CHAPTER 1
Introduction

Practice Questions

Problem 1.1
What is the difference between a long forward position and a short forward position?

When a trader enters into a long forward contract, she is agreeing to *buy* the underlying asset for a certain price at a certain time in the future. When a trader enters into a short forward contract, she is agreeing to *sell* the underlying asset for a certain price at a certain time in the future.

Problem 1.2.
Explain carefully the difference between hedging, speculation, and arbitrage.

A trader is *hedging* when she has an exposure to the price of an asset and takes a position in a derivative to offset the exposure. In a *speculation* the trader has no exposure to offset. She is betting on the future movements in the price of the asset. *Arbitrage* involves taking a position in two or more different markets to lock in a profit.

Problem 1.3.
What is the difference between entering into a long forward contract when the forward price is $50 and taking a long position in a call option with a strike price of $50?

In the first case the trader is obligated to buy the asset for $50. (The trader does not have a choice.) In the second case the trader has an option to buy the asset for $50. (The trader does not have to exercise the option.)

Problem 1.4.
Explain carefully the difference between selling a call option and buying a put option.

Selling a call option involves giving someone else the right to buy an asset from you. It gives you a payoff of

$$-\max(S_T - K, 0) = \min(K - S_T, 0)$$

Buying a put option involves buying an option from someone else. It gives a payoff of

$$\max(K - S_T, 0)$$

In both cases the potential payoff is $K - S_T$. When you write a call option, the payoff is negative or zero. (This is because the counterparty chooses whether to exercise.) When you buy a put option, the payoff is zero or positive. (This is because you choose whether to exercise.)

Problem 1.5.
An investor enters into a short forward contract to sell 100,000 British pounds for US dollars at an exchange rate of 1.4000 US dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.3900 and (b) 1.4200?

(a) The investor is obligated to sell pounds for 1.4000 when they are worth 1.3900. The gain is (1.4000-1.3900) $\times$100,000 = \$1,000.

(b) The investor is obligated to sell pounds for 1.4000 when they are worth 1.4200. The loss is (1.4200-1.4000)$\times$100,000 = \$2,000

Problem 1.6.
A trader enters into a short cotton futures contract when the futures price is 50 cents per pound. The contract is for the delivery of 50,000 pounds. How much does the trader gain or lose if the cotton price at the end of the contract is (a) 48.20 cents per pound; (b) 51.30 cents per pound?

(a) The trader sells for 50 cents per pound something that is worth 48.20 cents per pound. Gain $=(\$0.5000-\$0.4820)\times 50,000=\$900$.

(b) The trader sells for 50 cents per pound something that is worth 51.30 cents per pound. Loss $=(\$0.5130-\$0.5000)\times 50,000=\$650$.

Problem 1.7.
Suppose that you write a put contract with a strike price of \$40 and an expiration date in three months. The current stock price is \$41 and the contract is on 100 shares. What have you committed yourself to? How much could you gain or lose?

You have sold a put option. You have agreed to buy 100 shares for \$40 per share if the party on the other side of the contract chooses to exercise the right to sell for this price. The option will be exercised only when the price of stock is below \$40. Suppose, for example, that the option is exercised when the price is \$30. You have to buy at \$40 shares that are worth \$30; you lose \$10 per share, or \$1,000 in total. If the option is exercised when the price is \$20, you lose \$20 per share, or \$2,000 in total. The worst that can happen is that the price of the stock declines to almost zero during the three-month period. This highly unlikely event would cost you \$4,000. In return for the possible future losses, you receive the price of the option from the purchaser.

Problem 1.8.
What is the difference between the over-the-counter market and the exchange-traded market? What are the bid and offer quotes of a market maker in the over-the-counter market?

The over-the-counter market is a telephone- and computer-linked network of financial institutions, fund managers, and corporate treasurers where two participants can enter into any mutually acceptable contract. An exchange-traded market is a market organized by an exchange where traders either meet physically or communicate electronically and the contracts that can be traded have been defined by the exchange. When a market maker quotes a bid and an offer, the bid is the price at which the market maker is prepared to buy and the offer is the price at which the market maker is prepared to sell.

Problem 1.9.

You would like to speculate on a rise in the price of a certain stock. The current stock price is $29, and a three-month call with a strike of $30 costs $2.90. You have $5,800 to invest. Identify two alternative strategies, one involving an investment in the stock and the other involving investment in the option. What are the potential gains and losses from each?

One strategy would be to buy 200 shares. Another would be to buy 2,000 options. If the share price does well the second strategy will give rise to greater gains. For example, if the share price goes up to $40 you gain $[2{,}000 \times (\$40 - \$30)] - \$5{,}800 = \$14{,}200$ from the second strategy and only $200 \times (\$40 - \$29) = \$2{,}200$ from the first strategy. However, if the share price does badly, the second strategy gives greater losses. For example, if the share price goes down to $25, the first strategy leads to a loss of $200 \times (\$29 - \$25) = \$800,$ whereas the second strategy leads to a loss of the whole $5,800 investment. This example shows that options contain built in leverage.

Problem 1.10.

Suppose you own 5,000 shares that are worth $25 each. How can put options be used to provide you with insurance against a decline in the value of your holding over the next four months?

You could buy 50 put option contracts (each on 100 shares) with a strike price of $25 and an expiration date in four months. If at the end of four months the stock price proves to be less than $25, you can exercise the options and sell the shares for $25 each.

Problem 1.11.

When first issued, a stock provides funds for a company. Is the same true of an exchange-traded stock option? Discuss.

An exchange-traded stock option provides no funds for the company. It is a security sold by one investor to another. The company is not involved. By contrast, a stock when it is first issued is sold by the company to investors and does provide funds for the company.

Problem 1.12.

Explain why a futures contract can be used for either speculation or hedging.

If an investor has an exposure to the price of an asset, he or she can hedge with futures contracts. If the investor will gain when the price decreases and lose when the price increases, a long futures position will hedge the risk. If the investor will lose when the price decreases and gain when the price increases, a short futures position will hedge the risk. Thus either a long or a short futures position can be entered into for hedging purposes.
If the investor has no exposure to the price of the underlying asset, entering into a futures contract is speculation. If the investor takes a long position, he or she gains when the asset's price increases and loses when it decreases. If the investor takes a short position, he or she loses when the asset's price increases and gains when it decreases.

Problem 1.13.

Suppose that a March call option to buy a share for $50 costs $2.50 and is held until March. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram showing how the profit on a long position in the option depends on the stock price at the maturity of the option.

The holder of the option will gain if the price of the stock is above $52.50 in March. (This ignores the time value of money.) The option will be exercised if the price of the stock is above $50.00 in March. The profit as a function of the stock price is shown in Figure S1.1.

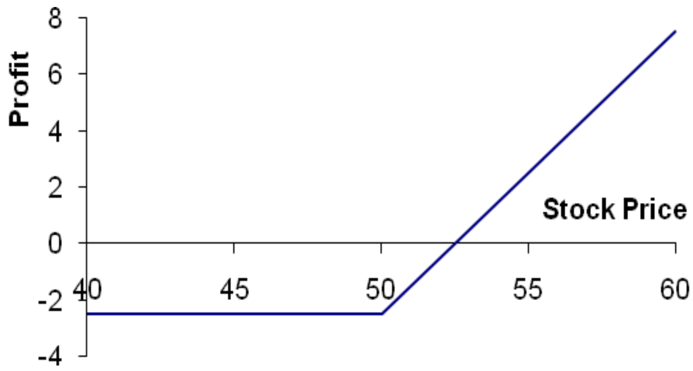

Figure S1.1 Profit from long position in Problem 1.13

Problem 1.14.

Suppose that a June put option to sell a share for $60 costs $4 and is held until June. Under what circumstances will the seller of the option (i.e., the party with a short position) make a profit? Under what circumstances will the option be exercised? Draw a diagram showing how the profit from a short position in the option depends on the stock price at the maturity of the option.

The seller of the option will lose money if the price of the stock is below $56.00 in June. (This ignores the time value of money.) The option will be exercised if the price of the stock is below $60.00 in June. The profit as a function of the stock price is shown in Figure S1.2.

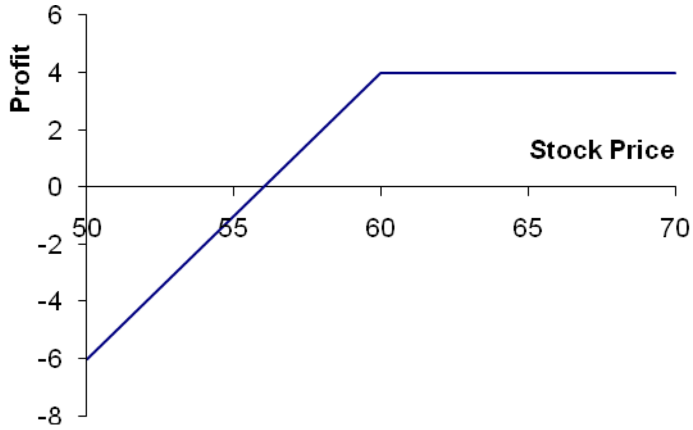

Figure S1.2 Profit from short position in Problem 1.14

Problem 1.15.
It is May and a trader writes a September call option with a strike price of $20. The stock price is $18, and the option price is $2. Describe the investor's cash flows if the option is held until September and the stock price is $25 at this time.

The trader has an inflow of $2 in May and an outflow of $5 in September. The $2 is the cash received from the sale of the option. The $5 is the result of the option being exercised. The investor has to buy the stock for $25 in September and sell it to the purchaser of the option for $20.

Problem 1.16.
A trader writes a December put option with a strike price of $30. The price of the option is $4. Under what circumstances does the trader make a gain?

The trader makes a gain if the price of the stock is above $26 at the time of exercise. (This ignores the time value of money.)

Problem 1.17.
A company knows that it is due to receive a certain amount of a foreign currency in four months. What type of option contract is appropriate for hedging?

A long position in a four-month put option can provide insurance against the exchange rate falling below the strike price. It ensures that the foreign currency can be sold for at least the strike price.

Problem 1.18.
A US company expects to have to pay 1 million Canadian dollars in six months. Explain how the exchange rate risk can be hedged using (a) a forward contract; (b) an option.

The company could enter into a long forward contract to buy 1 million Canadian dollars in six months. This would have the effect of locking in an exchange rate equal to the current forward exchange rate. Alternatively the company could buy a call option giving it the right (but not the obligation) to purchase 1 million Canadian dollars at a certain exchange rate in six months. This would provide insurance against a strong Canadian dollar in six months while still allowing the company to benefit from a weak Canadian dollar at that time.

Problem 1.19.
A trader enters into a short forward contract on 100 million yen. The forward exchange rate is $0.0080 per yen. How much does the trader gain or lose if the exchange rate at the end of the contract is (a) $0.0074 per yen; (b) $0.0091 per yen?

a) The trader sells 100 million yen for $0.0080 per yen when the exchange rate is $0.0074 per yen. The gain is 100×0.0006 millions of dollars or $60,000.
b) The trader sells 100 million yen for $0.0080 per yen when the exchange rate is $0.0091 per yen. The loss is 100×0.0011 millions of dollars or $110,000.

Problem 1.20.
The Chicago Board of Trade offers a futures contract on long-term Treasury bonds. Characterize the investors likely to use this contract.

Most investors will use the contract because they want to do one of the following:
a) Hedge an exposure to long-term interest rates.
b) Speculate on the future direction of long-term interest rates.
c) Arbitrage between the spot and futures markets for Treasury bonds.

This contract is discussed in Chapter 6.

Problem 1.21.
"Options and futures are zero-sum games." What do you think is meant by this statement?

The statement means that the gain (loss) to the party with the short position is equal to the loss (gain) to the party with the long position. In aggregate, the net gain to all parties is zero.

Problem 1.22.
Describe the profit from the following portfolio: a long forward contract on an asset and a long European put option on the asset with the same maturity as the forward contract and a strike price that is equal to the forward price of the asset at the time the portfolio is set up.

The terminal value of the long forward contract is:

$$S_T - F_0$$

where S_T is the price of the asset at maturity and F_0 is the delivery price, which is the same as the forward price of the asset at the time the portfolio is set up). The terminal value of the put option is:

$$\max(F_0 - S_T, 0)$$

The terminal value of the portfolio is therefore

$$S_T - F_0 + \max(F_0 - S_T, 0)$$
$$= \max(0, S_T - F_0]$$

This is the same as the terminal value of a European call option with the same maturity as the forward contract and a strike price equal to F_0. This result is illustrated in the Figure S1.3.

The profit equals the terminal value of the call option less the amount paid for the put option. (It does not cost anything to enter into the forward contract.

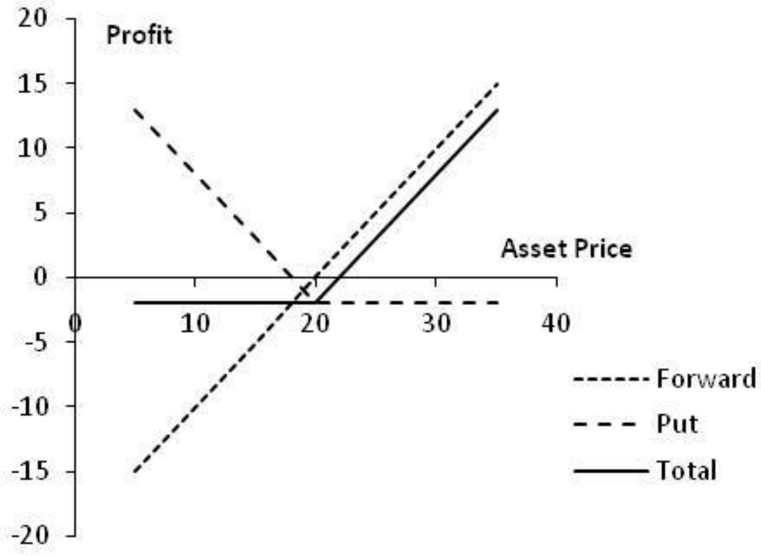

Figure S1.3 Profit from portfolio in Problem 1.22

Problem 1.23.

In the 1980s, Bankers Trust developed index currency option notes (ICONs). These are bonds in which the amount received by the holder at maturity varies with a foreign exchange rate. One example was its trade with the Long Term Credit Bank of Japan. The ICON specified that if the yen–U.S. dollar exchange rate, S_T, is greater than 169 yen per dollar at maturity (in 1995), the holder of the bond receives \$1,000. If it is less than 169 yen per dollar, the amount received by the holder of the bond is

$$1{,}000 - \max\left[0,\ 1{,}000\left(\frac{169}{S_T} - 1\right)\right]$$

When the exchange rate is below 84.5, nothing is received by the holder at maturity. Show that this ICON is a combination of a regular bond and two options.

Suppose that the yen exchange rate (yen per dollar) at maturity of the ICON is S_T. The payoff from the ICON is

$$\begin{array}{lll} 1{,}000 & \text{if} & S_T > 169 \\ 1{,}000 - 1{,}000\left(\dfrac{169}{S_T} - 1\right) & \text{if} & 84.5 \le S_T \le 169 \\ 0 & \text{if} & S_T < 84.5 \end{array}$$

When $84.5 \le S_T \le 169$ the payoff can be written

$$2{,}000 - \frac{169{,}000}{S_T}$$

The payoff from an ICON is the payoff from:

(a) A regular bond

(b) A short position in call options to buy 169,000 yen with an exercise price of 1/169

(c) A long position in call options to buy 169,000 yen with an exercise price of 1/84.5

This is demonstrated by the following table, which shows the terminal value of the various components of the position

	Bond	*Short Calls*	*Long Calls*	*Whole position*
$S_T > 169$	1000	0	0	1000
$84.5 \le S_T \le 169$	1000	$-169{,}000\left(\frac{1}{S_T} - \frac{1}{169}\right)$	0	$2000 - \frac{169{,}000}{S_T}$
$S_T < 84.5$	1000	$-169{,}000\left(\frac{1}{S_T} - \frac{1}{169}\right)$	$169{,}000\left(\frac{1}{S_T} - \frac{1}{84.5}\right)$	0

Problem 1.24.

On July 1, 2011, a company enters into a forward contract to buy 10 million Japanese yen on January 1, 2012. On September 1, 2011, it enters into a forward contract to sell 10 million Japanese yen on January 1, 2012. Describe the payoff from this strategy.

Suppose that the forward price for the contract entered into on July 1, 2011 is F_1 and that the forward price for the contract entered into on September 1, 2011 is F_2 with both F_1 and F_2 being measured as dollars per yen. If the value of one Japanese yen (measured in US dollars) is S_T on January 1, 2012, then the value of the first contract (in millions of dollars) at that time is

$$10(S_T - F_1)$$

while the value of the second contract (per yen sold) at that time is:

$$10(F_2 - S_T)$$

The total payoff from the two contracts is therefore

$$10(S_T - F_1) + 10(F_2 - S_T) = 10(F_2 - F_1)$$

Thus if the forward price for delivery on January 1, 2012 increased between July 1, 2011 and September 1, 2011 the company will make a profit. (Note that the yen/USD exchange rate is usually expressed as the number of yen per USD not as the number of USD per yen)

Problem 1.25.

Suppose that USD-sterling spot and forward exchange rates are as follows:

Spot	*1.4580*
90-day forward	*1.4556*
180-day forward	*1.4518*

What opportunities are open to an arbitrageur in the following situations?

(a) *A 180-day European call option to buy £1 for $1.42 costs 2 cents.*

(b) *A 90-day European put option to sell £1 for $1.49 costs 2 cents.*

(a) The arbitrageur buys a 180-day call option and takes a short position in a 180-day forward contract. If S_T is the terminal spot rate, the profit from the call option is

$\max(S_T - 1.42, 0) - 0.02$

The profit from the short forward contract is

$1.4518 - S_T$

The profit from the strategy is therefore

$\max(S_T - 1.42, 0) - 0.02 + 1.4518 - S_T$

or

$\max(S_T - 1.42, 0) + 1.4318 - S_T$

This is

$1.4318 - S_T$ when $S_T < 1.42$

0.118 when $S_T > 1.42$

This shows that the profit is always positive. The time value of money has been ignored in these calculations. However, when it is taken into account the strategy is still likely to be profitable in all circumstances. (We would require an extremely high interest rate for \$0.0118 interest to be required on an outlay of \$0.02 over a 180-day period.)

(b) The trader buys 90-day put options and takes a long position in a 90 day forward contract. If S_T is the terminal spot rate, the profit from the put option is

$\max(1.49 - S_T, 0) - 0.02$

The profit from the long forward contract is

$S_T - 1.4556$

The profit from this strategy is therefore

$\max(1.49 - S_T, 0) - 0.02 + S_T - 1.4556$

or

$\max(1.49 - S_T, 0) + S_T - 1.4756$

This is

$S_T - 1.4756$ when $S_T > 1.49$

0.0144 when $S_T < 1.49$

The profit is therefore always positive. Again, the time value of money has been ignored but is unlikely to affect the overall profitability of the strategy. (We would require interest rates to be extremely high for \$0.0144 interest to be required on an outlay of \$0.02 over a 90-day period.)

Further Questions

Problem 1.26 (Excel file)

Trader A enters into a forward contract to buy gold for \$1000 an ounce in one year. Trader B buys a call option to buy gold for \$1000 an ounce in one year. The cost of the option is \$100 an ounce. What is the difference between the positions of the traders? Show the profit per ounce as a function of the price of gold in one year for the two traders.

Trader A makes a profit of $S_T - 1000$ and Trader B makes a profit of max $(S_T - 1000, 0) - 100$ where S_T is the price of gold in one year. Trader A does better if S_T is above \$900 as indicated in Figure S1.4.

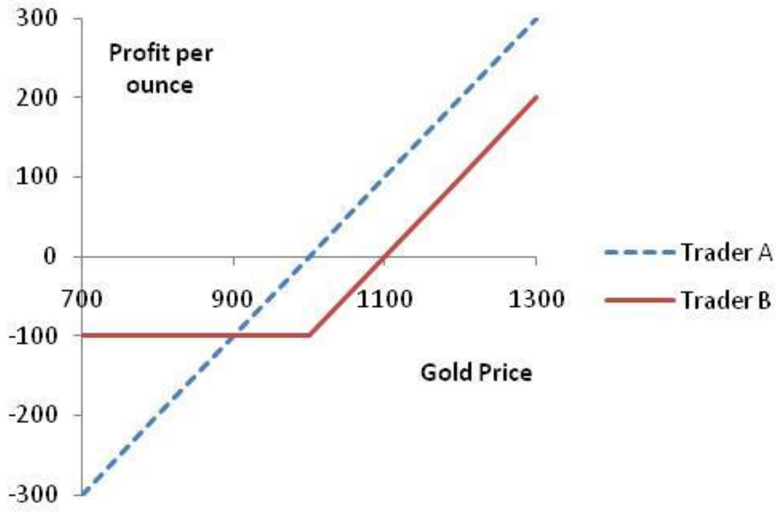

Figure S1.4: Profit to Trader A and Trader B in Problem 1.26

Problem 1.27
In March, a US investor instructs a broker to sell one July put option contract on a stock. The stock price is $42 and the strike price is $40. The option price is $3. Explain what the investor has agreed to. Under what circumstances will the trade prove to be profitable? What are the risks?

The investor has agreed to buy 100 shares of the stock for $40 in July (or earlier) if the party on the other side of the transaction chooses to sell. The trade will prove profitable if the option is not exercised or if the stock price is above $37 at the time of exercise. The risk to the investor is that the stock price plunges to a low level. For example, if the stock price drops to $1 by July (unlikely but possible), the investor loses $3,600. This is because the put options are exercised and $40 is paid for 100 shares when the value per share is $1. This leads to a loss of $3,900 which is only a little offset by the premium of $300 received for the options.

Problem 1.28
A US company knows it will have to pay 3 million euros in three months. The current exchange rate is 1.4500 dollars per euro. Discuss how forward and options contracts can be used by the company to hedge its exposure.

The company could enter into a forward contract obligating it to buy 3 million euros in three months for a fixed price (the forward price). The forward price will be close to but not exactly the same as the current spot price of 1.4500. An alternative would be to buy a call option giving the company the right but not the obligation to buy 3 million euros for a particular exchange rate (the strike price) in three months. The use of a forward contract locks in, at no cost, the exchange rate that will apply in three months. The use of a call option provides, at a cost, insurance against the exchange rate being higher than the strike price.

Problem 1.29 (Excel file)
A stock price is $29. An investor buys one call option contract on the stock with a strike price of $30 and sells a call option contract on the stock with a strike price of $32.50. The market prices of the options are $2.75 and $1.50, respectively. The options have the same maturity date. Describe the investor's position.

This is known as a bull spread and will be discussed in Chapter 11. The profit is shown in Figure S1.5.

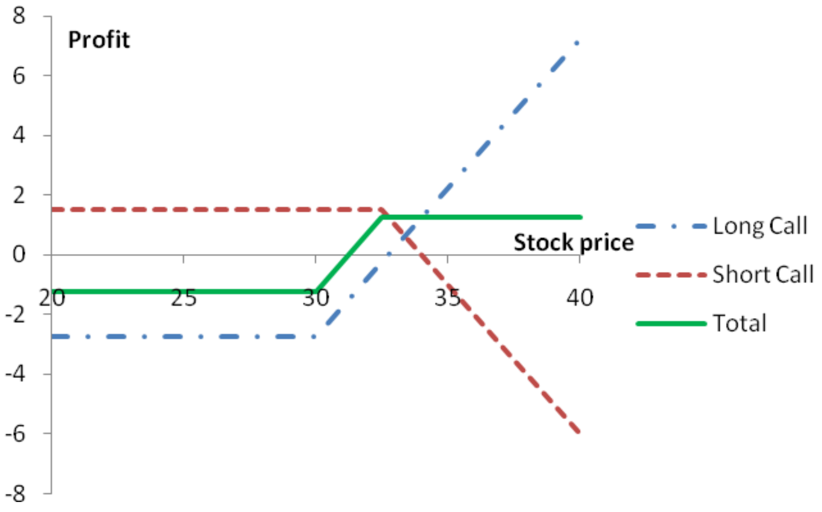

Figure S1.5: Profit in Problem 1.29

Problem 1.30
The price of gold is currently $1,000 per ounce. The forward price for delivery in one year is $1,200. An arbitrageur can borrow money at 10% per annum. What should the arbitrageur do? Assume that the cost of storing gold is zero and that gold provides no income.

