** Shrink the size of the problem instance, so that an equivalent smaller problem can then be solved by recursion.
  - ➢ The greedy operation is proved to be "safe": solving the smaller problem still leads to an optimal solution for the original problem.

- **Application.** ZIP file format that supports lossless data compression.
  - ➢ Its most common compression algorithm DEFLAME uses LZ77 and Huffman.
  - ➢ Document: https://pkware.cachefly.net/webdocs/APPNOTE/APPNOTE-6.2.0.txt

# 9. Optimal Caching

# Caching

- HD->memory->cache




| Memory type | Typical size | Typical access time |
|---|---|---|
| Processor registers | 128KB | 1 cycle |
| On-chip L1 cache | 32KB | 1-2 cycle(s) |
| On-chip L2 cache | 128KB | 8 cycles |
| Main memory (L3) dynamic RAM | MB or GB[1] | 30-42 cycles |
| Back-up memory (hard disk) (L4) | MB or GB | > 500 cycles |

[1] Size limited by the processor core addressing, for example a 32-bit processor without memory management can directly address 4GB of memory.

https://developer.arm.com/documentation/den0024/a/Caches

# Optimal Offline Caching

- **Offline caching.**
  - Cache with capacity to store *k* items.
  - Sequence of m item requests $d_1, d_2, \ldots, d_m$. ← offline: known sequence
  - Cache hit: item already in cache when requested.
  - Cache miss: item not already in cache when requested.
    - ✓Must bring requested item into cache, and evict some existing item if full.
- **Goal.** Eviction schedule that minimizes number of evictions.

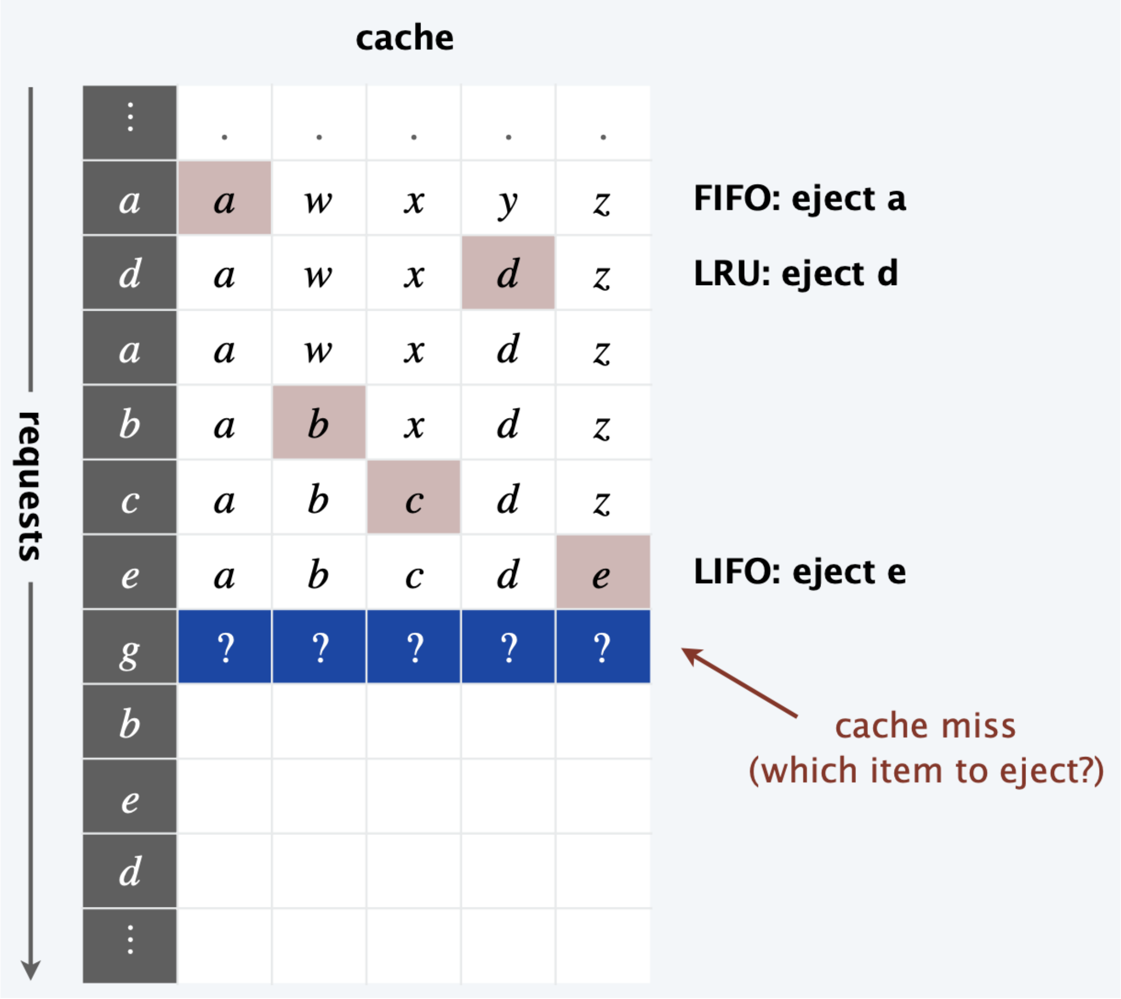
- **Ex:** $k = 2$, initial cache = ab, requests: a, b, c, b, c, a, a, b.
- Optimal eviction schedule: 2 cache misses.

