Assignment Week8

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4.14 As $\mathcal{F}^{-1}\{(1+j\omega)X(j\omega)\} = Ae^{-2t}u(t)$, $\mathcal{F}\{Ae^{-2t}u(t)\} = \frac{A}{2+j\omega} = (1+j\omega)X(j\omega)$

$\Rightarrow X(j\omega) = \frac{A}{(j\omega+1)(j\omega+2)}$. As $\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega = 2\pi \Rightarrow \int_{-\infty}^{\infty}\frac{A^2}{(1+\omega^2)(\omega^2+4)}d\omega = \frac{A^2}{3}\int_{-\infty}^{\infty}\frac{1}{\omega^2+1} - \frac{1}{\omega^2+4}d\omega$

$$= \frac{A^2}{3}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\tan^2\theta+1}d\tan\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4\tan^2\theta+4}d2\tan\theta\right)$$

$$= \frac{A^2}{3}\times\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\theta\, d\theta = \frac{A^2\pi}{6} = 2\pi$$

$\Rightarrow A^2 = 12$ as $x(t)$ is nonnegative, so $A = 2\sqrt{3}$

$\therefore X(j\omega) = 2\sqrt{3}\left(\frac{1}{j\omega+1} - \frac{1}{j\omega+2}\right)$, $\frac{1}{a+j\omega} \xrightarrow{\mathcal{F}^{-1}} e^{-at}u(t) \Rightarrow x(t) = 2\sqrt{3}e^{-t}u(t) - 2\sqrt{3}e^{-2t}u(t)$

4.25 (a) Observing $x(t)$, we find that, suppose $x'(t) = x(t+1)$, then we get $x'(t)$ is real and even, according to Conjugate Symmetry $X'(j\omega)$ is also real and even, thus $\measuredangle X'(j\omega) = 0$, and according to shifting property $X'(j\omega) = e^{j\omega}X(j\omega)$, so $\measuredangle X(j\omega) = -\omega$

(b) $X(j0) = \int_{-\infty}^{\infty}x(t)dt = \int_{-1}^{0}2\,dt + \int_{0}^{1}(2-t)dt + \int_{1}^{2}t\,dt + \int_{2}^{3}2\,dt$

$$= 2t\Big|_{-1}^{0} + (2t-\tfrac{1}{2}t^2)\Big|_{0}^{1} + \tfrac{1}{2}t^2\Big|_{1}^{2} + 2t\Big|_{2}^{3}$$

$$= 2 + \tfrac{3}{2} + 2 - \tfrac{1}{2} + 6 - 4$$

$$= 7$$

(c) As $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$, so $\int_{-\infty}^{\infty}X(j\omega)d\omega = 2\pi x(0) = 4\pi$

(d) take $Y(j\omega) = \frac{2\sin\omega}{\omega}e^{j2\omega}$

$\Rightarrow \int_{-\infty}^{\infty}X(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega = \int_{-\infty}^{\infty}X(j\omega)Y(j\omega)d\omega = 2\pi[x(t)*y(t)]\Big|_{t=0}$

and we know $x(t)\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases} \xrightarrow{\mathcal{F}} \frac{2\sin\omega T_1}{\omega}$, so $x(t)\begin{cases}1, & |t|<1\\ 0, & |t|>1\end{cases} \xrightarrow{\mathcal{F}} \frac{2\sin\omega}{\omega}$

and by shifting property $y(t) = x(t+2) \xrightarrow{\mathcal{F}} \frac{2\sin\omega}{\omega}e^{j2\omega}$

thus $y(t) = u(t+3) - u(t+1)$, so $\int_{-\infty}^{\infty}X(j\omega)\frac{2\sin\omega}{\omega}e^{j\omega}d\omega = 2\pi[x(t)*y(t)]\Big|_{t=0}$

