# Supplementary Materials of "Graphical Principal Component Analysis of Multivariate Functional Time Series"


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# A Technical Details

## A.1 Lemma 1

In this subsection, we introduce Lemma 1 for calculating the spectral densities of filtered scores in the proofs of Theorem 1 and Theorem 2.

To be more general, let $\{\boldsymbol{X}_j(\cdot); j \in \mathbb{Z}\}$ and $\{\boldsymbol{Y}_j(\cdot); j \in \mathbb{Z}\}$ be two collections of mean zero multivariate processes valued in $L^2([0,1], \mathbb{R}^{p_1})$ and $L^2([0,1], \mathbb{R}^{p_2})$, respectively. Besides, $\{\boldsymbol{\phi}_l^X(\cdot); l \in \mathbb{Z}\} \in \mathcal{H}([0,1], \mathbb{R}^{p_1})$ and $\{\boldsymbol{\phi}_l^Y(\cdot); l \in \mathbb{Z}\} \in \mathcal{H}([0,1], \mathbb{R}^{p_2})$ are two collections of functions. By their definitions, there exist $\boldsymbol{\psi}_X(t|\theta)$ and $\boldsymbol{\psi}_Y(t|\theta)$ s.t. $\boldsymbol{\psi}_X(t|\theta) = \sum_{l\in\mathbb{Z}} \boldsymbol{\phi}_l^X(t) \exp(\mathrm{i}\, l\theta)$ and $\boldsymbol{\psi}_Y(t|\theta) = \sum_{l\in\mathbb{Z}} \boldsymbol{\phi}_l^Y(t) \exp(\mathrm{i}\, l\theta)$ for each $\theta$, where "=" represents

$$\lim_{L\to\infty} \int_{-\pi}^{\pi} \int_0^1 \left| \boldsymbol{\psi}(t|\theta) - \sum_{|l|\le L} \boldsymbol{\phi}_l(t) \exp(\mathrm{i}\, l\theta) \right|^2 \mathrm{d}t\mathrm{d}\theta = 0$$

with $\int_{-\pi}^{\pi} \int_0^1 \left|\boldsymbol{\psi}(t|\theta)\right|^2 \mathrm{d}t\mathrm{d}\theta < \infty$.

**Lemma 1.** *Let $\boldsymbol{C}_g^{X,Y}(t,s) := \mathbb{E}\big\{\boldsymbol{X}_{j+g}(t)\boldsymbol{Y}_j^T(s)\big\}$ be the cross-covariance between $\boldsymbol{X}_{j+g}(\cdot)$ and $\boldsymbol{Y}_j(\cdot)$, and denote $[\boldsymbol{C}_g^{X,Y}(t,s)]_{i_1,i_2}$ as $C_{i_1,i_2,g}^{X,Y}(t,s)$, $\forall t,s \in [0,1]$. Assuming $\big\{C_{i_1,i_2,g}^{X,Y}(\cdot,\cdot); g \in \mathbb{Z}\big\} \in \mathcal{H}([0,1]^2, \mathbb{R})$ for $i_1 = 1, \cdots, p_1$ and $i_2 = 1, \cdots, p_2$, we define $\boldsymbol{f}^{X,Y}(\cdot,\cdot|\theta)$ as the spectral density kernel induced by $\boldsymbol{C}_g^{X,Y}(\cdot,\cdot)$. Accordingly, $C_g^{\xi} := \mathbb{E}\xi_{j+g}^X \xi_j^Y$ is free of $j$, where $\xi_j^X := \sum_{l\in\mathbb{Z}} \langle \boldsymbol{X}_{j-l}, \boldsymbol{\phi}_l^X \rangle_{p_1}$ and $\xi_j^Y := \sum_{l\in\mathbb{Z}} \langle \boldsymbol{Y}_{j-l}, \boldsymbol{\phi}_l^Y \rangle_{p_2}$. Furthermore, we have*

$$f^{\xi}(\theta) := \frac{1}{2\pi} \sum_{g\in\mathbb{Z}} C_g^{\xi} \exp(\mathrm{i}\, g\theta) = \int_0^1 \int_0^1 \{\boldsymbol{\psi}_X(t|\theta)\}^T \boldsymbol{f}^{X,Y}(t,s|\theta) \overline{\boldsymbol{\psi}_Y(s|\theta)}\, \mathrm{d}t\mathrm{d}s.$$

*Proof.* Notice that

$$\begin{aligned}
\mathbb{E}\xi_{j+g}^X \xi_j^Y &= \mathbb{E} \sum_{l_1\in\mathbb{Z}} \langle \boldsymbol{X}_{j+g-l_1}, \boldsymbol{\phi}_{l_1}^X \rangle_{p_1} \sum_{l_2\in\mathbb{Z}} \langle \boldsymbol{Y}_{j-l_2}, \boldsymbol{\phi}_{l_2}^Y \rangle_{p_2} \\
&= \sum_{l_1,l_2\in\mathbb{Z}} \int_0^1 \int_0^1 \{\boldsymbol{\phi}_{l_1}^X(t)\}^T \boldsymbol{C}_{l_2-l_1+g}^{X,Y}(t,s) \boldsymbol{\phi}_{l_2}^Y(s)\, \mathrm{d}t\mathrm{d}s. \qquad (1)
\end{aligned}$$

Hence, $\mathbb{E}\xi_{j+g}^X \xi_j^Y$ is independent of $j$. Furthermore, it can be shown that

$$\int_{-\pi}^{\pi} \int_0^1 \int_0^1 \{\boldsymbol{\psi}_X(t|\theta)\}^T \boldsymbol{f}^{X,Y}(t,s|\theta) \overline{\boldsymbol{\psi}_Y(s|\theta)}\, \mathrm{d}t\mathrm{d}s\mathrm{d}\theta < \infty.$$

Therefore, we have

$$\begin{aligned}
f^{\xi}(\theta) &\overset{(1)}{=} \frac{1}{2\pi} \sum_{g\in\mathbb{Z}} \sum_{l_1,l_2\in\mathbb{Z}} \int_0^1 \int_0^1 \{\boldsymbol{\phi}_{l_1}^X(t)\}^T \boldsymbol{C}_{l_2-l_1+g}^{X,Y}(t,s) \boldsymbol{\phi}_{l_2}^Y(s)\, \mathrm{d}t\mathrm{d}s \exp(\mathrm{i}\, g\theta) \\
&= \int_0^1 \int_0^1 \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{g\in\mathbb{Z}} \sum_{l_1,l_2\in\mathbb{Z}} \{\boldsymbol{\phi}_{l_1}^X(t)\}^T \boldsymbol{f}^{X,Y}(t,s|\theta_1) \exp\{-\mathrm{i}(l_2 - l_1 + g)\theta_1\}\, \boldsymbol{\phi}_{l_2}^Y(s) \exp(\mathrm{i}\, g\theta)\, \mathrm{d}\theta_1\mathrm{d}t\mathrm{d}s \\
&= \int_0^1 \int_0^1 \frac{1}{2\pi} \sum_{g\in\mathbb{Z}} \left[ \int_{-\pi}^{\pi} \{\boldsymbol{\psi}_X(t|\theta_1)\}^T \boldsymbol{f}^{X,Y}(t,s|\theta_1) \overline{\boldsymbol{\psi}_Y(s|\theta_1)} \exp(-\mathrm{i}\, g\theta_1)\, \mathrm{d}\theta_1 \right] \exp(\mathrm{i}\, g\theta)\, \mathrm{d}t\mathrm{d}s \\
&= \int_0^1 \int_0^1 \{\boldsymbol{\psi}_X(t|\theta)\}^T \boldsymbol{f}^{X,Y}(t,s|\theta) \overline{\boldsymbol{\psi}_Y(s|\theta)}\, \mathrm{d}t\mathrm{d}s,
\end{aligned}$$

by Tonelli's theorem. $\square$

One can similarly show that $\mathbb{E}\boldsymbol{X}_{j+g}(t)\xi_j^Y$ and $\mathbb{E}\xi_{j+g}^X\{\boldsymbol{Y}_j(s)\}^T$ are free of $j$ (denoted as $\boldsymbol{C}_g^{X,\xi}(t)$ and $\boldsymbol{C}_g^{\xi,Y}(s)$, respectively), and

$$\boldsymbol{f}^{X,\xi}(t|\theta) := \frac{1}{2\pi}\sum_{g\in\mathbb{Z}}\boldsymbol{C}_g^{X,\xi}(t)\exp(\mathrm{i}\, g\theta) = \int_0^1 \boldsymbol{f}^{X,Y}(t,s|\theta)\overline{\boldsymbol{\psi}_Y(s|\theta)}\ \mathrm{d}s,$$

$$\boldsymbol{f}^{\xi,Y}(s|\theta) := \frac{1}{2\pi}\sum_{g\in\mathbb{Z}}\boldsymbol{C}_g^{\xi,Y}(s)\exp(\mathrm{i}\, g\theta) = \int_0^1 \{\boldsymbol{\psi}_X(t|\theta)\}^T\boldsymbol{f}^{X,Y}(t,s|\theta)\ \mathrm{d}t.$$

These properties would also be applied in the proof of Theorem 1.

## A.2 Proof of Theorem 1

*Proof.* Recall that

$$\varepsilon_{i_1 j}^{\mathrm{p}} = \arg\min_{\varepsilon'_{i_1 j}\in L^2_{i_1,i_2}(\mathcal{T},\mathbb{R})}\int_0^1 \mathbb{E}\big\{\varepsilon_{i_1 j}(t) - \varepsilon'_{i_1 j}(t)\big\}^2\ \mathrm{d}t,$$

where $L^2_{i_1,i_2}(\mathcal{T},\mathbb{R})$ is the closure of all linear predictors on $\boldsymbol{\varepsilon}_{V_{-\{i_1,i_2\}},\cdot}$ in the sense of $L^2$-norm, i.e., $\forall\ \varepsilon'_{i_1 j}\in L^2_{i_1,i_2}(\mathcal{T},\mathbb{R})$ and $\forall\delta>0$, there exists $\{\boldsymbol{a}_{i_1,g}(\cdot,\cdot); g\in\mathbb{Z}\}\in\mathcal{H}([0,1]^2,\mathbb{R}^{p-2})$ s.t.

$$\int_0^1 \mathbb{E}\left\{\varepsilon'_{i_1 j}(t) - \sum_{g\in\mathbb{Z}}\langle\boldsymbol{a}_{i_1,g}(t,\cdot),\boldsymbol{\varepsilon}_{V_{-\{i_1,i_2\}},j+g}\rangle_{p-2}\right\}^2\ \mathrm{d}t<\delta.$$

We first prove the existence of $\varepsilon_{i_1 j}^{\mathrm{p}}$.

Assume $\boldsymbol{A}_i(t,s|\theta) = \sum_{g\in\mathbb{Z}}\boldsymbol{a}_{i,g}(t,s)\exp(\mathrm{i}\, g\theta)$ for $\{\boldsymbol{a}_{i_1,g}(\cdot,\cdot); g\in\mathbb{Z}\}\in\mathcal{H}([0,1]^2,\mathbb{R}^{p-2})$, and abbreviate $V_{-\{i_1,i_2\}}$ as $\mathcal{V}$ for ease of notation. Notice that

$$\begin{aligned}
&\mathbb{E}\left\{\varepsilon_{i_1 j}(t) - \sum_{g\in\mathbb{Z}}\langle\boldsymbol{a}_{i_1,g}(t,\cdot),\boldsymbol{\varepsilon}_{\mathcal{V},j+g}\rangle_{p-2}\right\}^2\\
=&C_1 - 2\int_0^1\sum_{g\in\mathbb{Z}}\{\boldsymbol{a}_{i_1,g}(t,s)\}^T\left[\boldsymbol{C}_g(s,t)\right]_{\mathcal{V},i_1}\ \mathrm{d}s\\
&+\int_0^1\int_0^1\sum_{g_1,g_2\in\mathbb{Z}}\{\boldsymbol{a}_{i_1,g_1}(t,s_1)\}^T\left[\boldsymbol{C}_{g_1-g_2}(s_1,s_2)\right]_{\mathcal{V},\mathcal{V}}\boldsymbol{a}_{i_1,g_2}(t,s_2)\ \mathrm{d}s_1\mathrm{d}s_2,
\end{aligned}\tag{2}$$

where $C_1$ contains the terms that are independent of $\boldsymbol{a}_{i_1,g}(\cdot,\cdot)$.