# Optimal Offline Caching

- **Greedy templates.**
  - LIFO. (last-in-first-out) Evict item brought in least recently.
  - FIFO. (first-in-first-out) Evict item brought in most recently.
  - LRU. (least-recently-used) Evict item whose most recent access was earliest.
  - LFU. (least-frequently-used) Evict item that was least frequently requested.

# Optimal Offline Caching

- **Greedy templates.**
  - LIFO. (last-in-first-out)
  - FIFO. (first-in-first-out)
  - LRU. (least-recently-used)
  - LFU. (least-frequently-used)



# Optimal Offline Caching: Farthest-In-Future

- **Farthest-in-future (FF).** Evict item in the cache that is not requested until farthest in the future.

current cache: | a | b | c | d | e | f |

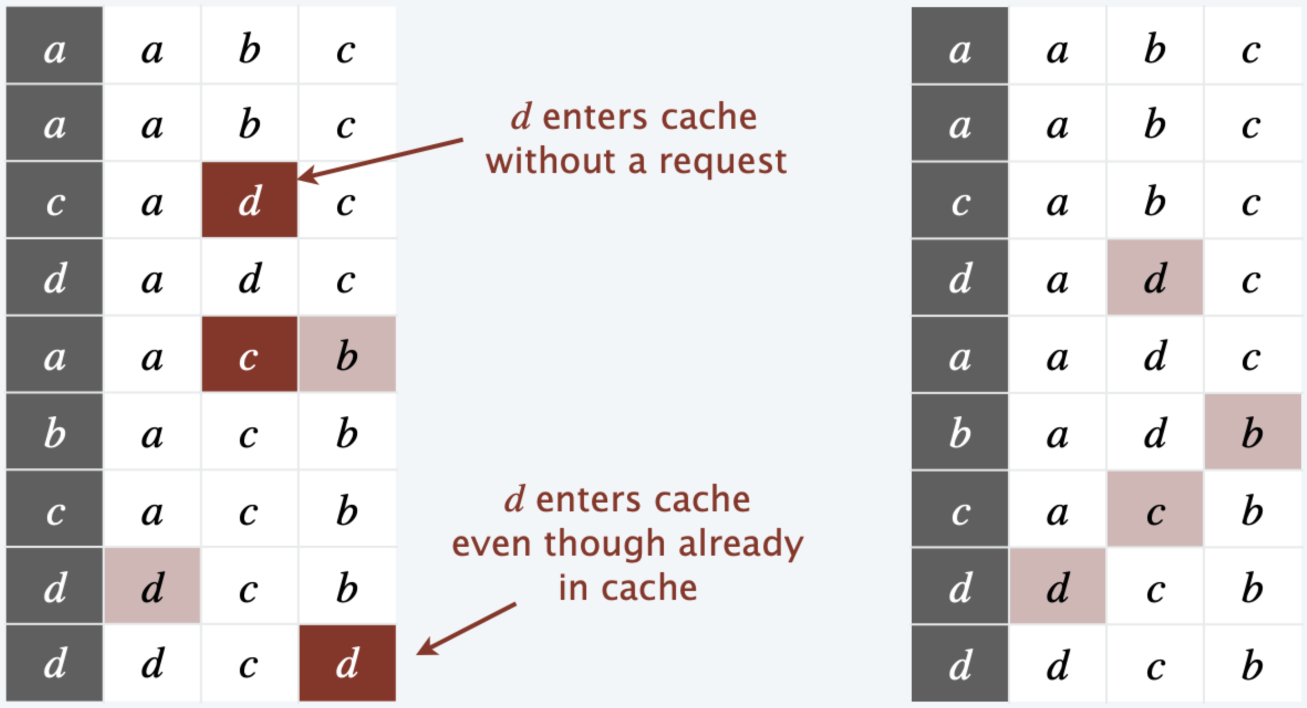
future queries: `g a b c e d a b b a c d e a f a d e f g h ...`

cache miss (g) — eject this one (f)

- **Theorem.** [Bellady, 1960s] FF is optimal eviction schedule.
- Algorithm and theorem are intuitive; proof is subtle (shown later).

