Algorithm Design and Analysis (H)

CS216

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(slides edited from Prof. Shiqi Yu)

About Myself

Shan CHEN (陈杉)

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- **Homepage:** https://shanchencrypto.github.io/
- **Research interests:** cryptography and information security
  - ➢ Current focus: applied cryptography

About the Course

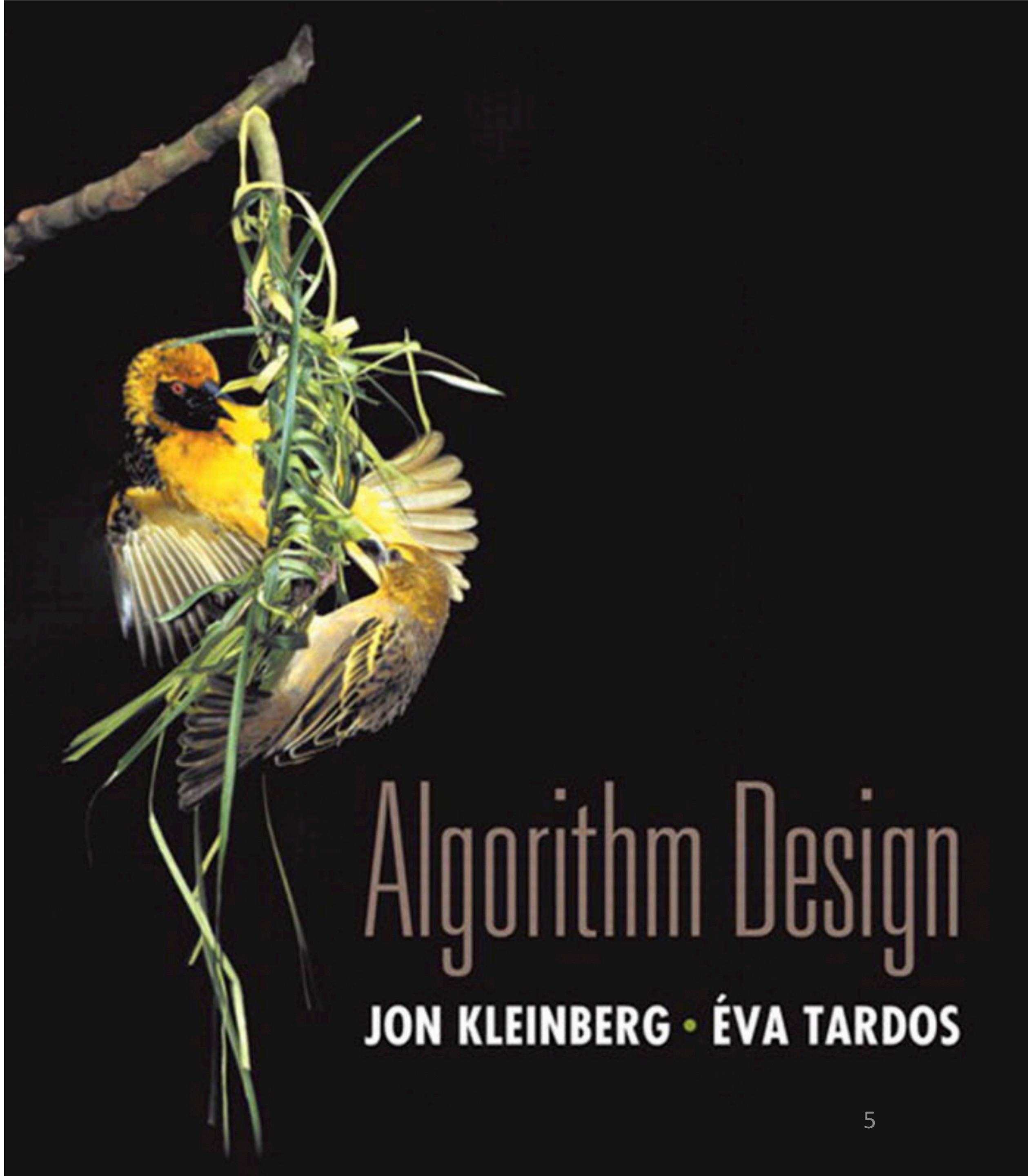
Grading Scheme and Teaching Assistants

- Grading Scheme:
  - ➢ **Assignments:** 20%
  - ➢ **Labs:** 40%
  - ➢ **Final exam:** 40%

