CS370 Notes

Minyang Jiang

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1 Floating Point Number Systems

1.1 Introduction

$$F(\beta, t, L, u)$$

continas:

$$\begin{gathered} 0 \text{ or } \pm 0.\beta_1\beta_2...\beta_t * \beta^d \\ \text{where } \beta_1 \neq 0 \\ 0 \leq \beta_i \leq \beta \\ L \leq d \leq u \end{gathered}$$

Two common systems used today - Base 2

1. Single precision: $F(2, 24, -126, 127)$
2. Double precision: $F(2, 53, -1022, 1023)$

Important concepts
If x is any real number then set $fl(x) =$ floating point representation of x
if we write:

$$x = \pm 0.x_1x_2x_3...x_tx_{t+1}... * \beta^d$$

$$fl(x) = \pm 0.x_1x_2x_3...x_t * \beta^d$$

relative error

$$\delta_x = \frac{fl(x) - x}{x}$$

$$\|\delta_x\| \leq ?$$

$$\begin{aligned} \frac{\|fl(x) - x\|}{\|x\|} &= \frac{0.00\ldots 0x_{t+1}\ldots * \beta^d}{0.x_1x_2\ldots x_{t+1}\ldots * \beta^d} \\ &= \frac{0.x_{t+1}x_{t+2}\ldots * \beta^{-t}}{x_1.x_2\ldots * \beta_{-1}} \\ \delta = \frac{fl(x) - x}{x} & \qquad \text{then } \|\delta\| \leq \epsilon = \begin{cases} \beta^{1-t} \\ \frac{\beta^{1-t}}{2} \end{cases} \\ fl(x) = x(1 + \delta) & \qquad |\delta| \leq \epsilon \end{aligned}$$

What about floating point arithmetic?
x, y real numbers, $x + y$ real
$x \oplus y =$addition inside floating pt system $= fl(fl(x) + fl(y))$

1.2 Analysis some errors in computation

Example. Addition

$$\left\| \frac{(x+y)-(x\oplus y)}{x+y} \right\| = \frac{\|(x+y)-fl(fl(x)+fl(y))\|}{x+y} = \frac{\|x+y-x(1+\delta)+y(1+\delta)\|}{x+y}$$
$$= \frac{\|x+y-(x+y+\delta_1 x+\delta_2 y+x\delta_3+y\delta_3+\delta_1\delta_3 x+\delta_2\delta_3 y)\|}{x+y}$$
$$\leq \frac{\|\delta_1 x\|+\|\delta_2 y\|+\|\delta_3 x\|+\|\delta_3 y\|+\delta_1\delta_3 x+\|\delta 2\delta_3 y\|}{\|x+y\|}$$
$$\leq \frac{(\|x\|+\|y\|)(2\epsilon+\epsilon^2)}{\|x+y\|}$$
$$\|\delta_1\| \leq \epsilon, \qquad \|\delta_2\| \leq \epsilon, \qquad \|\delta_3\| \leq \epsilon$$

$\Rightarrow$ if x and y have same sign then relative error of addition

$$\left\| \frac{x\oplus y-(x+y)}{x+y} \right\| \leq 2\epsilon+\epsilon^2$$

However if x and y have opposite sign, then you potentially have a problem particularly when $x+y\approx 0$, Situation is called **Catastrophic cancellation**

$$x = 0.x_1x_2\dots x_{t-1}x_tx_{t+1}\dots * \beta^d$$
$$y = -0.x_1x_2\dots x_{t-1}x_tx_{t+1}\dots * \beta^d$$
$$x+y = 0.00\dots 0??\dots * \beta^d$$

1.3 How about some algorithms?

1.3.1 Example

Given α, compute:

$$I_n = \int_0^1 \frac{x^n}{x+\alpha}dx \qquad n = 0,1,\dots,100\dots$$

Step 1

$$I_0 = \int_0^1 \frac{1}{x+\alpha}dx = Ln(x+\alpha) = Ln(1+\alpha)-Ln(\alpha) = Ln\left(\frac{1+\alpha}{\alpha}\right)$$

e.g.

$$\alpha = 0.5 \text{ then } I_0 = 1.098612288668\dots$$
$$\alpha = 2.0 \text{ then } I_0 = 0.405465108108\dots$$

Step 2 Notice:

$$I_{n+1} = \int_0^1 \frac{x^{n+1}}{x+\alpha}dx = \int_0^1 \frac{x^n(x+\alpha-\alpha}{x+\alpha}dx = \int_0^1 x^n dx - \alpha\int_0^1 \frac{x^n}{x+\alpha}dx$$
$$I_{n+1} = \frac{1}{n+1} - \alpha I_n$$

$$\begin{aligned}
&I_0 \\
&I_1 = 1 - \alpha I_0 \\
&I_2 = \frac{1}{2} - \alpha I_1 \\
&\vdots \\
I_{100} &= \frac{1}{100} - \alpha I_0 0
\end{aligned}$$

If $\alpha = 0.5$ then $I_{100} = 0.00664$
if $\alpha = 2.0$ then $I_{100} = 2.1 * 10^{22}$

$$\|I_n\| \leq \frac{1}{1+\alpha}$$

Let's analyze what is happening
<u>Math</u>:

$$I_0^{ex}, I_{n+1}^{ex} = \frac{1}{n+1} - \alpha I_n^{ex}$$

<u>CS</u>:

$$I_0^{app}, I_{n+1}^{app} = \frac{1}{n+1} - \alpha I_n^{aop}$$

At every step, therer is some error:

$$\begin{aligned}
e_0 &= I_0^{ex} - I_0^{app} \\
e_n &= I_0^{ex} - I_n^{app}
\end{aligned}$$