The arbitrageur should borrow money to buy a certain number of ounces of gold today and short forward contracts on the same number of ounces of gold for delivery in one year. This means that gold is purchased for $1000 per ounce and sold for $1200 per ounce. Assuming the cost of borrowed funds is less than 20% per annum this generates a riskless profit.

Problem 1.31.
The current price of a stock is $94, and three-month call options with a strike price of $95 currently sell for $4.70. An investor who feels that the price of the stock will increase is trying to decide between buying 100 shares and buying 2,000 call options (20 contracts). Both strategies involve an investment of $9,400. What advice would you give? How high does the stock price have to rise for the option strategy to be more profitable?

The investment in call options entails higher risks but can lead to higher returns. If the stock price stays at $94, an investor who buys call options loses $9,400 whereas an investor who buys shares neither gains nor loses anything. If the stock price rises to $120, the investor who buys call options gains

$$2000\times(120-95)-9400=\$40,600$$

An investor who buys shares gains

$$100 \times (120 - 94) = \$2,600$$

The strategies are equally profitable if the stock price rises to a level, *S*, where

$$100 \times (S - 94) = 2000(S - 95) - 9400$$

or

$$S = 100$$

The option strategy is therefore more profitable if the stock price rises above $100.

Problem 1.32.

On July 15, 2010, an investor owns 100 Google shares. As indicated in Table 1.3, the share price is about $497 and a December put option with a strike price $400 costs $27.30. The investor is comparing two alternatives to limit downside risk. The first involves buying one December put option contract with a strike price of $460. The second involves instructing a broker to sell the 100 shares as soon as Google's price reaches $460. Discuss the advantages and disadvantages of the two strategies.

The second alternative involves what is known as a stop or stop-loss order. It costs nothing and ensures that $46,000, or close to $46,000, is realized for the holding in the event the stock price ever falls to $460. The put option costs $2,730 and guarantees that the holding can be sold for $4,600 any time up to December. If the stock price falls marginally below $460 and then rises the option will not be exercised, but the stop-loss order will lead to the holding being liquidated. There are some circumstances where the put option alternative leads to a better outcome and some circumstances where the stop-loss order leads to a better outcome. If the stock price ends up below $460, the stop-loss order alternative leads to a better outcome because the cost of the option is avoided. If the stock price falls to $380 in November and then rises to $490 by December, the put option alternative leads to a better outcome. The investor is paying $2,730 for the chance to benefit from this second type of outcome.

Problem 1.33.

A bond issued by Standard Oil some time ago worked as follows. The holder received no interest. At the bond's maturity the company promised to pay $1,000 plus an additional amount based on the price of oil at that time. The additional amount was equal to the product of 170 and the excess (if any) of the price of a barrel of oil at maturity over $25. The maximum additional amount paid was $2,550 (which corresponds to a price of $40 per barrel). Show that the bond is a combination of a regular bond, a long position in call options on oil with a strike price of $25, and a short position in call options on oil with a strike price of $40.

Suppose S_T is the price of oil at the bond's maturity. In addition to $1000 the Standard Oil bond pays:

$$\begin{array}{rcc} S_T < \$25 & : & 0 \\ \$40 > S_T > \$25 & : & 170(S_T - 25) \\ S_T > \$40 & : & 2,550 \end{array}$$

This is the payoff from 170 call options on oil with a strike price of 25 less the payoff from 170 call options on oil with a strike price of 40. The bond is therefore equivalent to a regular bond plus a long position in 170 call options on oil with a strike price of $25 plus a short position in 170 call options on oil with a strike price of $40. The investor has what is termed a bull spread on oil. This is discussed in Chapter 11.

Problem 1.34.

Suppose that in the situation of Table 1.1 a corporate treasurer said: "I will have £1 million to sell in six months. If the exchange rate is less than 1.41, I want you to give me 1.41. If it is greater than 1.47 I will accept 1.47. If the exchange rate is between 1.41 and 1.47, I will sell the sterling for the exchange rate." How could you use options to satisfy the treasurer?

You sell the treasurer a put option on GBP with a strike price of 1.41 and buy from the treasurer a call option on GBP with a strike price of 1.47. Both options are on one million pounds and have a maturity of six months. This is known as a range forward contract and is discussed in Chapter 16.

Problem 1.35.

Describe how foreign currency options can be used for hedging in the situation considered in Section 1.7 so that (a) ImportCo is guaranteed that its exchange rate will be less than 1.4600, and (b) ExportCo is guaranteed that its exchange rate will be at least 1.4200. Use DerivaGem to calculate the cost of setting up the hedge in each case assuming that the exchange rate volatility is 12%, interest rates in the United States are 5% and interest rates in Britain are 5.7%. Assume that the current exchange rate is the average of the bid and offer in Table 1.1.

ImportCo should buy three-month call options on $10 million with a strike price of 1.4600. ExportCo should buy three-month put options on $10 million with a strike price of 1.4200. In this case the foreign exchange rate is 1.4409 (the average of the bid and offer quotes in Table 1.1.), the (domestic) risk-free rate is 5%, the foreign risk-free rate is 5.7%, the volatility is 12%, and the time to exercise is 0.25. Using the Equity_FX_Index_Futures_Options worksheet in the DerivaGem Options Calculator select Currency as the underlying and Black-Scholes European as the option type. The software shows that a call with a strike price of 1.46 is worth 0.0245 and a put with a strike of 1.42 is worth 0.0256. This means that the hedging would cost 0.0245×10,000,000 or $245,000 for ImportCo and 0.0256×30,000,000 or $768,000 for ExportCo.

Problem 1.36.

A trader buys a European call option and sells a European put option. The options have the same underlying asset, strike price and maturity. Describe the trader's position. Under what circumstances does the price of the call equal the price of the put?

The trader has a long European call option with strike price K and a short European put option with strike price K. Suppose the price of the underlying asset at the maturity of the option is S_T. If $S_T > K$, the call option is exercised by the investor and the put option expires worthless. The payoff from the portfolio is then $S_T - K$. If $S_T < K$, the call option expires worthless and the put option is exercised against the investor. The cost to the investor is $K - S_T$. Alternatively we can say that the payoff to the investor in this case is $S_T - K$ (a negative amount). In all cases, the payoff is $S_T - K$, the same as the payoff from the forward contract. The trader's position is equivalent to a forward contract with delivery price K.

Suppose that F is the forward price. If $K = F$, the forward contract that is created has zero value. Because the forward contract is equivalent to a long call and a short put, this shows that the price of a call equals the price of a put when the strike price is F.

CHAPTER 2
Mechanics of Futures Markets

Practice Questions

Problem 2.1.
Distinguish between the terms open interest and trading volume.

The *open interest* of a futures contract at a particular time is the total number of long positions outstanding. (Equivalently, it is the total number of short positions outstanding.) The *trading volume* during a certain period of time is the number of contracts traded during this period**.**

Problem 2.2.
What is the difference between a local and a futures commission merchant?

A futures *commission merchant* trades on behalf of a client and charges a commission. A *local* trades on his or her own behalf.

Problem 2.3.
Suppose that you enter into a short futures contract to sell July silver for $17.20 per ounce. The size of the contract is 5,000 ounces. The initial margin is $4,000, and the maintenance margin is $3,000. What change in the futures price will lead to a margin call? What happens if you do not meet the margin call?

There will be a margin call when $1,000 has been lost from the margin account. This will occur when the price of silver increases by 1,000/5,000 $=$ $0.20. The price of silver must therefore rise to $17.40 per ounce for there to be a margin call. If the margin call is not met, your broker closes out your position.

Problem 2.4.
Suppose that in September 2012 a company takes a long position in a contract on May 2013 crude oil futures. It closes out its position in March 2013. The futures price (per barrel) is $68.30 when it enters into the contract, $70.50 when it closes out its position, and $69.10 at the end of December 2012. One contract is for the delivery of 1,000 barrels. What is the company's total profit? When is it realized? How is it taxed if it is (a) a hedger and (b) a speculator? Assume that the company has a December 31 year-end.

The total profit is ($70.50 $-$ $68.30) $\times$ 1,000 $=$ $2,200. Of this $(\$69.10-\$68.30)\times 1{,}000$ or $800 is realized on a day-by-day basis between September 2012 and December 31, 2012. A further $(\$70.50-\$69.10)\times 1,000=\$1,400$ is realized on a day-by-day basis between January 1, 2013, and March 2013. A hedger would be taxed on the whole profit of $2,200 in 2013. A speculator would be taxed on $800 in 2012 and $1,400 in 2013.

Problem 2.5.
What does a stop order to sell at $2 mean? When might it be used? What does a limit order to sell at $2 mean? When might it be used?

A *stop order* to sell at $2 is an order to sell at the best available price once a price of $2 or less is reached. It could be used to limit the losses from an existing long position. A *limit order* to sell at $2 is an order to sell at a price of $2 or more. It could be used to instruct a broker that a short position should be taken, providing it can be done at a price more favorable than $2.

Problem 2.6.
What is the difference between the operation of the margin accounts administered by a clearing house and those administered by a broker?

The margin account administered by the clearing house is marked to market daily, and the clearing house member is required to bring the account back up to the prescribed level daily. The margin account administered by the broker is also marked to market daily. However, the account does not have to be brought up to the initial margin level on a daily basis. It has to be brought up to the initial margin level when the balance in the account falls below the maintenance margin level. The maintenance margin is usually about 75% of the initial margin.

Problem 2.7.
What differences exist in the way prices are quoted in the foreign exchange futures market, the foreign exchange spot market, and the foreign exchange forward market?

In futures markets, prices are quoted as the number of US dollars per unit of foreign currency. Spot and forward rates are quoted in this way for the British pound, euro, Australian dollar, and New Zealand dollar. For other major currencies, spot and forward rates are quoted as the number of units of foreign currency per US dollar.

Problem 2.8.
The party with a short position in a futures contract sometimes has options as to the precise asset that will be delivered, where delivery will take place, when delivery will take place, and so on. Do these options increase or decrease the futures price? Explain your reasoning.

These options make the contract less attractive to the party with the long position and more attractive to the party with the short position. They therefore tend to reduce the futures price.

Problem 2.9.
What are the most important aspects of the design of a new futures contract?

The most important aspects of the design of a new futures contract are the specification of the underlying asset, the size of the contract, the delivery arrangements, and the delivery months.

Problem 2.10.
Explain how margins protect investors against the possibility of default.

A margin is a sum of money deposited by an investor with his or her broker. It acts as a guarantee that the investor can cover any losses on the futures contract. The balance in the margin account is adjusted daily to reflect gains and losses on the futures contract. If losses are above a certain level, the investor is required to deposit a further margin. This system makes it unlikely that the investor will default. A similar system of margins makes it unlikely that the investor's broker will default on the contract it has with the clearing house member and unlikely that the clearing house member will default with the clearing house.

Problem 2.11.
A trader buys two July futures contracts on frozen orange juice. Each contract is for the delivery of 15,000 pounds. The current futures price is 160 cents per pound, the initial margin is $6,000 per contract, and the maintenance margin is $4,500 per contract. What price change would lead to a margin call? Under what circumstances could $2,000 be withdrawn from the margin account?

There is a margin call if more than $1,500 is lost on one contract. This happens if the futures price of frozen orange juice falls by more than 10 cents to below 150 cents per pound. $2,000 can be withdrawn from the margin account if there is a gain on one contract of $1,000. This will happen if the futures price rises by 6.67 cents to 166.67 cents per pound.

Problem 2.12.
Show that, if the futures price of a commodity is greater than the spot price during the delivery period, then there is an arbitrage opportunity. Does an arbitrage opportunity exist if the futures price is less than the spot price? Explain your answer.

If the futures price is greater than the spot price during the delivery period, an arbitrageur buys the asset, shorts a futures contract, and makes delivery for an immediate profit. If the futures price is less than the spot price during the delivery period, there is no similar perfect arbitrage strategy. An arbitrageur can take a long futures position but cannot force immediate delivery of the asset. The decision on when delivery will be made is made by the party with the short position. Nevertheless companies interested in acquiring the asset may find it attractive to enter into a long futures contract and wait for delivery to be made.

Problem 2.13.
Explain the difference between a market-if-touched order and a stop order.

A market-if-touched order is executed at the best available price after a trade occurs at a specified price or at a price more favorable than the specified price. A stop order is executed at the best available price after there is a bid or offer at the specified price or at a price less favorable than the specified price.

Problem 2.14.
Explain what a stop-limit order to sell at 20.30 with a limit of 20.10 means.

A stop-limit order to sell at 20.30 with a limit of 20.10 means that as soon as there is a bid at 20.30 the contract should be sold providing this can be done at 20.10 or a higher price.

Problem 2.15.
At the end of one day a clearing house member is long 100 contracts, and the settlement price is $50,000 per contract. The original margin is $2,000 per contract. On the following day the member becomes responsible for clearing an additional 20 long contracts, entered into at a price of $51,000 per contract. The settlement price at the end of this day is $50,200. How much does the member have to add to its margin account with the exchange clearing house?

The clearing house member is required to provide $20 \times \$2{,}000 = \$40{,}000$ as initial margin for the new contracts. There is a gain of $(50{,}200 - 50{,}000) \times 100 = \$20{,}000$ on the existing contracts. There is also a loss of $(51{,}000 - 50{,}200) \times 20 = \$16{,}000$ on the new contracts. The member must therefore add

$$40{,}000 - 20{,}000 + 16{,}000 = \$36{,}000$$

to the margin account.

Problem 2.16.
On July 1, 2012, a Japanese company enters into a forward contract to buy $1 million with yen on January 1, 2013. On September 1, 2012, it enters into a forward contract to sell $1 million on January 1, 2013. Describe the profit or loss the company will make in dollars as a function of the forward exchange rates on July 1, 2012 and September 1, 2012.

Suppose F_1 and F_2 are the forward exchange rates for the contracts entered into July 1, 2012 and September 1, 2012, and S is the spot rate on January 1, 2013. (All exchange rates are measured as yen per dollar). The payoff from the first contract is $(S - F_1)$ million yen and the payoff from the second contract is $(F_2 - S)$ million yen. The total payoff is therefore $(S - F_1) + (F_2 - S) = (F_2 - F_1)$ million yen.

Problem 2.17.

The forward price on the Swiss franc for delivery in 45 days is quoted as 1.1000. The futures price for a contract that will be delivered in 45 days is 0.9000. Explain these two quotes. Which is more favorable for an investor wanting to sell Swiss francs?

The 1.1000 forward quote is the number of Swiss francs per dollar. The 0.9000 futures quote is the number of dollars per Swiss franc. When quoted in the same way as the futures price the forward price is $1/1.1000 = 0.9091$. The Swiss franc is therefore more valuable in the forward market than in the futures market. The forward market is therefore more attractive for an investor wanting to sell Swiss francs.

Problem 2.18.

Suppose you call your broker and issue instructions to sell one July hogs contract. Describe what happens.

Live hog futures are traded on the Chicago Mercantile Exchange. The broker will request some initial margin. The order will be relayed by telephone to your broker's trading desk on the floor of the exchange (or to the trading desk of another broker). It will then be sent by messenger to a commission broker who will execute the trade according to your instructions. Confirmation of the trade eventually reaches you. If there are adverse movements in the futures price your broker may contact you to request additional margin.

Problem 2.19.

"Speculation in futures markets is pure gambling. It is not in the public interest to allow speculators to trade on a futures exchange." Discuss this viewpoint.

Speculators are important market participants because they add liquidity to the market. However, contracts must be useful for hedging as well as speculation. This is because regulators generally only approve contracts when they are likely to be of interest to hedgers as well as speculators.

Problem 2.20.

Live cattle futures trade with June, August, October, December, February, and April maturities. Why do you think that the open interest for the June contract is less than that for the August contract in Table 2.2?

Normally, the shorter the maturity of a contract is, the higher the open interest. However, traders tend to close out their positions in the month immediately before the maturity month. This means that the open interest for the closest maturity month can be less than that for the next closest maturity month

Problem 2.21.
What do you think would happen if an exchange started trading a contract in which the quality of the underlying asset was incompletely specified?

The contract would not be a success. Parties with short positions would hold their contracts until delivery and then deliver the cheapest form of the asset. This might well be viewed by the party with the long position as garbage! Once news of the quality problem became widely known no one would be prepared to buy the contract. This shows that futures contracts are feasible only when there are rigorous standards within an industry for defining the quality of the asset. Many futures contracts have in practice failed because of the problem of defining quality.

Problem 2.22.
"When a futures contract is traded on the floor of the exchange, it may be the case that the open interest increases by one, stays the same, or decreases by one." Explain this statement.

If both sides of the transaction are entering into a new contract, the open interest increases by one. If both sides of the transaction are closing out existing positions, the open interest decreases by one. If one party is entering into a new contract while the other party is closing out an existing position, the open interest stays the same.

Problem 2.23.
Suppose that on October 24, 2012, a company sells one April 2013 live-cattle futures contracts. It closes out its position on January 21, 2013. The futures price (per pound) is 91.20 cents when it enters into the contract, 88.30 cents when it closes out its position, and 88.80 cents at the end of December 2012. One contract is for the delivery of 40,000 pounds of cattle. What is the total profit? How is it taxed if the company is (a) a hedger and (b) a speculator? Assume that the company has a December 31 year end.

The total profit is

$$40,000\times(0.9120-0.8830)=\$1,160$$

If the company is a hedger this is all taxed in 2013. If it is a speculator

$$40,000\times(0.9120-0.8880)=\$960$$

is taxed in 2012 and

$$40,000\times(0.8880-0.8830)=\$200$$

is taxed in 2013.

Problem 2.24.
A cattle farmer expects to have 120,000 pounds of live cattle to sell in three months. The live-cattle futures contract traded by the CME Group is for the delivery of 40,000 pounds of cattle. How can the farmer use the contract for hedging? From the farmer's viewpoint, what are the pros and cons of hedging?

The farmer can short 3 contracts that have 3 months to maturity. If the price of cattle falls, the gain on the futures contract will offset the loss on the sale of the cattle. If the price of cattle rises, the gain on the sale of the cattle will be offset by the loss on the futures contract. Using futures contracts to hedge has the advantage that it can at no cost reduce risk to almost zero. Its disadvantage is that the farmer no longer gains from favorable movements in cattle prices.

Problem 2.25.
It is July 2011. A mining company has just discovered a small deposit of gold. It will take six months to construct the mine. The gold will then be extracted on a more or less continuous basis for one year. Futures contracts on gold are available with delivery months every two months from August 2011 to December 2012. Each contract is for the delivery of 100 ounces. Discuss how the mining company might use futures markets for hedging.

The mining company can estimate its production on a month by month basis. It can then short futures contracts to lock in the price received for the gold. For example, if a total of 3,000 ounces are expected to be produced in September 2011 and October 2011, the price received for this production can be hedged by shorting 30 October 2011 contracts.

Further Questions

Problem 2.26
Trader A enters into futures contracts to buy 1 million euros for 1.4 million dollars in three months. Trader B enters in a forward contract to do the same thing. The exchange (dollars per euro) declines sharply during the first two months and then increases for the third month to close at 1.4300. Ignoring daily settlement, what is the total profit of each trader? When the impact of daily settlement is taken into account, which trader does better?

The total profit of each trader in dollars is $0.03 \times 1{,}000{,}000 = 30{,}000$. Trader B's profit is realized at the end of the three months. Trader A's profit is realized day-by-day during the three months. Substantial losses are made during the first two months and profits are made during the final month. It is likely that Trader B has done better because Trader A had to finance its losses during the first two months.

Problem 2.27

Explain what is meant by open interest. Why does the open interest usually decline during the month preceding the delivery month? On a particular day, there were 2,000 trades in a particular futures contract. This means that there were 2000 buyers (going long) and 2000 sellers (going short). Of the 2,000 buyers, 1,400 were closing out positions and 600 were entering into new positions. Of the 2,000 sellers, 1,200 were closing out positions and 800 were entering into new positions. What is the impact of the day's trading on open interest?

Open interest is the number of contract outstanding. Many traders close out their positions just before the delivery month is reached. This is why the open interest declines during the month preceding the delivery month. The open interest went down by 600. We can see this in two ways. First, 1,400 shorts closed out and there were 800 new shorts. Second, 1,200 longs closed out and there were 600 new longs.

Problem 2.28

One orange juice future contract is on 15,000 pounds of frozen concentrate. Suppose that in September 2011 a company sells a March 2013 orange juice futures contract for 120 cents per pound. In December 2011 the futures price is 140 cents. In December 2012 the futures price is 110 cents. In February 2013 it is closed out at 125 cents. The company has a December year end. What is the company's profit or loss on the contract? How is it realized? What is the accounting and tax treatment of the transaction if the company is classified as a) a hedger and b) a speculator?

The price goes up during the time the company holds the contract from 120 to 125 cents per pound. Overall the company therefore takes a loss of $15{,}000 \times 0.05 = \$750$. If the company is classified as a hedger this loss is realized in 2013, If it is classified as a speculator it realizes a loss of $15{,}000 \times 0.20 = \$3000$ in 2011, a gain of $15{,}000 \times 0.30 = \$4{,}500$ in 2012, and a loss of $15{,}000 \times 0.15 = \$2{,}250$ in 2013.

Problem 2.29.

A company enters into a short futures contract to sell 5,000 bushels of wheat for 450 cents per bushel. The initial margin is $3,000 and the maintenance margin is $2,000. What price change would lead to a margin call? Under what circumstances could $1,500 be withdrawn from the margin account?

There is a margin call if $1000 is lost on the contract. This will happen if the price of wheat futures rises by 20 cents from 450 cents to 470 cents per bushel. $1500 can be withdrawn if the futures price falls by 30 cents to 420 cents per bushel.

Problem 2.30.

Suppose that there are no storage costs for crude oil and the interest rate for borrowing or lending is 5% per annum. How could you make money on May 26, 2010 by trading July 2010 and December 2010 contracts on crude oil? Use Table 2.2.

The July 2010 settlement price for oil is \$71.51 per barrel. The December 2010 settlement price for oil is \$75.23 per barrel. You could go long one July 2010 oil contract and short one December 2010 contract. In July 2010 you take delivery of the oil borrowing \$71.51 per barrel at 5% to meet cash outflows. The interest accumulated in five months is about $71.51\times0.05\times5/12$ or \$1.49. In December the oil is sold for \$75.23 per barrel which is more than the amount that has to be repaid on the loan. The strategy therefore leads to a profit. Note that this profit is independent of the actual price of oil in June 2010 or December 2010. It will be slightly affected by the daily settlement procedures.

Problem 2.31.

What position is equivalent to a long forward contract to buy an asset at K on a certain date and a put option to sell it for K on that date?

The long forward contract provides a payoff of $S_T - K$ where S_T is the asset price on the date and K is the delivery price. The put option provides a payoff of max $(K - S_T, 0)$. If $S_T > K$ the sum of the two payoffs is $S_T - K$. If $S_T < K$ the sum of the two payoffs is 0. The combined payoff is therefore max $(S_T - K, 0)$. This is the payoff from a call option. The equivalent position is therefore a call option.

Problem 2.32. (Excel file)

The author's Web page (www.rotman.utoronto.ca/~hull/data) contains daily closing prices for the crude oil futures contract and gold futures contract. (Both contracts are traded on NYMEX.) You are required to download the data and answer the following:

a) *How high do the maintenance margin levels for oil and gold have to be set so that there is a 1% chance that an investor with a balance slightly above the maintenance margin level on a particular day has a negative balance two days later? How high do they have to be for a 0.1% chance? Assume daily price changes are normally distributed with mean zero. Explain why the exchange might be interested in this calculation*
b) *Imagine an investor who starts with a long position in the oil contract at the beginning of the period covered by the data and keeps the contract for the whole of the period of time covered by the data. Margin balances in excess of the initial margin are withdrawn. Use the maintenance margin you calculated in part (a) for a 1% risk level and assume that the maintenance margin is 75% of the initial margin. Calculate the number of margin calls and the number of times the investor has a negative margin balance and therefore an incentive to walk away. Assume that all margin calls are met in your calculations. Repeat the calculations for an investor who starts with a short position in the gold*

contract.

