Computing 3D δR and δS

Similar with 2D, $\underline{R}^T \delta \underline{R} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$

$\underline{R}^T \delta \underline{F} = \underline{R}^T \delta \underline{R}\, \underline{S} + \delta \underline{S}$, call $\underline{R}^T \delta \underline{F}$ with $\underline{W}$

$$\Rightarrow \delta \underline{S} = \underline{W} - \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

$$= \begin{pmatrix} * & W_{12} + r_3 S_{22} - r_2 S_{23} & W_{13} + r_3 S_{23} - r_2 S_{33} \\ W_{21} - r_3 S_{11} + r_1 S_{13} & * & W_{23} - r_3 S_{13} + r_1 S_{33} \\ W_{31} + r_2 S_{11} - r_1 S_{12} & W_{32} + r_2 S_{12} - r_1 S_{22} & * \end{pmatrix}$$

The symmetry of $\delta \underline{S}$ implies
$$\begin{cases} W_{12} + r_3 S_{22} - r_2 S_{23} = W_{21} - r_3 S_{11} + r_1 S_{13} \\ W_{13} + r_3 S_{23} - r_2 S_{33} = W_{31} + r_2 S_{11} - r_1 S_{12} \\ W_{23} - r_3 S_{13} + r_1 S_{33} = W_{32} + r_2 S_{12} - r_1 S_{22} \end{cases}$$

i.e. $$\left(\underline{S} - tr(\underline{S})\, \underline{I}\right) \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} W_{23} - W_{32} \\ W_{31} - W_{13} \\ W_{12} - W_{21} \end{pmatrix}$$

$$\left| \underline{S} - tr(\Sigma)\, \underline{I} \right| = \left| \underline{V} \Sigma \underline{V}^T - tr(\Sigma)\, \underline{I} \right| = \left| \underline{V} \Sigma - tr(\Sigma) \underline{V} \right| \cdot \left| \underline{V}^T \right|$$

$$= |\underline{V}^T| \left| \begin{pmatrix} -(\sigma_2+\sigma_3)V_{11} & -(\sigma_1+\sigma_3)V_{12} & -(\sigma_1+\sigma_2)V_{13} \\ -(\sigma_2+\sigma_3)V_{21} & -(\sigma_1+\sigma_3)V_{22} & -(\sigma_1+\sigma_2)V_{23} \\ -(\sigma_2+\sigma_3)V_{31} & -(\sigma_1+\sigma_3)V_{32} & -(\sigma_1+\sigma_2)V_{33} \end{pmatrix} \right|$$

$$= |\underline{V}^T|\, |\underline{V}| \left| \begin{pmatrix} -(\sigma_2+\sigma_3) & & \\ & -(\sigma_1+\sigma_3) & \\ & & -(\sigma_1+\sigma_2) \end{pmatrix} \right|$$

It is 0 (Not-invertable) iff $\sigma_i + \sigma_j = 0$ for some $i \neq j,\ i, j \in [1, 3]$

otherwise, $\delta \underline{R} = \underline{R} \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$

$$\delta \underline{S} = \underline{R}^T \delta \underline{F} - \underline{R}^T \delta \underline{R}\, \underline{S} = \underline{R}^T \delta \underline{F} - \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \underline{S}$$