Define $\boldsymbol{A}_{i_1}^k(t|\theta) := \int_0^1\boldsymbol{A}_{i_1}(t,s|\theta)\psi_k(s|\theta)\ \mathrm{d}s$, a simple calculation shows that

$$\begin{aligned}
\int_0^1\sum_{g\in\mathbb{Z}}\{\boldsymbol{a}_{i_1,g}(t,s)\}^T\left[\boldsymbol{C}_g(s,t)\right]_{\mathcal{V},i_1}\ \mathrm{d}s &= \int_{-\pi}^{\pi}\int_0^1\{\boldsymbol{A}_{i_1}(t,s|\theta)\}^*\left[\boldsymbol{f}(s,t|\theta)\right]_{\mathcal{V},i_1}\ \mathrm{d}s\mathrm{d}\theta\\
&=\sum_{k=1}^{\infty}\int_{-\pi}^{\pi}\{\boldsymbol{A}_{i_1}^k(t|\theta)\}^*\left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},i_1}\psi_k(t|\theta)\ \mathrm{d}\theta,
\end{aligned}$$

and

$$\int_0^1 \int_0^1 \sum_{g_1,g_2\in\mathbb{Z}} \{\boldsymbol{a}_{i_1,g_1}(t,s_1)\}^T \left[\boldsymbol{C}_{g_1-g_2}(s_1,s_2)\right]_{\mathcal{V},\mathcal{V}} \boldsymbol{a}_{i_1,g_2}(t,s_2)\ \mathrm{d}s_1\mathrm{d}s_2$$

$$= \int_{-\pi}^{\pi} \left[\int_0^1 \int_0^1 \{\boldsymbol{A}_{i_1}(t,s_1|\theta)\}^* \left[\boldsymbol{f}(s_1,s_2|\theta)\right]_{\mathcal{V},\mathcal{V}} \boldsymbol{A}_{i_1}(t,s_2|\theta)\ \mathrm{d}s_1\mathrm{d}s_2\right]\ \mathrm{d}\theta$$

$$= \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \{\boldsymbol{A}_{i_1}^k(t|\theta)\}^* \left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},\mathcal{V}} \boldsymbol{A}_{i_1}^k(t|\theta)\ \mathrm{d}\theta.$$

Combine them into (2), we have

$$\mathbb{E}\left\{\varepsilon_{i_1j}(t) - \sum_{g\in\mathbb{Z}} \langle \boldsymbol{a}_{i_1,g}(t,\cdot), \boldsymbol{\varepsilon}_{\mathcal{V},j+g}\rangle_{p-2}\right\}^2$$

$$= \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \left[\{\boldsymbol{A}_{i_1}^k(t|\theta)\}^* \left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},\mathcal{V}} \boldsymbol{A}_{i_1}^k(t|\theta) - 2\{\boldsymbol{A}_{i_1}^k(t|\theta)\}^* \left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},i_1} \psi_k(t|\theta)\right]\ \mathrm{d}\theta + C_1.$$

For each $t$, $k$, and $\theta$, the integrand reaches its minimum when

$$\boldsymbol{A}_{i_1}^k(t|\theta) = \left\{\left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},\mathcal{V}}\right\}^{-1} \left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},i_1} \psi_k(t|\theta).$$

Recall $\boldsymbol{A}_{i_1}^k(t|\theta) = \int_0^1 \boldsymbol{A}_{i_1}(t,s|\theta)\psi_k(s|\theta)\ \mathrm{d}s$, we therefore define

$$\begin{aligned}
\boldsymbol{A}_{i_1}^K(t,s|\theta) &:= \sum_{k=1}^{K} \left\{\left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},\mathcal{V}}\right\}^{-1} \left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},i_1} \psi_k(t|\theta)\overline{\psi_k(s|\theta)}, \\
\boldsymbol{a}_{i_1,g}^K(t,s) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} \boldsymbol{A}_{i_1}^K(t,s|\theta) \exp(-\mathrm{i}\, g\theta)\ \mathrm{d}\theta
\end{aligned} \tag{3}$$

for $K < \infty$. Noting that

$$\left[\boldsymbol{\eta}_k(\theta)\right]_{i_1,\mathcal{V}} \left\{\left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},\mathcal{V}}\right\}^{-1} \left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},i_1} \leq \eta_{i_1,i_1,k}(\theta)$$

since $\boldsymbol{\eta}_k(\theta)$ is a strictly positive definite matrix, we obtain

$$\int_0^1 \mathbb{E}\left\{\sup_{K_1,K_2\geq K} \sum_{g\in\mathbb{Z}} \langle \boldsymbol{a}_{i_1,g}^{K_1}(t,\cdot) - \boldsymbol{a}_{i_1,g}^{K_2}(t,\cdot), \boldsymbol{\varepsilon}_{\mathcal{V},j+g}\rangle_{p-2}\right\}^2\ \mathrm{d}t$$

$$= \int_0^1 \int_{-\pi}^{\pi} \sup_{K_1,K_2\geq K} \int_0^1 \int_0^1 \{\boldsymbol{A}_{i_1}^{K_1}(t,s_1|\theta) - \boldsymbol{A}_{i_1}^{K_2}(t,s_1|\theta)\}^* \left[\boldsymbol{f}(s_1,s_2|\theta)\right]_{\mathcal{V},\mathcal{V}} \{\boldsymbol{A}_{i_1}^{K_1}(t,s_2|\theta) - \boldsymbol{A}_{i_1}^{K_2}(t,s_2|\theta)\}$$

$$\mathrm{d}s_1\mathrm{d}s_2\ \mathrm{d}\theta\mathrm{d}t$$

$$\leq \int_{-\pi}^{\pi} \sum_{k\geq K} \left[\boldsymbol{\eta}_k(\theta)\right]_{i_1,\mathcal{V}} \left\{\left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},\mathcal{V}}\right\}^{-1} \left[\boldsymbol{\eta}_k(\theta)\right]_{\mathcal{V},i_1}\ \mathrm{d}\theta$$

$$\leq \int_{-\pi}^{\pi} \sum_{k\geq K} \eta_{i_1,i_1,k}(\theta)\ \mathrm{d}\theta \to 0$$

as $K \to \infty$, i.e., $\{\sum_{g\in\mathbb{Z}} \langle \boldsymbol{a}^K_{i_1,g}(t,\cdot), \boldsymbol{\varepsilon}_{\mathcal{V},j+g}\rangle_{p-2}; K \in \mathbb{Z}\}$, $\forall t \in [0,1]$, is a Cauchy sequence in $\mathbb{R}$, a.s.. We define

$$\varepsilon^{\mathrm{p}}_{i_1 j}(t) = \lim_{K\to\infty} \sum_{g\in\mathbb{Z}} \langle \boldsymbol{a}^K_{i_1,g}(t,\cdot), \boldsymbol{\varepsilon}_{\mathcal{V},j+g}\rangle_{p-2}.$$

We can show that

$$\lim_{K\to\infty} \int_0^1 \mathbb{E}\left\{\varepsilon^{\mathrm{p}}_{i_1 j}(t) - \sum_{g\in\mathbb{Z}} \langle \boldsymbol{a}^K_{i_1,g}(t,\cdot), \boldsymbol{\varepsilon}_{\mathcal{V},j+g}\rangle_{p-2}\right\}^2 \mathrm{d}t = 0,$$

hence $\varepsilon^{p}_{i_1 j} \in L^2_{i_1,i_2}(\mathcal{T},\mathbb{R})$ and $\varepsilon^{\mathrm{p}}_{i_1 j} = \arg\min_{\varepsilon'_{i_1 j}\in L^2_{i_1,i_2}(\mathcal{T},\mathbb{R})} \int_0^1 \mathbb{E}\left\{\varepsilon_{i_1 j}(t) - \varepsilon'_{i_1 j}(t)\right\}^2 \mathrm{d}t$.

After that, we achieve the residual functional time series $\boldsymbol{\varepsilon}^{\mathrm{r}}_{i_1,\cdot}$ and $\boldsymbol{\varepsilon}^{\mathrm{r}}_{i_2,\cdot}$ by

$$\varepsilon^{\mathrm{r}}_{ij}(t) := \varepsilon_{ij}(t) - \varepsilon^{\mathrm{p}}_{ij}(t).$$

We then calculate the spectral density kernel of $\boldsymbol{\varepsilon}^{\mathrm{r}}_{i_1,\cdot}$ and $\boldsymbol{\varepsilon}^{\mathrm{r}}_{i_2,\cdot}$ by

$$\begin{aligned}
& f_{i_1,i_2|\cdot}(t,s|\theta) = \frac{1}{2\pi}\sum_{g\in\mathbb{Z}} \mathbb{E}\varepsilon^{\mathrm{r}}_{i_1(j+g)}(t)\varepsilon^{\mathrm{r}}_{i_2(j+g)}(s)\exp(\mathrm{i}\,g\theta) \\
=& \frac{1}{2\pi}\sum_{g\in\mathbb{Z}} \mathbb{E}\left\{\varepsilon_{i_1(j+g)}(t) - \varepsilon^{\mathrm{p}}_{i_1 j}(t)\right\}\left\{\varepsilon_{i_2 j}(s) - \varepsilon^{\mathrm{p}}_{i_2 j}(t)\right\}\cdot\exp(\mathrm{i}\,g\theta) \\
=& f_{i_1,i_2}(t,s|\theta) - \frac{1}{2\pi}\sum_{g\in\mathbb{Z}} D^{(1)}_g(t,s)\exp(\mathrm{i}\,g\theta) - \frac{1}{2\pi}\sum_{g\in\mathbb{Z}} D^{(2)}_g(t,s)\exp(\mathrm{i}\,g\theta) + \frac{1}{2\pi}\sum_{g\in\mathbb{Z}} D^{(3)}_g(t,s)\exp(\mathrm{i}\,g\theta),
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
D^{(1)}_g(t,s) &= \lim_{K\to\infty} \mathbb{E}\varepsilon_{i_1(j+g)}(t) \sum_{g_2\in\mathbb{Z}} \langle \boldsymbol{a}^K_{i_2,g_2}(s,\cdot), \boldsymbol{\varepsilon}_{\mathcal{V},j+g_2}\rangle_{p-2}, \\
D^{(2)}_g(t,s) &= \lim_{K\to\infty} \mathbb{E}\left\{\sum_{g_1\in\mathbb{Z}} \langle \boldsymbol{a}^K_{i_1,g_1}(t,\cdot), \boldsymbol{\varepsilon}_{\mathcal{V},j+g+g_1}\rangle_{p-2}\right\}\varepsilon_{i_2 j}(s), \\
D^{(3)}_g(t,s) &= \lim_{K_1\to\infty}\lim_{K_2\to\infty} \mathbb{E}\left\{\sum_{g_1\in\mathbb{Z}} \langle \boldsymbol{a}^{K_1}_{i_1,g_1}(t,\cdot), \boldsymbol{\varepsilon}_{\mathcal{V},j+g+g_1}\rangle_{p-2}\right\}\left\{\sum_{g_2\in\mathbb{Z}} \langle \boldsymbol{a}^{K_2}_{i_2,g_2}(s,\cdot), \boldsymbol{\varepsilon}_{\mathcal{V},j+g_2}\rangle_{p-2}\right\}.
\end{aligned}$$

