Extended Kalman filter algorithm

- At each time step t,
  - Prediction update

$$G_t = \left. \frac{\partial}{\partial x} g(x, u) \right|_{x=\mu_{t-1}}$$

$$\overline{\mu}_t = g(\mu_{t-1}, u_t)$$

$$\overline{\Sigma}_t = G_t \Sigma_{t-1} G_t^\top + R_t$$

  - Measurement update

$$H_t = \left. \frac{\partial}{\partial x} h(x) \right|_{x=\overline{\mu}_t}$$

$$K_t = \overline{\Sigma}_t H_t^\top (H_t \overline{\Sigma}_t H_t^\top + Q_t)^{-1}$$

$$\mu_t = \overline{\mu}_t + K_t (z_t - h(\overline{\mu}_t))$$

$$\Sigma_t = (I - K_t H_t) \overline{\Sigma}_t$$

Example: EKF localization

- Use sensor information to locate the vehicle in a known environment

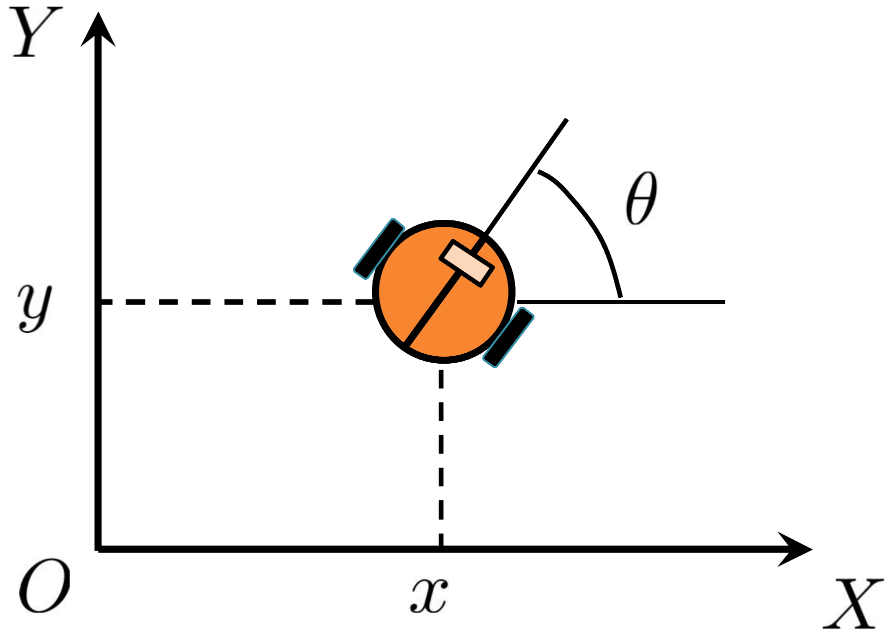
- Given
  - Control inputs and motion model
  - Sensor measurements and measurement model relative to the environment
  - Environment model (map)