# Reduced Eviction Schedules

- **Def.** A reduced schedule is a schedule that only inserts an item into the cache in a step when that item is requested and not yet in cache.




**an unreduced schedule**

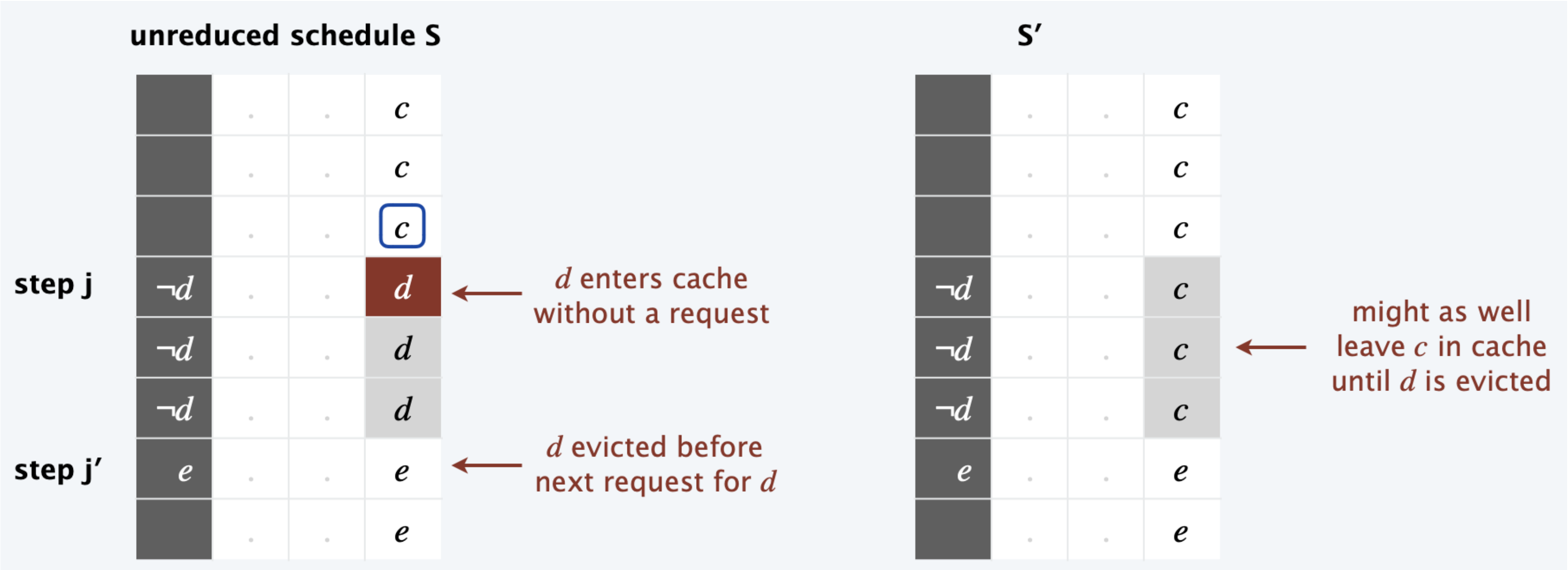
**a reduced schedule**

# Reduced Eviction Schedules

- **Claim.** Given any unreduced schedule S, can transform it into a reduced schedule S' with no more evictions.
- Pf. (by induction on number of steps *j*) [proof idea]
  - Base case: $j = 0$.
  - Case 1: *S* brings *d* into the cache in step *j* without a request.
  - Case 2: *S* brings *d* into the cache in step *j* even though *d* is in cache.
  - If multiple unreduced items in step *j*, apply each one in turn.
  - Deal with Case 1 before Case 2, as resolving Case 1 might trigger Case 2 ▪