According to Lemma 1 with $\boldsymbol{X}_j(\cdot)$, $\boldsymbol{Y}_j(\cdot)$ and $\{\boldsymbol{\phi}^Y_l(\cdot); l \in \mathbb{Z}\}$ taken as $\varepsilon_{i_1 j}$, $\boldsymbol{\varepsilon}_{\mathcal{V},j}$ and $\{\boldsymbol{a}^K_{i_2,-l}(s,\cdot); l \in \mathbb{Z}\}$ respectively, we have

$$\begin{aligned}
\frac{1}{2\pi}\sum_{g\in\mathbb{Z}} D^{(1)}_g(t,s)\exp(\mathrm{i}\,g\theta) &= \lim_{K\to\infty}\int_0^1 [\boldsymbol{f}(t,u|\theta)]_{i_1,\mathcal{V}}\ \boldsymbol{A}^K_{i_2}(s,u|\theta)\ \mathrm{d}u \\
&= \sum_{k=1}^{\infty} [\boldsymbol{\eta}_k(\theta)]_{i_1,\mathcal{V}}\left\{[\boldsymbol{\eta}_k(\theta)]_{\mathcal{V},\mathcal{V}}\right\}^{-1}[\boldsymbol{\eta}_k(\theta)]_{\mathcal{V},i_2}\overline{\psi_k(t|\theta)}\psi_k(s|\theta).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{2\pi}\sum_{g\in\mathbb{Z}} D_g^{(2)}(t,s)\exp(\mathrm{i}\,g\theta) &= \lim_{K\to\infty}\int_0^1 \{\boldsymbol{A}_{i_1}^K(t,u|\theta)\}^*\ [\boldsymbol{f}(u,s|\theta)]_{\mathcal{V},i_2}\ \mathrm{d}u,\\
&= \sum_{k=1}^{\infty} [\boldsymbol{\eta}_k(\theta)]_{i_1,\mathcal{V}} \left\{[\boldsymbol{\eta}_k(\theta)]_{\mathcal{V},\mathcal{V}}\right\}^{-1} [\boldsymbol{\eta}_k(\theta)]_{\mathcal{V},i_2}\ \overline{\psi_k(t|\theta)}\psi_k(s|\theta),\\
\frac{1}{2\pi}\sum_{g\in\mathbb{Z}} D_g^{(3)}(t,s)\exp(\mathrm{i}\,g\theta) &= \lim_{K_1\to\infty}\lim_{K_2\to\infty}\int_0^1\int_0^1 \{\boldsymbol{A}_{i_1}^{K_1}(t,u_1|\theta)\}^*\ [\boldsymbol{f}(u_1,u_2|\theta)]_{\mathcal{V},\mathcal{V}}\boldsymbol{A}_{i_2}^{K_2}(s,u_2|\theta)\ \mathrm{d}u_1\mathrm{d}u_2\\
&= \sum_{k=1}^{\infty} [\boldsymbol{\eta}_k(\theta)]_{i_1,\mathcal{V}} \left\{[\boldsymbol{\eta}_k(\theta)]_{\mathcal{V},\mathcal{V}}\right\}^{-1} [\boldsymbol{\eta}_k(\theta)]_{\mathcal{V},i_2}\ \overline{\psi_k(t|\theta)}\psi_k(s|\theta).
\end{aligned}$$

Plugging these three terms into (4), we obtain

$$f_{i_1,i_2|\cdot}(t,s|\theta) = \sum_{k=1}^{\infty}\sigma_{i_1,i_2,k}(\theta)\overline{\psi_k(t|\theta)}\psi_k(s|\theta),$$

where

$$\sigma_{i_1,i_2,k}(\theta) = \eta_{i_1,i_2,k}(\theta) - [\boldsymbol{\eta}_k(\theta)]_{i_1,\mathcal{V}}\left\{[\boldsymbol{\eta}_k(\theta)]_{\mathcal{V},\mathcal{V}}\right\}^{-1}[\boldsymbol{\eta}_k(\theta)]_{\mathcal{V},i_2}. \tag{5}$$

Noting that $\left\{[\boldsymbol{\Phi}_k(\theta)]_{i,i'}\right\}_{i,i'\in\{i_1,i_2\}}$ is the inverse matrix of $\left\{\sigma_{i,i',k}(\theta)\right\}_{i,i'\in\{i_1,i_2\}}$ by the block matrix inverse formula of $\boldsymbol{\eta}_k(\theta)$, we have

$$\sigma_{i_1,i_2,k}(\theta) = -\frac{[\boldsymbol{\Phi}_k(\theta)]_{i_1,i_2}}{[\boldsymbol{\Phi}_k(\theta)]_{i_1,i_1}[\boldsymbol{\Phi}_k(\theta)]_{i_2,i_2} - [\boldsymbol{\Phi}_k(\theta)]_{i_1,i_2}[\boldsymbol{\Phi}_k(\theta)]_{i_2,i_1}},$$

which completes the proof of equation (8) in the main text. □

## A.3 Proof of Theorem 2

The following lemma is useful in proving Theorem 2:

**Lemma 2.** *Under the weak stationarity* (1)*, the following conditions are equivalent:*
*(a) The dynamic weak separability* (4) *is satisfied with* $\eta_k(\theta)$*s being nonsingular.*
*(b) For each* $\theta$*, there exist orthonormal bases* $\{\boldsymbol{e}_{ki}(\theta)\in\mathbb{C}^p; i=1,\cdots,p\}$ *of* $\mathbb{C}^p$ *and* $\{\psi_k(\cdot|\theta); k\geq 1\}$ *of* $L^2([0,1],\mathbb{C})$ *such that the eigenfunctions of* $\boldsymbol{f}(t,s|\theta)$ *are given as* $\{\boldsymbol{e}_{ki}(\theta)\psi_k(\cdot|\theta); k\geq 1, i=1,\cdots,p\}$.

Lemma 2 can be proven by the spectral decomposition of $\boldsymbol{\eta}_k(\theta)$ in the dynamic weak separability (4). With this lemma, we present the proof of Theorem 2 as follows.

*Proof.* (a) ⇒ (b): Recall that

$$\phi_{kl}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi_k(t|\theta)\exp(-\mathrm{i}\,l\theta)\ \mathrm{d}\theta,$$

we have

$$\psi_k(t|\theta) = \sum_{l\in\mathbb{Z}} \phi_{kl}(t) \exp(\mathrm{i}\, l\theta)$$

for each $\theta$. Applying Lemma 1 with $\boldsymbol{X}_j(\cdot)$, $\boldsymbol{Y}_j(\cdot)$, $\{\boldsymbol{\phi}_l^X(\cdot); l \in \mathbb{Z}\}$ and $\{\boldsymbol{\phi}_l^Y(\cdot); l \in \mathbb{Z}\}$ being $\varepsilon_{i_1 j}$, $\varepsilon_{i_2 j}$, $\{\phi_{k_1 l}(\cdot); l \in \mathbb{Z}\}$ and $\{\phi_{k_2 l}(\cdot); l \in \mathbb{Z}\}$ respectively, we obtain that $\mathbb{E}\xi_{i_1(j+g)k_1}\xi_{i_2 j k_2}$ is independent of $j$, which is denoted as $C_{i_1,i_2,g}^{k_1,k_2}$. Let $f_{i_1,i_2}^{k_1,k_2}(\theta) := \frac{1}{2\pi}\sum_{g\in\mathbb{Z}} C_{i_1,i_2,g}^{k_1,k_2} \exp(\mathrm{i}\, g\theta)$ be the spectral density of $C_{i_1,i_2,g}^{k_1,k_2}$, it follows that

$$f_{i_1,i_2}^{k_1,k_2}(\theta) = \int_0^1\int_0^1 f_{i_1,i_2}(t,s|\theta)\psi_{k_1}(t|\theta)\overline{\psi_{k_2}(s|\theta)}\ \mathrm{d}t\mathrm{d}s.$$

Under the assumed dynamic weak separability (4), we thus obtain

$$f_{i_1,i_2}^{k_1,k_2}(\theta) = \mathbb{I}(k_1 = k_2)\eta_{i_1,i_2,k_1}(\theta).$$

In other words, $f_{i_1,i_2}^{k_1,k_2}(\theta) = 0$ for any $k_1 \neq k_2$ and $\theta \in [-\pi,\pi]$, and consequently, $C_{i_1,i_2,g}^{k_1,k_2} = 0$ for any $k_1 \neq k_2$ and $g \in \mathbb{Z}$. Thus, the resulting filtered scores $\xi_{ijk}$s are uncorrelated for different $k$s. Besides, we obtain that $\{\boldsymbol{\xi}_{\cdot,jk}; j \in \mathbb{Z}\}$ is a weakly stationary multivariate time series with the spectral density matrix $\boldsymbol{\eta}_k(\theta)$.

In the remaining, we prove that the dynamic multivariate Karhunen–Loève expansion degenerates as

$$\boldsymbol{\varepsilon}_j(t) = \sum_{k\geq 1}\sum_{l\in\mathbb{Z}} \phi_{kl}(t)\boldsymbol{\xi}_{\cdot,(j+l)k}.$$

Notice that

$$\begin{aligned}
\boldsymbol{\varepsilon}_j(t) &= \sum_{m\geq 1}\sum_{l\in\mathbb{Z}} \boldsymbol{\phi}_{ml}(t)\xi_{(j+l)m}\\
&= \sum_{m\geq 1}\sum_{l_1,l_2\in\mathbb{Z}} \boldsymbol{\phi}_{ml_1}(t)\int_0^1 \boldsymbol{\varepsilon}_{(j+l_1-l_2)}^T(s)\boldsymbol{\phi}_{ml_2}(s)\ \mathrm{d}s\\
&= \sum_{m\geq 1}\sum_{l_1,l_2\in\mathbb{Z}} \boldsymbol{\phi}_{ml_1}(t)\int_0^1 \boldsymbol{\varepsilon}_{l_2}^T(s)\boldsymbol{\phi}_{m(j+l_1-l_2)}(s)\ \mathrm{d}s\\
&= \sum_{m\geq 1}\frac{1}{2\pi}\sum_{l_1,l_2\in\mathbb{Z}} \boldsymbol{\phi}_{ml_1}(t)\int_0^1 \boldsymbol{\varepsilon}_{l_2}^T(s)\int_{-\pi}^{\pi} \boldsymbol{\delta}_m(s|\theta)\exp\{-\mathrm{i}(j+l_1-l_2)\theta\}\ \mathrm{d}\theta\ \mathrm{d}s\\
&= \sum_{m\geq 1}\frac{1}{2\pi}\sum_{l_2\in\mathbb{Z}}\int_{-\pi}^{\pi}\int_0^1 \boldsymbol{\varepsilon}_{l_2}(s)\{\boldsymbol{\delta}_m(t|\theta)\}^*\boldsymbol{\delta}_m(s|\theta)\exp\{-\mathrm{i}(j-l_2)\theta\}\ \mathrm{d}s\mathrm{d}\theta.
\end{aligned}$$

By Lemma 2, we have

$$\begin{aligned}
\boldsymbol{\varepsilon}_j(t) &= \sum_{k\geq 1}\sum_{i=1}^{p}\frac{1}{2\pi}\sum_{l_2\in\mathbb{Z}}\int_{-\pi}^{\pi}\int_0^1 \boldsymbol{\varepsilon}_{l_2}(s)\overline{\psi_k(t|\theta)}\{\boldsymbol{e}_{ki}(\theta)\}^*\boldsymbol{e}_{ki}(\theta)\psi_k(s|\theta)\exp\{-\mathrm{i}(j-l_2)\theta\}\ \mathrm{d}s\mathrm{d}\theta\\
&= \sum_{k\geq 1}\frac{1}{2\pi}\sum_{l_2\in\mathbb{Z}}\int_{-\pi}^{\pi}\int_0^1 \boldsymbol{\varepsilon}_{l_2}(s)\overline{\psi_k(t|\theta)}\psi_k(s|\theta)\exp\{-\mathrm{i}(j-l_2)\theta\}\ \mathrm{d}s\mathrm{d}\theta\\
&= \sum_{k\geq 1}\sum_{l\in\mathbb{Z}} \phi_{kl}(t)\boldsymbol{\xi}_{\cdot,(j+l)k}.
\end{aligned}$$

(b) $\Rightarrow$ (a): As before, it can be similarly shown that for all $i_1, i_2 \in V$,

$$\int_0^1 \int_0^1 \psi_{k_1}(t|\theta)\overline{\psi_{k_2}(s|\theta)} f_{i_1,i_2}(t,s|\theta)\ \mathrm{d}t\mathrm{d}s = 0$$

when $k_1 \neq k_2$, and $\int_0^1 \int_0^1 \psi_{k_1}(t|\theta)\overline{\psi_{k_2}(s|\theta)} f_{i_1,i_2}(t,s|\theta)\ \mathrm{d}t\mathrm{d}s = \eta_{i_1,i_2,k}(\theta)$ when $k_1 = k_2 = k$. Note that $\{\overline{\psi_{k_1}(t|\theta)}\psi_{k_2}(s|\theta); k_1, k_2 \geq 1\}$ are orthonormal basis functions of $L^2([0,1]^2, \mathbb{C})$, then

$$f_{i_1,i_2}(t,s|\theta) = \sum_{k\geq 1} \eta_{i_1,i_2,k_1}(\theta)\overline{\psi_k(t|\theta)}\psi_k(s|\theta)$$

for all $\theta$ and $i_1, i_2 \in V$ in the sense of $L^2$-norm.