- Find
  - The robot's current position

Example: EKF localization

- Two wheeled robot
  - Vehicle state and inputs

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

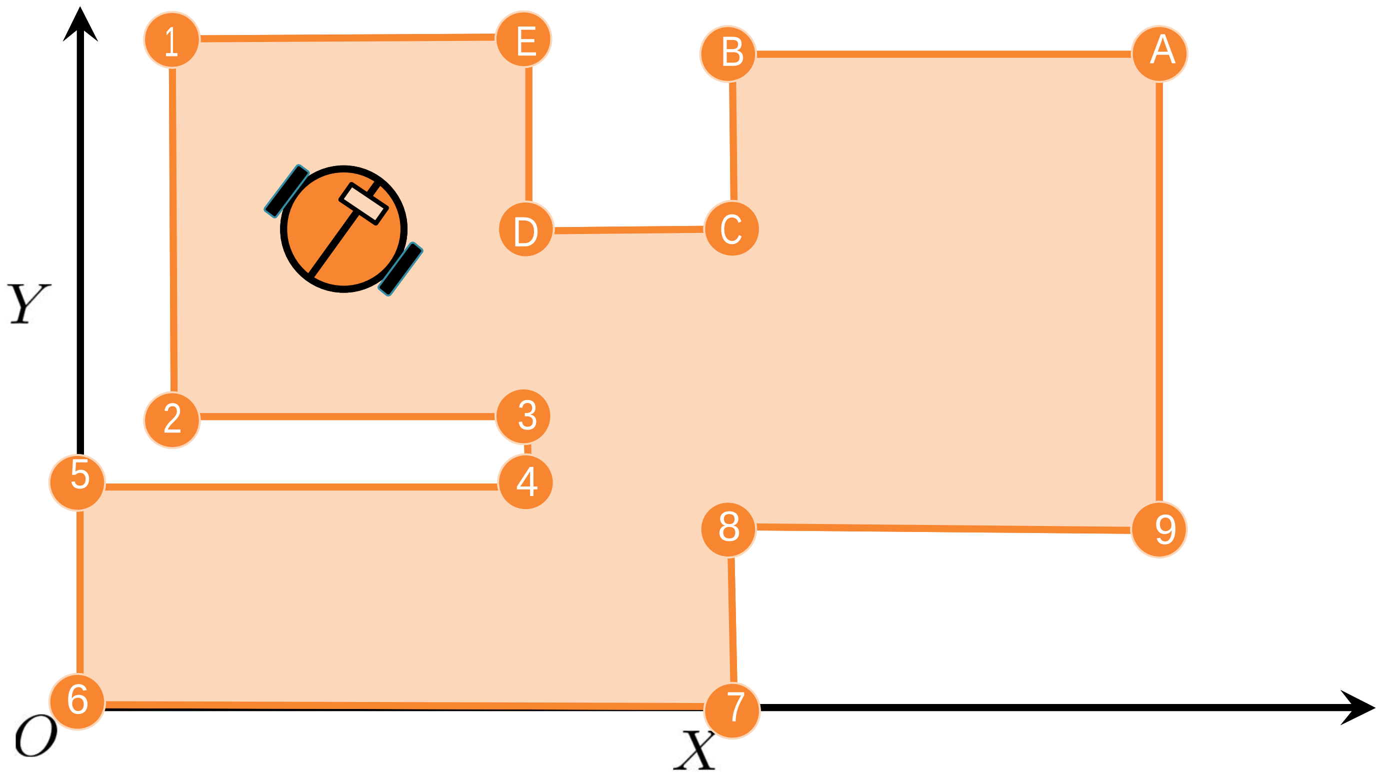


  - Motion model

$$\begin{bmatrix} x_{t,1} \\ x_{t,2} \\ x_{t,3} \end{bmatrix} = g(x_{t-1}, u_t) = \begin{bmatrix} x_{t-1,1} + u_{t,1} \cos x_{t-1,3} \Delta t \\ x_{t-1,2} + u_{t,1} \sin x_{t-1,3} \Delta t \\ x_{t-1,3} + u_{t,2} \Delta t \end{bmatrix}$$

Example: EKF localization

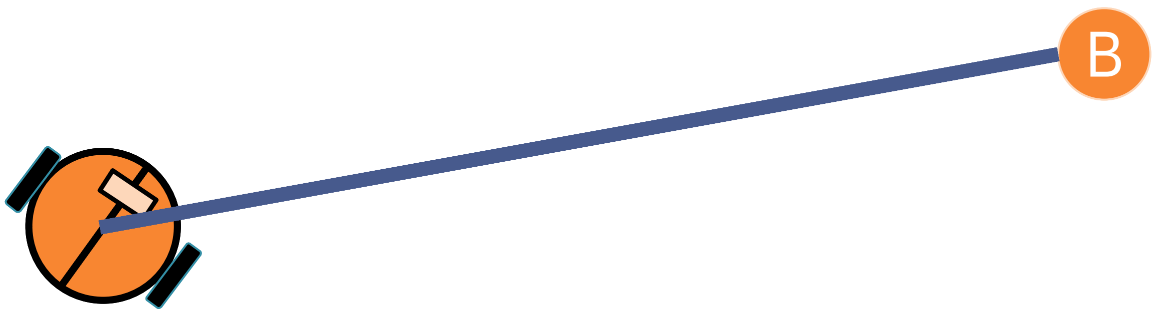
- Landmark (feature) map $m = \{m_1, \cdots, m_M\}$ $m_i = (m_{i,x}, m_{i,y})$