Note that the data for this problem in the 8th edition is different from that in the 7th edition.

a) For gold, the standard deviation of daily changes is \$15.184 per ounce or \$1518.4 per contract. For a 1% risk this means that the maintenance margin per contract should be set at $1518.4\times\sqrt{2}\times 2.3263$ or 4996 when rounded. For a 0.1% risk the maintenance margin per contract should be set at $1518.4\times\sqrt{2}\times 3.0902$ or 6,636 when rounded. For crude oil, the standard deviation of daily changes is \$1.5777 per barrel or \$1577.7 per contract. For a 1% risk, this means that the maintenance margin should be set at $1577.7\times\sqrt{2}\times 2.3263$ or 5,191 when rounded. For a 0.1% chance the maintenance margin should be set at $1577.7\times\sqrt{2}\times 3.0902$ or 6,895 when rounded. The exchange might be interested in these calculations because they indicate the chance of a trader who is just above the maintenance margin level at the beginning of the period having a negative margin level before funds have to be submitted to the broker.

b) For a 1% risk the initial margin is set at 6,921 for on crude oil. (This is the maintenance margin of 5,191 divided by 0.75.) As the spreadsheet shows, for a long investor in oil there are 157 margin calls and 9 times (out of 1039 days) where the investor is tempted to walk away. For a 1% risk the initial margin is set at 6,661 for gold. (This is 4,996 divided by 0.75.) As the spreadsheet shows, for a short investor in gold there are 81 margin calls and 4 times (out of 459 days) when the investor is tempted to walk away. When the 0.1% risk level is used there is 1 time when the oil investor might walk away and 2 times when the gold investor might do so.

CHAPTER 7
Swaps

Practice Questions

Problem 7.1.
Companies A and B have been offered the following rates per annum on a $20 million five-year loan:

	Fixed Rate	*Floating Rate*
Company A	5.0%	LIBOR+0.1%
Company B	6.4%	LIBOR+0.6%

Company A requires a floating-rate loan; company B requires a fixed-rate loan. Design a swap that will net a bank, acting as intermediary, 0.1% per annum and that will appear equally attractive to both companies.

A has an apparent comparative advantage in fixed-rate markets but wants to borrow floating. B has an apparent comparative advantage in floating-rate markets but wants to borrow fixed. This provides the basis for the swap. There is a 1.4% per annum differential between the fixed rates offered to the two companies and a 0.5% per annum differential between the floating rates offered to the two companies. The total gain to all parties from the swap is therefore $1.4-0.5=0.9\%$ per annum. Because the bank gets 0.1% per annum of this gain, the swap should make each of A and B 0.4% per annum better off. This means that it should lead to A borrowing at LIBOR -0.3% and to B borrowing at 6.0%. The appropriate arrangement is therefore as shown in Figure S7.1.

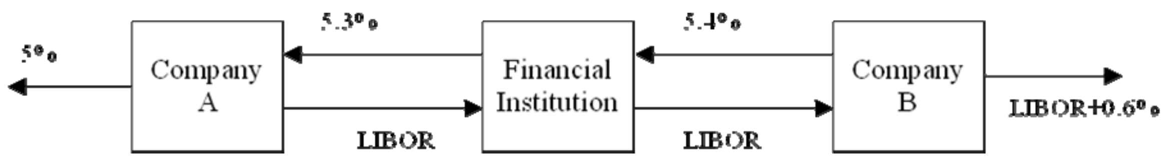

Figure S7.1 Swap for Problem 7.1

Problem 7.2.
Company X wishes to borrow U.S. dollars at a fixed rate of interest. Company Y wishes to borrow Japanese yen at a fixed rate of interest. The amounts required by the two companies are roughly the same at the current exchange rate. The companies have been quoted the following interest rates, which have been adjusted for the impact of taxes:

	Yen	*Dollars*
Company X	5.0%	9.6%
Company Y	6.5%	10.0%

Design a swap that will net a bank, acting as intermediary, 50 basis points per annum. Make the swap equally attractive to the two companies and ensure that all foreign exchange risk is assumed by the bank.

X has a comparative advantage in yen markets but wants to borrow dollars. Y has a comparative advantage in dollar markets but wants to borrow yen. This provides the basis for the swap. There is a 1.5% per annum differential between the yen rates and a 0.4% per annum differential between the dollar rates. The total gain to all parties from the swap is therefore $1.5-0.4=1.1\%$ per annum. The bank requires 0.5% per annum, leaving 0.3% per annum for each of X and Y. The swap should lead to X borrowing dollars at $9.6-0.3=9.3\%$ per annum and to Y borrowing yen at $6.5-0.3=6.2\%$ per annum. The appropriate arrangement is therefore as shown in Figure S7.2. All foreign exchange risk is borne by the bank.

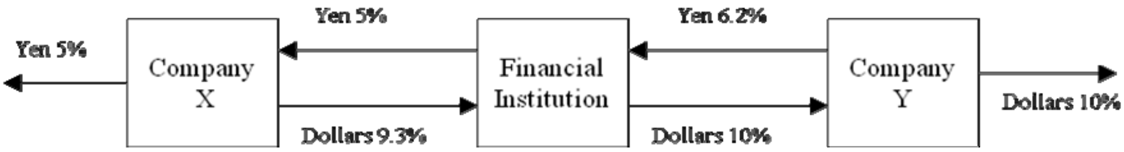

Figure S7.2 Swap for Problem 7.2

Problem 7.3.
A $100 million interest rate swap has a remaining life of 10 months. Under the terms of the swap, six-month LIBOR is exchanged for 7% per annum (compounded semiannually). The average of the bid–offer rate being exchanged for six-month LIBOR in swaps of all maturities is currently 5% per annum with continuous compounding. The six-month LIBOR rate was 4.6% per annum two months ago. What is the current value of the swap to the party paying floating? What is its value to the party paying fixed?

In four months $3.5 million ($=0.5\times0.07\times\$100$ million) will be received and $2.3 million ($=0.5\times0.046\times\$100$ million) will be paid. (We ignore day count issues.) In 10 months $3.5 million will be received, and the LIBOR rate prevailing in four months' time will be paid.
The value of the fixed-rate bond underlying the swap is
$3.5e^{-0.05\times4/12}+103.5e^{-0.05\times10/12}=\$102.718$ million
The value of the floating-rate bond underlying the swap is
$(100+2.3)e^{-0.05\times4/12}=\$100.609$ million
The value of the swap to the party paying floating is $\$102.718-\$100.609=\$2.109$ million.
The value of the swap to the party paying fixed is $-\$2.109$ million.
These results can also be derived by decomposing the swap into forward contracts. Consider the party paying floating. The first forward contract involves paying $2.3 million and receiving $3.5 million in four months. It has a value of $1.2e^{-0.05\times4/12}=\$1.180$ million. To value the second forward contract, we note that the forward interest rate is 5% per annum with continuous compounding, or 5.063% per annum with semiannual compounding. The

value of the forward contract is

$$100\times(0.07\times0.5-0.05063\times0.5)e^{-0.05\times10/12}=\$0.929 \text{ million}$$

The total value of the forward contracts is therefore $\$1.180+\$0.929=\$2.109$ million.

Problem 7.4.

Explain what a swap rate is. What is the relationship between swap rates and par yields?

A swap rate for a particular maturity is the average of the bid and offer fixed rates that a market maker is prepared to exchange for LIBOR in a standard plain vanilla swap with that maturity. The swap rate for a particular maturity is the LIBOR/swap par yield for that maturity.

Problem 7.5.

A currency swap has a remaining life of 15 months. It involves exchanging interest at 10% on £20 million for interest at 6% on \$30 million once a year. The term structure of interest rates in both the United Kingdom and the United States is currently flat, and if the swap were negotiated today the interest rates exchanged would be 4% in dollars and 7% in sterling. All interest rates are quoted with annual compounding. The current exchange rate (dollars per pound sterling) is 1.8500. What is the value of the swap to the party paying sterling? What is the value of the swap to the party paying dollars?

The swap involves exchanging the sterling interest of 20×0.10 or £2 million for the dollar interest of $30\times0.06=\$1.8$ million. The principal amounts are also exchanged at the end of the life of the swap. The value of the sterling bond underlying the swap is

$$\frac{2}{(1.07)^{1/4}}+\frac{22}{(1.07)^{5/4}}=22.182 \text{ million pounds}$$

The value of the dollar bond underlying the swap is

$$\frac{1.8}{(1.04)^{1/4}}+\frac{31.8}{(1.04)^{5/4}}=\$32.061 \text{ million}$$

The value of the swap to the party paying sterling is therefore

$$32.061-(22.182\times1.85)=-\$8.976 \text{ million}$$

The value of the swap to the party paying dollars is $+\$8.976$ million. The results can also be obtained by viewing the swap as a portfolio of forward contracts. The continuously compounded interest rates in sterling and dollars are 6.766% per annum and 3.922% per annum. The 3-month and 15-month forward exchange rates are $1.85e^{(0.03922-0.06766)\times0.25}=1.8369$ and $1.85e^{(0.03922-0.06766)\times1.25}=1.7854$. The values of the two forward contracts corresponding to the exchange of interest for the party paying sterling are therefore

$$(1.8-2\times1.8369)e^{-0.03922\times0.25}=-\$1.855 \text{ million}$$

and

$$(1.8-2\times1.7854)e^{-0.03922\times1.25}=-\$1.686 \text{ million}$$

The value of the forward contract corresponding to the exchange of principals is

$$(30-20\times1.7854)e^{-0.03922\times1.25}=-\$5.435 \text{ million}$$

The total value of the swap is $-\$1.855-\$1.686-\$5.435=-\$8.976$ million.

Problem 7.6.

Explain the difference between the credit risk and the market risk in a financial contract.

Credit risk arises from the possibility of a default by the counterparty. Market risk arises from movements in market variables such as interest rates and exchange rates. A complication is that the credit risk in a swap is contingent on the values of market variables. A company's position in a swap has credit risk only when the value of the swap to the company is positive.

Problem 7.7.
A corporate treasurer tells you that he has just negotiated a five-year loan at a competitive fixed rate of interest of 5.2%. The treasurer explains that he achieved the 5.2% rate by borrowing at six-month LIBOR plus 150 basis points and swapping LIBOR for 3.7%. He goes on to say that this was possible because his company has a comparative advantage in the floating-rate market. What has the treasurer overlooked?

The rate is not truly fixed because, if the company's credit rating declines, it will not be able to roll over its floating rate borrowings at LIBOR plus 150 basis points. The effective fixed borrowing rate then increases. Suppose for example that the treasurer's spread over LIBOR increases from 150 basis points to 200 basis points. The borrowing rate increases from 5.2% to 5.7%.

Problem 7.8.
Explain why a bank is subject to credit risk when it enters into two offsetting swap contracts.

At the start of the swap, both contracts have a value of approximately zero. As time passes, it is likely that the swap values will change, so that one swap has a positive value to the bank and the other has a negative value to the bank. If the counterparty on the other side of the positive-value swap defaults, the bank still has to honor its contract with the other counterparty. It is liable to lose an amount equal to the positive value of the swap.

Problem 7.9.
Companies X and Y have been offered the following rates per annum on a $5 million 10-year investment:

	Fixed Rate	*Floating Rate*
Company X	8.0%	LIBOR
Company Y	8.8%	LIBOR

Company X requires a fixed-rate investment; company Y requires a floating-rate investment. Design a swap that will net a bank, acting as intermediary, 0.2% per annum and will appear equally attractive to X and Y.

The spread between the interest rates offered to X and Y is 0.8% per annum on fixed rate investments and 0.0% per annum on floating rate investments. This means that the total apparent benefit to all parties from the swap is $0.8\% \text{per annum}$. Of this 0.2% per annum will go to the bank. This leaves 0.3% per annum for each of X and Y. In other words, company X should be able to get a fixed-rate return of 8.3% per annum while company Y should be able to get a floating-rate return LIBOR + 0.3% per annum. The required swap is shown in Figure S7.3. The bank earns 0.2%, company X earns 8.3%, and company Y earns LIBOR + 0.3%.

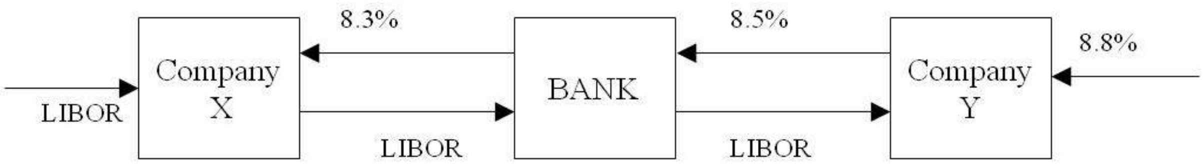

Figure S7.3 Swap for Problem 7.9

Problem 7.10.
A financial institution has entered into an interest rate swap with company X. Under the terms of the swap, it receives 10% per annum and pays six-month LIBOR on a principal of $10 million for five years. Payments are made every six months. Suppose that company X defaults on the sixth payment date (end of year 3) when the interest rate (with semiannual compounding) is 8% per annum for all maturities. What is the loss to the financial institution? Assume that six-month LIBOR was 9% per annum halfway through year 3.

At the end of year 3 the financial institution was due to receive $500,000 ($=0.5\times 10\%$ of $10 million) and pay $450,000 ($=0.5\times 9\%$ of $10 million). The immediate loss is therefore $50,000. To value the remaining swap we assume than forward rates are realized. All forward rates are 8% per annum. The remaining cash flows are therefore valued on the assumption that the floating payment is $0.5\times 0.08\times 10,000,000=\$400,000$ and the net payment that would be received is $500,000-400,000=\$100,000$. The total cost of default is therefore the cost of foregoing the following cash flows:

3 year:	$50,000
3.5 year:	$100,000
4 year:	$100,000
4.5 year:	$100,000
5 year:	$100,000

Discounting these cash flows to year 3 at 4% per six months we obtain the cost of the default as $413,000.

Problem 7.11.
Companies A and B face the following interest rates (adjusted for the differential impact of taxes):

	A	*B*
US dollars (floating rate)	LIBOR+0.5%	LIBOR+1.0%
Canadian dollars (fixed rate)	5.0%	6.5%

Assume that A wants to borrow U.S. dollars at a floating rate of interest and B wants to borrow Canadian dollars at a fixed rate of interest. A financial institution is planning to arrange a swap and requires a 50-basis-point spread. If the swap is equally attractive to A and B, what rates of interest will A and B end up paying?

Company A has a comparative advantage in the Canadian dollar fixed-rate market. Company B has a comparative advantage in the U.S. dollar floating-rate market. (This may be because of their tax positions.) However, company A wants to borrow in the U.S. dollar floating-rate

market and company B wants to borrow in the Canadian dollar fixed-rate market. This gives rise to the swap opportunity.
The differential between the U.S. dollar floating rates is 0.5% per annum, and the differential between the Canadian dollar fixed rates is 1.5% per annum. The difference between the differentials is 1% per annum. The total potential gain to all parties from the swap is therefore 1% per annum, or 100 basis points. If the financial intermediary requires 50 basis points, each of A and B can be made 25 basis points better off. Thus a swap can be designed so that it provides A with U.S. dollars at LIBOR + 0.25% per annum, and B with Canadian dollars at 6.25% per annum. The swap is shown in Figure S7.4.

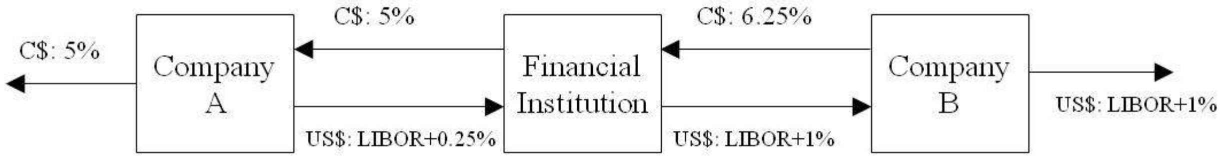

Figure S7.4 Swap for Problem 7.11

Principal payments flow in the opposite direction to the arrows at the start of the life of the swap and in the same direction as the arrows at the end of the life of the swap. The financial institution would be exposed to some foreign exchange risk which could be hedged using forward contracts.

Problem 7.12.
A financial institution has entered into a 10-year currency swap with company Y. Under the terms of the swap, the financial institution receives interest at 3% per annum in Swiss francs and pays interest at 8% per annum in U.S. dollars. Interest payments are exchanged once a year. The principal amounts are 7 million dollars and 10 million francs. Suppose that company Y declares bankruptcy at the end of year 6, when the exchange rate is \$0.80 per franc. What is the cost to the financial institution? Assume that, at the end of year 6, the interest rate is 3% per annum in Swiss francs and 8% per annum in U.S. dollars for all maturities. All interest rates are quoted with annual compounding.

When interest rates are compounded annually

$$F_0 = S_0\left(\frac{1+r}{1+r_f}\right)^T$$

where F_0 is the T-year forward rate, S_0 is the spot rate, r is the domestic risk-free rate, and r_f is the foreign risk-free rate. As $r = 0.08$ and $r_f = 0.03$, the spot and forward exchange rates at the end of year 6 are

Spot:	0.8000
1 year forward:	0.8388
2 year forward:	0.8796
3 year forward:	0.9223
4 year forward:	0.9670

The value of the swap at the time of the default can be calculated on the assumption that forward rates are realized. The cash flows lost as a result of the default are therefore as follows:

Year	*Dollar Paid*	*CHF Received*	*Forward Rate*	*Dollar Equiv of CHF Received*	*Cash Flow Lost*
6	560,000	300,000	0.8000	240,000	-320,000
7	560,000	300,000	0.8388	251,600	-308,400
8	560,000	300,000	0.8796	263,900	-296,100
9	560,000	300,000	0.9223	276,700	-283,300
10	7,560,000	10,300,000	0.9670	9,960,100	2,400,100

Discounting the numbers in the final column to the end of year 6 at 8% per annum, the cost of the default is $679,800.

Note that, if this were the only contract entered into by company Y, it would make no sense for the company to default at the end of year six as the exchange of payments at that time has a positive value to company Y. In practice, company Y is likely to be defaulting and declaring bankruptcy for reasons unrelated to this particular contract and payments on the contract are likely to stop when bankruptcy is declared.

Problem 7.13.

After it hedges its foreign exchange risk using forward contracts, is the financial institution's average spread in Figure 7.11 likely to be greater than or less than 20 basis points? Explain your answer.

The financial institution will have to buy 1.1% of the AUD principal in the forward market for each year of the life of the swap. Since AUD interest rates are higher than dollar interest rates, AUD is at a discount in forward markets. This means that the AUD purchased for year 2 is less expensive than that purchased for year 1; the AUD purchased for year 3 is less expensive than that purchased for year 2; and so on. This works in favor of the financial institution and means that its spread increases with time. The spread is always above 20 basis points.

Problem 7.14.

"Companies with high credit risks are the ones that cannot access fixed-rate markets directly. They are the companies that are most likely to be paying fixed and receiving floating in an interest rate swap." Assume that this statement is true. Do you think it increases or decreases the risk of a financial institution's swap portfolio? Assume that companies are most likely to default when interest rates are high.

Consider a plain-vanilla interest rate swap involving two companies X and Y. We suppose that X is paying fixed and receiving floating while Y is paying floating and receiving fixed. The quote suggests that company X will usually be less creditworthy than company Y. (Company X might be a BBB-rated company that has difficulty in accessing fixed-rate markets directly; company Y might be a AAA-rated company that has no difficulty accessing fixed or floating rate markets.) Presumably company X wants fixed-rate funds and company Y wants floating-rate funds.

The financial institution will realize a loss if company Y defaults when rates are high or if company X defaults when rates are low. These events are relatively unlikely since (a) Y is unlikely to default in any circumstances and (b) defaults are less likely to happen when rates are low. For the purposes of illustration, suppose that the probabilities of various events are as follows:

Default by Y:	0.001
Default by X:	0.010
Rates high when default occurs:	0.7
Rates low when default occurs:	0.3

The probability of a loss is
$0.001\times0.7+0.010\times0.3=0.0037$
If the roles of X and Y in the swap had been reversed the probability of a loss would be
$0.001\times0.3+0.010\times0.7=0.0073$
Assuming companies are more likely to default when interest rates are high, the above argument shows that the observation in quotes has the effect of decreasing the risk of a financial institution's swap portfolio. It is worth noting that the assumption that defaults are more likely when interest rates are high is open to question. The assumption is motivated by the thought that high interest rates often lead to financial difficulties for corporations. However, there is often a time lag between interest rates being high and the resultant default. When the default actually happens interest rates may be relatively low.

Problem 7.15.
Why is the expected loss from a default on a swap less than the expected loss from the default on a loan with the same principal?

In an interest-rate swap a financial institution's exposure depends on the difference between a fixed-rate of interest and a floating-rate of interest. It has no exposure to the notional principal. In a loan the whole principal can be lost.

Problem 7.16.
A bank finds that its assets are not matched with its liabilities. It is taking floating-rate deposits and making fixed-rate loans. How can swaps be used to offset the risk?

The bank is paying a floating-rate on the deposits and receiving a fixed-rate on the loans. It can offset its risk by entering into interest rate swaps (with other financial institutions or corporations) in which it contracts to pay fixed and receive floating.

Problem 7.17.
Explain how you would value a swap that is the exchange of a floating rate in one currency for a fixed rate in another currency.

The floating payments can be valued in currency A by (i) assuming that the forward rates are realized, and (ii) discounting the resulting cash flows at appropriate currency A discount rates. Suppose that the value is V_A. The fixed payments can be valued in currency B by discounting them at the appropriate currency B discount rates. Suppose that the value is V_B. If Q is the current exchange rate (number of units of currency A per unit of currency B), the value of the swap in currency A is V_A-QV_B. Alternatively, it is $V_A/Q-V_B$ in currency B.

Problem 7.18.
The LIBOR zero curve is flat at 5% (continuously compounded) out to 1.5 years. Swap rates for 2- and 3-year semiannual pay swaps are 5.4% and 5.6%, respectively. Estimate the LIBOR zero rates for maturities of 2.0, 2.5, and 3.0 years. (Assume that the 2.5-year swap

rate is the average of the 2- and 3-year swap rates.)

The two-year swap rate is 5.4%. This means that a two-year LIBOR bond paying a semiannual coupon at the rate of 5.4% per annum sells for par. If R_2 is the two-year LIBOR zero rate

$$2.7e^{-0.05\times0.5}+2.7e^{-0.05\times1.0}+2.7e^{-0.05\times1.5}+102.7e^{-R_2\times2.0}=100$$

Solving this gives $R_2=0.05342$. The 2.5-year swap rate is assumed to be 5.5%. This means that a 2.5-year LIBOR bond paying a semiannual coupon at the rate of 5.5% per annum sells for par. If $R_{2.5}$ is the 2.5-year LIBOR zero rate

$$2.75e^{-0.05\times0.5}+2.75e^{-0.05\times1.0}+2.75e^{-0.05\times1.5}+2.75e^{-0.05342\times2.0}+102.75e^{-R_{2.5}\times2.5}=100$$

Solving this gives $R_{2.5}=0.05442$. The 3-year swap rate is 5.6%. This means that a 3-year LIBOR bond paying a semiannual coupon at the rate of 5.6% per annum sells for par. If R_3 is the three-year LIBOR zero rate

$$2.8e^{-0.05\times0.5}+2.8e^{-0.05\times1.0}+2.8e^{-0.05\times1.5}+2.8e^{-0.05342\times2.0}+2.8e^{-0.05442\times2.5}$$
$$+102.8e^{-R_3\times3.0}=100$$

Solving this gives $R_3=0.05544$. The zero rates for maturities 2.0, 2.5, and 3.0 years are therefore 5.342%, 5.442%, and 5.544%, respectively.