$\square$

**Remark**: For given $k$, note that given any arbitrary complex numbers $\{\gamma_k(\theta); \theta \in [-\pi, \pi]\}$ on the complex unit circle, we always have

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \phi_{kl}(t)\boldsymbol{\xi}_{\cdot,(j+l)k} &= \frac{1}{2\pi}\sum_{l_2\in\mathbb{Z}} \int_{-\pi}^{\pi}\int_0^1 \boldsymbol{\varepsilon}_{l_2}(s)\overline{\psi_k(t|\theta)}\psi_k(s|\theta)\exp\{-\mathrm{i}(j-l_2)\theta\}\ \mathrm{d}s\mathrm{d}\theta \\ &= \frac{1}{2\pi}\sum_{l_2\in\mathbb{Z}} \int_{-\pi}^{\pi}\int_0^1 \boldsymbol{\varepsilon}_{l_2}(s)\overline{\psi_k(t|\theta)\gamma_k(\theta)}\psi_k(s|\theta)\gamma_k(\theta)\exp\{-\mathrm{i}(j-l_2)\theta\}\ \mathrm{d}s\mathrm{d}\theta.\end{aligned}$$

Specifically, we recall that the extracted $\phi_{kl}(\cdot)$ and $\boldsymbol{\xi}_{\cdot,jk}$ cannot be uniquely determined, since any eigenfunction in the form $\psi_k(t|\theta)\gamma_k(\theta)$ can be used in the definition of (12) in the main text. However, the optimal reconstructed process $\sum_{l\in\mathbb{Z}} \phi_{kl}(t)\boldsymbol{\xi}_{\cdot,(j+l)k}$ exists uniquely.

## A.4 Proof of Theorem 3

We first define some notions about linear operators. We define the Hilbert-Schmidt norm and the trace norm for an operator $\mathcal{T}: L^2([0,1], \mathbb{C}) \to L^2([0,1], \mathbb{C})$ as

$$||\mathcal{T}||_{\mathrm{HS}} = \sqrt{\sum_{k=1}^{\infty} ||\mathcal{T}e_k||^2},$$

$$\mathrm{tr}(\mathcal{T}) = \sum_{k=1}^{\infty} \langle (\mathcal{T}^*\mathcal{T})^{1/2} e_k, e_k\rangle,$$

where we abuse notations $\mathrm{tr}(\cdot)$ and $(\cdot)^*$ to denote the trace norm and the adjoint operation for an operator, and $\{e_k(\cdot); k = 1, 2, \cdots\}$ is any orthonormal bases of $L^2([0,1], \mathbb{C})$.

Letting $G(t,s|\theta) := \sum_{i=1}^{p} f_{i,i}(t,s|\theta)$, we define an integral operator associated with the kernel $G(t,s|\theta)$ as

$$\mathcal{G}_\theta u := \int_0^1 G(t,s|\theta)u(t)\ \mathrm{d}t,\ \forall u \in L^2([0,1], \mathbb{C}).$$

For each $\theta$, we assume that $G(t,s|\theta)$ is a continuous kernel on $[0,1]^2$. By the fact that

$$G(t,s|\theta) = \overline{G(s,t|\theta)} \text{ and } \langle \mathcal{G}_\theta u, u\rangle \geq 0,\ \forall u \in L^2([0,1],\mathbb{C}),$$

$\mathcal{G}_\theta$ is a non-negative definite compact operator, hence admitting a spectral decomposition by Mercer's Theorem. Using the notation in the dynamic weak separability (4), we have

$$\mathcal{G}_\theta = \sum_{k=1}^{\infty} \operatorname{tr}\big\{\boldsymbol{\eta}_k(\theta)\big\}\psi_k(\theta)\otimes\psi_k(\theta),$$

where $\psi_k(\theta)$ denotes $\psi_k(t|\theta)$ by omitting it functional nature, and $\psi_k(\theta)\otimes\psi_k(\theta)$ is a tensor product operator between $L^2([0,1],\mathbb{C})$s satisfying

$$\big\{\psi_k(\theta)\otimes\psi_k(\theta)\big\}u = \psi_k(\theta)\langle\psi_k(\theta), u\rangle,$$

$\forall u \in L^2([0,1],\mathbb{C})$. Besides, $\sum_{k=1}^{K}\operatorname{tr}(\boldsymbol{\eta}_k(\theta))\psi_k(\theta)\otimes\psi_k(\theta)$ is the optimal rank-$K$ approximation of $\mathcal{G}_\theta$ under $||\cdot||_{\mathrm{HS}}$. Detailed proof of the above properties can be found in Section 4 of Hsing & Eubank (2015). We now prove Theorem 3 based on these definitions and properties.

*Proof.* By the definition of $\tilde{\boldsymbol{\varepsilon}}_j^K$, we have

$$\mathbb{E}||\boldsymbol{\varepsilon}_j - \tilde{\boldsymbol{\varepsilon}}_j^K||_p^2 = \mathbb{E}\int_0^1 \sum_{i=1}^{p}\left\{\varepsilon_{ij}(t) - \sum_{k\leq K}\sum_{l\in\mathbb{Z}}\tilde{\phi}_{kl}(t)\tilde{\xi}_{i(j+l)k}\right\}^2 \mathrm{d}t$$

$$= \int_0^1\left[\sum_{i=1}^{p} C_{i,i,0}(t,t) - 2\left\{\sum_{i=1}^{p}\sum_{k\leq K}\sum_{l\in\mathbb{Z}}\mathbb{E}\varepsilon_{ij}(t)\tilde{\phi}_{kl}(t)\tilde{\xi}_{i(j+l)k}\right\} + \sum_{i=1}^{p}\mathbb{E}\left\{\sum_{k\leq K}\sum_{l\in\mathbb{Z}}\tilde{\phi}_{kl}(t)\tilde{\xi}_{i(j+l)k}\right\}^2\right]\mathrm{d}t.$$

We first notice

$$\sum_{i=1}^{p}\int_0^1 C_{i,i,0}(t,t)\ \mathrm{d}t = \int_{-\pi}^{\pi}\int_0^1 G(t,t|\theta)\ \mathrm{d}t\mathrm{d}\theta = \int_{-\pi}^{\pi}\operatorname{tr}(\mathcal{G}_\theta)\ \mathrm{d}\theta. \tag{6}$$

Also, we note that

$$\begin{aligned}
\sum_{i=1}^{p}\sum_{k\leq K}\sum_{l\in\mathbb{Z}}\mathbb{E}\varepsilon_{ij}(t)\tilde{\phi}_{kl}(t)\tilde{\xi}_{i(j+l)k} &= \sum_{i=1}^{p}\sum_{k\leq K}\sum_{l_1,l_2\in\mathbb{Z}}\tilde{\phi}_{kl_1}(t)\int_0^1 C_{i,i,l_1-l_2}(s,t)\tilde{\phi}_{kl_2}(s)\ \mathrm{d}s\\
&= \sum_{i=1}^{p}\sum_{k\leq K}\sum_{l_1,l_2\in\mathbb{Z}}\tilde{\phi}_{kl_1}(t)\int_0^1\int_{-\pi}^{\pi} f_{i,i}(s,t|\theta)\exp\{\mathrm{i}(l_2-l_1)\theta\}\tilde{\phi}_{kl_2}(s)\ \mathrm{d}\theta\mathrm{d}s\\
&= \sum_{i=1}^{p}\sum_{k\leq K}\overline{\tilde{\psi}_k(t|\theta)}\int_0^1\int_{-\pi}^{\pi} f_{i,i}(s,t|\theta)\tilde{\psi}_k(s|\theta)\ \mathrm{d}\theta\mathrm{d}s\\
&= \int_0^1\int_{-\pi}^{\pi}\int_0^1 G(s,t|\theta)Q_K(t,s|\theta)\ \mathrm{d}s\mathrm{d}\theta,
\end{aligned}$$

where $Q_K(t,s|\theta)=\sum_{k\leq K}\overline{\tilde{\psi}_k(t|\theta)}\tilde{\psi}_k(s|\theta)$. For each $K$ and $\theta$, we define $\mathcal{Q}_{K,\theta}$ as the integral operator associated with the kernel $Q_K(t,s|\theta)$, and denote $\circ$ as the multiplication between operators. It follows that

$$\begin{aligned}2\sum_{i=1}^{p}\int_0^1\sum_{k\leq K}\sum_{l\in\mathbb{Z}}\mathbb{E}\varepsilon_{ij}(t)\tilde{\phi}_{kl}(t)\tilde{\xi}_{i(j+l)k}\ \mathrm{d}t &= \int_{-\pi}^{\pi}\int_0^1\Big\{G(s,t|\theta)Q_K(t,s|\theta)+Q_K(t,s|\theta)G(s,t|\theta)\Big\}\ \mathrm{d}s\mathrm{d}\theta\\ &\leq \int_{-\pi}^{\pi}\mathrm{tr}(\mathcal{G}_\theta\circ\mathcal{Q}_{K,\theta}+\mathcal{Q}_{K,\theta}^*\circ\mathcal{G}_\theta^*)\ \mathrm{d}\theta. \qquad (7)\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{i=1}^{p}\int_0^1\mathbb{E}\left\{\sum_{k\leq K}\sum_{l\in\mathbb{Z}}\tilde{\phi}_{kl}(t)\tilde{\xi}_{i(j+l)k}\right\}^2\ \mathrm{d}t\\ &=\int_{-\pi}^{\pi}\left\{\int_0^1\int_0^1\int_0^1\overline{Q_K(t,u|\theta)}G(s,u|\theta)Q_K(t,s|\theta)\ \mathrm{d}s\mathrm{d}u\mathrm{d}t\right\}\mathrm{d}\theta\\ &=\int_{-\pi}^{\pi}\mathrm{tr}(\mathcal{Q}_{K,\theta}^*\circ\mathcal{G}_\theta\circ\mathcal{Q}_{K,\theta})\ \mathrm{d}\theta. \qquad (8)\end{aligned}$$

Combining the results (6) to (8), we have

$$\begin{aligned}\mathbb{E}||\boldsymbol{\varepsilon}_j-\tilde{\boldsymbol{\varepsilon}}_j^K||_p^2 &\geq \int_{-\pi}^{\pi}\mathrm{tr}(\mathcal{G}_\theta-\mathcal{G}_\theta\circ\mathcal{Q}_{K,\theta}-\mathcal{Q}_{K,\theta}^*\circ\mathcal{G}_\theta^*+\mathcal{Q}_{K,\theta}^*\circ\mathcal{G}_\theta\circ\mathcal{Q}_{K,\theta})\ \mathrm{d}\theta\\ &=\int_{-\pi}^{\pi}\mathrm{tr}\left\{\left(\mathcal{G}_\theta^{1/2}-\mathcal{G}_\theta^{1/2}\circ\mathcal{Q}_{K,\theta}\right)^*\circ\left(\mathcal{G}_\theta^{1/2}-\mathcal{G}_\theta^{1/2}\circ\mathcal{Q}_{K,\theta}\right)\right\}\ \mathrm{d}\theta\\ &=\int_{-\pi}^{\pi}\left\|\mathcal{G}_\theta^{1/2}-\mathcal{G}_\theta^{1/2}\circ\mathcal{Q}_{K,\theta}\right\|_{\mathrm{HS}}^2\ \mathrm{d}\theta,\end{aligned}$$

where $\mathcal{G}_\theta^{1/2}=\sum_{k=1}^{\infty}\sqrt{\mathrm{tr}(\boldsymbol{\eta}_k(\theta))}\psi_k(\theta)\otimes\psi_k(\theta)$. This indicates that $\mathbb{E}||\boldsymbol{\varepsilon}_j-\tilde{\boldsymbol{\varepsilon}}_j^K||_p^2$ reaches its minimum

$$\int_{-\pi}^{\pi}\sum_{k>K}\mathrm{tr}\left\{\boldsymbol{\eta}_k(\theta)\right\}\ \mathrm{d}\theta,$$

which equals to $\mathbb{E}||\boldsymbol{\varepsilon}_j-\boldsymbol{\varepsilon}_j^K||_p^2$, when $\mathcal{Q}_{K,\theta}$ is given by $\sum_{k\leq K}\psi_k(\theta)\otimes\psi_k(\theta)$, i.e. $\tilde{\psi}_k(t|\theta)=\psi_k(t|\theta)$, $t\in[0,1]$ and $\theta\in[-\pi,\pi]$. □

## A.5 Proof of Theorem 4

*Proof.* $(a)\Rightarrow(b)$: When weak separability (5) is achieved, we can express $\psi_k(t|\theta)$ in the dynamic weak separability (4) as $\varphi_k(t)$ multiplied by any complex number $\gamma_k(\theta)$ with a length of one. Consequently, (b) is obtained based on (12) in the main text.