Example: EKF localization

- Measurement model
  - Relative range and / or bearing to nearby feature m_i, regardless of heading
  - For this example, assume measurement of the closest feature only

$$\begin{bmatrix} z_{t,1} \\ z_{t,2} \end{bmatrix} = h(x_t) = \begin{bmatrix} \arctan\left(\frac{m_{i,y} - x_{t,2}}{m_{i,x} - x_{t,1}}\right) - x_{t,3} \\ \sqrt{(m_{i,x} - x_{t,1})^2 + (m_{i,y} - x_{t,2})^2} \end{bmatrix} \begin{matrix} \text{bearing} \\ \text{range} \end{matrix}$$



Example: EKF localization

- Recall Extended Kalman Filter Algorithm
  1. Prediction update

$$G_t = \left. \frac{\partial}{\partial x} g(x, u) \right|_{x=\mu_{t-1}}$$

$$\overline{\mu}_t = g(\mu_{t-1}, u_t)$$

$$\overline{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$$

  2. Measurement update

$$H_t = \left. \frac{\partial}{\partial x} h(x) \right|_{x=\overline{\mu}_t}$$

$$K_t = \overline{\Sigma}_t H_t^T (H_t \overline{\Sigma}_t H_t^T + Q_t)^{-1}$$

$$\mu_t = \overline{\mu}_t + K_t (z_t - h(\overline{\mu}_t))$$

$$\Sigma_t = (I - K_t H_t) \overline{\Sigma}_t$$

Example: EKF localization

- Linearization of the motion model

$$x_t = \begin{bmatrix} x_{t,1} \\ x_{t,2} \\ x_{t,3} \end{bmatrix} = g(x_{t-1}, u_t) = \begin{bmatrix} x_{t-1,1} + u_{t,1} \cos x_{t-1,3} \Delta t \\ x_{t-1,2} + u_{t,1} \sin x_{t-1,3} \Delta t \\ x_{t-1,3} + u_{t,2} \Delta t \end{bmatrix}$$

$$\Downarrow$$

$$\frac{\partial}{\partial x_{t-1}} g(x_{t-1}, u_t) = \begin{bmatrix} 1 & 0 & -u_{t,1} \sin x_{t-1,3} \Delta t \\ 0 & 1 & u_{t,1} \cos x_{t-1,3} \Delta t \\ 0 & 0 & 1 \end{bmatrix}$$

Example: EKF localization

- Linearization of measurement model

$$z_t = \begin{bmatrix} z_{t,1} \\ z_{t,2} \end{bmatrix} = h(x_t) = \begin{bmatrix} \arctan\left(\frac{m_{i,y}-x_{t,2}}{m_{i,x}-x_{t,1}}\right) - x_{t,3} \\ \sqrt{(m_{i,x}-x_{t,1})^2 + (m_{i,y}-x_{t,2})^2} \end{bmatrix}$$

$$\frac{\partial}{\partial x_t} h(x_t) = \begin{bmatrix} \frac{(m_{i,y}-x_{t,2})}{q} & \frac{-(m_{i,x}-x_{t,1})}{q} & -1 \\ \frac{-(m_{i,x}-x_{t,1})}{\sqrt{q}} & \frac{-(m_{i,y}-x_{t,2})}{\sqrt{q}} & 0 \end{bmatrix}$$

where $q = (m_{i,x} - x_{t,1})^2 + (m_{i,y} - x_{t,2})^2$

Example: EKF localization

$$H_t = \frac{\partial}{\partial x} h(x)\Big|_{x=\bar{\mu}_t}$$

$$H_t = \begin{bmatrix} \frac{(m_{i,y}-\bar{\mu}_{t,2})}{q} & \frac{-(m_{i,x}-\bar{\mu}_{t,1})}{q} & -1 \\ \frac{-(m_{i,x}-\bar{\mu}_{t,1})}{\sqrt{q}} & \frac{-(m_{i,y}-\bar{\mu}_{t,2})}{\sqrt{q}} & 0 \end{bmatrix}$$

$$q = (m_{i,x} - \bar{\mu}_{t,1})^2 + (m_{i,y} - \bar{\mu}_{t,2})^2$$

Example: EKF localization

- Till now we assume at each time step, the robot can only measure a single landmark that is closest to it.