- Teaching Assistants:
  - ➢ **Assignments:** Ran ZHANG (张然) 12011511
  - ➢ **Labs:** Yiqian HUANG (黄弋骞) 12012911 Boao LI (李博翱) 12011407

Resources

- **Textbook:** Algorithm Design
- **Sakai:** CS 216 1 spring2023
- **QQ group (Q&A):** 669358781



About Plagiarism

- **Assignment 0** (deadline **Feb 28th**)
  - ➢ Please fill out the Undergraduate Students Declaration Form, submit it on Sakai with your handwritten signature. Otherwise, your course grade will be 0.

南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

计算机科学与工程系
Department of Computer Science and Engineering

本科生作业承诺书

本人__________（学号____________）本学期已选修计算机科学与工程系__________课程。本人已阅读并了解《南方科技大学计算机科学与工程系本科生作业抄袭学术不端行为的认定标准及处理办法》制度中关于禁止本科生作业抄袭的相关规定，并承诺自觉遵守其规定。

About Plagiarism

- **Plagiarism Policy:**
  - ➢ If an undergraduate assignment is found to be plagiarized, the first time the score of the assignment will be 0.
  - ➢ The second time the score of the course will be 0.
  - ➢ If a student does not sign the Assignment Declaration Form or cheats in the course, including regular assignments, midterms, final exams, etc., in addition to the grade penalty, the student will not be allowed to enroll in the two CS majors through 1+3, and cannot receive any recommendation for postgraduate admission exam exemption and all other academic awards.
  - ➢ As it may be difficult when two assignments are identical or nearly identical who actually wrote it, the policy will apply to BOTH students, unless one confesses having copied without the knowledge of the other.