$(b)\Rightarrow(c)$: By the inverse Fourier transformation of $\phi_{kl}(\cdot)$, we have

$$\psi_k(t|\theta)=\sum_{l\in\mathbb{Z}}\phi_{kl}(t)\exp(\mathrm{i}\,l\theta)=\varphi_k(t)\sum_{l\in\mathbb{Z}}c_l\exp(\mathrm{i}\,l\theta):=\varphi_k(t)\cdot\gamma_k(\theta).$$

Since $||\psi_k(\cdot|\theta)|| = 1$ for all $\theta$, it follows that $|\gamma_k(\theta)| = 1$ for all $\theta$. Consequently, $\psi_k(t|\theta)$ in the dynamic weak separability (4) can be taken as $\varphi_k(t)$, and then we have

$$\phi_{k0}(t) = \varphi_k(t)$$

and $\phi_{kl}(t) = 0$ when $l \neq 0$, for $t \in [0,1]$ and $K \geq 1$, by taking $\psi_k(t|\theta)$ in (12) in the main text as $\varphi_k(t)$. The optimal dynamic representation (15) in the main text then degenerates as $\sum_{k=1}^{K} \varphi_k(t)\xi_{ijk}$ with $\xi_{ijk} = \langle \varepsilon_{ij}, \varphi_k \rangle$.

$(c) \Rightarrow (a)$: (c) is equivalent that for $t \in [0,1]$ and $K \geq 1$, $\phi_{k0}(t) = \varphi_k(t)$ and $\phi_{kl}(t) = 0$ when $l \neq 0$, we then obtain $\psi_k(t|\theta)$ by the inverse Fourier transformation of $\phi_{kl}(t)$

$$\psi_k(t|\theta) = \sum_{l \in \mathbb{Z}} \phi_{kl}(t) \exp(\mathrm{i}\, l\theta) = \varphi_k(t).$$

Consequently, we achieve the weak separability (5). $\square$

## A.6 Proof of Theorem 5

Following Hörmann & Kokoszka (2010), we define the $L^4$-$m$-approximablility for the MFTS $\{\boldsymbol{\varepsilon}_j; j \in \mathbb{Z}\}$. Let $\mathcal{S}$ be a measurable space. The random sequence $\{\boldsymbol{\varepsilon}_j; j \in \mathbb{Z}\}$ satisfying $\sup_{i \in V} \mathbb{E}||\varepsilon_{ij}||^4 < \infty$ is $L^4$-$m$-approximable if there exists a measurable function $f$ s.t.

$$\boldsymbol{\varepsilon}_j = f(\vartheta_j, \vartheta_{j-1}, \cdots)$$

with i.i.d. $\mathcal{S}$-valued random elements $\vartheta_j$, and

$$\sup_{i \in V} \sum_{m=1}^{\infty} \left( \mathbb{E} \left\| \varepsilon_{im} - \varepsilon_{im}^{(m)} \right\|^4 \right)^{1/4} < \infty,$$

where $\varepsilon_{ij}^{(m)}$ is the $i^{th}$ component of

$$\boldsymbol{\varepsilon}_j^{(m)} = f(\vartheta_j, \vartheta_{j-1}, \cdots, \vartheta_{j-m+1}, \vartheta'_{j-m}, \vartheta'_{j-m-1}, \cdots),$$

with $\vartheta'_j$ being a independent copy of $\vartheta_j$.

It can also be shown that, under the $L^4$-$m$-approximability, $\{\varepsilon_j; j \in \mathbb{Z}\}$ is a strictly stationary sequence satisfying $\sup_{i_1, i_2 \in V} \sum_{g \in \mathbb{Z}} \|C_{i_1,i_2,g}\|_2 < \infty$; the proof is analogous to Lemma 4.1 in Hörmann & Kokoszka (2010).

**Lemma 3.** *Assume that $\{\boldsymbol{\varepsilon}_j; j \in \mathbb{Z}\}$ is $L^4$-$m$-approximable, then for $\forall i_1, i_2 \in V$,*

$$\mathbb{E} \left\| \hat{C}_{i_1,i_2,g} - \mathbb{E}\hat{C}_{i_1,i_2,g} \right\| \leq U J^{-1/2},$$

*where $U$ is a constant that is independent to $g$, $J$, $p$, $i_1$ and $i_2$.*

*Proof.* This lemma generalizes the results of Lemma 4 in Hörmann et al. (2015) to MFTS. For simplicity, we just prove the case that $g \geq 0$. Note that

$$
\begin{aligned}
&\mathbb{E}\int_0^1\int_0^1 \left|\hat{C}_{i_1,i_2,g}(t,s) - \mathbb{E}\hat{C}_{i_1,i_2,g}(t,s)\right|^2 \,\mathrm{d}t\mathrm{d}s \\
=&\int_0^1\int_0^1 \mathrm{Var}\left[J^{-1}\sum_{j=1}^{J-g}\left\{\varepsilon_{i_1(j+g)}(t)\varepsilon_{i_2j}(s) - C_{i_1,i_2,g}(t,s)\right\}\right]\mathrm{d}t\mathrm{d}s.
\end{aligned}
$$

For fixed $i_1$, $i_2$ and $g$, we define

$$
Y_j(t,s) := \varepsilon_{i_1(j+g)}(t)\varepsilon_{i_2j}(s) - C_{i_1,i_2,g}(t,s),
$$

and also abbreviate $Y_j(t,s)$ as $Y_j$ by omitting its functional nature. Since $\{\boldsymbol{\varepsilon}_j; j\in\mathbb{Z}\}$ is $L^4$-$m$-approximable, it follows that $\{Y_j(t,s); j\in\mathbb{Z}\}$ is a stationary sequence, hence

$$
\mathrm{Var}\left\{J^{-1}\sum_{j=1}^{J-g}Y_j(t,s)\right\} = J^{-1}\sum_{|l|<J-g}\left(1-\frac{|l|+g}{J}\right)\mathrm{Cov}\big\{Y_1(t,s), Y_{l+1}(t,s)\big\},
$$

and so

$$
J\cdot\mathrm{Var}\left\{J^{-1}\sum_{j=1}^{J-g}Y_j(t,s)\right\} \leq \left|\mathrm{Cov}\big\{Y_1(t,s),Y_1(t,s)\big\}\right| + 2\sum_{l=1}^{\infty}\left|\mathrm{Cov}\big\{Y_1(t,s),Y_{1+l}(t,s)\big\}\right|.
$$

In what follows, we show that

$$
\int_0^1\int_0^1\left\{\left|\mathrm{Cov}\big\{Y_1(t,s),Y_1(t,s)\big\}\right| + 2\sum_{l=1}^{\infty}\left|\mathrm{Cov}\big\{Y_1(t,s),Y_{1+l}(t,s)\big\}\right|\right\}\mathrm{d}t\mathrm{d}s
$$

is finite and independent to $g$, $J$, $p$, $i_1$ and $i_2$, which consequently implies Lemma 3.

By the Cauchy–Schwarz inequality,

$$
\int_0^1\int_0^1\left|\mathrm{Cov}\left\{Y_1(t,s),Y_1(t,s)\right\}\right|\mathrm{d}t\mathrm{d}s = \mathbb{E}\left\|Y_1\right\|_2^2 \leq \mathbb{E}||\varepsilon_{i_1(1+g)}||^2||\varepsilon_{i_21}||^2 < \infty. \tag{9}
$$

On the other hand, define

$$
Y_j^{(m)}(t,s) := \varepsilon_{i_1(j+g)}^{(m)}(t)\varepsilon_{i_2j}^{(m)}(s) - C_{i_1,i_2,g}(t,s), \quad \forall i_1,i_2\in V,
$$

where $(\varepsilon_{1(j+g)}^{(m)},\cdots,\varepsilon_{pj}^{(m)})^T := \boldsymbol{\varepsilon}_j^{(m)}$ is a copy of $\boldsymbol{\varepsilon}_j = (\varepsilon_{1(j+g)},\cdots,\varepsilon_{pj})^T$ according to definition of the $L^4$-$m$-approximability of $\{\boldsymbol{\varepsilon}_j; j\in\mathbb{Z}\}$. It follows that, for $l>0$, $Y_1(t,s)$ and $Y_{1+l}^{(l)}(t,s)$ are independent, and

$$
\begin{aligned}
&\left|\mathrm{Cov}\big\{Y_1(t,s),Y_{1+l}(t,s)\big\}\right| = \left|\mathrm{Cov}\left\{Y_1(t,s),Y_{1+l}(t,s)-Y_{1+l}^{(l)}(t,s)\right\}\right| \\
\leq&\left[\mathrm{Var}\left\{Y_1(t,s)\right\}\right]^{1/2}\left[\mathrm{Var}\big\{Y_{1+l}(t,s)-Y_{1+l}^{(l)}(t,s)\big\}\right]^{1/2},
\end{aligned}
$$

Again, by the Cauchy–Schwarz inequality, it suffices to show that $\sum_{l=1}^{\infty}\sqrt{\mathbb{E}\left\|Y_{1+l}-Y_{1+l}^{(l)}\right\|_2^2}$ is finite and independent to $g$, $J$, $p$, $i_1$ and $i_2$. Noting the inequality

$$|ab-cd|^2 \le 2a^2(b-d)^2+2d^2(a-c)^2,$$

we have

$$\begin{aligned}
&\mathbb{E}\|Y_{l+1}-Y_{l+1}^{(l)}\|_2^2=\mathbb{E}\int_0^1\int_0^1\left\{\varepsilon_{i_1(l+1+g)}(t)\varepsilon_{i_2(l+1)}(s)-\varepsilon_{i_1(l+1+g)}^{(l)}(t)\varepsilon_{i_2(l+1)}^{(l)}(s)\right\}^2\ \mathrm{d}t\mathrm{d}s\\
\le&2\mathbb{E}\left\|\varepsilon_{i_1(l+1+g)}\right\|^2\left\|\varepsilon_{i_2(l+1)}-\varepsilon_{i_2(l+1)}^{(l)}\right\|^2+2\mathbb{E}\left\|\varepsilon_{i_2(l+1)}^{(l)}\right\|^2\left\|\varepsilon_{i_1(l+1+g)}-\varepsilon_{i_1(l+1+g)}^{(l)}\right\|^2\\
\le&2\sqrt{\mathbb{E}\left\|\varepsilon_{i_1 1}\right\|^4\mathbb{E}\left\|\varepsilon_{i_2(l+1)}-\varepsilon_{i_2(l+1)}^{(l)}\right\|^4}+2\sqrt{\mathbb{E}\left\|\varepsilon_{i_2 1}\right\|^4\mathbb{E}\left\|\varepsilon_{i_1(l+1)}-\varepsilon_{i_1(l+1)}^{(l)}\right\|^4}.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
&\sum_{l=1}^{\infty}\sqrt{\mathbb{E}||Y_{1+l}-Y_{1+l}^{(l)}||_2^2}\\
\le&\sqrt{2}\left(\mathbb{E}\left\|\varepsilon_{i_1 1}\right\|^4\right)^{1/4}\sum_{l=1}^{\infty}\left(\mathbb{E}\left\|\varepsilon_{i_2(l+1)}-\varepsilon_{i_2(l+1)}^{(l)}\right\|^4\right)^{1/4}+\sqrt{2}\left(\mathbb{E}\left\|\varepsilon_{i_2 1}\right\|^4\right)^{1/4}\sum_{l=1}^{\infty}\left(\mathbb{E}\left\|\varepsilon_{i_1(l+1)}-\varepsilon_{i_1(l+1)}^{(l)}\right\|^4\right)^{1/4}.
\end{aligned}$$

The $L^4$-$m$-approximability implies that $\sum_{l=1}^{\infty}\left(\mathbb{E}\left\|\varepsilon_{i(l+1)}-\varepsilon_{i(l+1)}^{(l)}\right\|_p^4\right)^{1/4}<\infty,\ i=i_1,i_2$. Therefore,

$$\int_0^1\int_0^1\left\{2\sum_{l=1}^{\infty}\left|\mathrm{Cov}\left(Y_1(t,s),Y_{1+l}(t,s)\right)\right|\right\}\ \mathrm{d}t\mathrm{d}s<\infty. \tag{10}$$

Combing (9) and (10), we obtain that there exists a constant $U$ independent to $g$, $J$, $p$, $i_1$ and $i_2$ s.t.

$$\mathbb{E}\int_0^1\int_0^1\left|\hat{C}_{i_1,i_2,g}(t,s)-\mathbb{E}\hat{C}_{i_1,i_2,g}(t,s)\right|^2\ \mathrm{d}t\mathrm{d}s\le U^2/J.$$