Further Questions

Problem 7.19
(a) Company A has been offered the rates shown in Table 7.3. It can borrow for three years at 6.45%. What floating rate can it swap this fixed rate into?
(b) Company B has been offered the rates shown in Table 7.3. It can borrow for 5 years at LIBOR plus 75 basis points. What fixed rate can it swap this floating rate into?

(a) Company A can pay LIBOR and receive 6.21% for three years. It can therefore exchange a loan at 6.45% into a loan at LIBOR plus 0.24% or LIBOR plus 24 basis points
(b) Company B can receive LIBOR and pay 6.51% for five years. It can therefore exchange a loan at LIBOR plus 0.75% for a loan at 7.26%.

Problem 7.20
(a) Company X has been offered the rates shown in Table 7.3. It can invest for four years at 5.5%. What floating rate can it swap this fixed rate into?
(b) Company Y has been offered the rates shown in Table 7.3. It can invest for 10 years at LIBOR minus 50 basis points. What fixed rate can it swap this floating rate into?

(a) Company X can pay 6.39% for four years and receive LIBOR. It can therefore exchange the investment at 5.5% for an investment at LIBOR minus 0.89% or LIBOR minus 89 basis points.
(b) Company Y can receive 6.83% and pay LIBOR for 10 years. It can therefore exchange an investment at LIBOR minus 0.5% for an investment at 6.33%.

Problem 7.21.
The one-year LIBOR rate is 10% with annual compounding. A bank trades swaps where a fixed rate of interest is exchanged for 12-month LIBOR with payments being exchanged annually. Two- and three-year swap rates (expressed with annual compounding) are 11%

and 12% per annum. Estimate the two- and three-year LIBOR zero rates.

The two-year swap rate implies that a two-year LIBOR bond with a coupon of 11% sells for par. If R_2 is the two-year zero rate

$$11/1.10 + 111/(1+R)^2 = 100$$

so that $R_2 = 0.1105$. The three-year swap rate implies that a three-year LIBOR bond with a coupon of 12% sells for par. If R_3 is the three-year zero rate

$$12/1.10 + 12/1.1105^2 + 112/(1+R_3)^3 = 100$$

so that $R_3 = 0.1217$. The two- and three-year rates are therefore 11.05% and 12.17% with annual compounding.

Problem 7.22.
Company A, a British manufacturer, wishes to borrow U.S. dollars at a fixed rate of interest. Company B, a US multinational, wishes to borrow sterling at a fixed rate of interest. They have been quoted the following rates per annum (adjusted for differential tax effects):

	Sterling	*US Dollars*
Company A	*11.0%*	*7.0%*
Company B	*10.6%*	*6.2%*

Design a swap that will net a bank, acting as intermediary, 10 basis points per annum and that will produce a gain of 15 basis points per annum for each of the two companies.

The spread between the interest rates offered to A and B is 0.4% (or 40 basis points) on sterling loans and 0.8% (or 80 basis points) on U.S. dollar loans. The total benefit to all parties from the swap is therefore
$80 - 40 = 40$ basis points
It is therefore possible to design a swap which will earn 10 basis points for the bank while making each of A and B 15 basis points better off than they would be by going directly to financial markets. One possible swap is shown in Figure S7.5. Company A borrows at an effective rate of 6.85% per annum in U.S. dollars.
Company B borrows at an effective rate of 10.45% per annum in sterling. The bank earns a 10-basis-point spread. The way in which currency swaps such as this operate is as follows. Principal amounts in dollars and sterling that are roughly equivalent are chosen. These principal amounts flow in the opposite direction to the arrows at the time the swap is initiated. Interest payments then flow in the same direction as the arrows during the life of the swap and the principal amounts flow in the same direction as the arrows at the end of the life of the swap.
Note that the bank is exposed to some exchange rate risk in the swap. It earns 65 basis points in U.S. dollars and pays 55 basis points in sterling. This exchange rate risk could be hedged using forward contracts.

Figure S7.5 One Possible Swap for Problem 7.22

Problem 7.23.

Under the terms of an interest rate swap, a financial institution has agreed to pay 10% per annum and receive three-month LIBOR in return on a notional principal of $100 million with payments being exchanged every three months. The swap has a remaining life of 14 months. The average of the bid and offer fixed rates currently being swapped for three-month LIBOR is 12% per annum for all maturities. The three-month LIBOR rate one month ago was 11.8% per annum. All rates are compounded quarterly. What is the value of the swap?

The swap can be regarded as a long position in a floating-rate bond combined with a short position in a fixed-rate bond. The correct discount rate is 12% per annum with quarterly compounding or 11.82% per annum with continuous compounding.
Immediately after the next payment the floating-rate bond will be worth $100 million. The next floating payment ($ million) is

$$0.118\times 100\times 0.25 = 2.95$$

The value of the floating-rate bond is therefore

$$102.95e^{-0.1182\times 2/12} = 100.941$$

The value of the fixed-rate bond is

$$2.5e^{-0.1182\times 2/12} + 2.5e^{-0.1182\times 5/12} + 2.5e^{-0.1182\times 8/12}$$
$$+2.5e^{-0.1182\times 11/12} + 102.5e^{-0.1182\times 14/12} = 98.678$$

The value of the swap is therefore

$$100.941 - 98.678 = \$2.263\text{million}$$

As an alternative approach we can value the swap as a series of forward rate agreements. The calculated value is

$$(2.95-2.5)e^{-0.1182\times 2/12} + (3.0-2.5)e^{-0.1182\times 5/12}$$
$$+(3.0-2.5)e^{0.1182\times 8/12} + (3.0-2.5)e^{-0.1182\times 11/12}$$
$$+(3.0-2.5)e^{-0.1182\times 14/12} = \$2.263\text{ million}$$

which is in agreement with the answer obtained using the first approach.

Problem 7.24.

Suppose that the term structure of interest rates is flat in the United States and Australia. The USD interest rate is 7% per annum and the AUD rate is 9% per annum. The current value of the AUD is 0.62 USD. In a swap agreement, a financial institution pays 8% per annum in AUD and receives 4% per annum in USD. The principals in the two currencies are $12 million USD and 20 million AUD. Payments are exchanged every year, with one exchange having just taken place. The swap will last two more years. What is the value of the swap to the financial institution? Assume all interest rates are continuously compounded.

The financial institution is long a dollar bond and short a USD bond. The value of the dollar bond (in millions of dollars) is

$$0.48e^{-0.07\times 1} + 12.48e^{-0.07\times 2} = 11.297$$

The value of the AUD bond (in millions of AUD) is

$$1.6e^{-0.09\times 1} + 21.6e^{-0.09\times 2} = 19.504$$

The value of the swap (in millions of dollars) is therefore

$$11.297 - 19.504\times 0.62 = -0.795$$

or –$795,000.
As an alternative we can value the swap as a series of forward foreign exchange contracts.

The one-year forward exchange rate is $0.62e^{-0.02} = 0.6077$. The two-year forward exchange rate is $0.62e^{-0.02\times2} = 0.5957$. The value of the swap in millions of dollars is therefore

$$(0.48-1.6\times0.6077)e^{-0.07\times1} + (12.48-21.6\times0.5957)e^{-0.07\times2} = -0.795$$

which is in agreement with the first calculation.

Problem 7.25.

Company X is based in the United Kingdom and would like to borrow $50 million at a fixed rate of interest for five years in U.S. funds. Because the company is not well known in the United States, this has proved to be impossible. However, the company has been quoted 12% per annum on fixed-rate five-year sterling funds. Company Y is based in the United States and would like to borrow the equivalent of $50 million in sterling funds for five years at a fixed rate of interest. It has been unable to get a quote but has been offered U.S. dollar funds at 10.5% per annum. Five-year government bonds currently yield 9.5% per annum in the United States and 10.5% in the United Kingdom. Suggest an appropriate currency swap that will net the financial intermediary 0.5% per annum.

There is a 1% differential between the yield on sterling and dollar 5-year bonds. The financial intermediary could use this differential when designing a swap. For example, it could (a) allow company X to borrow dollars at 1% per annum less than the rate offered on sterling funds, that is, at 11% per annum and (b) allow company Y to borrow sterling at 1% per annum more than the rate offered on dollar funds, that is, at $11\frac{1}{2}$% per annum. However, as shown in Figure S7.6, the financial intermediary would not then earn a positive spread.

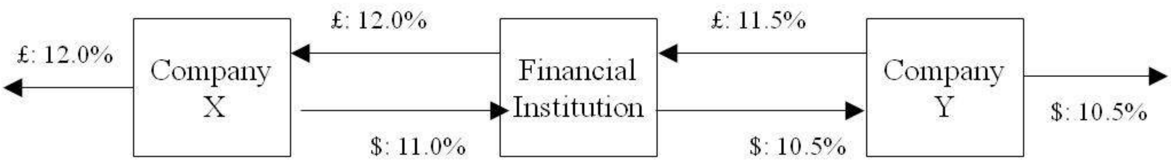

Figure S7.6 First attempt at designing swap for Problem 7.25

To make 0.5% per annum, the financial intermediary could add 0.25% per annum, to the rates paid by each of X and Y. This means that X pays 11.25% per annum, for dollars and Y pays 11.75% per annum, for sterling and leads to the swap shown in Figure S7.7. The financial intermediary would be exposed to some foreign exchange risk in this swap. This could be hedged using forward contracts.

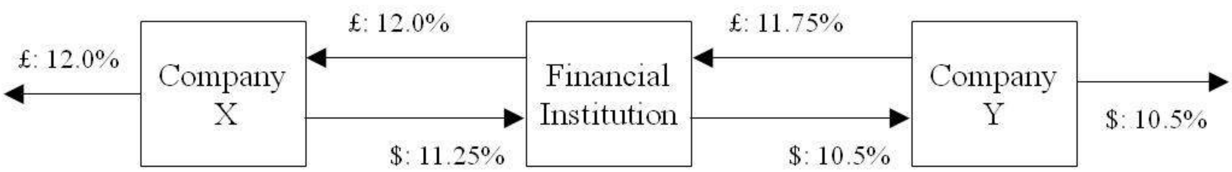

Figure S7.7 Final swap for Problem 7.25

CHAPTER 9
Mechanics of Options Markets

Practice Questions

Problem 9.1.
An investor buys a European put on a share for \$3. The stock price is \$42 and the strike price is \$40. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

The investor makes a profit if the price of the stock on the expiration date is less than \$37. In these circumstances the gain from exercising the option is greater than \$3. The option will be exercised if the stock price is less than \$40 at the maturity of the option. The variation of the investor's profit with the stock price in Figure S9.1.

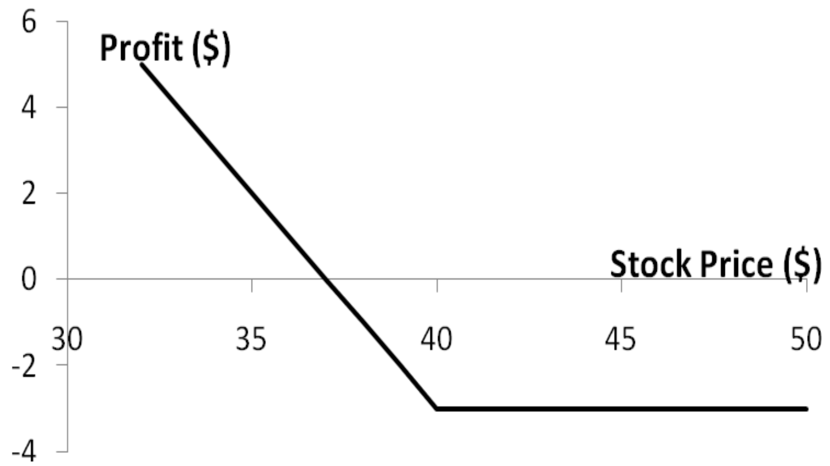

Figure S9.1 Investor's profit in Problem 9.1

Problem 9.2.
An investor sells a European call on a share for \$4. The stock price is \$47 and the strike price is \$50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

The investor makes a profit if the price of the stock is below \$54 on the expiration date. If the stock price is below \$50, the option will not be exercised, and the investor makes a profit of \$4. If the stock price is between \$50 and \$54, the option is exercised and the investor makes a profit between \$0 and \$4. The variation of the investor's profit with the stock price is as shown in Figure S9.2.

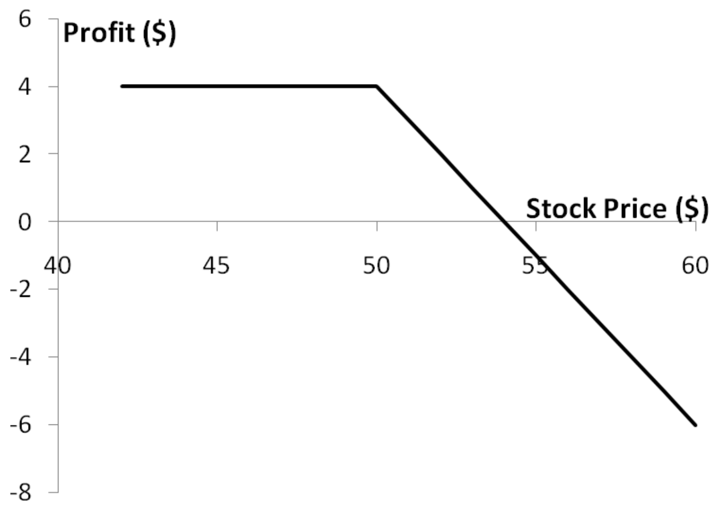

Figure S9.2 Investor's profit in Problem 9.2

Problem 9.3.
An investor sells a European call option with strike price of K and maturity T and buys a put with the same strike price and maturity. Describe the investor's position.

The payoff to the investor is

$$-\max(S_T - K, 0) + \max(K - S_T, 0)$$

This is $K - S_T$ in all circumstances. The investor's position is the same as a short position in a forward contract with delivery price K.

Problem 9.4.
Explain why margins are required when clients write options but not when they buy options.

When an investor buys an option, cash must be paid up front. There is no possibility of future liabilities and therefore no need for a margin account. When an investor sells an option, there are potential future liabilities. To protect against the risk of a default, margins are required.

Problem 9.5.
A stock option is on a February, May, August, and November cycle. What options trade on (a) April 1 and (b) May 30?

On April 1 options trade with expiration months of April, May, August, and November. On May 30 options trade with expiration months of June, July, August, and November.

Problem 9.6.
A company declares a 2-for-1 stock split. Explain how the terms change for a call option with a strike price of $60.

The strike price is reduced to \$30, and the option gives the holder the right to purchase twice as many shares.

Problem 9.7.
"Employee stock options issued by a company are different from regular exchange-traded call options on the company's stock because they can affect the capital structure of the company." Explain this statement.

The exercise of employee stock options usually leads to new shares being issued by the company and sold to the employee. This changes the amount of equity in the capital structure. When a regular exchange-traded option is exercised no new shares are issued and the company's capital structure is not affected.

Problem 9.8.
A corporate treasurer is designing a hedging program involving foreign currency options. What are the pros and cons of using (a) the NASDAQ OMX and (b) the over-the-counter market for trading?

The NASDAQ OMX offers options with standard strike prices and times to maturity. Options in the over-the-counter market have the advantage that they can be tailored to meet the precise needs of the treasurer. Their disadvantage is that they expose the treasurer to some credit risk. Exchanges organize their trading so that there is virtually no credit risk.

Problem 9.9.
Suppose that a European call option to buy a share for \$100.00 costs \$5.00 and is held until maturity. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.

Ignoring the time value of money, the holder of the option will make a profit if the stock price at maturity of the option is greater than \$105. This is because the payoff to the holder of the option is, in these circumstances, greater than the \$5 paid for the option. The option will be exercised if the stock price at maturity is greater than \$100. Note that if the stock price is between \$100 and \$105 the option is exercised, but the holder of the option takes a loss overall. The profit from a long position is as shown in Figure S9.3.

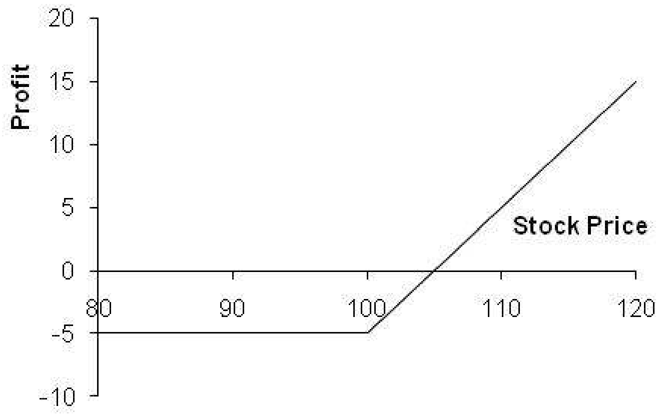

Figure S9.3 Profit from long position in Problem 9.9

Problem 9.10.
Suppose that a European put option to sell a share for $60 costs $8 and is held until maturity. Under what circumstances will the seller of the option (the party with the short position) make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a short position in the option depends on the stock price at maturity of the option.

Ignoring the time value of money, the seller of the option will make a profit if the stock price at maturity is greater than $52.00. This is because the cost to the seller of the option is in these circumstances less than the price received for the option. The option will be exercised if the stock price at maturity is less than $60.00. Note that if the stock price is between $52.00 and $60.00 the seller of the option makes a profit even though the option is exercised. The profit from the short position is as shown in Figure S9.4.

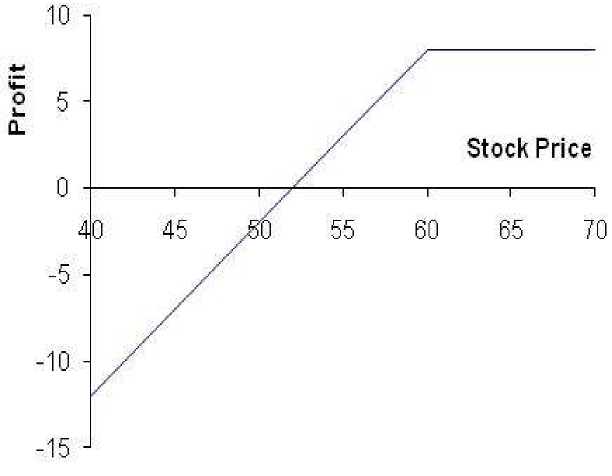

Figure S9.4 Profit from short position in Problem 9.10

Problem 9.11.
Describe the terminal value of the following portfolio: a newly entered-into long forward contract on an asset and a long position in a European put option on the asset with the same maturity as the forward contract and a strike price that is equal to the forward price of the asset at the time the portfolio is set up. Show that the European put option has the same value as a European call option with the same strike price and maturity.

The terminal value of the long forward contract is:

$$S_T - F_0$$

where S_T is the price of the asset at maturity and F_0 is the forward price of the asset at the time the portfolio is set up. (The delivery price in the forward contract is also F_0.)
The terminal value of the put option is:

$$\max(F_0 - S_T, 0)$$

The terminal value of the portfolio is therefore

$$S_T - F_0 + \max(F_0 - S_T, 0)$$
$$= \max(0, S_T - F_0]$$

This is the same as the terminal value of a European call option with the same maturity as the forward contract and an exercise price equal to F_0. This result is illustrated in the Figure S9.5.

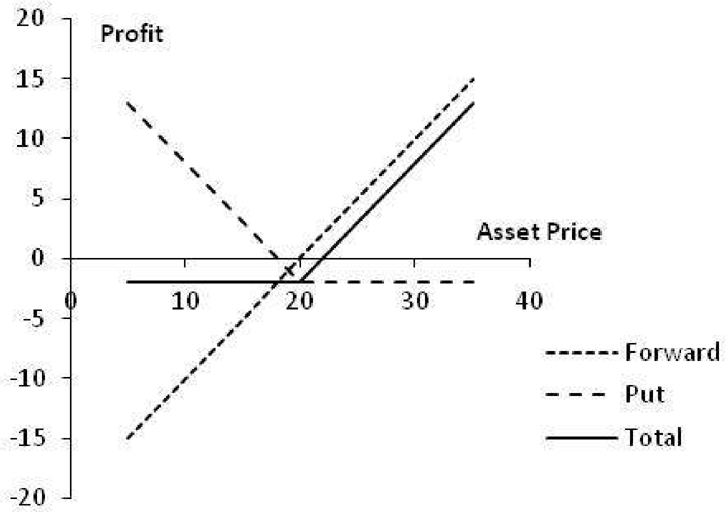

Figure S9.5 Profit from portfolio in Problem 9.11

We have shown that the forward contract plus the put is worth the same as a call with the same strike price and time to maturity as the put. The forward contract is worth zero at the time the portfolio is set up. It follows that the put is worth the same as the call at the time the portfolio is set up.

Problem 9.12.
A trader buys a call option with a strike price of \$45 and a put option with a strike price of \$40. Both options have the same maturity. The call costs \$3 and the put costs \$4. Draw a diagram showing the variation of the trader's profit with the asset price.

Figure S9.6 shows the variation of the trader's position with the asset price. We can divide the alternative asset prices into three ranges:

a) When the asset price less than \$40, the put option provides a payoff of $40-S_T$ and the call option provides no payoff. The options cost \$7 and so the total profit is $33-S_T$.
b) When the asset price is between \$40 and \$45, neither option provides a payoff. There is a net loss of \$7.
c) When the asset price greater than \$45, the call option provides a payoff of S_T-45 and the put option provides no payoff. Taking into account the \$7 cost of the options, the total profit is S_T-52.

The trader makes a profit (ignoring the time value of money) if the stock price is less than \$33 or greater than \$52. This type of trading strategy is known as a strangle and is discussed in Chapter 11.

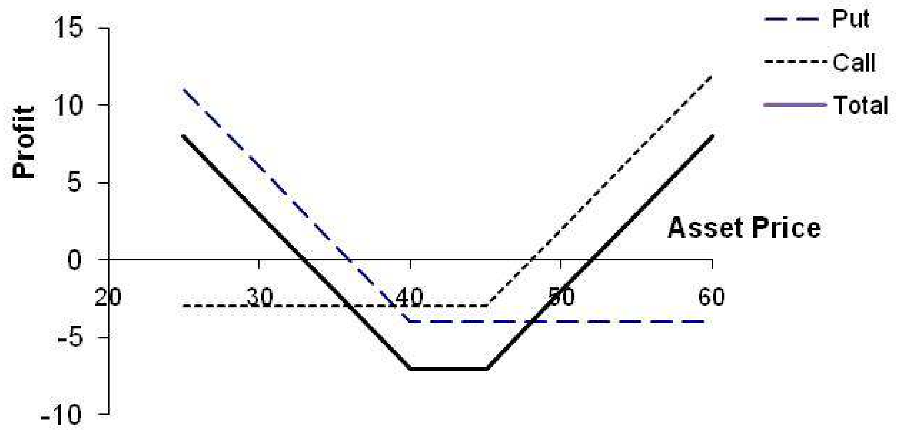

Figure S9.6 Profit from trading strategy in Problem 9.12

Problem 9.13.
Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date.

The holder of an American option has all the same rights as the holder of a European option and more. It must therefore be worth at least as much. If it were not, an arbitrageur could short the European option and take a long position in the American option.

Problem 9.14.
Explain why an American option is always worth at least as much as its intrinsic value.

The holder of an American option has the right to exercise it immediately. The American option must therefore be worth at least as much as its intrinsic value. If it were not an arbitrageur could lock in a sure profit by buying the option and exercising it immediately.

Problem 9.15.
Explain carefully the difference between writing a put option and buying a call option.

Writing a put gives a payoff of $\min(S_T - K, 0)$. Buying a call gives a payoff of $\max(S_T - K, 0)$. In both cases the potential payoff is $S_T - K$. The difference is that for a written put the counterparty chooses whether you get the payoff (and will allow you to get it only when it is negative to you). For a long call you decide whether you get the payoff (and you choose to get it when it is positive to you.)

Problem 9.16.
The treasurer of a corporation is trying to choose between options and forward contracts to hedge the corporation's foreign exchange risk. Discuss the advantages and disadvantages of each.

Forward contracts lock in the exchange rate that will apply to a particular transaction in the future. Options provide insurance that the exchange rate will not be worse than some level. The advantage of a forward contract is that uncertainty is eliminated as far as possible. The disadvantage is that the outcome with hedging can be significantly worse than the outcome with no hedging. This disadvantage is not as marked with options. However, unlike forward contracts, options involve an up-front cost.