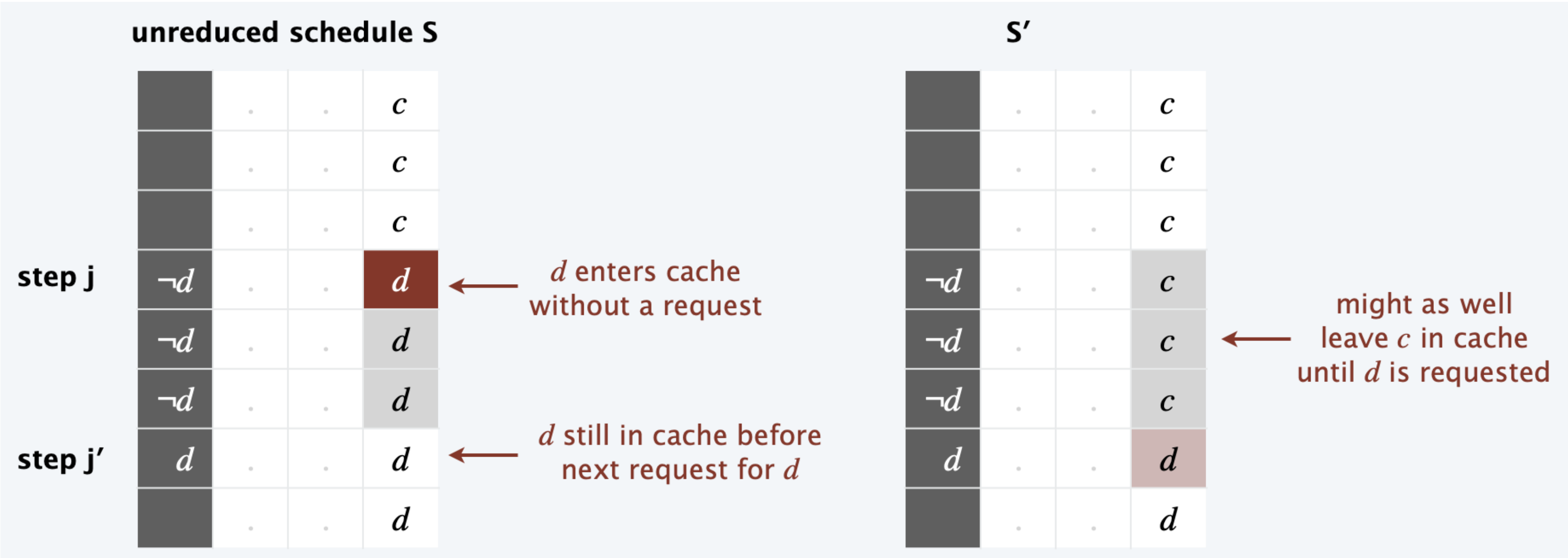
# Reduced Eviction Schedules

- **Claim.** Given any unreduced schedule S, can transform it into a reduced schedule S' with no more evictions.
- Pf. (by induction on number of steps *j*) [*d* is inserted in step *j*]
  - Let *c* be the item *S* evicts when it brings *d* into the cache.
  - Case 1a: *d* evicted before next request for *d*.