Thus Lemma 3 holds. □

**Proof of Theorem 5**: Note that

$$\begin{aligned}
&2\pi\left\|f_{i_1,i_2,\theta}-\hat{f}_{i_1,i_2,\theta}\right\|_2=\left\|\sum_{g\in\mathbb{Z}}C_{i_1,i_2,g}\exp(\mathrm{i}g\theta)-\sum_{|g|\le r}\left(1-\frac{|g|}{r}\right)\hat{C}_{i_1,i_2,g}\exp(\mathrm{i}g\theta)\right\|_2\\
\le&\left\|\sum_{|g|\le r}\left(1-\frac{|g|}{r}\right)\left(C_{i_1,i_2,g}-\hat{C}_{i_1,i_2,g}\right)\exp(\mathrm{i}g\theta)\right\|_2+\left\|\frac{1}{r}\sum_{|g|\le r}|g|C_{i_1,i_2,g}\exp(\mathrm{i}g\theta)\right\|_2+\left\|\sum_{|g|>r}C_{i_1,i_2,g}\exp(\mathrm{i}g\theta)\right\|_2\\
\le&\sum_{|g|\le r}\left(1-\frac{|g|}{r}\right)\left\|C_{i_1,i_2,g}-\hat{C}_{i_1,i_2,g}\right\|_2+\frac{1}{r}\sum_{|g|\le r}|g|\left\|C_{i_1,i_2,g}\right\|_2+\sum_{|g|>r}\left\|C_{i_1,i_2,g}\right\|_2.
\end{aligned}$$

The last two terms converge to 0 as $r \to \infty$ by the fact that $\sum_{g \in \mathbb{Z}} \|C_{i_1,i_2,g}\|_2 < \infty$ according to Kronecker's lemma. For the first term, by the triangle inequality, we have

$$
\begin{aligned}
&\sum_{|g| \le r} \left(1 - \frac{|g|}{r}\right) \mathbb{E} \left\| C_{i_1,i_2,g} - \hat{C}_{i_1,i_2,g} \right\|_2 \\
\le &\sum_{|g| \le r} \left(1 - \frac{|g|}{r}\right) \mathbb{E} \left\| \hat{C}_{i_1,i_2,g} - \mathbb{E}\hat{C}_{i_1,i_2,g} \right\|_2 + \sum_{|g| \le r} \left(1 - \frac{|g|}{r}\right) \mathbb{E} \left\| \mathbb{E}\hat{C}_{i_1,i_2,g} - C_{i_1,i_2,g} \right\|_2 \\
\le &\sum_{|g| \le r} \left(1 - \frac{|g|}{r}\right) \mathbb{E} \left\| \hat{C}_{i_1,i_2,g} - \mathbb{E}\hat{C}_{i_1,i_2,g} \right\|_2 + \frac{1}{J} \sum_{|g| \le r} |g| \left\| C_{i_1,i_2,g} \right\|_2,
\end{aligned}
$$

where the second term converges to 0 as $J, r \to \infty$ and $r = o(J)$. Besides, Lemma 3 implies that

$$
\sum_{|g| \le r} \left(1 - \frac{|g|}{r}\right) \mathbb{E} \left\| C_{i_1,i_2,g} - \tilde{C}_{i_1,i_2,g} \right\|_2 \le \frac{Ur}{\sqrt{J}},
$$

and this rate does not depend on $\theta$, $i_1$, and $i_2$. Given $r = o(J^{1/2})$, the statement of Theorem 5 follows directly.

## A.7 Proof of Theorem 6

*Proof.* Recall that $\mathcal{G}_\theta$ is the integral operator associated with the kernel

$$
\sum_{i=1}^{p} f_{i,i}(t, s|\theta) = \sum_{k=1}^{\infty} \nu_k(\theta) \overline{\psi_k(t|\theta)} \psi_k(s|\theta),
$$

where $\nu_k(\theta) = \operatorname{tr}\left\{\boldsymbol{\eta}_k(\theta)\right\}$. Similarly, let $\hat{\mathcal{G}}_\theta$ be the integral operator associated with the kernel $\sum_{i=1}^{p} \hat{f}_{i,i}(t, s|\theta) = \sum_{k=1}^{\infty} \hat{\nu}_k(\theta) \overline{\hat{\psi}_k(t|\theta)} \hat{\psi}_k(s|\theta)$. Let

$$
C := \inf_{\theta \in [-\pi,\pi], k' \neq k} \left| \frac{1}{p} \sum_{i=1}^{p} \left\{ \eta_{i,i,k'}(\theta) - \eta_{i,i,k}(\theta) \right\} \right| > 0,
$$

we have

$$
\left\| \psi_k(\cdot|\theta) - \hat{\psi}_k(\cdot|\theta) \right\| \le 2\sqrt{2} \cdot \frac{\left\| \mathcal{G}_\theta - \hat{\mathcal{G}}_\theta \right\|_\infty}{pC}, \ \forall \theta \in [-\pi, \pi],
$$

by Lemma 4.3 in Bosq (2000), where $|| \cdot ||_\infty$ is the operator norm of a linear operator given as

$$
||\mathcal{T}||_\infty = \sup_{||x|| \le 1} ||\mathcal{T}x||,
$$

with $x \in L^2([0,1],\mathbb{C})$ and $\mathcal{T}$ being an operator between $L^2([0,1],\mathbb{C})$s. Accordingly,

$$\begin{aligned}
\sup_{l\in\mathbb{Z}} \left\|\phi_{kl} - \hat{\phi}_{kl}\right\| \le \int_{-\pi}^{\pi} \left\|\psi_k(\cdot|\theta) - \hat{\psi}_k(\cdot|\theta)\right\| \, \mathrm{d}\theta \le \frac{2\sqrt{2}}{Cp} \int_{-\pi}^{\pi} \left\|\mathcal{G}_\theta - \hat{\mathcal{G}}_\theta\right\|_\infty \, \mathrm{d}\theta \\
= \frac{2\sqrt{2}}{Cp} \int_{-\pi}^{\pi} \sup_{||x||\le 1} \sqrt{\int_0^1 \left| \int_0^1 \left[ \sum_{i=1}^{p} \left\{ f_{i,i}(t,s|\theta) - \hat{f}_{i,i}(t,s|\theta) \right\} \right] x(t) \, \mathrm{d}t \right|^2 \, \mathrm{d}s} \, \mathrm{d}\theta \\
\le \frac{2\sqrt{2}}{C} \int_{-\pi}^{\pi} \left\| \frac{1}{p} \sum_{i=1}^{p} \left( f_{i,i,\theta} - \hat{f}_{i,i,\theta} \right) \right\| \, \mathrm{d}\theta, \text{ a.s..}
\end{aligned}$$

Define $\mathcal{C}_{i_1,i_2}(\theta) = \sqrt{\|f_{i_1,i_1,\theta} - \hat{f}_{i_1,i_1,\theta}\| \cdot \|f_{i_2,i_2,\theta} - \hat{f}_{i_2,i_2,\theta}\|}$. Note that

$$\begin{aligned}
\sqrt{\left\{ \int_{-\pi}^{\pi} \left\| \frac{1}{p} \sum_{i=1}^{p} \left( f_{i,i,\theta} - \hat{f}_{i,i,\theta} \right) \right\| \, \mathrm{d}\theta \right\}^2} &\le \sqrt{\frac{\sum_{1\le i_1,i_2\le p} \int_{-\pi}^{\pi} \mathcal{C}^2_{i_1,i_2}(\theta) \, \mathrm{d}\theta}{p^2}} \\
&\le \frac{\sqrt{\sum_{i=1}^{p} \int_{-\pi}^{\pi} \mathcal{C}^2_{i,i}(\theta) \, \mathrm{d}\theta}}{p} + \frac{\sqrt{\sum_{1\le i_1\ne i_2\le p} \int_{-\pi}^{\pi} \mathcal{C}^2_{i_1,i_2}(\theta) \, \mathrm{d}\theta}}{p}, \ a.s..
\end{aligned}$$

By Theorem 5, we have

$$\begin{aligned}
\frac{\sqrt{\sum_{i=1}^{p} \int_{-\pi}^{\pi} \mathcal{C}^2_{i,i}(\theta) \, \mathrm{d}\theta}}{p} &\le \sqrt{\frac{2\pi}{p}} \cdot \sup_{i\in V} \mathcal{C}_{i,i} \\
&= \frac{1}{\sqrt{p}} \cdot O_{\mathrm{p}} \left\{ \frac{r}{\sqrt{J}} + \sup_{i\in V} \left( \frac{1}{r} \sum_{|g|\le r} |g| \, \|C_{i,i,g}\| + \sum_{|g|>r} \|C_{i,i,g}\| \right) \right\}, a.s..
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathbb{E} \frac{\sqrt{\sum_{1\le i_1\ne i_2\le p} \int_{-\pi}^{\pi} \mathcal{C}^2_{i_1,i_2}(\theta) \, \mathrm{d}\theta}}{p} &\le \sqrt{2\pi} \cdot \mathbb{E} \sup_{1\le i_1\ne i_2\le p} \mathcal{C}_{i_1,i_2} \\
&= \sqrt{2\pi} \cdot \varrho_{p,J,r} \cdot \mathbb{E} \sup_{i\in V} \mathcal{C}_{i,i} \\
&= \varrho_{p,J,r} \cdot O \left\{ \frac{r}{\sqrt{J}} + \sup_{i\in V} \left( \frac{1}{r} \sum_{|g|\le r} |g| \, \|C_{i,i,g}\| + \sum_{|g|>r} \|C_{i,i,g}\| \right) \right\}.
\end{aligned}$$

Combining the above results, we conclude that

$$\sup_{l\in\mathbb{Z}} \left\|\phi_{kl} - \hat{\phi}_{kl}\right\| = O_{\mathrm{p}} \left[ \left( \frac{1}{\sqrt{p}} + \varrho_{p,J,r} \right) \cdot \left\{ \frac{r}{\sqrt{J}} + \sup_{i\in V} \left( \frac{1}{r} \sum_{|g|\le r} |g| \cdot \left\|C_{i,i,g}\right\| + \sum_{|g|>r} \left\|C_{i,i,g}\right\| \right) \right\} \right].$$

$\square$

**Remark**: We can prove that $\hat{\eta}_{i_1,i_2,k}(\theta)$ is a consistent estimate of $\eta_{i_1,i_2,k}(\theta)$ for $i_1, i_2 \in V$. Note that

$$
\begin{aligned}
& |\eta_{i_1,i_2,k}(\theta) - \hat{\eta}_{i_1,i_2,k}(\theta)| \\
\leq & \int_0^1 \int_0^1 \left\{ f_{i_1,i_2}(t,s|\theta)\psi_k(t|\theta)\overline{\psi_k(s|\theta)} - \hat{f}_{i_1,i_2}(t,s|\theta)\hat{\psi}_k(t|\theta)\overline{\hat{\psi}_k(s|\theta)} \right\} \, \mathrm{d}t\mathrm{d}s \\
\leq & \int_0^1 \int_0^1 \left\{ f_{i_1,i_2}(t,s|\theta) - \hat{f}_{i_1,i_2}(t,s|\theta) \right\} \psi_k(t|\theta)\overline{\psi_k(s|\theta)} \, \mathrm{d}t\mathrm{d}s \\
& + \int_0^1 \int_0^1 \hat{f}_{i_1,i_2}(t,s|\theta) \left\{ \psi_k(t|\theta) - \hat{\psi}_k(t|\theta) \right\} \overline{\psi_k(s|\theta)} \, \mathrm{d}t\mathrm{d}s \\
& + \int_0^1 \int_0^1 \hat{f}_{i_1,i_2}(t,s|\theta)\hat{\psi}_k(t|\theta) \left\{ \overline{\psi_k(s|\theta)} - \overline{\hat{\psi}_k(s|\theta)} \right\} \, \mathrm{d}t\mathrm{d}s \\
\leq & \left\| \hat{f}_{i_1,i_2,\theta} - f_{i_1,i_2,\theta} \right\|_2 + 2 \left\| \hat{f}_{i_1,i_2,\theta} \right\|_2 \left\| \psi_k(\theta) - \hat{\psi}_k(\theta) \right\| \\
\leq & \sup_{\theta \in [-\pi,\pi]} \left\| \hat{f}_{i_1,i_2,\theta} - f_{i_1,i_2,\theta} \right\|_2 + 2 \left( \sup_{\theta \in [-\pi,\pi]} \left\| \hat{f}_{i_1,i_2,\theta} \right\|_2 \right) \left( \sup_{\theta \in [-\pi,\pi]} \left\| \psi_k(\theta) - \hat{\psi}_k(\theta) \right\| \right).
\end{aligned}
$$

Since

$$
\begin{aligned}
\sup_{\theta \in [-\pi,\pi]} \left\| \hat{f}_{i_1,i_2,\theta} - f_{i_1,i_2,\theta} \right\|_2 &= o_{\mathrm{p}}(1), \\
\sup_{\theta \in [-\pi,\pi]} \left\| \psi_k(\theta) - \hat{\psi}_k(\theta) \right\| &= o_{\mathrm{p}}(1), \\
\sup_{\theta \in [-\pi,\pi]} \left\| \hat{f}_{i_1,i_2,\theta} \right\|_2 &= O_{\mathrm{p}}(1),
\end{aligned}
$$

it follows that

$$
\sup_{\theta \in [-\pi,\pi]} |\eta_{i_1,i_2,k}(\theta) - \hat{\eta}_{i_1,i_2,k}(\theta)| = o_{\mathrm{p}}\left(1\right), \text{ as } J \to \infty.
$$