Problem 9.17.
Consider an exchange-traded call option contract to buy 500 shares with a strike price of $40 and maturity in four months. Explain how the terms of the option contract change when there is

a) *A 10% stock dividend*
b) *A 10% cash dividend*
c) *A 4-for-1 stock split*

a) The option contract becomes one to buy $500 \times 1.1 = 550$ shares with an exercise price $40/1.1 = 36.36$.
b) There is no effect. The terms of an options contract are not normally adjusted for cash dividends.
c) The option contract becomes one to buy $500 \times 4 = 2,000$ shares with an exercise price of $40/4 = \$10$.

Problem 9.18.
"If most of the call options on a stock are in the money, it is likely that the stock price has risen rapidly in the last few months." Discuss this statement.

The exchange has certain rules governing when trading in a new option is initiated. These mean that the option is close-to-the-money when it is first traded. If all call options are in the money it is therefore likely that the stock price has increased since trading in the option began.

Problem 9.19.
What is the effect of an unexpected cash dividend on (a) a call option price and (b) a put option price?

An unexpected cash dividend would reduce the stock price on the ex-dividend date. This stock price reduction would not be anticipated by option holders. As a result there would be a reduction in the value of a call option and an increase the value of a put option. (Note that the terms of an option are adjusted for cash dividends only in exceptional circumstances.)

Problem 9.20.
Options on General Motors stock are on a March, June, September, and December cycle. What options trade on (a) March 1, (b) June 30, and (c) August 5?

a) March, April, June and September
b) July, August, September, December
c) August, September, December, March.

Longer dated options may also trade.

Problem 9.21.
Explain why the market maker's bid-offer spread represents a real cost to options investors.

A "fair" price for the option can reasonably be assumed to be half way between the bid and the offer price quoted by a market maker. An investor typically buys at the market maker's offer and sells at the market maker's bid. Each time he or she does this there is a hidden cost equal to half the bid-offer spread.

Problem 9.22.
A United States investor writes five naked call option contracts. The option price is \$3.50, the strike price is \$60.00, and the stock price is \$57.00. What is the initial margin requirement?

The two calculations are necessary to determine the initial margin. The first gives

$$500 \times (3.5 + 0.2 \times 57 - 3) = 5{,}950$$

The second gives

$$500 \times (3.5 + 0.1 \times 57) = 4{,}600$$

The initial margin is the greater of these, or \$5,950. Part of this can be provided by the initial amount of $500 \times 3.5 = \$1{,}750$ received for the options.

Further Questions

Problem 9.23.
The price of a stock is \$40. The price of a one-year European put option on the stock with a strike price of \$30 is quoted as \$7 and the price of a one-year European call option on the stock with a strike price of \$50 is quoted as \$5. Suppose that an investor buys 100 shares, shorts 100 call options, and buys 100 put options. Draw a diagram illustrating how the investor's profit or loss varies with the stock price over the next year. How does your answer change if the investor buys 100 shares, shorts 200 call options, and buys 200 put options?

Figure S9.7 shows the way in which the investor's profit varies with the stock price in the

first case. For stock prices less than $30 there is a loss of $1,200. As the stock price increases from $30 to $50 the profit increases from –$1,200 to $800. Above $50 the profit is $800. Students may express surprise that a call which is $10 out of the money is less expensive than a put which is $10 out of the money. This could be because of dividends or the crashophobia phenomenon discussed in Chapter 19.
Figure S9.8 shows the way in which the profit varies with stock price in the second case. In this case the profit pattern has a zigzag shape. The problem illustrates how many different patterns can be obtained by including calls, puts, and the underlying asset in a portfolio.

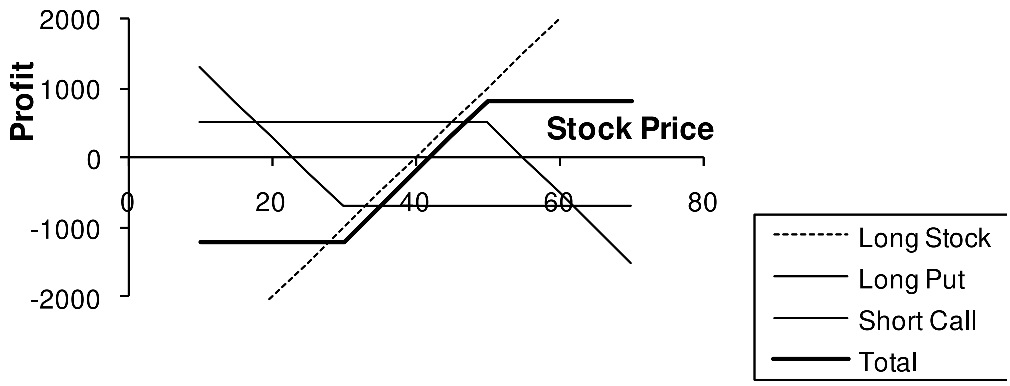

Figure S9.7 Profit in first case considered Problem 9.23

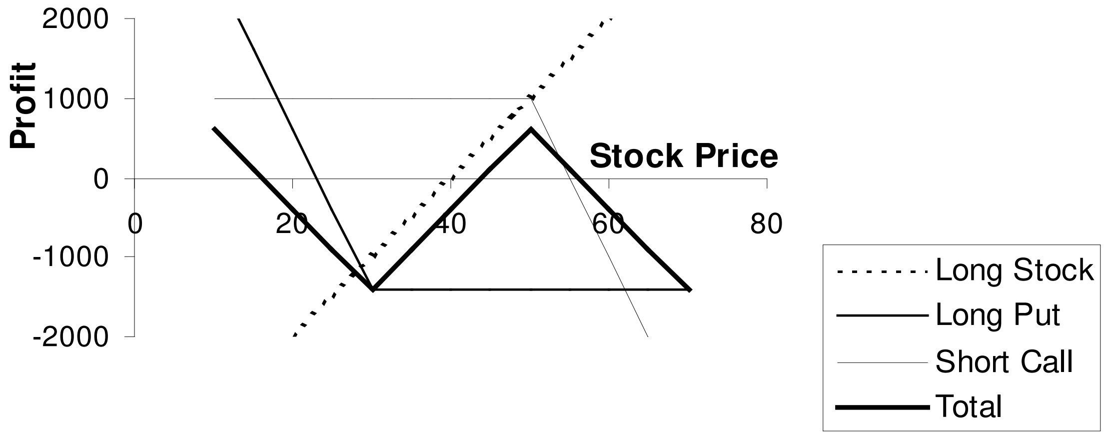

Figure S9.8 Profit for the second case considered Problem 9.23

Problem 9.24.

"If a company does not do better than its competitors but the stock market goes up, executives do very well from their stock options. This makes no sense" Discuss this viewpoint. Can you think of alternatives to the usual executive stock option plan that take the viewpoint into account.

Executive stock option plans account for a high percentage of the total remuneration received

by executives. When the market is rising fast, many corporate executives do very well out of their stock option plans — even when their company does worse than its competitors. Large institutional investors have argued that executive stock options should be structured so that the payoff depends how the company has performed relative to an appropriate industry index. In a regular executive stock option the strike price is the stock price at the time the option is issued. In the type of relative-performance stock option favored by institutional investors, the strike price at time t is $S_0 I_t / I_0$ where S_0 is the company's stock price at the time the option is issued, I_0 is the value of an equity index for the industry in which the company operates at the time the option is issued, and I_t is the value of the index at time t. If the company's performance equals the performance of the industry, the options are always at-the-money. If the company outperforms the industry, the options become in the money. If the company underperforms the industry, the options become out of the money. Note that a relative performance stock option can provide a payoff when both the market and the company's stock price decline.
Relative performance stock options clearly provide a better way of rewarding senior management for superior performance. Some companies have argued that, if they introduce relative performance options when their competitors do not, they will lose some of their top management talent.

Problem 9.25.
Use DerivaGem to calculate the value of an American put option on a nondividend paying stock when the stock price is \$30, the strike price is \$32, the risk-free rate is 5%, the volatility is 30%, and the time to maturity is 1.5 years. (Choose Binomial American for the "option type" and 50 time steps.)

a. *What is the option's intrinsic value?*
b. *What is the option's time value?*
c. *What would a time value of zero indicate? What is the value of an option with zero time value?*
d. *Using a trial and error approach calculate how low the stock price would have to be for the time value of the option to be zero.*

DerivaGem shows that the value of the option is 4.57. The option's intrinsic value is $32-30=2.00$. The option's time value is therefore $4.57-2.00=2.57$. A time value of zero would indicate that it is optimal to exercise the option immediately. In this case the value of the option would equal its intrinsic value. When the stock price is 20, DerivaGem gives the value of the option as 12, which is its intrinsic value. When the stock price is 25, DerivaGem gives the value of the options as 7.54, indicating that the time value is still positive ($=0.54$). Keeping the number of time steps equal to 50, trial and error indicates the time value disappears when the stock price is reduced to 21.6 or lower. (With 500 time steps this estimate of how low the stock price must become is reduced to 21.3.)

Problem 9.26.
On July 20, 2004 Microsoft surprised the market by announcing a \$3 dividend. The ex-dividend date was November 17, 2004 and the payment date was December 2, 2004. Its stock price at the time was about \$28. It also changed the terms of its employee stock options so that each exercise price was adjusted downward to

Pre-dividend Exercise Price $\times \frac{\text{ClosingPrice}-\$3.00}{\text{ClosingPrice}}$

The number of shares covered by each stock option outstanding was adjusted upward to

Number of Shares Pre-dividend $\times \frac{\text{ClosingPrice}}{\text{ClosingPrice} - \$3.00}$

"Closing Price" means the official NASDAQ closing price of a share of Microsoft common stock on the last trading day before the ex-dividend date.

Evaluate this adjustment. Compare it with the system used by exchanges to adjust for extraordinary dividends (see Business Snapshot 9.1).

Suppose that the closing stock price is $28 and an employee has 1000 options with a strike price of $24. Microsoft's adjustment involves changing the strike price to $24 \times 25/28 = 21.4286$ and changing the number of options to $1000 \times 28/25 = 1{,}120$. The system used by exchanges would involve keeping the number of options the same and reducing the strike price by $3 to $21.

The Microsoft adjustment is more complicated than that used by the exchange because it requires a knowledge of the Microsoft's stock price immediately before the stock goes ex-dividend. However, arguably it is a better adjustment than the one used by the exchange. Before the adjustment the employee has the right to pay $24,000 for Microsoft stock that is worth $28,000. After the adjustment the employee also has the option to pay $24,000 for Microsoft stock worth $28,000. Under the adjustment rule used by exchanges the employee would have the right to buy stock worth $25,000 for $21,000. If the volatility of Microsoft remains the same this is a less valuable option.

One complication here is that Microsoft's volatility does not remain the same. It can be expected to go up because some cash (a zero risk asset) has been transferred to shareholders. The employees therefore have the same basic option as before but the volatility of Microsoft can be expected to increase. The employees are slightly better off because the value of an option increases with volatility.

CHAPTER 11
Trading Strategies Involving Options

Practice Questions

Problem 11.1.
What is meant by a protective put? What position in call options is equivalent to a protective put?

A protective put consists of a long position in a put option combined with a long position in the underlying shares. It is equivalent to a long position in a call option plus a certain amount of cash. This follows from put–call parity:

$$p+S_0=c+Ke^{-rT}+D$$

Problem 11.2.
Explain two ways in which a bear spread can be created.

A bear spread can be created using two call options with the same maturity and different strike prices. The investor shorts the call option with the lower strike price and buys the call option with the higher strike price. A bear spread can also be created using two put options with the same maturity and different strike prices. In this case, the investor shorts the put option with the lower strike price and buys the put option with the higher strike price.

Problem 11.3.
When is it appropriate for an investor to purchase a butterfly spread?

A butterfly spread involves a position in options with three different strike prices (K_1, K_2, and K_3). A butterfly spread should be purchased when the investor considers that the price of the underlying stock is likely to stay close to the central strike price, K_2.

Problem 11.4.
Call options on a stock are available with strike prices of \$15, $\$17\frac{1}{2}$, and \$20 and expiration dates in three months. Their prices are \$4, \$2, and $\$\frac{1}{2}$, respectively. Explain how the options can be used to create a butterfly spread. Construct a table showing how profit varies with stock price for the butterfly spread.

An investor can create a butterfly spread by buying call options with strike prices of \$15 and \$20 and selling two call options with strike prices of $\$17\frac{1}{2}$. The initial investment is $4+\frac{1}{2}-2\times 2=\$\frac{1}{2}$. The following table shows the variation of profit with the final stock price:

Stock Price, S_T	*Profit*
$S_T < 15$	$-\frac{1}{2}$
$15 < S_T < 17\frac{1}{2}$	$(S_T - 15) - \frac{1}{2}$
$17\frac{1}{2} < S_T < 20$	$(20 - S_T) - \frac{1}{2}$
$S_T > 20$	$-\frac{1}{2}$

Problem 11.5.
What trading strategy creates a reverse calendar spread?

A reverse calendar spread is created by buying a short-maturity option and selling a long-maturity option, both with the same strike price.

Problem 11.6.
What is the difference between a strangle and a straddle?

Both a straddle and a strangle are created by combining a long position in a call with a long position in a put. In a straddle the two have the same strike price and expiration date. In a strangle they have different strike prices and the same expiration date.

Problem 11.7.
A call option with a strike price of \$50 costs \$2. A put option with a strike price of \$45 costs \$3. Explain how a strangle can be created from these two options. What is the pattern of profits from the strangle?

A strangle is created by buying both options. The pattern of profits is as follows:

Stock Price, S_T	*Profit*
$S_T < 45$	$(45 - S_T) - 5$
$45 < S_T < 50$	-5
$S_T > 50$	$(S_T - 50) - 5$

Problem 11.8.
Use put–call parity to relate the initial investment for a bull spread created using calls to the initial investment for a bull spread created using puts.

A bull spread using calls provides a profit pattern with the same general shape as a bull spread using puts (see Figures 11.2 and 11.3 in the text). Define p_1 and c_1 as the prices of put and call with strike price K_1 and p_2 and c_2 as the prices of a put and call with strike price K_2. From put-call parity

$$p_1 + S = c_1 + K_1 e^{-rT}$$

$$p_2 + S = c_2 + K_2 e^{-rT}$$

Hence:

$$p_1 - p_2 = c_1 - c_2 - (K_2 - K_1)e^{-rT}$$

This shows that the initial investment when the spread is created from puts is less than the initial investment when it is created from calls by an amount $(K_2 - K_1)e^{-rT}$. In fact as mentioned in the text the initial investment when the bull spread is created from puts is negative, while the initial investment when it is created from calls is positive.

The profit when calls are used to create the bull spread is higher than when puts are used by $(K_2 - K_1)(1 - e^{-rT})$. This reflects the fact that the call strategy involves an additional risk-free investment of $(K_2 - K_1)e^{-rT}$ over the put strategy. This earns interest of $(K_2 - K_1)e^{-rT}(e^{rT} - 1) = (K_2 - K_1)(1 - e^{-rT})$.

Problem 11.9.

Explain how an aggressive bear spread can be created using put options.

An aggressive bull spread using call options is discussed in the text. Both of the options used have relatively high strike prices. Similarly, an aggressive bear spread can be created using put options. Both of the options should be out of the money (that is, they should have relatively low strike prices). The spread then costs very little to set up because both of the puts are worth close to zero. In most circumstances the spread will provide zero payoff. However, there is a small chance that the stock price will fall fast so that on expiration both options will be in the money. The spread then provides a payoff equal to the difference between the two strike prices, $K_2 - K_1$.

Problem 11.10.

Suppose that put options on a stock with strike prices \$30 and \$35 cost \$4 and \$7, respectively. How can the options be used to create (a) a bull spread and (b) a bear spread? Construct a table that shows the profit and payoff for both spreads.

A bull spread is created by buying the \$30 put and selling the \$35 put. This strategy gives rise to an initial cash inflow of \$3. The outcome is as follows:

Stock Price	*Payoff*	*Profit*
$S_T \geq 35$	0	3
$30 \leq S_T < 35$	$S_T - 35$	$S_T - 32$
$S_T < 30$	−5	−2

A bear spread is created by selling the \$30 put and buying the \$35 put. This strategy costs \$3 initially. The outcome is as follows:

Stock Price	*Payoff*	*Profit*
$S_T \geq 35$	0	−3
$30 \leq S_T < 35$	$35 - S_T$	$32 - S_T$
$S_T < 30$	5	2

Problem 11.11.

Use put–call parity to show that the cost of a butterfly spread created from European puts is identical to the cost of a butterfly spread created from European calls.

Define c_1, c_2, and c_3 as the prices of calls with strike prices K_1, K_2 and K_3. Define p_1, p_2 and p_3 as the prices of puts with strike prices K_1, K_2 and K_3. With the usual notation

$$c_1 + K_1 e^{-rT} = p_1 + S$$

$$c_2 + K_2 e^{-rT} = p_2 + S$$

$$c_3 + K_3 e^{-rT} = p_3 + S$$

Hence

$$c_1 + c_3 - 2c_2 + (K_1 + K_3 - 2K_2)e^{-rT} = p_1 + p_3 - 2p_2$$

Because $K_2 - K_1 = K_3 - K_2$, it follows that $K_1 + K_3 - 2K_2 = 0$ and

$$c_1 + c_3 - 2c_2 = p_1 + p_3 - 2p_2$$

The cost of a butterfly spread created using European calls is therefore exactly the same as the cost of a butterfly spread created using European puts.

Problem 11.12.

A call with a strike price of \$60 costs \$6. A put with the same strike price and expiration date costs \$4. Construct a table that shows the profit from a straddle. For what range of stock prices would the straddle lead to a loss?

A straddle is created by buying both the call and the put. This strategy costs \$10. The profit/loss is shown in the following table:

Stock Price	*Payoff*	*Profit*
$S_T > 60$	$S_T - 60$	$S_T - 70$
$S_T \le 60$	$60 - S_T$	$50 - S_T$

This shows that the straddle will lead to a loss if the final stock price is between \$50 and \$70.

Problem 11.13.

Construct a table showing the payoff from a bull spread when puts with strike prices K_1 and K_2 are used $(K_2 > K_1)$.

The bull spread is created by buying a put with strike price K_1 and selling a put with strike price K_2. The payoff is calculated as follows:

Stock Price	*Payoff from Long Put*	*Payoff from Short Put*	*Total Payoff*
$S_T \ge K_2$	0	0	0
$K_1 < S_T < K_2$	0	$S_T - K_2$	$-(K_2 - S_T)$
$S_T \le K_1$	$K_1 - S_T$	$S_T - K_2$	$-(K_2 - K_1)$

Problem 11.14.

An investor believes that there will be a big jump in a stock price, but is uncertain as to the direction. Identify six different strategies the investor can follow and explain the differences *among them.*

Possible strategies are:

Strangle
Straddle
Strip
Strap
Reverse calendar spread
Reverse butterfly spread

The strategies all provide positive profits when there are large stock price moves. A strangle is less expensive than a straddle, but requires a bigger move in the stock price in order to provide a positive profit. Strips and straps are more expensive than straddles but provide bigger profits in certain circumstances. A strip will provide a bigger profit when there is a large downward stock price move. A strap will provide a bigger profit when there is a large upward stock price move. In the case of strangles, straddles, strips and straps, the profit increases as the size of the stock price movement increases. By contrast in a reverse calendar spread and a reverse butterfly spread there is a maximum potential profit regardless of the size of the stock price movement.

Problem 11.15.
How can a forward contract on a stock with a particular delivery price and delivery date be created from options?

Suppose that the delivery price is K and the delivery date is T. The forward contract is created by buying a European call and selling a European put when both options have strike price K and exercise date T. This portfolio provides a payoff of $S_T - K$ under all circumstances where S_T is the stock price at time T. Suppose that F_0 is the forward price. If $K = F_0$, the forward contract that is created has zero value. This shows that the price of a call equals the price of a put when the strike price is F_0.

Problem 11.16.
"A box spread comprises four options. Two can be combined to create a long forward position and two can be combined to create a short forward position." Explain this statement.

A box spread is a bull spread created using calls and a bear spread created using puts. With the notation in the text it consists of a) a long call with strike K_1, b) a short call with strike K_2, c) a long put with strike K_2, and d) a short put with strike K_1. a) and d) give a long forward contract with delivery price K_1; b) and c) give a short forward contract with delivery price K_2. The two forward contracts taken together give the payoff of $K_2 - K_1$.

Problem 11.17.
What is the result if the strike price of the put is higher than the strike price of the call in a strangle?

The result is shown in Figure S11.1. The profit pattern from a long position in a call and a put when the put has a higher strike price than a call is much the same as when the call has a higher strike price than the put. Both the initial investment and the final payoff are much higher in the first case

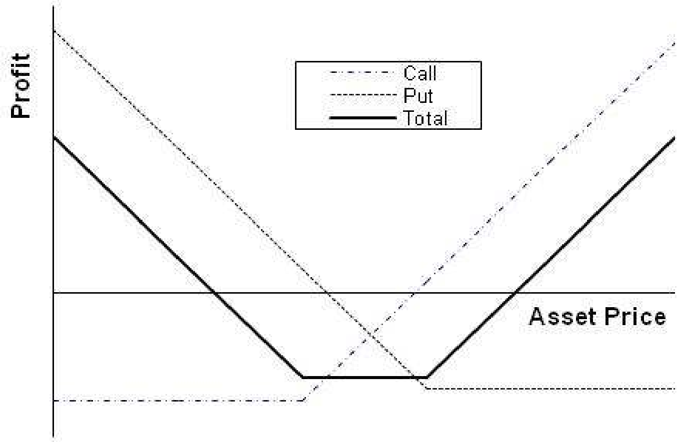

Figure S11.1 Profit Pattern in Problem 11.17

Problem 11.18.

One Australian dollar is currently worth \$0.64. A one-year butterfly spread is set up using European call options with strike prices of \$0.60, \$0.65, and \$0.70. The risk-free interest rates in the United States and Australia are 5% and 4% respectively, and the volatility of the exchange rate is 15%. Use the DerivaGem software to calculate the cost of setting up the butterfly spread position. Show that the cost is the same if European put options are used instead of European call options.

To use DerivaGem select the first worksheet and choose Currency as the Underlying Type. Select Black--Scholes European as the Option Type. Input exchange rate as 0.64, volatility as 15%, risk-free rate as 5%, foreign risk-free interest rate as 4%, time to exercise as 1 year, and exercise price as 0.60. Select the button corresponding to call. Do not select the implied volatility button. Hit the Enter key and click on calculate. DerivaGem will show the price of the option as 0.0618. Change the exercise price to 0.65, hit Enter, and click on calculate again. DerivaGem will show the value of the option as 0.0352. Change the exercise price to 0.70, hit Enter, and click on calculate. DerivaGem will show the value of the option as 0.0181.

Now select the button corresponding to put and repeat the procedure. DerivaGem shows the values of puts with strike prices 0.60, 0.65, and 0.70 to be 0.0176, 0.0386, and 0.0690, respectively.

The cost of setting up the butterfly spread when calls are used is therefore

$$0.0618 + 0.0181 - 2 \times 0.0352 = 0.0095$$

The cost of setting up the butterfly spread when puts are used is

$$0.0176 + 0.0690 - 2 \times 0.0386 = 0.0094$$

Allowing for rounding errors these two are the same.

Problem 11.19

An index provides a dividend yield of 1% and has a volatility of 20%. The risk-free interest rate is 4%. How long does a principal-protected note, created as in Example 11.1, have to *last for it to be profitable to the bank? Use DerivaGem.*

Assume that the investment in the index is initially \$100. (This is a scaling factor that makes no difference to the result.) DerivaGem can be used to value an option on the index with the index level equal to 100, the volatility equal to 20%, the risk-free rate equal to 4%, the dividend yield equal to 1%, and the exercise price equal to 100. For different times to maturity, T, we value a call option (using Black-Scholes European) and the amount available to buy the call option, which is $100\text{-}100e^{-0.04\times T}$. Results are as follows:

Time to maturity, T	Funds Available	Value of Option
1	3.92	9.32
2	7.69	13.79
5	18.13	23.40
10	32.97	33.34
11	35.60	34.91

This table shows that the answer is between 10 and 11 years. Continuing the calculations we find that if the life of the principal-protected note is 10.35 year or more, it is profitable for the bank. (Excel's Solver can be used in conjunction with the DerivaGem functions to facilitate calculations.)

