A B-Spline Curve Fitting Approach by Implementing the Parameter Correction Terms

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***Abstract*- Fitting a set of points with a B-Spline curve is a usual CADG application, which remains an open problem due to the choice of parameter values. The crucial point is to find optimal parameter values which lead to an optimal approximation curve. Since these parameter values are only a first guess, parameter correction can be used to improve parameterization. This paper discusses iterative solutions in least-squares B-Spline curve fitting sense. And the initial results are presented.**

1. Introduction

Smoothing a set of points P_i in R^3 with a curve is a usual CAGD application. The smoothing curve can be a Bézier curve if a low degree curve can provide a solution. A NURBS curve can also be considered but the weights largely increase the degrees of freedom of this difficult problem without significant benefits. This explains the widely use of B-spline curves.

In parametric approximation of curves an ordered set of points P_i (x_i, y_i, z_i) is given and a class of basis functions is prescribed. The given points are parameterized, for each P_i, a parameter value t_i is assigned. The key question in parameterization is to assign good parameter values to the data points, to locate points on the optimal smooth curve, which are close to the data points. Many solutions have been proposed for choosing the parameter values [10], [11], [12], but no solution can be considered as the best choice in any situation.

Let P be a set of ordered and distinct points P_i $(i=0,..,N)$ in R^3 to be approximated by a B-Spline curve.

$$C(t) = \sum_{i=0}^{n} N_{i,p}(t) d_i \qquad (1)$$

where the $N_{i,p}(t)$ are the $n+1$ normalized B-Spline basis functions of order p (degree $p-1$) defined on knot vector U with
$U = \{0,...,0, u_{p+1},...,u_{r-p-1},1,...,1\}$
The unknowns are the control points d_i's.

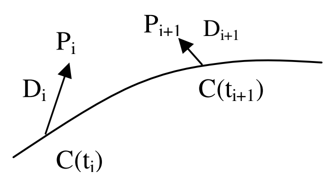

Figure 1. The given points P_i and the error vectors D_i with respect to given parameterization t_i.

The basic aim here is to fit a B-Spline curve that will approximate N measured data points in a Least Squares Method sense. This leads to minimization problem: find an optimal set of parameter values t_i producing an optimal approximating spline $C(t)$ with minimal distances to the points P_i. The objective function is

$$D_i = \sum_{i=0}^{N} \left\| P_i - C(t_i) \right\|^2 \rightarrow \min \qquad (2)$$

where $C(t_i)$ is the B-Spline curve point at t_i and P_i is the corresponding measured data point. When the t_i are fixed, the numerical best formulation to compute the control points is an over determined $(n+1)*(n+1)$ linear system solved by a least square method.

In general, error vectors D_i are not orthogonal to the approximation curve $C(t)$. Thus we have a linear problem, which exactly means that we do not have to minimize the shortest distances between the given points and the approximation curve. We obtain a situation as described in Fig.1. This paper discusses mainly how this situation can be improved. So we implement the parameter correction terms for each parameter values t_i on the approximation curve. This improvement has two steps: Initial parameterization and reparameterization steps. The reparametrization step, or so-called parameter correction, has been used by various authors [1], [6] ,[7] Rogers [6] used a Gauss-Newton method, and Hoschek [1] used a simplified Newton method.

Three new approaches have been recently proposed [14], [15], [16] in order to deal with the same approximation problem. The first one considers the problem of equation (3) as a global problem. The unknowns are the control points and the parameter values. The minimization problem is solved with a Levenberg–Marquardt method. The second paper [15] is proposed in a larger framework of optimal control but can be directly applied to the same problem. It also considers the global problem of equation (3) but takes into account that variables d_i and t_i are linked with a state equation. The minimization is split into two searches both based on gradient calculations. This implies smaller systems and as a result better computing times than in [14]. The third one also deals with a new improvement of Hoschek's method providing better results with a higher speed of convergence. Their solution is to optimize the minimization problem with respect to t by applying an unconstrained nonlinear optimization method.