<u>Notice:</u>

$$\begin{aligned}
e_{n+1} &= I_{n+1}^{ex} - I_{n+1}^{app} \\
&= \left(\frac{1}{n+1} - \alpha I_n^{ex}\right) - \left(\frac{1}{n+1} - \alpha I_n^{app}\right) \\
&= -\alpha I_n^{ex} + \alpha I_n^{app} \\
&= -\alpha\left(I_n^{ex} - I_n^{app}\right) \\
&= -\alpha e_n \\
&= (-\alpha)^{n+1} e_0
\end{aligned}$$

If $\|\alpha\| \leq 1$ then $\|e_n\| \to 0$ as $n \to \inf$
If $\|\alpha\| \geq 1$ then $\|e_n\| \to \inf$ as $n \to \inf$
What to do when $\|\alpha\| \geq 1$:

$$\begin{aligned}
I_n &= \frac{1}{\alpha(n+1)} - \frac{1}{\alpha} I_{n+1} \\
e_n &= -\frac{1}{\alpha} e_{n+1}
\end{aligned}$$

If $\|\alpha\| \geq 1$ then work backwards, e.g. I_{100}, Do $I_{200}, I_{199} \ldots$

2 Interpolation

Given n points $(x_1, y_1), \ldots, (x_N, y_N)$ x_i distinct $x_1 < x_2 < \ldots < x_N$
$y = p(x)$, p should be 'nice'
'nice':

- polynomial; piecewise polynomial;

2.1 Polynomial Interpolation

Given n points (x_i, y_i) $i = 1, 2, \ldots, n$
Find a polynomial having degree $< n$ satisfying $p(x_i) = y_i$
Example
$(-1, 3), (1, 1), (2, 2)$

$$
\begin{aligned}
p(x) &= c_0 + c_1 x + c_2 x^2 \\
p(-1) &= c_0 - c_1 + c_2 = 3 \\
p(1) &= c_0 + c_1 + c_2 = 1 \\
p(2) &= c_0 + 2c_1 + 4c_2 = 2
\end{aligned}
$$

$$
\begin{bmatrix} 1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \end{bmatrix}
$$

$$
p(x) = \frac{4}{3} - x + \frac{2}{3} x^2
$$

Given n points $(x_1, y_1), \ldots, (x_N, y_N)$ x_i distinct $x_1 < x_2 < \ldots < x_N$

1) Does there exist a polynomial $p(x)$ of degree ¡ n which interpolate the n points?
2) If it exists then is it unique?

$$
(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)
$$

$$
p(x) = c_1 + c_2 x + \ldots + c_n x^{n-1}
$$

n unknowns
n equations

$$
\begin{aligned}
c_1 + c_2 x_1 + \ldots + c_n x_1^{n-1} &= y_1 \\
c_1 + c_2 x_2 + \ldots + c_n x_2^{n-1} &= y_2 \\
&\vdots \\
c_1 + c_2 x_n + \ldots + c_n x_n^{n-1} &= y_n
\end{aligned}
$$

Vamdeimonde matrix

$$
V \cdot \vec{c} = \vec{y}
$$

$$
det(V) = \Pi_{i>j}(x_i - x_j) \neq 0 \qquad \text{since all } x_i \text{ distinct}
$$

2.2 Lagrange Form of Interpolating Polynomial

$$\begin{aligned} p(x) &= c_1 + c_2 x + c_3 x^2 + \ldots + c_n x^{n-1} \\ &= y_1 L_1(x) + y_2 L_2(x) + \ldots + y_n L_n(x) \end{aligned}$$

where each $L_i(x)$ is a polyominal of degree $< n$ which satisfies $L_i(x_i) = 1, L_i(x_j) = 0$ if $i \neq j$
$L_i(x) \equiv$ Lagrange polynominal

$$L_i(x) = \frac{(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$$

2.3 Hermite Interpolation

Points: $(x_L, y_L), (x_R, y_R)$
Derivative values: S_L, S_R
$S(x)$ degree < 4, $S(X_L) = Y_L, S(X_R) = Y_R, S'(X_L) = S_L, S'(X_R) = S_R$, Find $S(X)$

$$\begin{aligned} S(x) =& c_1 + c_2(x - x_L) + c_3(x - x_L)^2 + c_4(x - x_L)^3 \\ & \begin{bmatrix} 1 & 0 & 0 & 0 & y_L \\ 0 & 1 & 0 & 0 & s_L \\ 1 & \Delta x & \Delta x^2 & \Delta x^3 & y_R \\ 0 & 1 & 2\Delta x & 3\Delta x^2 & s_R \end{bmatrix} \\ =& \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 2 & 4 & 8 & 4 \\ 0 & 1 & 4 & 12 & -1 \end{bmatrix} \\ y'_L =& \frac{y_R - y_L}{x_R - x_L} = slope \\ c_1 =& y_L \\ c_2 =& s_L \\ c_3 =& \frac{3y'_L - 2s_L - s_R}{\Delta x} \\ c_4 =& \frac{s_L + s_R - 2y'_L}{\delta x^2} \end{aligned}$$