Further Questions

Problem 11.20.
Three put options on a stock have the same expiration date and strike prices of \$55, \$60, and \$65. The market prices are \$3, \$5, and \$8, respectively. Explain how a butterfly spread can be created. Construct a table showing the profit from the strategy. For what range of stock prices would the butterfly spread lead to a loss?

A butterfly spread is created by buying the \$55 put, buying the \$65 put and selling two of the \$60 puts. This costs $3+8-2\times 5=\$1$ initially. The following table shows the profit/loss from the strategy.

Stock Price	*Payoff*	*Profit*
$S_T \geq 65$	0	-1
$60 \leq S_T < 65$	$65-S_T$	$64-S_T$
$55 \leq S_T < 60$	S_T-55	S_T-56
$S_T < 55$	0	-1

The butterfly spread leads to a loss when the final stock price is greater than \$64 or less than \$56.

Problem 11.21.
A diagonal spread is created by buying a call with strike price K_2 and exercise date T_2 and

selling a call with strike price K_1 *and exercise date* T_1 $(T_2 > T_1)$. *Draw a diagram showing the profit when (a)* $K_2 > K_1$ *and (b)* $K_2 < K_1$.

There are two alternative profit patterns for part (a). These are shown in Figures S11.2 and S11.3. In Figure S11.2 the long maturity (high strike price) option is worth more than the short maturity (low strike price) option. In Figure S11.3 the reverse is true. There is no ambiguity about the profit pattern for part (b). This is shown in Figure S11.4.

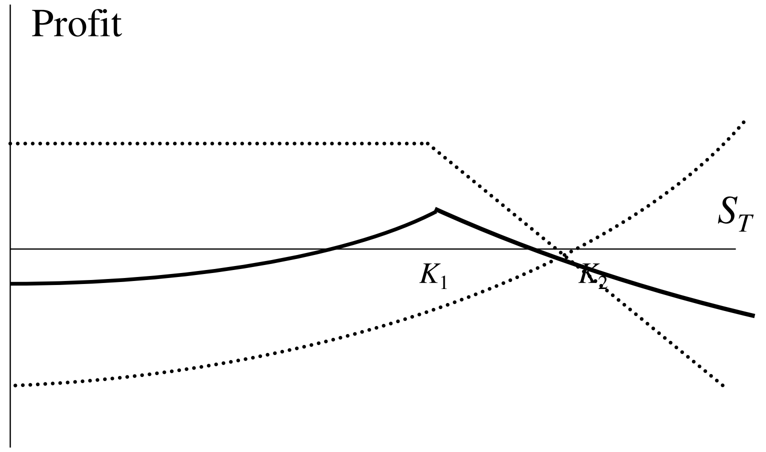

Figure S11.2: Investor's Profit/Loss in Problem 11.21a when long maturity call is worth more than short maturity call

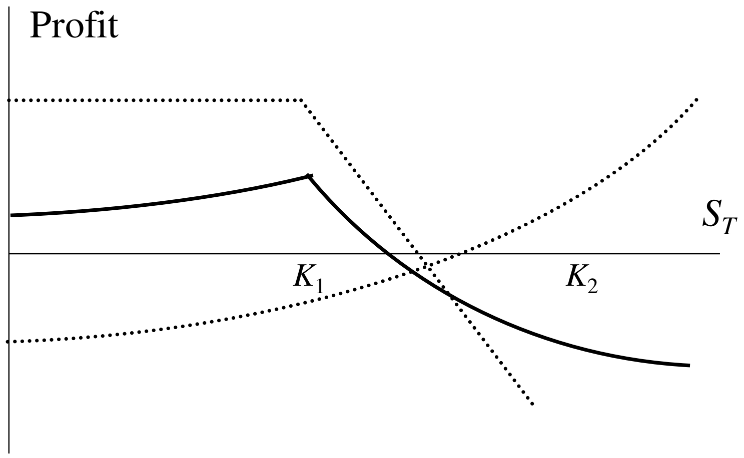

Figure S11.3 Investor's Profit/Loss in Problem 11.21b when short maturity call is worth more than long maturity call

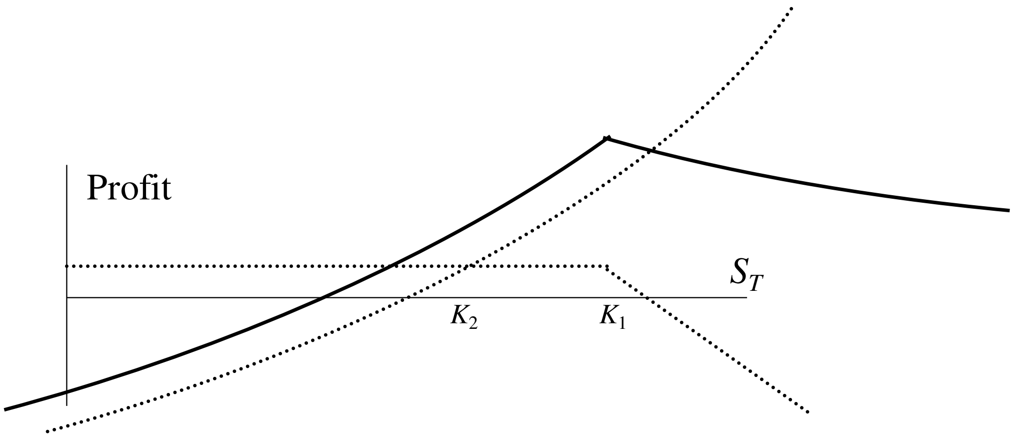

Figure S11.4 Investor's Profit/Loss in Problem 11.21b

Problem 11.22.

Draw a diagram showing the variation of an investor's profit and loss with the terminal stock price for a portfolio consisting of

a. *One share and a short position in one call option*
b. *Two shares and a short position in one call option*
c. *One share and a short position in two call options*
d. *One share and a short position in four call options*

In each case, assume that the call option has an exercise price equal to the current stock price.

The variation of an investor's profit/loss with the terminal stock price for each of the four strategies is shown in Figure S11.5. In each case the dotted line shows the profits from the components of the investor's position and the solid line shows the total net profit.

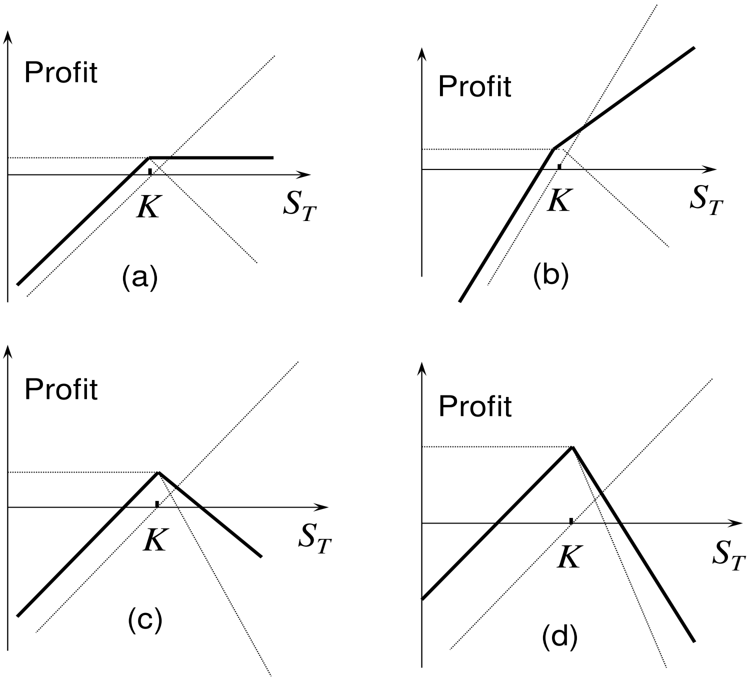

Figure S11.5 Answer to Problem 11.22

Problem 11.23.

Suppose that the price of a non-dividend-paying stock is \$32, its volatility is 30%, and the risk-free rate for all maturities is 5% per annum. Use DerivaGem to calculate the cost of setting up the following positions. In each case provide a table showing the relationship between profit and final stock price. Ignore the impact of discounting.

a. *A bull spread using European call options with strike prices of \$25 and \$30 and a maturity of six months.*
b. *A bear spread using European put options with strike prices of \$25 and \$30 and a maturity of six months*
c. *A butterfly spread using European call options with strike prices of \$25, \$30, and \$35 and a maturity of one year.*
d. *A butterfly spread using European put options with strike prices of \$25, \$30, and \$35 and a maturity of one year.*
e. *A straddle using options with a strike price of \$30 and a six-month maturity.*
f. *A strangle using options with strike prices of \$25 and \$35 and a six-month maturity.*

In each case provide a table showing the relationship between profit and final stock price. Ignore the impact of discounting.

(a) A call option with a strike price of 25 costs 7.90 and a call option with a strike price of 30 costs 4.18. The cost of the bull spread is therefore $7.90 - 4.18 = 3.72$. The profits ignoring the impact of discounting are

Stock Price Range	*Profit*
$S_T \le 25$	-3.72
$25 < S_T < 30$	$S_T - 28.72$
$S_T \ge 30$	1.28

(b) A put option with a strike price of 25 costs 0.28 and a put option with a strike price of 30 costs 1.44. The cost of the bear spread is therefore $1.44 - 0.28 = 1.16$. The profits ignoring the impact of discounting are

Stock Price Range	*Profit*
$S_T \le 25$	$+3.84$
$25 < S_T < 30$	$28.84 - S_T$
$S_T \ge 30$	-1.16

(c) Call options with maturities of one year and strike prices of 25, 30, and 35 cost 8.92, 5.60, and 3.28, respectively. The cost of the butterfly spread is therefore $8.92 + 3.28 - 2 \times 5.60 = 1.00$. The profits ignoring the impact of discounting are

Stock Price Range	*Profit*
$S_T \le 25$	-1.00
$25 < S_T < 30$	$S_T - 26.00$
$30 \le S_T < 35$	$34.00 - S_T$

(d) Put options with maturities of one year and strike prices of 25, 30, and 35 cost 0.70, 2.14, 4.57, respectively. The cost of the butterfly spread is therefore $0.70 + 4.57 - 2 \times 2.14 = 0.99$. Allowing for rounding errors, this is the same as in (c). The profits are the same as in (c).

(e) A call option with a strike price of 30 costs 4.18. A put option with a strike price of 30 costs 1.44. The cost of the straddle is therefore $4.18 + 1.44 = 5.62$. The profits ignoring the impact of discounting are

Stock Price Range	*Profit*
$S_T \le 30$	$24.38 - S_T$
$S_T > 30$	$S_T - 35.62$

(f) A six-month call option with a strike price of 35 costs 1.85. A six-month put option with a strike price of 25 costs 0.28. The cost of the strangle is therefore $1.85 + 0.28 = 2.13$. The profits ignoring the impact of discounting are

Stock Price Range	*Profit*
$S_T \le 25$	$22.87 - S_T$
$25 < S_T < 35$	-2.13
$S_T \ge 35$	$S_T - 37.13$

Problem 11.24
What trading position is created from a long strangle and a short straddle when both have the same time to maturity? Assume that the strike price in the straddle is halfway between the two strike prices of the strangle.

A butterfly spread (together with a cash position) is created.

Problem 11.25 (Excel file)
Describe the trading position created in which a call option is bought with strike price K_1 and a put option is sold with strike price K_2 when both have the same time to maturity and $K_2 > K_1$. What does the position become when $K_1 = K_2$?

The position is as shown in the diagram below (for $K_1 = 25$ and $K_2 = 35$). It is known as a range forward and is discussed further in Chapter 16. When $K_1 = K_2$, the position becomes a regular long forward.

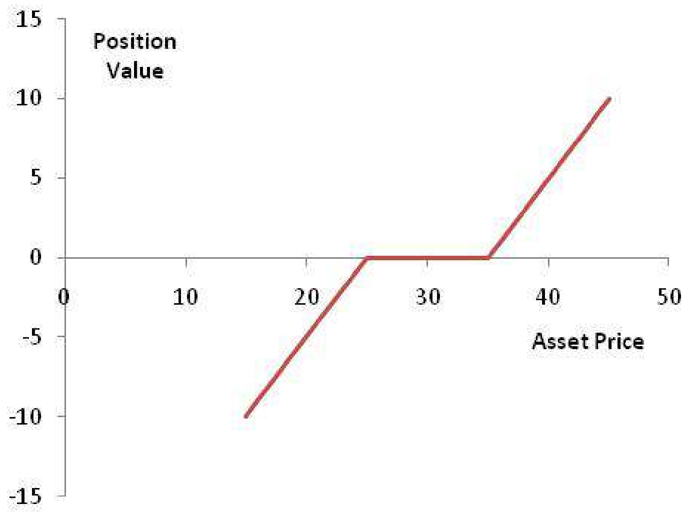

Figure S11.6 Trading position in Problem 11.25

Problem 11.26
A bank decides to create a five-year principal-protected note on a non-dividend-paying stock by offering investors a zero-coupon bond plus a bull spread created from calls. The risk-free rate is 4% and the stock price volatility is 25%. The low strike price option in the bull spread is at the money. What is the maximum ratio of the higher strike price to the lower strike price in the bull spread? Use DerivaGem.

Assume that the amount invested is 100. (This is a scaling factor.) The amount available to create the option is $100\text{-}100e^{-0.04\times5}$=18.127. The cost of the at-the money option can be calculated from DerivaGem by setting the stock price equal to 100, the volatility equal to 25%, the risk-free interest rate equal to 4%, the time to exercise equal to 5 and the exercise price equal to 100. It is 30.313. We therefore require the option given up by the investor to be worth at least 30.313−18.127 = 12.186. Results obtained are as follows:

Strike	*Option Value*

125	21.12
150	14.71
175	10.29
165	11.86

Continuing in this way we find that the strike must be set below 163.1. The ratio of the high strike to the low strike must therefore be less than 1.631 for the bank to make a profit. (Excel's Solver can be used in conjunction with the DerivaGem functions to facilitate calculations.)

CHAPTER 14
The Black-Scholes-Merton Model

Practice Questions

Problem 14.1.
What does the Black–Scholes–Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the continuously compounded rate of return on the stock during the year?

The Black–Scholes–Merton option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed.

Problem 14.2.
The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\Delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

Problem 14.3.
Explain the principle of risk-neutral valuation.

The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

Problem 14.4.
Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$50N(-0.0917)e^{-0.1\times 0.25} - 50N(-0.2417)$$

$$= 50 \times 0.4634e^{-0.1\times 0.25} - 50 \times 0.4045 = 2.37$$

or \$2.37.

Problem 14.5.
What difference does it make to your calculations in Problem 14.4 if a dividend of \$1.50 is expected in two months?

In this case we must subtract the present value of the dividend from the stock price before using Black–Scholes-Merton. Hence the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667\times 0.1} = 48.52$$

As before $K = 50$, $r = 0.1$, $\sigma = 0.3$, and $T = 0.25$. In this case

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1\times 0.25} - 48.52N(-0.0414)$$

$$= 50 \times 0.5432e^{-0.1\times 0.25} - 48.52 \times 0.4835 = 3.03$$

or \$3.03.

Problem 14.6.
What is implied volatility? How can it be calculated?

The implied volatility is the volatility that makes the Black–Scholes-Merton price of an option equal to its market price. It is calculated using an iterative procedure.

Problem 14.7.
A stock price is currently \$40. Assume that the expected return from the stock is 15% and its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a two-year period?

In this case $\mu = 0.15$ and $\sigma = 0.25$. From equation (14.7) the probability distribution for the rate of return over a two-year period with continuous compounding is:

$$\varphi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25^2}{2}\right)$$

i.e.,

$$\varphi(0.11875, 0.03125)$$

The expected value of the return is 11.875% per annum and the standard deviation is 17.7% per annum.

Problem 14.8.
A stock price has an expected return of 16% and a volatility of 35%. The current price is \$38.

a) *What is the probability that a European call option on the stock with an exercise price* *of \$40 and a maturity date in six months will be exercised?*

b) *What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?*

a) The required probability is the probability of the stock price being above \$40 in six months time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \varphi(\ln 38 + (0.16 - \frac{0.35^2}{2})0.5, 0.35^2 \times 0.5)$$

i.e.,

$$\ln S_T \sim \varphi(3.687, 0.247^2)$$

Since $\ln 40 = 3.689$, the required probability is

$$1 - N\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - N(0.008)$$

From normal distribution tables N(0.008) = 0.5032 so that the required probability is 0.4968.

b) In this case the required probability is the probability of the stock price being less than \$40 in six months time. It is

$$1 - 0.4968 = 0.5032$$

Problem 14.9.

Using the notation in the chapter, prove that a 95% confidence interval for S_T is between

$S_0 e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}}$ and $S_0 e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$

From equation (14.3):

$$\ln S_T \sim \varphi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T]$$

95% confidence intervals for $\ln S_T$ are therefore

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T are therefore

$$e^{\ln S_0 + (\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \quad \text{and} \quad e^{\ln S_0 + (\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

i.e.

$$S_0 e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \quad \text{and} \quad S_0 e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

Problem 14.10.

A portfolio manager announces that the average of the returns realized in each of the last 10 years is 20% per annum. In what respect is this statement misleading?

This problem relates to the material in Section 14.3. The statement is misleading in that a certain sum of money, say \$1000, when invested for 10 years in the fund would have realized

a return (with annual compounding) of less than 20% per annum.
The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

Problem 14.11.
Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a derivative that pays off a dollar amount equal to ln S_T at time T where S_T denotes the values of the stock price at time T.

a) *Use risk-neutral valuation to calculate the price of the derivative at time t in term of the stock price, S, at time t*
b) *Confirm that your price satisfies the differential equation (14.16)*

a) At time *t*, the expected value of ln S_T is from equation (14.3)

$$\ln S + (\mu - \sigma^2 / 2)(T - t)$$

In a risk-neutral world the expected value of ln S_T is therefore

$$\ln S + (r - \sigma^2 / 2)(T - t)$$

Using risk-neutral valuation the value of the derivative at time *t* is

$$e^{-r(T-t)}[\ln S + (r - \sigma^2 / 2)(T - t)]$$

b) If

$$f = e^{-r(T-t)}[\ln S + (r - \sigma^2 / 2)(T - t)]$$

then

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}[\ln S + (r - \sigma^2 / 2)(T - t)] - e^{-r(T-t)}\left(r - \sigma^2 / 2\right)$$

$$\frac{\partial f}{\partial S} = \frac{e^{-r(T-t)}}{S}$$

$$\frac{\partial^2 f}{\partial S^2} = -\frac{e^{-r(T-t)}}{S^2}$$

The left-hand side of the Black-Scholes-Merton differential equation is

$$e^{-r(T-t)}\left[r\ln S + r(r - \sigma^2 / 2)(T - t) - (r - \sigma^2 / 2) + r - \sigma^2 / 2\right]$$
$$= e^{-r(T-t)}\left[r\ln S + r(r - \sigma^2 / 2)(T - t)\right]$$
$$= rf$$

Hence the differential equation is satisfied.

Problem 14.12.
Consider a derivative that pays off S_T^n at time T where S_T is the stock price at that time. When the stock price follows geometric Brownian motion, it can be shown that its price at time t $(t \leq T)$ has the form

$$h(t,T)S^n$$

where S is the stock price at time t and h is a function only of t and T.
(a) By substituting into the Black–Scholes–Merton partial differential equation derive an ordinary differential equation satisfied by $h(t,T)$.
(b) What is the boundary condition for the differential equation for $h(t,T)$?

(c) Show that

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

where r is the risk-free interest rate and σ is the stock price volatility.

If $G(S,t) = h(t,T)S^n$ then $\partial G/\partial t = h_t S^n$, $\partial G/\partial S = hnS^{n-1}$, and $\partial^2 G/\partial S^2 = hn(n-1)S^{n-2}$ where $h_t = \partial h/\partial t$. Substituting into the Black–Scholes–Merton differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$

The equation

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively we can solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The solution to this is

$$\ln h = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)](T-t)$$

or

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

Problem 14.13.

What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$

$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$\begin{aligned} &52N(0.5365) - 50e^{-0.12\times0.25}N(0.3865) \\ &= 52\times0.7042 - 50e^{-0.03}\times0.6504 \\ &= 5.06 \end{aligned}$$

or \$5.06.

Problem 14.14.

What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility *is 35% per annum, and the time to maturity is six months?*

In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$\begin{aligned} &70e^{-0.05\times0.5}N(0.0809) - 69N(-0.1666) \\ &= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338 \\ &= 6.40 \end{aligned}$$

or \$6.40.

Problem 14.15.

Consider an American call option on a stock. The stock price is \$70, the time to maturity is eight months, the risk-free rate of interest is 10% per annum, the exercise price is \$65, and the volatility is 32%. A dividend of \$1 is expected after three months and again after six months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

Using the notation of Section 14.12, $D_1 = D_2 = 1$, $K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1\times0.1667}) = 1.07$, and $K(1 - e^{-r(t_2-t_1)}) = 65(1 - e^{-0.1\times0.25}) = 1.60$. Since

$$D_1 < K(1 - e^{-r(T-t_2)})$$

and

$$D_2 < K(1 - e^{-r(t_2-t_1)})$$

It is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

Problem 14.16.

A call option on a non-dividend-paying stock has a market price of \$2.50. The stock price is \$15, the exercise price is \$13, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be calculated using an iterative procedure.

A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.396 or 39.6% per annum.

The implied volatility can also be calculated using DerivaGem. Select equity as the Underlying Type in the first worksheet. Select Black-Scholes European as the Option Type. Input stock price as 15, the risk-free rate as 5%, time to exercise as 0.25, and exercise price as 13. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Select the implied volatility button. Input the Price as 2.5 in the second half of the option data table. Hit the *Enter* key and click on calculate. DerivaGem will show the volatility of the option as 39.64%.

Problem 14.17.

With the notation used in this chapter

(a) *What is* $N'(x)$ *?*

(b) *Show that* $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, *where* S *is the stock price at time* t

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

(c) *Calculate* $\partial d_1 / \partial S$ *and* $\partial d_2 / \partial S$.

(d) *Show that when*

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

where c *is the price of a call option on a non-dividend-paying stock.*

(e) *Show that* $\partial c / \partial S = N(d_1)$.

(f) *Show that the* c *satisfies the Black–Scholes–Merton differential equation.*

(g) *Show that* c *satisfies the boundary condition for a European call option, i.e., that* $c = \max(S-K, 0)$ *as* $t \longrightarrow T$

(a) Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

(b)

$$N'(d_1) = N'(d_2 + \sigma\sqrt{T-t})$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{d_2^2}{2} - \sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

$$= N'(d_2)\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

Because

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

it follows that

$$\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

which is the required result.

(c)

$$d_1 = \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln S - \ln K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Hence

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Similarly

$$d_2 = \frac{\ln S - \ln K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Therefore:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

(d)

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

From (b):

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

Since

$$d_1 - d_2 = \sigma\sqrt{T-t}$$

$$\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = \frac{\partial}{\partial t}(\sigma\sqrt{T-t})$$

$$= -\frac{\sigma}{2\sqrt{T-t}}$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

(e) From differentiating the Black–Scholes–Merton formula for a call price we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{dS}$$

From the results in (b) and (c) it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

(f) Differentiating the result in (e) and using the result in (c), we obtain

$$\frac{\partial^2 c}{\partial S^2} = N'(d_1)\frac{\partial d_1}{\partial S}$$

$$= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$$

From the results in d) and e)

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

$$+rSN(d_1) + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$$

$$= r[SN(d_1) - Ke^{-r(T-t)}N(d_2)]$$

$$= rc$$

This shows that the Black–Scholes-Merton formula for a call option does indeed satisfy the Black–Scholes–Merton differential equation

(g) Consider what happens in the formula for c in part (d) as t approaches T. If $S > K$, d_1 and d_2 tend to infinity and $N(d_1)$ and $N(d_2)$ tend to 1. If $S < K$, d_1 and d_2 tend to zero. It follows that the formula for c tends to $\max(S-K, 0)$.