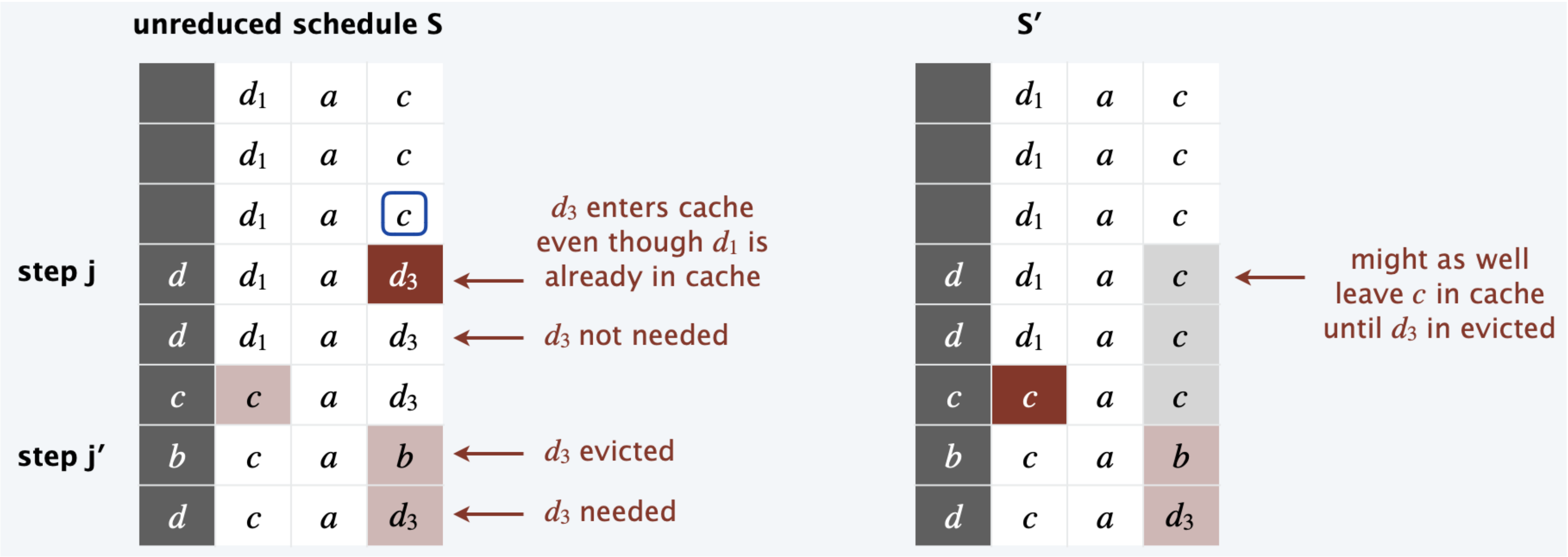
# Reduced Eviction Schedules

- **Claim.** Given any unreduced schedule S, can transform it into a reduced schedule S' with no more evictions.
- Pf. (by induction on number of steps *j*) [*d* is inserted in step *j*]
  - ➢ Let *c* be the item *S* evicts when it brings *d* into the cache.
  - ➢ Case 1b: next request for *d* occurs before *d* is evicted.



# Reduced Eviction Schedules

- **Claim.** Given any unreduced schedule S, can transform it into a reduced schedule S' with no more evictions.
- Pf. (by induction on number of steps *j*) [*d* is inserted in step *j*]
  - ➢ Let *c* be the item *S* evicts when it brings *d* into the cache.
  - ➢ Case 2a: *d* evicted before it is needed.



# Reduced Eviction Schedules

- **Claim.** Given any unreduced schedule S, can transform it into a reduced schedule S' with no more evictions.
- Pf. (by induction on number of steps *j*) [*d* is inserted in step *j*]
  - Let *c* be the item *S* evicts when it brings *d* into the cache.
  - Case 2b: *d* needed before it is evicted.

**unreduced schedule S**

| | | | | | |
|---|---|---|---|---|---|
| | | $d_1$ | $a$ | $c$ | |
| | | $d_1$ | $a$ | $c$ | |
| | | $d_1$ | $a$ | $\boxed{c}$ | |
| step j | $d$ | $d_1$ | $a$ | $d_3$ | ← $d_3$ enters cache even though $d_1$ is already in cache |
| | $d$ | $d_1$ | $a$ | $d_3$ | ← $d_3$ not needed |
| | $c$ | $c$ | $a$ | $d_3$ | |
| | $a$ | $c$ | $a$ | $d_3$ | |
| step j’ | $d$ | $c$ | $a$ | $d_3$ | ← $d_3$ needed |

**S’**

| | | | | |
|---|---|---|---|---|
| | $d_1$ | $a$ | $c$ | |
| | $d_1$ | $a$ | $c$ | |
| | $d_1$ | $a$ | $c$ | |
| $d$ | $d_1$ | $a$ | $c$ | ← might as well leave $c$ in cache until $d_3$ in needed |
| $d$ | $d_1$ | $a$ | $c$ | |
| $c$ | $c$ | $a$ | $c$ | |
| $a$ | $c$ | $a$ | $c$ | |
| $d$ | $c$ | $a$ | $d_3$ | |

# Farthest-In-Future: Greedy Analysis

- **Theorem.** Farthest-in-future (FF) is optimal eviction algorithm.
- Pf. Follows directly from the following invariant.
- **Invariant.** There exists an optimal reduced schedule $S$ that has the same eviction schedule as $S_{FF}$ through the first $j$ steps.
- Pf. (by induction on number of steps $j$)
  - ➢ **Basis Step**: $j = 0$.
  - ➢ **Inductive Step:** Let $S$ be reduced schedule that satisfies invariant through $j$ steps. We produce $S'$ that satisfies invariant after $j + 1$ steps.
    - ✓Let $d$ denote the item requested in step $j + 1$.
    - ✓Since $S$ and $S_{FF}$ have agreed up until now, they have the same cache contents before step $j + 1$.