2. Correction Terms for Reparameterization

In all parameterization strategies the distance vectors are generally not perpendicular to the surface. This means we have minimized an arbitrary error due to the parameterization. Now we will change the parameterization with the aim to approximate the shortest distances, which means we will construct a sequence of new parameter values t_i with the goal that the corresponding error vectors D_i converge in general to the normals of the approximation curve. Therefore we replace the curve at each point $C(t_i)$ by the tangent as described in Fig.2, by projecting the error vector D_i on the tangent and obtain Δt_i as a measure for changing the parameter values t_i in direction of the parameter values of the perpendicular from P_i to $C(t_i)$

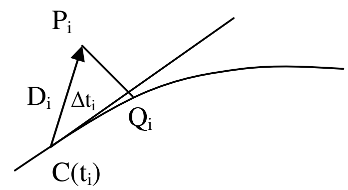

Figure 2. Projection of the error vector D_i on the tangent of $C(t)$ at the point $C(t_i)$ and the foot Q_i of the perpendicular from P_i . Δt_i is a measure for parameter correction.

Hoschek [1] initially proposed a correction value Δt_i, assuming that the length of the overall parameter interval is given by $t_N - t_0$ and that μ is the approximate length of the curve (approximation to the polygon P_i)

$$\Delta t_i = \left\langle D_i, \frac{\dot{C}_i}{\left\|\dot{C}_i\right\|} \right\rangle \frac{t_N - t_0}{\;} \tag{3}$$

In [6] it is proposed to locally minimize the error vector using the first terms in the Taylor expansions by taking the absolute value and differentiating, we get

$$\Delta t_i = \left\langle D_i, \frac{\dot{C}_i}{\left\|\dot{C}_i\right\|^2} \right\rangle \tag{4}$$

Another method for minimizing the local error vector was suggested by [8], [9]. The idea is to minimize $D_i^2 = (P_i - C(t_i))^2$, which after differentiation leads to

$$f := \left\langle D_i, \dot{C}(t) \right\rangle = 0 \tag{5}$$

Now if the well known Newton iteration formula is used to compute a zero of f , we are led to the correction formula

$$\Delta t_i = -\frac{f}{\dot{f}} = \frac{-\langle D_i, \dot{C}_i \rangle}{\langle D_i, \ddot{C}_i \rangle - \langle D_i, D_i \rangle} \tag{6}$$

The new parameter values are defined by $\tilde{t}_i = t_i + \Delta t_i$

3. Algorithm

The algorithm consists of the following two separate parts:

- initial parameterization
- fitting/reparametrization loop

To accomplish the first step of algorithm, initial parameterization must be performed. Centripetal, chord-length, etc. methods can be chosen as the initial parameterization method. The chord-length parameterization is chosen for our example in Fig.3. and the initial results represented in Table 1 and Table 2. We proposed the influence the initial parameterization methods on the initial approximation in Table 3. The second part in the algorithm is the iteration loop in which linear fitting steps alternate with reparameterization steps. In reparameterization stage, the parameter values are improved when we replace the old parameters by the new parameter values which are found after implementing parameter correction formula. However, the parameter correction is permissible if the new error vector, $\tilde{D}_i < D_i$ else the parameter correction is overshooting the foot of the perpendicular from P_i on the approximation curve $C(t)$, then t_i will be corrected only by $\Delta t_i/2$. Then fitting step is repeated until stopping criteria achieved.

4. Implementation and Discussion

Three parameter correction terms are used and compared with the initial approximation curve. The implementation performed on Sun Blade 2000 workstation which has 900-MHz UltraSPARC III Cu CPU (8MB L2 cache). C Programming Language and Sun Performance Library routines are used. Since the approximation problem arises to $Ax=b$ system, which is sparse, we need to solve this sparse system. So we used sparse solver routines due to this sparsity. The different methods are analyzed with respect to the root mean square error and the maximum error defined as:

$$E_{mean} = \sqrt{\frac{1}{NDP+1}\sum_{i=0}^{NDP} \left\| C(t_i) - P_i \right\|^2} \tag{7}$$

$$E_{\max} = \underset{i=0,\ldots,NDP}{\mathrm{Max}} \left\| C(t_i) - P_i \right\| \tag{8}$$

Our example corresponds to a set of points of an isodepth line from a submarine relief (provided by E. Saux, M. Daniel) in Fig.3. We approximate to data set with the C^2 continuous cubic B-Spline curves. The average knot vectors and chord-length parameterization are used for $n+1$ control points. The approximation errors are presented in Tables 1 and 2. The programs have been stopped after number of iterations to be sure that no more significant error improvements can be obtained, the deviation of residual vector is considered.