Problem 14.18..

Show that the Black–Scholes–Merton formulas for call and put options satisfy put–call parity.

The Black–Scholes–Merton formula for a European call option is

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

so that

$$c + Ke^{-rT} = S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}$$

or

$$c + Ke^{-rT} = S_0N(d_1) + Ke^{-rT}[1 - N(d_2)]$$

or

$$c + Ke^{-rT} = S_0N(d_1) + Ke^{-rT}N(-d_2)$$

The Black–Scholes–Merton formula for a European put option is

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

so that

$$p + S_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0$$

or

$$p + S_0 = Ke^{-rT}N(-d_2) + S_0[1 - N(-d_1)]$$

or

$$p + S_0 = Ke^{-rT}N(-d_2) + S_0N(d_1)$$

This shows that the put–call parity result

$$c + Ke^{-rT} = p + S_0$$

holds.

Problem 14.19.
A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black–Scholes–Merton?

Stock Price	*Maturity = 3 months*	*Maturity = 6 months*	*Maturity = 12 months*
45	7.00	8.30	10.50
50	3.50	5.20	7.50
55	1.60	2.90	5.10

Using DerivaGem we obtain the following table of implied volatilities

Stock Price	*Maturity = 3 months*	*Maturity = 6 months*	*Maturity = 12 months*
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

To calculate first number, select equity as the Underlying Type in the first worksheet. Select Black-Scholes European as the Option Type. Input stock price as 50, the risk-free rate as 5%, time to exercise as 0.25, and exercise price as 45. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Select the implied volatility button. Input the Price as 7.0 in the second half of the option data table. Hit the *Enter* key and click on calculate. DerivaGem will show the volatility of the option as 37.78%. Change the strike price and time to exercise and recompute to calculate the rest of the numbers in the table.
The option prices are not exactly consistent with Black–Scholes–Merton. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock. This phenomenon is discussed in Chapter 19.

Problem 14.20.
Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

Black's approach in effect assumes that the holder of option must decide at time zero whether it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T. In fact the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n, it can still be exercised at time T.
It appears that Black's approach should understate the true option value. This is because the holder of the option has more alternative strategies for deciding when to exercise the option than the two strategies implicitly assumed by the approach. These alternative strategies add value to the option.
However, this is not the whole story! The standard approach to valuing either an American or a European option on a stock paying a single dividend applies the volatility to the stock price

less the present value of the dividend. (The procedure for valuing an American option is explained in Chapter 20.) Black's approach when considering exercise just prior to the dividend date applies the volatility to the stock price itself. Black's approach therefore assumes more stock price variability than the standard approach in some of its calculations. In some circumstances it can give a higher price than the standard approach.

Problem 14.21.

Consider an American call option on a stock. The stock price is $50, the time to maturity is 15 months, the risk-free rate of interest is 8% per annum, the exercise price is $55, and the volatility is 25%. Dividends of $1.50 are expected in 4 months and 10 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Calculate the price of the option.

With the notation in the text

$$D_1 = D_2 = 1.50, \quad t_1 = 0.3333, \quad t_2 = 0.8333, \quad T = 1.25, \quad r = 0.08 \quad \text{and} \quad K = 55$$

$$K\left[1 - e^{-r(T-t_2)}\right] = 55(1 - e^{-0.08\times 0.4167}) = 1.80$$

Hence

$$D_2 < K\left[1 - e^{-r(T-t_2)}\right]$$

Also:

$$K\left[1 - e^{-r(t_2-t_1)}\right] = 55(1 - e^{-0.08\times 0.5}) = 2.16$$

Hence:

$$D_1 < K\left[1 - e^{-r(t_2-t_1)}\right]$$

It follows from the conditions established in Section 14.12 that the option should never be exercised early.

The present value of the dividends is

$$1.5e^{-0.3333\times 0.08} + 1.5e^{-0.8333\times 0.08} = 2.864$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.864 = 47.136, \quad K = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$$

$$d_1 = \frac{\ln(47.136/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, \quad N(d_2) = 0.3692$$

and the call price is

$$47.136 \times 0.4783 - 55e^{-0.08\times 1.25} \times 0.3692 = 4.17$$

or $4.17.

Problem 14.22.

Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$. *What is an expression for the* *value of a derivative that pays off $100 if the price of a stock at time T is greater than K ?*

The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T. In a risk neutral world

$$\ln S_T \sim \varphi[\ln S_0 + (r - \sigma^2/2)T, \sigma^2 T]$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$1 - N\left[\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N\left[\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N(d_2)$$

The expected value at time T in a risk neutral world of a derivative security which pays off $100 when $S_T > K$ is therefore

$$100N(d_2)$$

From risk neutral valuation the value of the security at time t is

$$100e^{-rT}N(d_2)$$

Problem 14.23.

Show that S^{-2r/σ^2} could be the price of a traded security.

If $f = S^{-2r/\sigma^2}$ then

$$\frac{\partial f}{\partial S} = -\frac{2r}{\sigma^2} S^{-2r/\sigma^2 - 1}$$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2} + 1\right) S^{-2r/\sigma^2 - 2}$$

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rS^{-2r/\sigma^2} = rf$$

This shows that the Black–Scholes equation is satisfied. S^{-2r/σ^2} could therefore be the price of a traded security.

Problem 14.24.

A company has an issue of executive stock options outstanding. Should dilution be taken into account when the options are valued? Explain you answer.

The answer is no. If markets are efficient they have already taken potential dilution into account in determining the stock price. This argument is explained in Business Snapshot

14.3.

Problem 14.25.
A company's stock price is $50 and 10 million shares are outstanding. The company is considering giving its employees three million at-the-money five-year call options. Option exercises will be handled by issuing more shares. The stock price volatility is 25%, the five-year risk-free rate is 5% and the company does not pay dividends. Estimate the cost to the company of the employee stock option issue.

The Black-Scholes-Merton price of the option is given by setting $S_0 = 50$, $K = 50$, $r = 0.05$, $\sigma = 0.25$, and $T = 5$. It is 16.252. From an analysis similar to that in Section 14.10 the cost to the company of the options is

$$\frac{10}{10+3} \times 16.252 = 12.5$$

or about $12.5 per option. The total cost is therefore 3 million times this or $37.5 million. If the market perceives no benefits from the options the stock price will fall by $3.75.

Further Questions

Problem 14.26
A stock price is currently $50. Assume that the expected return from the stock is 18% per annum and its volatility is 30% per annum. What is the probability distribution for the stock price in two years? Calculate the mean and standard deviation of the distribution. Determine the 95% confidence interval.

In this case $S_0 = 50$, $\mu = 0.18$ and $\sigma = 0.30$. The probability distribution of the stock price in two years, S_T, is lognormal and is, from equation (14.3), given by:

$$\ln S_T \sim \varphi\left[\ln 50 + \left(0.18 - \frac{0.09}{2}\right)2,\ 0.3^2 \times 2\right]$$

i.e.,

$$\ln S_T \sim \varphi(4.18,\ 0.42^2)$$

The mean stock price is from equation (14.3)

$$50e^{0.18\times 2} = 50e^{0.36} = 71.67$$

and the standard deviation is

$$50e^{0.18\times 2}\sqrt{e^{0.09\times 2} - 1} = 31.83$$

95% confidence intervals for $\ln S_T$ are

$$4.18 - 1.96 \times 0.42 \quad \text{and} \quad 4.18 + 1.96 \times 0.42$$

i.e.,

$$3.35 \quad \text{and} \quad 5.01$$

These correspond to 95% confidence limits for S_T of

$$e^{3.35} \quad \text{and} \quad e^{5.01}$$

i.e.,

$$28.52 \quad \text{and} \quad 150.44$$

Problem 14.27. (Excel file)

Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:
30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0,
32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2
Estimate the stock price volatility. What is the standard error of your estimate?

The calculations are shown in the table below

$$\sum u_i = 0.09471 \quad \sum u_i^2 = 0.01145$$

and an estimate of standard deviation of weekly returns is:

$$\sqrt{\frac{0.01145}{13} - \frac{0.09471^2}{14 \times 13}} = 0.02884$$

The volatility per annum is therefore $0.02884\sqrt{52} = 0.2079$ or 20.79%. The standard error of this estimate is

$$\frac{0.2079}{\sqrt{2 \times 14}} = 0.0393$$

or 3.9% per annum.

Week	*Closing Stock Price ($)*	*Price Relative* $= S_i / S_{i-1}$	*Weekly Return* $u_i = \ln(S_i / S_{i-1})$
1	30.2		
2	32.0	1.05960	0.05789
3	31.1	0.97188	–0.02853
4	30.1	0.96785	–0.03268
5	30.2	1.00332	0.00332
6	30.3	1.00331	0.00331
7	30.6	1.00990	0.00985
8	33.0	1.07843	0.07551
9	32.9	0.99697	–0.00303
10	33.0	1.00304	0.00303
11	33.5	1.01515	0.01504
12	33.5	1.00000	0.00000
13	33.7	1.00597	0.00595
14	33.5	0.99407	–0.00595
15	33.2	0.99104	–0.00900

Problem 14.28.

A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T.

(a) *Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t. (Hint: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in section 14.1.)*

(b) *Confirm that your price satisfies the differential equation (14.16).*

(a) The expected value of the security is $E[(S_T)^2]$ From equations (14.4) and (14.5), at time t :

$$E(S_T) = Se^{\mu(T-t)}$$

$$\text{var}(S_T) = S^2 e^{2\mu(T-t)}[e^{\sigma^2(T-t)} - 1]$$

Since $\text{var}(S_T) = E[(S_T)^2] - [E(S_T)]^2$, it follows that $E[(S_T)^2] = \text{var}(S_T) + [E(S_T)]^2$ so that

$$E[(S_T)^2] = S^2 e^{2\mu(T-t)}[e^{\sigma^2(T-t)} - 1] + S^2 e^{2\mu(T-t)}$$

$$= S^2 e^{(2\mu+\sigma^2)(T-t)}$$

In a risk-neutral world $\mu = r$ so that

$$\hat{E}[(S_T)^2] = S^2 e^{(2r+\sigma^2)(T-t)}$$

Using risk-neutral valuation, the value of the derivative security at time t is

$$e^{-r(T-t)}\hat{E}[(S_T)^2]$$

$$= S^2 e^{(2r+\sigma^2)(T-t)} e^{-r(T-t)}$$

$$= S^2 e^{(r+\sigma^2)(T-t)}$$

(b) If:

$$f = S^2 e^{(r+\sigma^2)(T-t)}$$

$$\frac{\partial f}{\partial t} = -S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)}$$

$$\frac{\partial f}{\partial S} = 2Se^{(r+\sigma^2)(T-t)}$$

$$\frac{\partial^2 f}{\partial S^2} = 2e^{(r+\sigma^2)(T-t)}$$

The left-hand side of the Black-Scholes–Merton differential equation is:

$$-S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)} + 2rS^2 e^{(r+\sigma^2)(T-t)} + \sigma^2 S^2 e^{(r+\sigma^2)(T-t)}$$

$$= rS^2 e^{(r+\sigma^2)(T-t)}$$

$$= rf$$

Hence the Black-Scholes equation is satisfied.

Problem 14.29.

Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5% per annum, the volatility is 25% per annum, and the time to maturity is four months.

a. *What is the price of the option if it is a European call?*
b. *What is the price of the option if it is an American call?*
c. *What is the price of the option if it is a European put?*
d. *Verify that put–call parity holds.*

In this case $S_0 = 30$, $K = 29$, $r = 0.05$, $\sigma = 0.25$ and $T = 4/12$

$$d_1 = \frac{\ln(30/29) + (0.05 + 0.25^2/2) \times 4/12}{0.25\sqrt{0.3333}} = 0.4225$$

$$d_2 = \frac{\ln(30/29) + (0.05 - 0.25^2/2) \times 4/12}{0.25\sqrt{0.3333}} = 0.2782$$

$$N(0.4225) = 0.6637, \quad N(0.2782) = 0.6096$$

$$N(-0.4225) = 0.3363, \quad N(-0.2782) = 0.3904$$

a. The European call price is

$$30 \times 0.6637 - 29e^{-0.05\times4/12} \times 0.6096 = 2.52$$

or \$2.52.

b. The American call price is the same as the European call price. It is \$2.52.

c. The European put price is

$$29e^{-0.05\times4/12} \times 0.3904 - 30 \times 0.3363 = 1.05$$

or \$1.05.

d. Put-call parity states that:

$$p + S = c + Ke^{-rT}$$

In this case $c = 2.52$, $S_0 = 30$, $K = 29$, $p = 1.05$ and $e^{-rT} = 0.9835$ and it is easy to verify that the relationship is satisfied,

Problem 14.30.

Assume that the stock in Problem 14.29 is due to go ex-dividend in 1.5 months. The expected dividend is 50 cents.

a. *What is the price of the option if it is a European call?*
b. *What is the price of the option if it is a European put?*
c. *If the option is an American call, are there any circumstances when it will be exercised early?*

a. The present value of the dividend must be subtracted from the stock price. This gives a new stock price of:

$$30 - 0.5e^{-0.125\times0.05} = 29.5031$$

and

$$d_1 = \frac{\ln(29.5031/29) + (0.05 + 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.3068$$

$$d_2 = \frac{\ln(29.5031/29) + (0.05 - 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.1625$$

$$N(d_1) = 0.6205; \quad N(d_2) = 0.5645$$

The price of the option is therefore

$$29.5031 \times 0.6205 - 29e^{-0.05\times4/12} \times 0.5645 = 2.21$$

or \$2.21.

b. Because

$$N(-d_1) = 0.3795, \quad N(-d_2) = 0.4355$$

the value of the option when it is a European put is

$$29e^{-0.05\times 4/12} \times 0.4355 - 29.5031 \times 0.3795 = 1.22$$

or $1.22.

c. If t_1 denotes the time when the dividend is paid:

$$K(1 - e^{-r(T-t_1)}) = 29(1 - e^{-0.05\times 0.2083}) = 0.3005$$

This is less than the dividend. Hence the option should be exercised immediately before the ex-dividend date for a sufficiently high value of the stock price.

Problem 14.31.

Consider an American call option when the stock price is \$18, the exercise price is \$20, the time to maturity is six months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends are expected during the life of the option, with ex-dividend dates at the end of two months and five months. Assume the dividends are 40 cents. Use Black's approximation and the DerivaGem software to value the option. Suppose now that the dividend is D on each ex-dividend date. Use the results in the Appendix to determine how high D can be without the American option being exercised early.

We first value the option assuming that it is not exercised early, we set the time to maturity equal to 0.5. There is a dividend of 0.4 in 2 months and 5 months. Other parameters are $S_0 = 18$, $K = 20$, $r = 10\%$, $\sigma = 30\%$. DerivaGem gives the price as 0.7947. We next value the option assuming that it is exercised at the five-month point just before the final dividend. DerivaGem gives the price as 0.7668. The price given by Black's approximation is therefore 0.7947. (DerivaGem also shows that the American option price calculated using the binomial model with 100 time steps is 0.8243.)
It is never optimal to exercise the option immediately before the first ex-dividend date when

$$D_1 \le K[1 - e^{-r(t_2-t_1)}]$$

where D_1 is the size of the first dividend, and t_1 and t_2 are the times of the first and second dividend respectively. Hence we must have:

$$D_1 \le 20[1 - e^{-(0.1\times 0.25)}]$$

that is,

$$D_1 \le 0.494$$

It is never optimal to exercise the option immediately before the second ex-dividend date when:

$$D_2 \le K(1 - e^{-r(T-t_2)})$$

where D_2 is the size of the second dividend. Hence we must have:

$$D_2 \le 20(1 - e^{-0.1\times 0.0833})$$

that is,

$$D_2 \le 0.166$$

It follows that the dividend can be as high as 16.6 cents per share without the American option being worth more than the corresponding European option.

CHAPTER 25
Exotic Options

Practice Questions

Problem 25.1.
Explain the difference between a forward start option and a chooser option.

A forward start option is an option that is paid for now but will start at some time in the future. The strike price is usually equal to the price of the asset at the time the option starts. A chooser option is an option where, at some time in the future, the holder chooses whether the option is a call or a put.

Problem 25.2.
Describe the payoff from a portfolio consisting of a floating lookback call and a floating lookback put with the same maturity.

A floating lookback call provides a payoff of $S_T - S_{\min}$. A floating lookback put provides a payoff of $S_{\max} - S_T$. A combination of a floating lookback call and a floating lookback put therefore provides a payoff of $S_{\max} - S_{\min}$.

Problem 25.3.
Consider a chooser option where the holder has the right to choose between a European call and a European put at any time during a two-year period. The maturity dates and strike prices for the calls and puts are the same regardless of when the choice is made. Is it ever optimal to make the choice before the end of the two-year period? Explain your answer.

No, it is never optimal to choose early. The resulting cash flows are the same regardless of when the choice is made. There is no point in the holder making a commitment earlier than necessary. This argument applies when the holder chooses between two American options providing the options cannot be exercised before the 2-year point. If the early exercise period starts as soon as the choice is made, the argument does not hold. For example, if the stock price fell to almost nothing in the first six months, the holder would choose a put option at this time and exercise it immediately.

Problem 25.4.
Suppose that c_1 and p_1 are the prices of a European average price call and a European average price put with strike price K and maturity T, c_2 and p_2 are the prices of a European average strike call and European average strike put with maturity T, and c_3 and p_3 are the prices of a regular European call and a regular European put with strike price K and maturity T. Show that

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

The payoffs from c_1, c_2, c_3, p_1, p_2, p_3 are, respectively, as follows:

$$\max(\bar{S} - K, 0)$$

$$\max(S_T - \bar{S}, 0)$$
$$\max(S_T - K, 0)$$
$$\max(K - \bar{S}, 0)$$
$$\max(\bar{S} - S_T, 0)$$
$$\max(K - S_T, 0)$$

The payoff from $c_1 - p_1$ is always $\bar{S} - K$; The payoff from $c_2 - p_2$ is always $S_T - \bar{S}$; The payoff from $c_3 - p_3$ is always $S_T - K$; It follows that

$$c_1 - p_1 + c_2 - p_2 = c_3 - p_3$$

or

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

Problem 25.5.

The text derives a decomposition of a particular type of chooser option into a call maturing at time T_2 and a put maturing at time T_1. Derive an alternative decomposition into a call maturing at time T_1 and a put maturing at time T_2.

Substituting for c, put-call parity gives

$$\max(c, p) = \max\left[p, p + S_1 e^{-q(T_2 - T_1)} - K e^{-r(T_2 - T_1)}\right]$$

$$= p + \max\left[0, S_1 e^{-q(T_2 - T_1)} - K e^{-r(T_2 - T_1)}\right]$$

This shows that the chooser option can be decomposed into

1. A put option with strike price K and maturity T_2; and
2. $e^{-q(T_2 - T_1)}$ call options with strike price $K e^{-(r-q)(T_2 - T_1)}$ and maturity T_1.

Problem 25.6.

Section 25.8 gives two formulas for a down-and-out call. The first applies to the situation where the barrier, H, is less than or equal to the strike price, K. The second applies to the situation where $H \geq K$. Show that the two formulas are the same when $H = K$.

Consider the formula for c_{do} when $H \geq K$

$$c_{\text{do}} = S_0 N(x_1) e^{-qT} - K e^{-rT} N(x_1 - \sigma\sqrt{T}) - S_0 e^{-qT} (H / S_0)^{2\lambda} N(y_1)$$

$$+ K e^{-rT} (H / S_0)^{2\lambda - 2} N(y_1 - \sigma\sqrt{T})$$

Substituting $H = K$ and noting that

$$\lambda = \frac{r - q + \sigma^2 / 2}{\sigma^2}$$

we obtain $x_1 = d_1$ so that

$$c_{\text{do}} = c - S_0 e^{-qT} (H / S_0)^{2\lambda} N(y_1) + K e^{-rT} (H / S_0)^{2\lambda - 2} N(y_1 - \sigma\sqrt{T})$$

The formula for c_{di} when $H \leq K$ is

$$c_{\text{di}} = S_0 e^{-qT} (H / S_0)^{2\lambda} N(y) - K e^{-rT} (H / S_0)^{2\lambda - 2} N(y - \sigma\sqrt{T})$$

Since $c_{\text{do}} = c - c_{\text{di}}$

$$c_{\mathrm{do}} = c - S_0 e^{-qT} (H / S_0)^{2\lambda} N(y) + Ke^{-rT} (H / S_0)^{2\lambda-2} N(y - \sigma\sqrt{T})$$

From the formulas in the text $y_1 = y$ when $H = K$. The two expression for c_{do} are therefore equivalent when $H = K$

Problem 25.7.
Explain why a down-and-out put is worth zero when the barrier is greater than the strike price.

The option is in the money only when the asset price is less than the strike price. However, in these circumstances the barrier has been hit and the option has ceased to exist.

Problem 25.8.
Suppose that the strike price of an American call option on a non-dividend-paying stock grows at rate g. Show that if g is less than the risk-free rate, r, it is never optimal to exercise the call early.

The argument is similar to that given in Chapter 10 for a regular option on a non-dividend-paying stock. Consider a portfolio consisting of the option and cash equal to the present value of the terminal strike price. The initial cash position is

$$Ke^{gT-rT}$$

By time τ ($0 \le \tau \le T$), the cash grows to

$$Ke^{-r(T-\tau)+gT} = Ke^{g\tau} e^{-(r-g)(T-\tau)}$$

Since $r > g$, this is less than $Ke^{g\tau}$ and therefore is less than the amount required to exercise the option. It follows that, if the option is exercised early, the terminal value of the portfolio is less than S_T. At time T the cash balance is Ke^{gT}. This is exactly what is required to exercise the option. If the early exercise decision is delayed until time T, the terminal value of the portfolio is therefore

$$\max[S_T, Ke^{gT}]$$

This is at least as great as S_T. It follows that early exercise cannot be optimal.

Problem 25.9.
How can the value of a forward start put option on a non-dividend-paying stock be calculated if it is agreed that the strike price will be 10% greater than the stock price at the time the option starts?

When the strike price of an option on a non-dividend-paying stock is defined as 10% greater that the stock price, the value of the option is proportional to the stock price. The same argument as that given in the text for forward start options shows that if t_1 is the time when the option starts and t_2 is the time when it finishes, the option has the same value as an option starting today with a life of $t_2 - t_1$ and a strike price of 1.1 times the current stock price.

Problem 25.10.
If a stock price follows geometric Brownian motion, what process does $A(t)$ follow where $A(t)$ is the arithmetic average stock price between time zero and time t?

Assume that we start calculating averages from time zero. The relationship between $A(t+\Delta t)$ and $A(t)$ is

$$A(t+\Delta t)\times(t+\Delta t)=A(t)\times t+S(t)\times\Delta t$$

where $S(t)$ is the stock price at time t and terms of higher order than Δt are ignored. If we continue to ignore terms of higher order than Δt, it follows that

$$A(t+\Delta t)=A(t)\left[1-\frac{\Delta t}{t}\right]+S(t)\frac{\Delta t}{t}$$

Taking limits as Δt tends to zero

$$dA(t)=\frac{S(t)-A(t)}{t}dt$$

The process for $A(t)$ has a stochastic drift and no dz term. The process makes sense intuitively. Once some time has passed, the change in S in the next small portion of time has only a second order effect on the average. If S equals A the average has no drift; if $S>A$ the average is drifting up; if $S<A$ the average is drifting down.

Problem 25.11.

Explain why delta hedging is easier for Asian options than for regular options.

In an Asian option the payoff becomes more certain as time passes and the delta always approaches zero as the maturity date is approached. This makes delta hedging easy. Barrier options cause problems for delta hedgers when the asset price is close to the barrier because delta is discontinuous.

Problem 25.12.