- How to incorporate more than one measurement?

Example: EKF localization

- Append different measurements
- Assume that the i-th measurement z_t^i at time t corresponds to the j-th landmark m_j in the map, the measurement model is the stack of

$$H_t^i = \frac{\partial}{\partial x} h_i(x)\bigg|_{x=\bar{\mu}_t} = \begin{bmatrix} \frac{(m_{j,y} - x_{t,2})}{q} & \frac{-(m_{j,x} - x_{t,1})}{q} & -1 \\ \frac{-(m_{j,x} - x_{t,1})}{\sqrt{q}} & \frac{-(m_{j,y} - x_{t,2})}{\sqrt{q}} & 0 \end{bmatrix}$$

$$q = (m_{j,x} - \bar{\mu}_{t,1})^2 + (m_{j,y} - \bar{\mu}_{t,2})^2$$

$$\begin{bmatrix} H_t^1 \\ \vdots \\ H_t^k \end{bmatrix}$$

Example: EKF localization

- The resulting H_t matrix would be of size $2k \times 3$
- The size of $H_t \bar{\Sigma}_t H_t^T$ is of the size $2k \times 2k$
- The noise matrix Q_t is (each Q is dim-2)

$$\begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix}$$

$$H_t = \left. \frac{\partial}{\partial x} h(x) \right|_{x=\overline{\mu}_t}$$

$$K_t = \overline{\Sigma}_t H_t^T (H_t \overline{\Sigma}_t H_t^T + Q_t)^{-1}$$

$$\mu_t = \overline{\mu}_t + K_t(z_t - h(\overline{\mu}_t))$$

$$\Sigma_t = (I - K_t H_t)\overline{\Sigma}_t$$

- To compute K matrix, we need to perform an inverse operation for a $2k \times 2k$ matrix – very expensive for large k

Example: EKF localization

- Recall: The more landmarks tracked, the better the resolution; But the larger the matrix inverse operation at each time step

- Fortunately, we can make another reasonable assumption: the sensor measurements are conditionally independent

$$p(z_t \mid x_t, m) = \prod_i p(z_t^i \mid x_t, m)$$

- This assumption allows us to incrementally add the information, as if there was zero motion in between measurements

Example: EKF localization

Data: $(\mu_{t-1}, \Sigma_{t-1}), u_t, z_t, m$
Result: (μ_t, Σ_t)
$\overline{\mu}_t = g(u_t, \mu_{t-1})$;
$\overline{\Sigma}_t = G_t \Sigma_{t-1} {G_t}^T + R_t$;

foreach $z_t^i = (r_t^i, \phi_t^i)^T$ **do** // for all landmarks that can be seen by the robot

$$\hat{z}_t^i = \begin{bmatrix} \arctan\left(\frac{m_{j,y} - \bar{\mu}_{t,2}}{m_{j,x} - \bar{\mu}_{t,1}}\right) - \bar{\mu}_{t,3} \\ \sqrt{(m_{j,x} - \bar{\mu}_{t,1})^2 + (m_{j,y} - \bar{\mu}_{t,2})^2} \end{bmatrix}$$

$S_t^i = H_t^i \, \overline{\Sigma}_t \, [H_t^i]^T + Q_t$;
$K_t^i = \overline{\Sigma}_t \, [H_t^i]^T \, [S_t^i]^{-1}$;
$\overline{\mu}_t = \overline{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i)$;
$\overline{\Sigma}_t = (I - K_t^i H_t^i) \, \overline{\Sigma}_t$;

end

$\mu_t = \overline{\mu}_t$ and $\Sigma_t = \overline{\Sigma}_t$;
Return (μ_t, Σ_t)