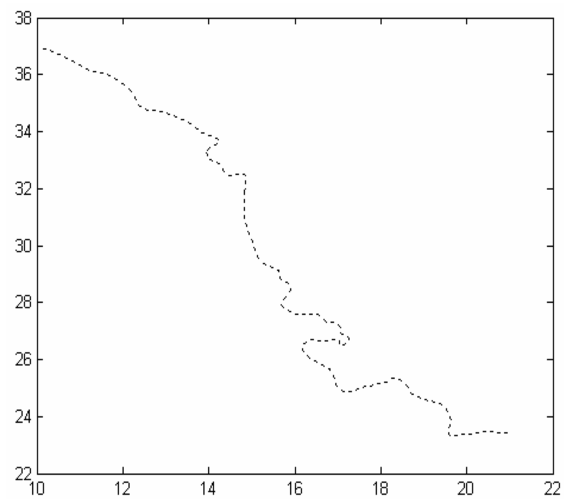

Figure 3. Given data set with 211 points, an isodepth line

Mean and maximum errors obtained from parameter correction implementation are greatly reduced in comparison with the initial approximation. Improvements also are shown in Table 1 and Table 2.

Table 1 Root mean square error for different correction terms and approximating curves.

Emean	n+1=25	n+1=50	n+1=100
Initial Approximation	0.07878	0.02704	0.00564
Using eq. (3)	0.04096	0.01170	0.00304
Using eq. (4)	0.04101	0.01179	0.00301
Using eq. (6)	0.04088	0.01176	0.00300

Table 2 Maximum error for different correction terms and approximating curves.

Emax	n+1=25	n+1=50	n+1=100
Initial Approximation	0.25539	0.08668	0.01807
Using eq. (3)	0.13884	0.05880	0.00917
Using eq. (4)	0.13657	0.05868	0.00919
Using eq. (6)	0.13582	0.05852	0.00918

The initial parameterization method has influence on the initial approximation but do not affect the results of iterative reparameterization solution significantly, for example the result for Universal Parameterization method very close to Centripetal Parameterization method for equation (6) in Table 3.

Fig.4 represents the graph of the convergence rate by using parameter correction term in equation (6). It illustrates the root mean square error and maximum error for $n+1=50$, it is obvious that errors rapidly decrease. Program was ran for 50 iterations, but actually there was no significant change in mean error value after 35th iteration.

Table 3 Influence of the different initial parameterization methods on the reparameterization

Emean (n+1=50)	Chord-Length	Centripetal	Universal
Initial Approximation	0.02704	0.03262	0.04894
Using eq. (3)	0.01170	0.01135	0.01099
Using eq. (4)	0.01179	0.01057	0.01028
Using eq. (6)	0.01176	0.01040	0.01022

5. Conclusions

In this paper we implement a reparameterization method to achieve more optimum parameterization which has very important role in B-Spline approximation problem. The improvement of these methods is efficient. Our example is performed for different parameter correction terms and initial parameterization methods, results are presented. The reparameterization method can also be implemented to surfaces analogously.

In combination with the simple least squares fit, parameter correction may not be stable depending on the point distribution. As the first approximation could be far from the optimal situation parameter correction may fail. It can be stated that the parameter correction together with the approximation including energy terms can be more robust [13].

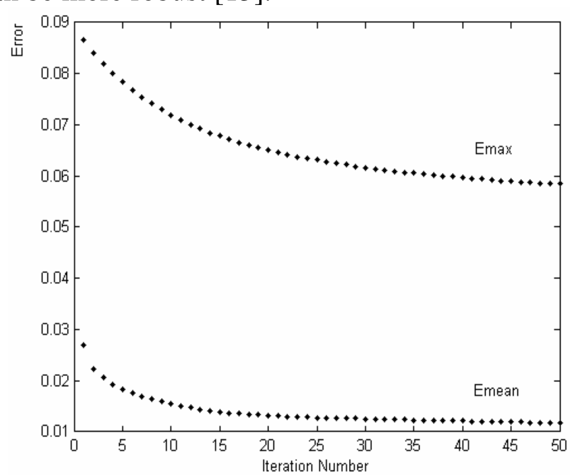

Figure 4. Mean and maximum error for 50 control points using eq.(6)

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