Calculate the price of a one-year European option to give up 100 ounces of silver in exchange for one ounce of gold. The current prices of gold and silver are \$380 and \$4, respectively; the risk-free interest rate is 10% per annum; the volatility of each commodity price is 20%; and the correlation between the two prices is 0.7. Ignore storage costs.

The value of the option is given by the formula in the text

$$V_0e^{-q_2T}N(d_1)-U_0e^{-q_1T}N(d_2)$$

where

$$d_1=\frac{\ln(V_0/U_0)+(q_1-q_2+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2=d_1-\sigma\sqrt{T}$$

and

$$\sigma=\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$$

In this case, $V_0=380$, $U_0=400$, $q_1=0$, $q_2=0$, $T=1$, and

$$\sigma=\sqrt{0.2^2+0.2^2-2\times0.7\times0.2\times0.2}=0.1549$$

Because $d_1=-0.2537$ and $d_2=-0.4086$, the option price is

$$380N(-0.2537)-400N(-0.4086)=15.38$$

or \$15.38.

Problem 25.13.

Is a European down-and-out option on an asset worth the same as a European down-and-out *option on the asset's futures price for a futures contract maturing at the same time as the option?*

No. If the future's price is above the spot price during the life of the option, it is possible that the spot price will hit the barrier when the futures price does not.

Problem 25.14.

Answer the following questions about compound options

(a) *What put–call parity relationship exists between the price of a European call on a call and a European put on a call? Show that the formulas given in the text satisfy the relationship.*

(b) *What put–call parity relationship exists between the price of a European call on a put and a European put on a put? Show that the formulas given in the text satisfy the relationship.*

(a) The put–call relationship is

$$cc + K_1 e^{-rT_1} = pc + c$$

where cc is the price of the call on the call, pc is the price of the put on the call, c is the price today of the call into which the options can be exercised at time T_1, and K_1 is the exercise price for cc and pc. The proof is similar to that in Chapter 10 for the usual put–call parity relationship. Both sides of the equation represent the values of portfolios that will be worth $\max(c, K_1)$ at time T_1. Because

$$M(a,b;\rho) = N(a) - M(a,-b;-\rho) = N(b) - M(-a,b,;-\rho)$$

and

$$N(x) = 1 - N(-x)$$

we obtain

$$cc - pc = Se^{-qT_2} N(b_1) - K_2 e^{-rT_2} N(b_2) - K_1 e^{-rT_1}$$

Since

$$c = Se^{-qT_2} N(b_1) - K_2 e^{-rT_2} N(b_2)$$

put–call parity is consistent with the formulas

(b) The put–call relationship is

$$cp + K_1 e^{-rT_1} = pp + p$$

where cp is the price of the call on the put, pp is the price of the put on the put, p is the price today of the put into which the options can be exercised at time T_1, and K_1 is the exercise price for cp and pp. The proof is similar to that in Chapter 10 for the usual put–call parity relationship. Both sides of the equation represent the values of portfolios that will be worth $\max(p, K_1)$ at time T_1. Because

$$M(a,b;\rho) = N(a) - M(a,-b;-\rho) = N(b) - M(-a,b,;-\rho)$$

and

$$N(x) = 1 - N(-x)$$

it follows that

$$cp - pp = -Se^{-qT_2} N(-b_1) + K_2 e^{-rT_2} N(-b_2) - K_1 e^{-rT_1}$$

Because

$$p = -Se^{-qT_2} N(-b_1) + K_2 e^{-rT_2} N(-b_2)$$

put–call parity is consistent with the formulas.

Problem 25.15.
Does a floating lookback call become more valuable or less valuable as we increase the frequency with which we observe the asset price in calculating the minimum?

As we increase the frequency we observe a more extreme minimum which increases the value of a floating lookback call.

Problem 25.16.
Does a down-and-out call become more valuable or less valuable as we increase the frequency with which we observe the asset price in determining whether the barrier has been crossed? What is the answer to the same question for a down-and-in call?

As we increase the frequency with which the asset price is observed, the asset price becomes more likely to hit the barrier and the value of a down-and-out call goes down. For a similar reason the value of a down-and-in call goes up. The adjustment mentioned in the text, suggested by Broadie, Glasserman, and Kou, moves the barrier further out as the asset price is observed less frequently. This increases the price of a down-and-out option and reduces the price of a down-and-in option.

Problem 25.17.
Explain why a regular European call option is the sum of a down-and-out European call and a down-and-in European call. Is the same true for American call options?

If the barrier is reached the down-and-out option is worth nothing while the down-and-in option has the same value as a regular option. If the barrier is not reached the down-and-in option is worth nothing while the down-and-out option has the same value as a regular option. This is why a down-and-out call option plus a down-and-in call option is worth the same as a regular option. A similar argument cannot be used for American options.

Problem 25.18.
What is the value of a derivative that pays off $100 in six months if the S&P 500 index is greater than 1,000 and zero otherwise? Assume that the current level of the index is 960, the risk-free rate is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 20%.

This is a cash-or-nothing call. The value is $100N(d_2)e^{-0.08\times0.5}$ where

$$d_2 = \frac{\ln(960/1000)+(0.08-0.03-0.2^2/2)\times0.5}{0.2\times\sqrt{0.5}} = -0.1826$$

Since $N(d_2) = 0.4276$ the value of the derivative is $41.08.

Problem 25.19.
In a three-month down-and-out call option on silver futures the strike price is $20 per ounce and the barrier is $18. The current futures price is $19, the risk-free interest rate is 5%, and the volatility of silver futures is 40% per annum. Explain how the option works and calculate its value. What is the value of a regular call option on silver futures with the same terms? What is the value of a down-and-in call option on silver futures with the same terms?

This is a regular call with a strike price of \$20 that ceases to exist if the futures price hits \$18. With the notation in the text $H = 18$, $K = 20$, $S = 19$, $r = 0.05$, $\sigma = 0.4$, $q = 0.05$, $T = 0.25$. From this $\lambda = 0.5$ and

$$y = \frac{\ln[18^2 / (19 \times 20)]}{0.4\sqrt{0.25}} + 0.5 \times 0.4\sqrt{0.25} = -0.69714$$

The value of a down-and-out call plus a down-and-in call equals the value of a regular call. Substituting into the formula given when $H < K$ we get $c_{di} = 0.4638$. The regular Black-Scholes-Merton formula gives $c = 1.0902$. Hence $c_{do} = 0.6264$. (These answers can be checked with DerivaGem.)

Problem 25.20.
A new European-style floating lookback call option on a stock index has a maturity of nine months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use DerivaGem to value the option.

DerivaGem shows that the value is 53.38. Note that the Minimum to date and Maximum to date should be set equal to the current value of the index for a new deal. (See material on DerivaGem at the end of the book.)

Problem 25.21.
Estimate the value of a new six-month European-style average price call option on a non-dividend-paying stock. The initial stock price is \$30, the strike price is \$30, the risk-free interest rate is 5%, and the stock price volatility is 30%.

We can use the analytic approximation given in the text.

$$M_1 = \frac{(e^{0.05 \times 0.5} - 1) \times 30}{0.05 \times 0.5} = 30.378$$

Also $M_2 = 936.9$ so that $\sigma = 17.41\%$. The option can be valued as a futures option with $F_0 = 30.378$, $K = 30$, $r = 5\%$, $\sigma = 17.41\%$, and $t = 0.5$. The price is 1.637.

Problem 25.22.
Use DerivaGem to calculate the value of:
(a) *A regular European call option on a non-dividend-paying stock where the stock price is \$50, the strike price is \$50, the risk-free rate is 5% per annum, the volatility is 30%, and the time to maturity is one year.*
(b) *A down-and-out European call which is as in (a) with the barrier at \$45.*
(c) *A down-and-in European call which is as in (a) with the barrier at \$45.*
Show that the option in (a) is worth the sum of the values of the options in (b) and (c).

The price of a regular European call option is 7.116. The price of the down-and-out call option is 4.696. The price of the down-and-in call option is 2.419.
The price of a regular European call is the sum of the prices of down-and-out and down-and-in options.

Problem 25.23.
Explain adjustments that have to be made when $r = q$ for a) the valuation formulas for

lookback call options in Section 25.10 and b) the formulas for M_1 and M_2 in Section 25.12.

When $r = q$ in the expression for a floating lookback call in Section 25.10 $a_1 = a_3$ and $Y_1 = \ln(S_0 / S_{\min})$ so that the expression for a floating lookback call becomes

$$S_0 e^{-qT} N(a_1) - S_{\min} e^{-rT} N(a_2)$$

As q approaches r in Section 25.12 we get

$$M_1 = S_0$$

$$M_2 = \frac{2e^{\sigma^2 T} S_0^2}{\sigma^4 T^2} - \frac{2S_0^2}{T^2} \frac{1+\sigma^2 T}{\sigma^4}$$

Problem 25.24.
Value the variance swap in Example 25.4 of Section 25.15 assuming that the implied volatilities for options with strike prices 800, 850, 900, 950, 1,000, 1,050, 1,100, 1,150, 1,200 are 20%, 20.5%, 21%, 21.5%, 22%, 22.5%, 23%, 23.5%, 24%, respectively.

In this case, DerivaGem shows that $Q(K_1) = 0.1772$, $Q(K_2) = 1.1857$, $Q(K_3) = 4.9123$, $Q(K_4) = 14.2374$, $Q(K_5) = 45.3738$, $Q(K_6) = 35.9243$, $Q(K_7) = 20.6883$, $Q(K_8) = 11.4135$, $Q(K_9) = 6.1043$. $\hat{E}(\bar{V}) = 0.0502$. The value of the variance swap is \$0.51 million.

Further Questions

Problem 25.25.
What is the value in dollars of a derivative that pays off \$10,000 in one year provided that the dollar–sterling exchange rate is greater than 1.5000 at that time? The current exchange rate is 1.4800. The dollar and sterling interest rates are 4% and 8% per annum respectively. The volatility of the exchange rate is 12% per annum.

It is instructive to consider two different ways of valuing this instrument. From the perspective of a sterling investor it is a cash or nothing put. The variables are $S_0 = 1/1.48 = 0.6757$, $K = 1/1.50 = 0.6667$, $r = 0.08$, $q = 0.04$, $\sigma = 0.12$, and $T = 1$. The derivative pays off if the exchange rate is less than 0.6667. The value of the derivative is $10{,}000 N(-d_2) e^{-0.08\times 1}$ where

$$d_2 = \frac{\ln(0.6757/0.6667) + (0.08 - 0.04 - 0.12^2/2)}{0.12} = 0.3852$$

Since $N(-d_2) = 0.3501$, the value of the derivative is $10{,}000 \times 0.3501 \times e^{-0.08} = 3{,}231$ or 3,231. In dollars this is $3{,}231 \times 1.48 = \$4782$

From the perspective of a dollar investor the derivative is an asset or nothing call. The variables are $S_0 = 1.48$, $K = 1.50$, $r = 0.04$, $q = 0.08$, $\sigma = 0.12$ and $T = 1$. The value is $10{,}000 N(d_1) e^{-0.08\times 1}$ where

$$d_1 = \frac{\ln(1.48/1.50) + (0.04 - 0.08 + 0.12^2/2)}{0.12} = -0.3852$$

$N(d_1) = 0.3500$ and the value of the derivative is as before

$10,000 \times 1.48 \times 0.3500 \times e^{-0.08} = 4782$ or $4,782.

Problem 25.26.

Consider an up-and-out barrier call option on a non-dividend-paying stock when the stock price is 50, the strike price is 50, the volatility is 30%, the risk-free rate is 5%, the time to maturity is one year, and the barrier at $80. Use the software to value the option and graph the relationship between (a) the option price and the stock price, (b) the delta and the option price, (c) the option price and the time to maturity, and (d) the option price and the volatility. Provide an intuitive explanation for the results you get. Show that the delta, gamma, theta, and vega for an up-and-out barrier call option can be either positive or negative.

The price of the option is 3.528.

(a) The option price is a humped function of the stock price with the maximum option price occurring for a stock price of about $57. If you could choose the stock price there would be a trade off. High stock prices give a high probability that the option will be knocked out. Low stock prices give a low potential payoff. For a stock price less than $57 delta is positive (as it is for a regular call option); for a stock price greater than $57, delta is negative.

(b) Delta increases up to a stock price of about 45 and then decreases. This shows that gamma can be positive or negative.

(c) The option price is a humped function of the time to maturity with the maximum option price occurring for a time to maturity of 0.5 years. This is because too long a time to maturity means that the option has a high probability of being knocked out; too short a time to maturity means that the option has a low potential payoff. For a time to maturity less than 0.5 years theta is negative (as it is for a regular call option); for a time to maturity greater than 0.5 years the theta of the option is positive.

(d) The option price is also a humped function of volatility with the maximum option price being obtained for a volatility of about 20%. This is because too high a volatility means that the option has a high probability of being knocked out; too low volatility means that the option has a low potential payoff. For volatilities less than 20% vega is positive (as it is for a regular option); for volatilities above 20% vega is negative.

Problem 25.27.

Sample Application F in the DerivaGem Application Builder Software considers the static options replication example in Section 25.16. It shows the way a hedge can be constructed using four options (as in Section 25.16) and two ways a hedge can be constructed using 16 options.

(a) *Explain the difference between the two ways a hedge can be constructed using 16 options.*

(b) *Explain intuitively why the second method works better.*

(c) *Improve on the four-option hedge by changing Tmat for the third and fourth options.*

(d) *Check how well the 16-option portfolios match the delta, gamma, and vega of the barrier option.*

(a) Both approaches use a one call option with a strike price of 50 and a maturity of 0.75. In the first approach the other 15 call options have strike prices of 60 and equally spaced times to maturity. In the second approach the other 15 call options have strike prices of 60, but the spacing between the times to maturity decreases as the maturity of the barrier option is approached. The second approach to setting times to maturity produces a better hedge. This is because the chance of the barrier being hit at time t is an increasing function of t. As t increases it therefore becomes more important to replicate the barrier at time t.

(b) By using either trial and error or the Solver tool we see that we come closest to matching the price of the barrier option when the maturities of the third and fourth options are changed from 0.25 and 0.5 to 0.39 and 0.65.

(c) To calculate delta for the two 16-option hedge strategies it is necessary to change the last argument of EPortfolio from 0 to 1 in cells L42 and X42. To calculate delta for the barrier option it is necessary to change the last argument of BarrierOption in cell F12 from 0 to 1. To calculate gamma and vega the arguments must be changed to 2 and 3, respectively. The delta, gamma, and vega of the barrier option are –0.0221, –0.0035, and –0.0254. The delta, gamma, and vega of the first 16-option portfolio are –0.0262, –0.0045, and –0.1470. The delta, gamma, and vega of the second 16-option portfolio are –0.0199, –0.0037, and –0.1449. The second of the two 16-option portfolios provides Greek letters that are closest to the Greek letters of the barrier option. Interestingly neither of the two portfolios does particularly well on vega.

Problem 25.28

Consider a down-and-out call option on a foreign currency. The initial exchange rate is 0.90, the time to maturity is two years, the strike price is 1.00, the barrier is 0.80, the domestic risk-free interest rate is 5%, the foreign risk-free interest rate is 6%, and the volatility is 25% per annum. Use DerivaGem to develop a static option replication strategy involving five options.

A natural approach is to attempt to replicate the option with positions in:

(a) A European call option with strike price 1.00 maturing in two years
(b) A European put option with strike price 0.80 maturing in two years
(c) A European put option with strike price 0.80 maturing in 1.5 years
(d) A European put option with strike price 0.80 maturing in 1.0 years
(e) A European put option with strike price 0.80 maturing in 0.5 years

The first option can be used to match the value of the down-and-out-call for $t = 2$ and $S > 1.00$. The others can be used to match it at the following (t, S) points: (1.5, 0.80) (1.0, 0.80), (0.5, 0.80), (0.0, 0.80). Following the procedure in the text, we find that the required positions in the options are as shown in the following table.

Option Type	*Strike Price*	*Maturity (yrs)*	*Position*
Call	1.0	2.00	+1.0000
Put	0.8	2.00	−0.1255
Put	0.8	1.50	−0.1758
Put	0.8	1.00	−0.0956
Put	0.8	0.50	−0.0547

The value of the portfolio initially is 0.482. This is only a little less than the value of the down-and-out-option which is 0.488. This example is different from the example in the text in a number of ways. Put options and call options are used in the replicating portfolio. The value of the replicating portfolio converges to the value of the option from below rather than from above. Also, even with relatively few options, the value of the replicating portfolio is close to the value of the down-and-out option.

Problem 25.29.

Suppose that a stock index is currently 900. The dividend yield is 2%, the risk-free rate is 5%, *and the volatility is 40%. Use the results in the appendix to calculate the value of a one-year average price call where the strike price is 900 and the index level is observed at the end of each quarter for the purposes of the averaging. Compare this with the price calculated by DerivaGem for a one-year average price option where the price is observed continuously. Provide an intuitive explanation for any differences between the prices.*

In this case $M_1 = 917.07$ and so that the option can be valued as an option on futures where the futures price is 917.07 and volatility is $\sqrt{\ln(904028.7/917.07^2}$ or 26.88%. The value of the option is 100.74. DerivaGem gives the price as 86.77 (set option type =Asian). The higher price for the first option arises because the average is calculated from prices at times 0.25, 0.50, 0.75, and 1.00. The mean of these times is 0.625. By contrast the corresponding mean when the price is observed continuously is 0.50. The later a price is observed the more uncertain it is and the more it contributes to the value of the option.

Problem 25.30.
Use the DerivaGem Application Builder software to compare the effectiveness of daily delta hedging for (a) the option considered in Tables 18.2 and 18.3 and (b) an average price call with the same parameters. Use Sample Application C. For the average price option you will find it necessary to change the calculation of the option price in cell C16, the payoffs in cells H15 and H16, and the deltas (cells G46 to G186 and N46 to N186). Carry out 20 Monte Carlo simulation runs for each option by repeatedly pressing F9. On each run record the cost of writing and hedging the option, the volume of trading over the whole 20 weeks and the volume of trading between weeks 11 and 20. Comment on the results.

For the regular option the theoretical price is 239,599. For the average price option the theoretical price is 115,259. My 20 simulation runs (40 outcomes because of the antithetic calculations) gave results as shown in the following table.

	Regular Call	Ave Price Call
Ave Hedging Cost	247,628	114,837
SD Hedging Cost	17,833	12,123
Ave Trading Vol (20 wks)	412,440	291,237
Ave Trading Vol (last 10 wks)	187,074	51,658

These results show that the standard deviation of using delta hedging for an average price option is lower than that for a regular option. However, using the criterion in Chapter 18 (standard deviation divided by value of option) hedge performance is better for the regular option. Hedging the average price option requires less trading, particularly in the last 10 weeks. This is because we become progressively more certain about the average price as the maturity of the option is approached.

Problem 25.31
In the DerivaGem Application Builder Software modify Sample Application D to test the effectiveness of delta and gamma hedging for a call on call compound option on a 100,000 units of a foreign currency where the exchange rate is 0.67, the domestic risk-free rate is 5%, the foreign risk-free rate is 6%, the volatility is 12%. The time to maturity of the first option is 20 weeks, and the strike price of the first option is 0.015. The second option matures 40 weeks from today and has a strike price of 0.68. Explain how you modified the cells. Comment on hedge effectiveness.

The value of the option is 1093. It is necessary to change cells F20 and F46 to 0.67. Cells G20 to G39 and G46 to G65 must be changed to calculate delta of the compound option. Cells H20 to H39 and H46 to H65 must be changed to calculate gamma of the compound option. Cells I20 to I40 and I46 to I66 must be changed to calculate the Black–Scholes price of the call option expiring in 40 weeks. Similarly cells J20 to J40 and J46 to J66 must be changed to calculate the delta of this option; cells K20 to K40 and K46 to K66 must be changed to calculate the gamma of the option. The payoffs in cells N9 and N10 must be calculated as MAX(I40-0.015,0)*100000 and MAX(I66-0.015,0)*100000. Delta plus gamma hedging works relatively poorly for the compound option. On 20 simulation runs the cost of writing and hedging the option ranged from 200 to 2500.

Problem 25.32.

Outperformance certificates (also called "sprint certificates", "accelerator certificates", or "speeders") are offered to investors by many European banks as a way of investing in a company's stock. The initial investment equals the company's stock price, S_0*. If the stock price goes up between time 0 and time* T*, the investor gains k times the increase at time* T *where* k *is a constant greater than 1.0. However, the stock price used to calculate the gain at time* T *is capped at some maximum level* M*. If the stock price goes down the investor's loss is equal to the decrease. The investor does not receive dividends.*

(a) *Show that the outperformance certificate is a package.*

(b) *Calculate using DerivaGem the value of a one-year outperformance certificate when the stock price is 50 euros,* $k = 1.5$*,* M *= 70 euros, the risk-free rate is 5%, and the stock price volatility is 25%. Dividends equal to 0.5 euros are expected in 2 months, 5 month, 8 months, and 11 months.*

a) The outperformance certificate is equivalent provides a return on an initial investment equal to the stock price consisting of

(i) A long position in k one-year European call options on the stock with a strike price equal to the current stock price.

(ii) A short position in k one-year European call options on the stock with a strike price equal to M

(iii) A short position in one European one-year put option on the stock with a strike price equal to the current stock price.

b) In this case the present value of the three parts to the gain are

(i) $1.5 \times 5.0056 = 7.5084$

(ii) $-1.5 \times 0.6339 = 0.9509$

(iii) –4.5138

The total of these is $7.5084 - 0.9509 - 4.5138 = 2.0437$.

The present value of the return of the initial investment is $50e^{-0.05\times1}$=47.56. The total present values of what will be received is therefore 49.6. This is less than the initial investment of 50.

Problem 25.33.

Carry out the analysis in Example 25.4 of Section 25.15 to value the variance swap on the assumption that the life of the swap is 1 month rather than 3 months.

In this case, $F_0 = 1022.55$ and DerivaGem shows that $Q(K_1) = 0.0366$, $Q(K_2) = 0.2858$,

$Q(K_3) = 1.5822$, $Q(K_4) = 6.3708$, $Q(K_5) = 30.3864$, $Q(K_6) = 16.9233$, $Q(K_7) = 4.8180$, $Q(K_8) = 0.8639$, and $Q_9 = 0.0863$. $\hat{E}(\overline{V}) = 0.0661$. The value of the variance swap is \$2.09 million.

Problem 25.34
What is the relationship between a regular call option, a binary call option, and a gap call option?

With the notation in the text, a regular call option with strike price K_2 plus a binary call option that pays off $K_2 - K_1$ is a gap call option that pays off $S_T - K_1$ when $S_T > K_2$.

Problem 25.35
Produce a formula for valuing a cliquet option where an amount Q is invested to produce a payoff at the end of n periods. The return each period is the greater of the return on an index (excluding dividends) and zero.

Suppose that there are n periods each of length τ, the risk-free interest rate is r, the dividend yield on the index is q, and the volatility of the index is σ. The value of the investment is

$$e^{-rn\tau} Q\hat{E}\left[\prod_{i=1}^{n} \max(1 + R_i, 1)\right]$$

where R_i is the return in period i and as usual $\hat{E}$ denotes expected value in a risk-neutral world. Because (assuming efficient markets) the returns in successive periods are independent, this is

$$e^{-rn\tau} Q\prod_{i=1}^{n} \{\hat{E}[\max(1 + R_i, 1)]\}$$

$$= e^{-rn\tau} Q\prod_{i=1}^{n} \left\{\hat{E}\left[1 + \max\left(\frac{S_i - S_{i-1}}{S_{i-1}}, 0\right)\right]\right\}$$

where S_i is the value of the index at the end of the ith period.
From Black-Scholes-Merton the risk-neutral expectation at time $(i\text{-}1)\tau$ of $\max(S_i - S_{i\text{-}1}, 0)$ is

$$e^{(r-q)\tau} S_{i-1} N(d_1) - S_{i-1} N(d_2)$$

where

$$d_1 = \frac{r - q + \sigma^2 \tau / 2}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{r - q - \sigma^2 \tau / 2}{\sigma\sqrt{\tau}}$$

The value of the investment is therefore

$$e^{-rn\tau} Q\left[1 + e^{(r-q)\tau} N(d_1) - N(d_2)\right]^n$$