# Farthest-In-Future: Greedy Analysis

- **Theorem.** Farthest-in-future (FF) is optimal eviction algorithm.
- Pf. Follows directly from the following invariant.
- **Invariant.** There exists an optimal reduced schedule $S$ that has the same eviction schedule as $S_{FF}$ through the first $j$ steps.
- Pf. (by induction on number of steps $j$) [$d$ is requested in step $j$ + 1]
  - **Inductive Step:** Let $S$ be reduced schedule that satisfies invariant through $j$ steps. We produce $S'$ that satisfies invariant after $j$ + 1 steps.
  - Case 1: $d$ is already in the cache.
    ✓$S' = S$ satisfies invariant.
  - Case 2: $d$ is not in the cache and $S$ and $S_{FF}$ evicts the same item.
    ✓$S' = S$ satisfies invariant.

# Farthest-In-Future: Greedy Analysis

- Pf. (continued)
  - Case 3: $d$ is not in the cache; $S_{FF}$ evicts $e$; $S$ evicts $f \neq e$.
    - ✓Begin construction of $S'$ from $S$ by evicting $e$ instead of $f$.

| | S | | | S' | | |
|---|---|---|---|---|---|---|
| j | same | e | f | same | e | f |
| j+1 | same | e | d | same | d | f |

    - ✓Now $S'$ agrees with $S_{FF}$ on first $j + 1$ requests; we show that having $f$ in cache is no worse than having $e$ in cache.

must involve e or f or both

    - ✓Let $S'$ behave the same as $S$ until $S'$ is forced to take a different action, because (Case 3a/3b) *either $e$ or $f$ is requested* or because (Case 3c) *$S$ evicts $e$*.

# Farthest-In-Future:  Greedy Analysis

- Pf.  (continued) [Let $j'$ be the first step after $j + 1$ that $S$ and $S'$ must take different actions, and let $g$ be the item requested at step $j'$.]
  - Case 3: $d$ is not in the cache; $S_{FF}$ evicts $e$; $S$ evicts $f \neq e$; $S'$ evicts $e$.

| j' | same | e | | same | f |
|---|---|---|---|---|---|
| | S | | | S' | |

S' agrees with $S_{FF}$ through step j+1

  - Case 3a:  $g = e$.
    - ✓Can't happen with FF since there must be a request for $f$ before $e$.
  - Case 3b:  $g = f$.
    - ✓Element $f$ is not in cache of $S$, so let $e'$ be the element that $S$ evicts.
    - ✓If $e' = e$, $S'$ accesses $f$ from cache; now $S$ and $S'$ have same cache.
    - ✓If $e' \neq e$, $S'$ evicts $e'$ and brings $e$ into cache; now $S$ and $S'$ have the same cache.
    - ✓Let $S'$ behave exactly like $S$ for remaining requests.

S' is no longer reduced, but can be transformed into a reduced schedule that agrees with $S_{FF}$ through step j+1

# Farthest-In-Future: Greedy Analysis

- Pf. (continued) [Let $j'$ be the first step after $j + 1$ that $S$ and $S'$ must take different actions, and let $g$ be the item requested at step $j'$.]
  - Case 3: $d$ is not in the cache; $S_{FF}$ evicts $e$; $S$ evicts $f \neq e$; $S'$ evicts $e$.

| | | | | |
|---|---|---|---|---|
| j' | same | e | same | f |
| | S | | S' | |

  - Case 3c: $S$ evicts $e$ (and $g \neq e, f$).
    - ✓Make $S'$ evict $f$ .

| | | | | |
|---|---|---|---|---|
| j' | same | g | same | g |
| | S | | S' | |

    - ✓Now $S$ and $S'$ have the same cache.
    - ✓Let $S'$ behave exactly like $S$ for the remaining requests. ▪

# Optimal Caching: Summary

- **Online vs. offline caching.**
  - Offline: full sequence of requests is known a priori.
  - Online (reality): requests are not known in advance.
  - Caching is among most fundamental online problems in CS.
- **LRU.** Evict item whose most recent access was earliest.

FF with direction of time reversed!

- **Theorem.** FF is optimal offline eviction algorithm.
  - Provides basis for understanding and analyzing online algorithms.
  - LIFO can be arbitrarily bad.
  - LRU is k-competitive, i.e., for any sequence of requests $R$, LRU($R$) ≤ $k$ FF(R) + $k$.
  - Randomized marking is $O(\log k)$-competitive.

shown later in Section 13.8