# B Implementation Details

## B.1 Pre-smoothing for Contaminated Data

To pre-smoothing the functional data, we use a collection of basis functions to approximate $X_{ij}(\cdot)$ as $X_{ij}(\cdot) \approx \boldsymbol{d}_{ij}^T \boldsymbol{B}(\cdot)$, where $\boldsymbol{B}(\cdot) = (B_1(\cdot), \cdots, B_H(\cdot))^T$ are the B-spline functions of degree three, with knots placed at $\{t_z; z = 1, \cdots, Z\}$. Each vector $\boldsymbol{d}_{ij}$ is estimated by smoothing splines (Gu 2002) based on $\{(t_z, Y_{ijz}); z = 1, \cdots, Z\}$, with $\lambda_{ij}^{\mathrm{b}}$ being the tuning parameter to penalize the integrated squared second derivative of the resulting estimate. We choose $\lambda_{ij}^{\mathrm{b}}$ for $X_{ij}(\cdot)$ by generalized cross-validation (Gu 2002). After that, we calculate $\boldsymbol{D} = \int_0^1 \boldsymbol{B}(t)\{\boldsymbol{B}(t)\}^T \, \mathrm{d}t$ and replace $\boldsymbol{d}_{ij}$ and $\boldsymbol{B}(t)$ by $\boldsymbol{D}^{1/2}\boldsymbol{d}_{ij}$ and $\boldsymbol{D}^{-1/2}\boldsymbol{B}(t)$, respectively. This is equivalent to transforming the basis functions to being mutually orthogonal for the pre-smoothing of functional data.

After the pre-smoothing step, we estimate $\mu_i(\cdot)$ by $\hat{\mu}_i(\cdot) = \bar{\boldsymbol{d}}_i^T \boldsymbol{B}(\cdot)$ with $\bar{\boldsymbol{d}}_i = \frac{1}{J}\sum_{j=1}^J \boldsymbol{d}_{ij}$. Furthermore, $\sigma_i^2$ is estimated as $\hat{\sigma}_i^2 = \frac{1}{J}\sum_{j=1}^J \hat{\sigma}_{ij}^2$, where

$$\hat{\sigma}_{ij}^2 = \frac{1}{Z - \mathrm{tr}_s(\lambda_{ij}^{\mathrm{b}})} \sum_{z=1}^{Z} \left\{Y_{ijz} - \boldsymbol{d}_{ij}^T \boldsymbol{B}(t_z)\right\}^2$$

as proposed in Gu (2002), with $\mathrm{tr}_s(\lambda_{ij,\mathrm{b}})$ being the trace of the corresponding smoothing matrix with tuning parameter $\lambda_{ij}^{\mathrm{b}}$. Accordingly, $\hat{C}_{i_1,i_2,d}(t,s)$ and $\hat{f}_{i_1,i_2}(t,s|\theta)$ are obtained as

$$\begin{aligned}\hat{C}_{i_1,i_2,g}(t,s) &= \{\boldsymbol{B}(t)\}^T \boldsymbol{C}_{i_1,i_2,g}^d \ \boldsymbol{B}(s),\\ \hat{f}_{i_1,i_2}(t,s|\theta) &= \{\boldsymbol{B}(t)\}^T \boldsymbol{F}_{i_1,i_2}^d(\theta) \ \boldsymbol{B}(s),\end{aligned}$$

where $\boldsymbol{C}_{i_1,i_2,g}^d = \frac{1}{J}\sum_{j=1}^{J-g}(\boldsymbol{d}_{i_1(j+g)} - \bar{\boldsymbol{d}}_{i_1})(\boldsymbol{d}_{i_2 j} - \bar{\boldsymbol{d}}_{i_2})^T$ and $\boldsymbol{F}_{i_1,i_2}^d(\theta) = \frac{1}{2\pi}\sum_{|g|\le r}(1 - \frac{|g|}{r})\boldsymbol{C}_{i_1,i_2,g}^d \exp(\mathrm{i}\, g\theta)$. Thereafter, we set the eigenvalue $\hat{\nu}_k(\theta)$ and eigenfunction $\hat{\psi}_k(t|\theta)$ of $\sum_{i=1}^p \hat{f}_{i,i}(t,s|\theta)$ as $\hat{\nu}_k^d(\theta)$ and $\left\{\boldsymbol{\psi}_k^d(\theta)\right\}^T \boldsymbol{B}(t)$, respectively, where $\hat{\nu}_k^d(\theta)$ and $\boldsymbol{\psi}_k^d(\theta)$ are the $k^{th}$ eigenvalue and eigenvector of $\sum_{i=1}^p \boldsymbol{F}_{i,i}^d(\theta)$.

Let $\{\theta_m; m = 1, \cdots, M\}$ be some grid points that are equally spaced in $[-\pi, \pi]$ for some sufficiently large $M$, and define

$$\boldsymbol{\phi}_{kl}^d := \frac{1}{M}\sum_{m=1}^{M} \boldsymbol{\psi}_k^d(\theta_m)\exp(-\mathrm{i}\, l\theta_m).$$

We then calculate $\hat{\phi}_{kl}(\cdot)$ as $\left(\boldsymbol{\phi}_{kl}^d\right)^T \boldsymbol{B}(\cdot)$. Besides, for each $\theta$, the $(i_1, i_2)^{th}$ element of the eigen-matrix $\boldsymbol{\eta}_k(\theta)$ is estimated by

$$\hat{\eta}_{i_1,i_2,k}(\theta) = \left\{\boldsymbol{\psi}_k^d(\theta)\right\}^T \boldsymbol{F}_{i_1,i_2}^d(\theta)\overline{\boldsymbol{\psi}_k^d(\theta)}.$$

Let $\hat{\boldsymbol{\eta}}_k(\theta) = \{\hat{\eta}_{i_1,i_2,k}(\theta)\}_{1\le i_1,i_2\le p}$.

## B.2 Truncation Rules

In this subsection, we establish the truncation rules for the $K$ and $L_k$s in equation (23) in the main text. Let $K_{\max}$ be the upper bound of $K$ and define $\boldsymbol{\zeta}_{jk}(t) = \sum_{l\in\mathbb{Z}} \phi_{kl}(t)\boldsymbol{\xi}_{\cdot,(j+l)k}$. Notice that

$$\mathbb{E}||\boldsymbol{\varepsilon}_j||_p^2 = \sum_{i=1}^{p} \int_0^1 C_{i,i,0}(t,t)\ \mathrm{d}t = \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \nu_k(\theta)\ \mathrm{d}\theta \quad \text{and} \quad \mathbb{E}||\boldsymbol{\zeta}_{jk}||_p^2 = \int_{-\pi}^{\pi} \nu_k(\theta)\ \mathrm{d}\theta,$$

we use $\int_{-\pi}^{\pi} \hat{\nu}_k(\theta)\ \mathrm{d}\theta$ to measure the variance explained by $\boldsymbol{\zeta}_{jk}(\cdot)$. One method to choose a suitable $K$ is to pick a $K$ within the range $K \leq K_{\max}$ that maximizes the ratio of variance explained

$$\frac{\int_{-\pi}^{\pi} \hat{\nu}_k(\theta)\ \mathrm{d}\theta}{\int_{-\pi}^{\pi} \hat{\nu}_{k+1}(\theta)\ \mathrm{d}\theta}.$$

Alternatively, we can take $K$ as the minimal value such that the fraction of variance explained

$$\frac{\sum_{k\leq K} \int_{-\pi}^{\pi} \hat{\nu}_k(\theta)\ \mathrm{d}\theta}{\sum_{i=1}^{p} \int_0^1 \hat{C}_{i,i,0}(t,t)\ \mathrm{d}t}$$

is larger than a given value. The ratio or fraction of variance explained can be approximated by the pre-smoothing of functional data in Part B.1.

For the value of $L_k$, we refer to (14) in the main text and choose $L_k$ such that the cumulative norm $\sum_{|l|\leq L_k} ||\hat{\phi}_{kl}||^2$ is close to 1, for example, $\sum_{|l|\leq L_k} ||\hat{\phi}_{kl}||^2$ is greater than 0.99. We choose a upper bound $L_{\max}$ for $L_k$ and select $L_k$ within the range $0 \leq L_k \leq L_{\max}$. When $\sum_{|l|\leq L_k} ||\hat{\phi}_{kl}||^2$ dose not obtain 0.99, we simply set that $L_k = L_{\max}$. Moreover, note that

$$\phi_{k(l+l_s)}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \psi_k(t|\theta) \exp(-\mathrm{i}\, l_s\theta) \right\} \exp(-\mathrm{i}\, l\theta)\ \mathrm{d}\theta,$$

$\forall l_s \in \mathbb{Z}$, where $\exp(-\mathrm{i}\, l_s\theta)$ can be viewed as a multiplicative factor for $\psi_k(\cdot|\theta)$. We conclude that the functional filters $\{\phi_{k(l+l_s)}(\cdot); l \in \mathbb{Z}\}$ with arbitrary $l_s \in \mathbb{Z}$ does not alter the reconstructed process $\boldsymbol{\zeta}_{jk}(t)$; see the Remark in Part A.3 for details. Therefore, to obtain a smaller $L_k$, we instead shift the largest term $||\hat{\phi}_{kl}||^2$ with respect to (w.r.t.) $l$ to the zero location, and then add the adjacent terms of which until the cumulative norm meets the threshold.

The above truncation rules can be applied to WDFPCA and GDFPCA. For the truncation rules of the univariate FPCAs (SFPCA and DFPCA), WSFPCA and GSFPCA, we can similarly define the ratio or fraction of variance explained, and the cumulative norm for the selection of $K_i$s or $L_{k,i}$s in Table 1 in the main text; refer to Ramsay & Silvermann (2005), Hörmann et al. (2015), and Zapata et al. (2022) for further details.

## B.3 Gradient Ascend Algorithm

We denote $\hat{\boldsymbol{\Sigma}}$, $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{\Phi}}$ as the estimates of $\boldsymbol{\Sigma}$, $\boldsymbol{\mu}$, $\boldsymbol{\phi}$, $\boldsymbol{\Phi}$ given the pre-smoothing of the contaminated data in Part B.1. Accordingly, one can be shown that the gradient of

$f_d\left(\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_K | \boldsymbol{Y}, \hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\Phi}}\right)$ w.r.t. $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_K$ is

$$\frac{\partial f_d}{\partial \boldsymbol{\xi}_k}(\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_K | \boldsymbol{Y}, \hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\Phi}})$$
$$= -\sum_{j=1}^{J} \left[ \hat{\boldsymbol{\Sigma}}^{-1} \left( \sum_{k'=1}^{K} \boldsymbol{\xi}_{k'} \boldsymbol{A}_{k'j} \boldsymbol{B} - \tilde{\boldsymbol{Y}}_j \right) (\boldsymbol{A}_{kj} \boldsymbol{B})^T + \mathrm{Re}\left\{ \hat{\boldsymbol{\Phi}}_k(\theta_j) \boldsymbol{\xi}_k \boldsymbol{\rho}_k(\theta_j) \boldsymbol{\rho}_k(\theta_j)^* \right\} \right],$$

where $\tilde{\boldsymbol{Y}}_j$ is a $p \times Z$ matrix with the $(i,z)^{th}$ element being the residual $Y_{ijz} - \hat{\mu}_i(t_z)$, $\boldsymbol{A}_{kj}$ is a $(J + 2L_k) \times H$ matrix with the $(j + l + L_k)^{th}$ row being the functional filters $\left(\boldsymbol{\phi}_{kl}^d\right)^T$ for $|l| \leq L_k$ and 0 otherwise, $\boldsymbol{B}$ is an $H \times Z$ matrix of the spline basis with the $z^{th}$ column being $\boldsymbol{B}(t_z)$, and $\mathrm{Re}(\cdot)$ is the operation to extract the real part of a complex matrix.

To implement the gradient ascend algorithm for score extractions, we initialize the scores for optimization as the projected scores in Hörmann et al. (2015), which are the filtered variables extracted from equation (11) in the main text, by replacing $\tilde{\phi}_{kl}(\cdot)$ with $\hat{\phi}_{kl}(\cdot)$ for $|l| \leq L_k$ and 0 otherwise, and by setting $\xi_{ijk}$ and $\varepsilon_{ij}$ as 0 for $j > J$ or $j < 1$.