About Plagiarism

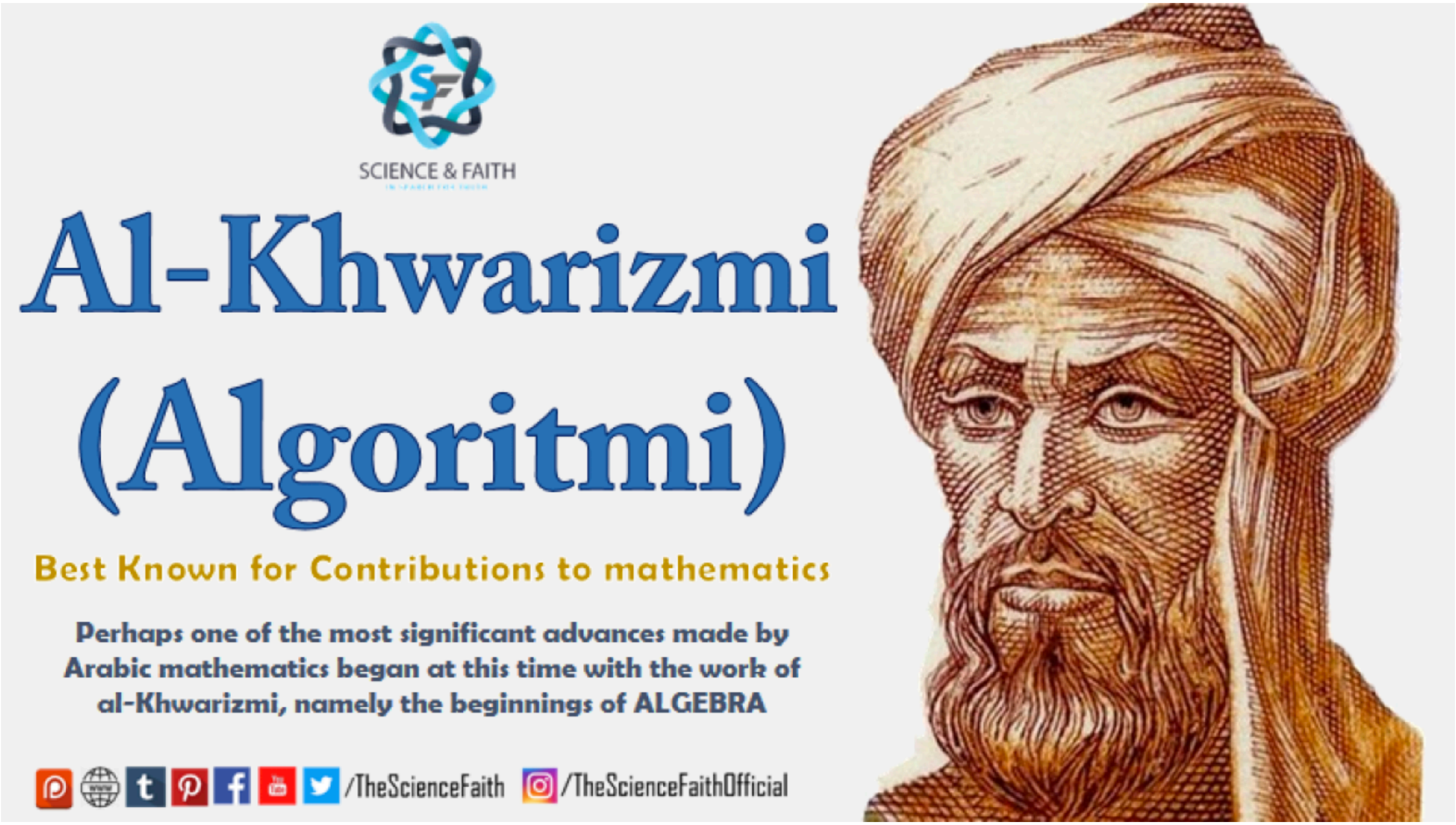
- **What is OK, and What is not OK?**
  - ➢ It's OK to work on an assignment with a friend, and think together about the program structure, share ideas and even the global logic. At the time of actually writing the code, you should write it alone.
  - ➢ It's OK to use in an assignment a piece of code found on the web, as long as you indicate in a comment where it was found and don't claim it as your own work.
  - ➢ It's OK to help friends debug their programs (you'll probably learn a lot yourself by doing so).
  - ➢ It's OK to show your code to friends to explain the logic, as long as the friends write their code on their own later.
  - ➢ It's NOT OK to take the code of a friend, make a few cosmetic changes (comments, some variable names) and pass it as your own work.

Introduction

Some Representative Problems

Algorithms

- An algorithm is a finite sequence of precise instructions for performing a computation or for solving a problem.



A First Problem: Stable Matching

Stable Matching

- David Gale and Lloyd Shapley asked a question in 1962
  - ➢ Could one design a college admissions process, or job recruiting process, that was self-enforcing (stable)?

Formulating the Problem

- A “bare-bones” version:
  - ➢ n applicants
  - ➢ n companies
- Let's look at one-to-one matching first, e.g., marriage:
  - ➢ n men
  - ➢ n women

Some Definitions

- Matching
  - ➢ $M = \{m_1, m_2, \ldots, m_n\}$, $W = \{w_1, w_2, \ldots, w_n\}$
  - ➢ All n^2 possible pairs: $M \times W$
- Perfect matching: everyone is matched monogamously.
  - ➢ Each man gets exactly one woman.
  - ➢ Each woman gets exactly one man.

Some Definitions

- Stability: no incentive for any pair of participants to undermine assignment by joint action.
  - ➢ In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
  - ➢ Unstable pair m-w could each improve by eloping.
- Stable matching: perfect matching with no unstable pairs.