$$= 2\pi\int_{-\infty}^{\infty}x(\tau)y(-\tau)d\tau$$

$$= 2\pi\int_{1}^{3}x(\tau)d\tau$$

$$= 2\pi\left(\int_{1}^{2}\tau\, d\tau + \int_{2}^{3}2\,d\tau\right)$$

$$= 2\pi\left(\tfrac{1}{2}\tau^2\Big|_{1}^{2} + 2\tau\Big|_{2}^{3}\right)$$

$$= 2\pi(2 - \tfrac{1}{2} + 6 - 4)$$

$$= 7\pi$$

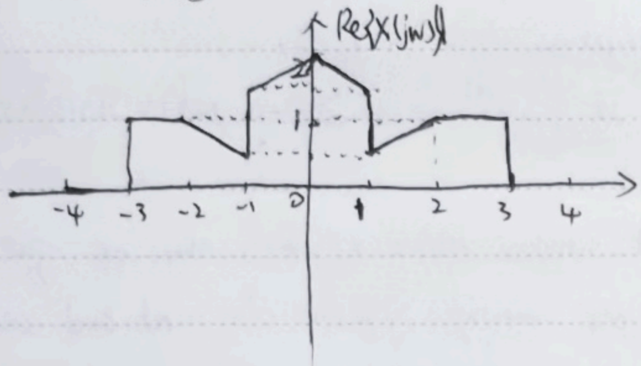
(e) $\int_{-\infty}^{\infty}|X(jw)|^2 dw = 2\pi\int_{-\infty}^{\infty}|x(t)|^2 dt = 2\pi\left(\int_{-1}^{0} 4dt + \int_0^1 (2-t)^2 dt + \int_1^2 t^2 dt + \int_2^3 4dt\right)$

$= 2\pi\left(4t\Big|_{-1}^{0} + (\frac{1}{3}t^3 - 2t^2 + 4t)\Big|_0^1 + \frac{1}{3}t^3\Big|_1^2 + 4t\Big|_2^3\right)$

$= 2\pi\left(4 + \frac{1}{3} - 2 + 4 + \frac{8}{3} - \frac{1}{3} + 12 - 8\right)$

$= \frac{76}{3}\pi$

(f) the inverse Fourier transform of $Re\{X(jw)\}$ is $Ev\{x(t)\} = \frac{x(t)+x(-t)}{2} = \begin{cases} 1, & -3 \le t < -2 \\ -\frac{t}{2}, & -2 \le t < -1 \\ \frac{t}{2}+2, & -1 \le t < 0 \\ -\frac{t}{2}+2, & 0 \le t < 1 \\ \frac{t}{2}, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ 0, & \text{else} \end{cases}$



4.31 (a) it equals to prove $F\{x(t) * h_1(t)\} = F\{x(t) * h_2(t)\} = F\{x(t) * h_3(t)\}$

$F\{x(t)\} = F\{\cos t\} = \pi[\delta(w-1) + \delta(w+1)]$

$F\{h_1(t)\} = F\{u(t)\} = \frac{1}{jw} + \pi\delta(w)$

$F\{h_2(t)\} = F\{-2\delta(t) + 5e^{-2t}u(t)\} = -2 + \frac{5}{2+jw}$

$F\{h_3(t)\} = F\{2te^{-t}u(t)\} = 2\frac{1}{(1+jw)^2}$

And $F\{x(t) * h_1(t)\} = F\{x(t)\}\cdot F\{h_1(t)\} = \pi[\delta(w-1) + \delta(w+1)](\frac{1}{jw} + \pi\delta(w))$

$= \begin{cases} \pi j, & w=-1 \\ -\pi j, & w=1 \\ 0, & \text{else} \end{cases}$

$F\{x(t) * h_2(t)\} = F\{x(t)\}\cdot F\{h_2(t)\} = \pi[\delta(w-1) + \delta(w+1)]\cdot(-2 + \frac{5}{2+jw})$

$= \begin{cases} \pi(-2 + \frac{5}{2-j}), & w=-1 \\ \pi(-2 + \frac{5}{2+j}), & w=1 \\ 0, & \text{else} \end{cases}$

$= \begin{cases} \pi j, & w=-1 \\ -\pi j, & w=1 \\ 0, & \text{else} \end{cases}$

$F\{x(t) * h_3(t)\} = F\{x(t)\}\cdot F\{h_3(t)\} = \pi[\delta(w-1) + \delta(w+1)](\frac{2}{(jw+1)^2})$

$= \begin{cases} \pi\frac{2}{(-j+1)^2}, & w=-1 \\ \pi\frac{2}{(j+1)^2}, & w=1 \\ 0, & \text{else} \end{cases} = \begin{cases} \pi j, & w=-1 \\ -\pi j, & w=1 \\ 0, & \text{else} \end{cases}$

(b) take $h_4(t) = (h_1(t) + h_2(t))/2 = \frac{1}{2}u(t) - \delta(t) + \frac{5}{2}e^{-2t}u(t)$