## B.4 Model Complexity of AIC

Recall that we select the $\lambda_k$ in $\hat{\boldsymbol{\Phi}}_{k,\lambda_k}$ by the AIC of Whittle likelihood. To determine the model complexity term $\mathrm{df}(\lambda_k)$ in AIC, we utilize the B-spline basis functions with knots $\{\theta_j; \theta_j \in [0, \pi]\}$ to fit $\{\hat{\boldsymbol{\Phi}}_{k,\lambda_k}(\theta_j); \theta_j \in [0, \pi]\}$, i.e.,

$$\mathrm{Re}([\hat{\boldsymbol{\Phi}}_{k,\lambda_k}(\theta_j)]_{i_1,i_2}) \approx \sum_{u=1}^{U} a_{u,i_1,i_2}^{Re} B_u(\theta_j),$$
$$\mathrm{Im}([\hat{\boldsymbol{\Phi}}_{k,\lambda_k}(\theta_j)]_{i_1,i_2}) \approx \sum_{u=1}^{U} a_{u,i_1,i_2}^{Im} B_u(\theta_j),$$

where $\mathrm{Im}(\cdot)$ is the operation to extract the imaginary part of a complex number, $\{B_u(\cdot); u \leq U\}$ are the used B-spline functions on $[0, \pi]$, and $\{a_{u,i_1,i_2}^{Re}, a_{u,i_1,i_2}^{Im}; u \leq U,\ i_1, i_2 \in V\}$ are the coefficients estimated by the smoothing spline (Gu 2002) with a tuning parameter to control the second-order smoothness. We choose the tuning parameter for the smoothing spline s.t. the R-square of their total fitting is close to 0.95. Hence, the trace of the corresponding smoothing matrix, denoted as $v_{\lambda_k}$, could be used to approximate the degree of smoothness of $\hat{\boldsymbol{\Phi}}_{k,\lambda_k}(\theta)$ w.r.t. $\theta$. Let $n_{\lambda_k}$ be the number of non-zero elements in $\mathrm{Re}\left\{\hat{\boldsymbol{\Phi}}_{k,\lambda_k}(\theta_1)\right\}$. Then the model complexity $\mathrm{df}(\lambda_k)$ in AIC is calculated as $n_{\lambda_k} v_{\lambda_k}$.

# C Supporting Results

## C.1 Additional Simulation Studies

The case with $J = 60$ in the simulation study of the main text is presented in Table 1.

Table 1: The NMSE($q$) of different FPCAs when the dynamic weak separability is achieved. We highlight the best performance in each setting in bold (excluding (G)DFPCA and (G)SFPCA).

| NMSE($q$) (%) | | | $\kappa = 0$ | | | | $\kappa = 3$ | | | | $\kappa = 6$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $q = 1$ | $q = 2$ | $q = 3$ | $q = 4$ | $q = 1$ | $q = 2$ | $q = 3$ | $q = 4$ | $q = 1$ | $q = 2$ | $q = 3$ | $q = 4$ |
| $p = 30$ | $J = 60$ | SFPCA | 57.28 | 37.56 | 25.93 | 18.87 | 57.38 | 37.59 | 25.93 | 18.84 | 57.69 | 37.76 | 25.83 | 18.64 |
| | | WSFPCA | 63.35 | 41.69 | 26.05 | 15.33 | 63.05 | 41.33 | 25.99 | 15.29 | 62.53 | 41.11 | 25.67 | 15.47 |
| | | GSFPCA | 63.22 | 41.43 | 25.58 | 14.61 | 62.93 | 41.07 | 25.52 | 14.59 | 62.39 | 40.82 | 25.17 | 14.71 |
| | | (G)SFPCA | 63.22 | 41.43 | 25.58 | 14.61 | 62.90 | 41.02 | 25.45 | 14.50 | 62.32 | 40.67 | 24.94 | 14.42 |
| | | DFPCA | **51.15** | **30.46** | 20.59 | 16.83 | **51.26** | **30.49** | 20.61 | 16.84 | **51.54** | **30.75** | 20.76 | 16.93 |
| | | WDFPCA | 61.08 | 37.92 | 20.75 | 9.10 | 60.34 | 37.07 | 20.38 | 9.32 | 58.52 | 35.48 | 19.49 | 9.94 |
| | | GDFPCA | 61.00 | 37.21 | **19.71** | **7.58** | 60.19 | 36.24 | **19.20** | **7.55** | 58.28 | 34.52 | **18.05** | **7.74** |
| | | (G)DFPCA | 61.00 | 37.21 | 19.71 | 7.59 | 60.17 | 36.21 | 19.16 | 7.49 | 58.22 | 34.42 | 17.91 | 7.59 |
| $p = 60$ | $J = 60$ | SFPCA | 57.26 | 37.43 | 25.68 | 18.57 | 57.08 | 37.27 | 25.60 | 18.52 | 57.74 | 37.44 | 25.35 | 18.02 |
| | | WSFPCA | 63.17 | 41.65 | 26.10 | 15.36 | 63.00 | 41.42 | 25.97 | 15.29 | 59.61 | 38.02 | 24.14 | 15.06 |
| | | GSFPCA | 63.04 | 41.39 | 25.62 | 14.64 | 62.87 | 41.15 | 25.50 | 14.58 | 59.47 | 37.74 | 23.68 | 14.41 |
| | | (G)SFPCA | 63.04 | 41.39 | 25.62 | 14.64 | 62.85 | 41.12 | 25.45 | 14.52 | 59.36 | 37.51 | 23.33 | 13.94 |
| | | DFPCA | **51.24** | **30.41** | 20.58 | 16.83 | **51.19** | **30.44** | 20.60 | 16.86 | **51.34** | 29.71 | 19.79 | 16.00 |
| | | WDFPCA | 61.06 | 37.88 | 20.71 | 8.80 | 60.79 | 37.57 | 20.56 | 8.85 | 53.26 | 29.60 | 16.89 | 10.36 |
| | | GDFPCA | 60.97 | 37.36 | **19.88** | **7.57** | 60.70 | 36.99 | **19.65** | **7.52** | 52.51 | **27.94** | **14.48** | **7.14** |
| | | (G)DFPCA | 60.98 | 37.36 | 19.88 | 7.57 | 60.69 | 36.97 | 19.62 | 7.49 | 52.44 | 27.80 | 14.28 | 6.87 |

We also compare the variances of the estimated functional filters from different methods. It should be noted that the functional filter $\phi_{ikl}(\cdot)$ in DFPCA (see Table 1 in the main text) cannot be uniquely identified. To calculate the variances of the estimated functional filters, we define

$$S\{t, \theta; \phi_{ikl}, l \in \mathbb{Z}\} := \left| \sum_{l \in \mathbb{Z}} \phi_{ikl}(t) \exp(\mathrm{i}\, l\theta) \right|.$$

where $S\{t, \theta; \phi_{ikl}, l \in \mathbb{Z}\}$ for $t \in [0, 1]$ and $\theta \in [-\pi, \pi]$ combines the information of the functional filter in the $k^{th}$ component. Note that $S\{t, \theta; \phi_{ikl}, l \in \mathbb{Z}\} = |\psi_k(t|\theta)|$. Consequently, the value of $S\{t, \theta; \phi_{ikl}, l \in \mathbb{Z}\}$ is unique regardless of the choice of the functional filters.

Next, we apply

$$\hat{S}\{t,\theta;\hat{\phi}_{ikl},|l|\leq L_{k,i}\} := \left|\sum_{|l|\leq L_{k,i}} \hat{\phi}_{ikl}(t)\exp(\mathrm{i}\, l\theta)\right|$$

to approximate $S\{t,\theta;\phi_{ikl},l\in\mathbb{Z}\}$, and use

$$\frac{1}{2\pi p}\sum_{i=1}^{p}\int_0^1\int_{-\pi}^{\pi}\mathrm{Var}\left\{\hat{S}\{t,\theta;\hat{\phi}_{ikl},|l|\leq L_{k,i}\}\right\}\mathrm{d}\theta\mathrm{d}t$$

to reflect the average variance of the estimated functional filters by the univariate DFPCA. This method is referred to as the univariate approach. Similarly, we use

$$\frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi}\mathrm{Var}\left\{\hat{S}\{t,\theta;\hat{\phi}_{kl},|l|\leq L_k\}\right\}\mathrm{d}\theta\mathrm{d}t$$

to reflect the variance of $\{\hat{\phi}_{kl};|l|\leq L_k\}$ estimated from multiple individual series in the graph, referred to as the graphical approach in the following. Table 2 presents the variances for the estimated functional filters in the first component from the above two approaches. These results are obtained from simulation studies that satisfy the dynamic weak separability. According to these results, we observe a significant reduction in the variances of the functional filters estimated using the graphical approach compared to the univariate approach.

Table 2: The variances of the estimated functional filters when the dynamic weak separability is achieved.

| Variance | | $\kappa=0$ | | | $\kappa=3$ | | | $\kappa=6$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $J=20$ | $J=40$ | $J=60$ | $J=20$ | $J=40$ | $J=60$ | $J=20$ | $J=40$ | $J=60$ |
| $p=30$ | Univariate | 0.210 | 0.194 | 0.179 | 0.209 | 0.194 | 0.178 | 0.209 | 0.194 | 0.180 |
| | Graphical | 0.067 | 0.048 | 0.037 | 0.107 | 0.090 | 0.076 | 0.164 | 0.138 | 0.124 |
| $p=60$ | Univariate | 0.208 | 0.191 | 0.176 | 0.209 | 0.192 | 0.177 | 0.205 | 0.192 | 0.177 |
| | Graphical | 0.035 | 0.024 | 0.016 | 0.053 | 0.034 | 0.027 | 0.180 | 0.169 | 0.159 |

## C.2 Assessment of Dynamic Weak Separability

Noting that the dynamic weak separability (4) is equivalent to that

$$\Omega_{k_1,k_2} := \sum_{i_1,i_2=1}^{p} \int_{-\pi}^{\pi} \left| \int_0^1 \int_0^1 f_{i_1,i_2}(t,s|\theta)\psi_{k_1}(t|\theta)\overline{\psi_{k_2}(s|\theta)} \, \mathrm{d}t\mathrm{d}s \right|^2 \mathrm{d}\theta = 0$$

for all $k_1 \neq k_2$, we therefore use the empirical estimate of $\Omega_{k_1,k_2}$, denoted as $\hat{\Omega}_{k_1,k_2}$, to assess the dynamic separability condition (4) for our dataset. The results are illustrated in Figure 1, which show that the $\hat{\Omega}_{k_1,k_2}$s for $k_1 \neq k_2$ are negligible compared to $\hat{\Omega}_{k,k}$s. This indicates certain reliability of the dynamic weak separability (4) for the PM2.5 data.

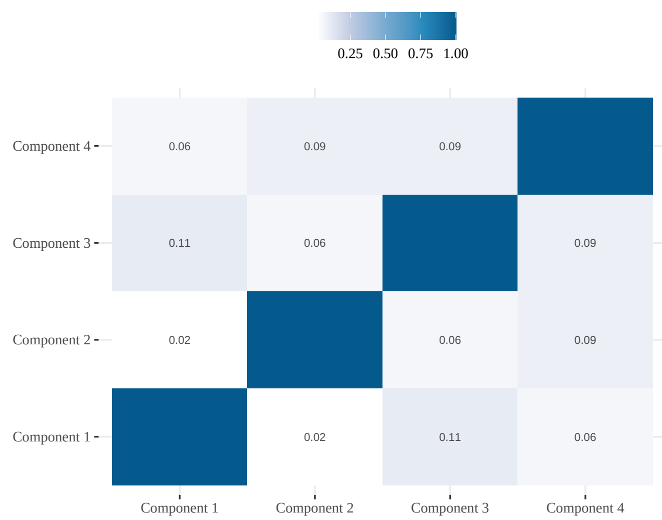


Figure 1: Diagram of $\hat{\Omega}_{k_1,k_2}/\sqrt{\hat{\Omega}_{k_1,k_1}\hat{\Omega}_{k_2,k_2}}$ for the first four components, where $\hat{\Omega}_{k_1,k_2} = \sum_{i_1,i_2=1}^{p} \int_{-\pi}^{\pi} \left| \int_0^1 \int_0^1 \hat{f}_{i_1,i_2}(t,s|\theta)\hat{\psi}_{k_1}(t|\theta)\overline{\hat{\psi}_{k_2}(s|\theta)} \, \mathrm{d}t\mathrm{d}s \right|^2 \mathrm{d}\theta$.

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