Some Definitions

- Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.
- Example: n=2
  - ➢ Case 1: [m1: w1 > w2; m2: w1 > w2; w1: m1 > m2; w2: m1 > m2]
    - ✓ Unique stable matching: (m1-w1, m2-w2)
    - ✓ The other perfect matching (m1-w2, m2-w1) has an unstable pair (m1, w1)
  - ➢ Case 2: [m1: w1 > w2; m2: w2 > w1; w1: m2 > m1; w2: m1 > m2]
    - ✓ Both perfect matchings are stable:
      - (m1-w1, m2-w2): both men are happy
      - (m1-w2, m2-w1): both women are happy

Questions

- Do stable matchings always exist?
- How can we find a stable matching?

The Gale-Shapley Algorithm

- The Gale-Shapley algorithm [Gale-Shapley 1962]. Intuitive method that guarantees to find a stable matching.
  - ➢ Also known as the propose-and-reject / delayed acceptance algorithm

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Proof of Correctness: Termination

- Observation 1. Men propose to women in decreasing order of preference.
- Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

- Claim. Algorithm terminates after at most n^2 iterations of while loop.
- Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪

Proof of Correctness: Perfection

- Claim. All men and women get matched.
- Pf. (by contradiction)
  - ➢ Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
  - ➢ Then some woman, say Amy, is not matched upon termination.
  - ➢ By Observation 2, Amy was never proposed to.
  - ➢ But Zeus proposes to everyone, since he ends up unmatched. ▪

Proof of Correctness: Stability

- Claim. No unstable pairs.
- Pf. (by contradiction)
  - ➢ Suppose Z-A is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.
  - ➢ Case 1: Z never proposed to A.
    - ✓ Z prefers B to A.
    - ✓ Z-A is stable.
  - ➢ Case 2: Z proposed to A.
    - ✓ A rejected Z (right away or later)
    - ✓ A prefers Y to Z.
    - ✓ Z-A is stable.
  - ➢ In either case Z-A is stable, a contradiction. ▪

men propose in decreasing order of preference

women only trade up

S*
Zeus-Bertha
Yancey-**Amy**
. . .

Proof Summary and Questions

- Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.
- Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

- Q. How to implement GS algorithm efficiently?
- Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

- Efficient implementation. We describe $O(n^2)$ time implementation.
- Representing men and women.
  - ➢ Assume men and women are each named 1, …, n.
- Engagements.
  - ➢ Maintain a list of free men, e.g., in a queue.
  - ➢ Maintain two arrays `wife[m]`, and `husband[w]`.
    - ✓ set entry to `0` if unmatched
    - ✓ if m matches w then `wife[m]=w` and `husband[w]=m`
- Men proposing.
  - ➢ For each man, maintain a list of women, ordered by preference.
  - ➢ Maintain an array `count[m]` that counts the number of proposals made by man `m`.

Efficient Implementation

- Women rejecting/accepting.
  - ➢ How do we judge if woman w prefers man m to man m' efficiently?
  - ➢ For each woman, create an inverse mapping from men to preference orders.
  - ➢ Constant time access for each query after O(n) preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man 3 to 6

since `inverse[3]` < `inverse[6]`

2 7

Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?
- An instance with two stable matchings.
  - ➢ X-A, Y-B, Z-C.
  - ➢ Y-A, X-B, Z-C.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?
- Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.
- Def. Man-optimal assignment: each man receives best valid partner.
- Claim. All executions of GS yield man-optimal assignment, which is a stable matching! Surprising...
  - ➢ No reason to believe that man-optimal assignment is perfect, let alone stable.
  - ➢ Simultaneously best for every man.