As $x(t) * h_1(t) = x(t) * h_2(t) = y(t)$, so $x(t) * h_4(t) = x(t) * \left(\frac{h_1(t)}{2} + \frac{h_2(t)}{2}\right)$

$$= \frac{1}{2}x(t) * h_1(t) + \frac{1}{2}x(t) * h_2(t)$$

$$= \frac{1}{2}y(t) + \frac{1}{2}y(t) = y(t)$$

4.33

(a) take Fourier Transform on both sides, we get

$$(j\omega)^2Y(j\omega) + 6(j\omega)Y(j\omega) + 8Y(j\omega) = 2X(j\omega)$$

take $x(t) = \delta(t)$:

$$(j\omega)^2H(j\omega) + 6(j\omega)H(j\omega) + 8H(j\omega) = 2$$

$$\Rightarrow H(j\omega) = \frac{2}{(j\omega)^2 + 6j\omega + 8} = \frac{1}{j\omega+2} - \frac{1}{j\omega+4}$$

$$\Rightarrow h(t) = F^{-1}\{H(j\omega)\} = (e^{-2t} - e^{-4t})u(t)$$

(b) $\because (j\omega)^2Y(j\omega) + 6(j\omega)Y(j\omega) + 8Y(j\omega) = \frac{2}{(j\omega+2)^2}$

$$\Rightarrow Y(j\omega) = \left(\frac{1}{j\omega+2} - \frac{1}{j\omega+4}\right)\frac{1}{(j\omega+2)^2} = \frac{1}{(j\omega+2)^3} - \frac{1}{2(j\omega+2)^2} + \frac{1}{4(j\omega+2)} - \frac{1}{4(j\omega+4)}$$

$$\Rightarrow y(t) = F^{-1}\{Y(j\omega)\} = \frac{t^2}{2}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t) + \frac{1}{4}e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t)$$

$$= \left(\frac{t^2}{2} - \frac{t}{2} + \frac{1}{4}\right)e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t)$$

(c) take Fourier Transform on both sides, we get,

$$(j\omega)^2Y(j\omega) + \sqrt{2}(j\omega)Y(j\omega) + Y(j\omega) = 2(j\omega)^2X(j\omega) - 2X(j\omega)$$

take $x(t) = \delta(t)$:

$$(j\omega)^2H(j\omega) + \sqrt{2}j\omega H(j\omega) + H(j\omega) = 2(j\omega)^2 - 2$$

$$\Rightarrow H(j\omega) = \frac{2(j\omega)^2 - 2}{(j\omega)^2 + \sqrt{2}j\omega + 1} = 2 - \frac{2\sqrt{2}j\omega + 4}{(j\omega)^2 + \sqrt{2}j\omega + 1} = 2 - \frac{2\sqrt{2}j\omega + 4}{\left(j\omega - \frac{-\sqrt{2}-\sqrt{2}j}{2}\right)\left(j\omega - \frac{-\sqrt{2}+\sqrt{2}j}{2}\right)}$$

$$= 2 - \frac{\sqrt{2}-\sqrt{2}j}{j\omega - \frac{-\sqrt{2}+\sqrt{2}j}{2}} - \frac{\sqrt{2}+\sqrt{2}j}{j\omega - \frac{-\sqrt{2}-\sqrt{2}j}{2}}$$

$$\Rightarrow h(t) = F^{-1}\{H(j\omega)\} = 2\delta(t) - (\sqrt{2}-\sqrt{2}j)e^{\frac{-\sqrt{2}+\sqrt{2}j}{2}t}u(t) - (\sqrt{2}+\sqrt{2})je^{\frac{-\sqrt{2}-\sqrt{2}j}{2}t}u(t)$$

4.35

(a) $|H(j\omega)| = \left|\frac{a-j\omega}{a+j\omega}\right| = 1$, $\measuredangle H(j\omega) = \arctan\frac{-\omega}{a} - \arctan\frac{\omega}{a}$

$= -2\arctan\frac{\omega}{a}$

$h(t) = F^{-1}\{H(j\omega)\} = F^{-1}\{\frac{2a}{j\omega+a} - 1\} = 2ae^{-at}u(t) - \delta(t)$

(b) $\therefore$ $h(t) = 2e^{-t}u(t) - \delta(t)$, $H(j\omega) = \frac{1-j\omega}{1+j\omega} = e^{j\measuredangle H(j\omega)\omega} = e^{-j\omega 2\arctan\omega}$

$x(t) = \cos\frac{t}{\sqrt{3}} + \cos t + \cos\sqrt{3}t$

$\Rightarrow$ For $\cos\frac{t}{\sqrt{3}}$, shift is $\measuredangle H(j\frac{1}{\sqrt{3}}) = -2\arctan\frac{1}{\sqrt{3}} = -\frac{\pi}{3}$

For $\cos t$, shift is $\measuredangle H(j) = -2\arctan 1 = -\frac{\pi}{2}$

For $\cos\sqrt{3}t$, shift is $\measuredangle H(j\sqrt{3}) = -2\arctan\sqrt{3} = -\frac{2}{3}\pi$

$\therefore$ $y(t) = \cos(\frac{t}{\sqrt{3}} - \frac{\pi}{3}) + \cos(t - \frac{\pi}{2}) + \cos(\sqrt{3}t - \frac{2}{3}\pi)$