Man Optimality

- Claim. GS matching S* is man-optimal.
- Pf. (by contradiction)
  - ➢ Suppose S* is not man-optimal, i.e., some man is not paired with his best valid partner. Since men proposed in decreasing order of preference, some man is rejected by his valid partner.
  - ➢ Let Y be first such man, and let A be first valid partner of Y that rejects him.
  - ➢ When Y is rejected, A forms (or reaffirms) engagement with a man, say Z, whom she prefers to Y. We know Z was not rejected by any valid partner at this point, so Z prefers A to any other valid partners.
  - ➢ Let S be a stable matching where Y and A are matched. Let B be Z's valid partner in S. From above, Z prefers A to B.
  - ➢ And A prefers Z to Y, so Z-A is unstable in S, a contradiction! ▪

S
Yancey-Amy
Zeus-Bertha
. . .

Woman Pessimality

- **Q.** Does man-optimality come at the expense of the women?
- Def. Woman-pessimal assignment: each woman receives worst valid partner.

- Claim. GS finds woman-pessimal stable matching S*.
- Pf. (by contradiction)
  - ➢ Suppose Z-A matched in S*, but Z is not the worst valid partner for A.
  - ➢ There exists a stable matching S in which A is paired with a man, say Y, whom she likes less than Z. Let B be Z's valid partner in S.
  - ➢ Man-optimality: Z prefers A to B.
  - ➢ And A prefers Z to Y, so Z-A is unstable in S, a contradiction! ▪

S
Yancey-Amy
Zeus-Bertha
. . .

Stable Matching Summary

- **Stable matching problem.** Given preference profiles of n men and n women, find a stable matching.

no man and woman both prefer to be with each other than assigned partner

- **Gale-Shapley algorithm.** Finds a stable matching in $O(n^2)$ time.

- **Man optimality.** In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exists some stable matching where m and w are matched

- **Woman pessimality.** In version of GS where men propose, each woman receives worst valid partner.

Extension: Matching Residents to Hospitals

- Extension: Men ≈ hospitals, Women ≈ medical residents.
  - ➢ Variant 1. Some participants declare others as unacceptable.
  - ➢ Variant 2. Unequal number of hospitals and residents.
  - ➢ Variant 3. Limited polygamy.

resident A unwilling to work in Cleveland

hospital X wants to hire 3 residents

- Def. A matching S is unstable if there exist hospital h and resident r such that all the following holds:
  - ➢ h and r are acceptable to each other;
  - ➢ either h does not have all its places filled, or h prefers r to at least one of its assigned residents;
  - ➢ either r is unmatched, or r prefers h to her assigned hospital.
- GS algorithm can be adapted to find stable matchings in this setting.

Men/Women ≠ Hospitals/Residents

- Men/Women: one-to-one matching
- Hospitals/Residents: one-to-many matching
- For around 20 years, most people thought these problems had very similar properties.
  - ➢ [Roth 1982] Any algorithm (e.g. GS) that yields a man-optimal stable matching implies that truth telling is the dominant strategy for men.
  - ➢ [Roth 1985] None of the algorithms for hospitals/residents implies that truth-telling is the dominant strategy for hospitals.

Real-World Application: NRMP

- NRMP. (National Resident Matching Program)
  - ➢ The algorithm is an extension to GS, but was in practical use before GS!
  - ➢ Original use in 1950s, just after WWII. ← predates computer usage
  - ➢ Initial version does not handle couples and other special cases.
  - ➢ The full algorithm was adopted and used since late 1990s.
- Rural hospital dilemma:
  - ➢ Certain hospitals (mainly in rural areas) are unpopular and declared unacceptable by many residents.
  - ➢ How can we find stable matchings that benefit "rural hospitals"?
- Rural Hospital Theorem [Roth 1986]. Rural hospitals get exactly same residents in every stable matching!

More on Stable Matching...

- If you are interested, please find how stable matching is extended to non-bipartite graphs:
  - ➢ Stable roommates problem (for which a stable matching may not exist!)
  - ➢ Irving's algorithm

Lessons Learned

- Powerful ideas learned in course:
  - ➢ Formulate a clean and simple statement of the problem.
  - ➢ Propose a clean and efficient algorithm.
  - ➢ Prove its correctness and bound the time it takes to terminate.

- Potentially deep social impacts. [legal disclaimer]