COMPOSING POWER SERIES
consider the power series $A(x) = \sum_{n=0}^{\infty} nx^n$ and $B(x) = \sum_{n=0}^{\infty}(n^2+1)x^n$
What are the coneefficients of x^0 and x^1 in $B(A(x))$ and $A(B(x))$

$$\begin{aligned} B(A(x)) &= \sum_{n=0}^{\infty}\{(n^2+1)(A(x))^n\} \\ &= \sum_{n=0}^{\infty}\{(n^2+1)(\sum_{j=0}^{\infty} jx^j)^n\} \\ &= (1+0)x^0 + (2*1)x^1 + \dots \end{aligned}$$

What is the constant term (cofficient of x^1) in each $(\sum_{j=0}^{\infty} jx^j)^n$?

$$\sum_{j=0}^{\infty} jx^j = 0x^0 + 1x^1 + 2x^2 + \dots = x(1+2x+3x^2+\dots)$$

$[x^0]\sum_{j=0}^{\infty} jx^j = 0,\ [x^0](\sum_{j=0}^{\infty} jx^j)^2 = 0$ for all $n \geq 1,\ [x^0](\sum_{j=0}^{\infty} jx^j)^n = 0$

$$\sum_{n=0}^{\infty}\{(n^2+1)(\sum_{j=0}^{\infty} jx^j)^n\} = 1*1 + 2*(\sum_{j=0}^{\infty} jx^j)^1 + 5*(\sum_{j=0}^{\infty} jx^j)^2 + 10*(\sum_{j=0}^{\infty} jx^j)^3 + \dots$$

coefficient of x^1:$= 0 + 2*1_0 + 0 + 0 + \dots$

$$A(B(x)) = \sum_{n=0}^{\infty}\{n(\sum_{j=0}^{\infty}(j^2+1)x^j)^n\}$$

$$\sum_{j=0}^{\infty}(j^2+1)x^j = 1x^0 + 2*x^1 + 5*x^3 + 10x^3 + \dots$$

$$(\sum_{j=0}^{\infty}(j^2+1)x^j)^2 = 1*1x^0 + 4x^1 + 14x^2 + \dots$$

$$[x^0](\sum_{j=0}^{\infty}(j^2+1)x^j)^n) = 1$$

$$[x^1](\sum_{j=0}^{\infty}(j^2+1)x^j)^n) = 2n$$

$$n = 2 : (1x^0 + 2*x^1 + 5*x^3 + 10x^3 + \dots)(1x^0 + 2*x^1 + 5*x^3 + 10x^3 + \dots) = 1x^0 + (1*2+2*1)x^1 + \dots$$

$$[x^0]A(B(x)) = 0*1 + 1*1 + 2+1 + 3*1 + \dots = \sum_{n_0}^{\infty} n*1$$

What is the cofficient of x^2 in $A(B(x))$ where $A(x) = \frac{1}{1-x}$ and $B(x) = x + 2x^2$?

$$\begin{aligned} A(B(x)) &= \frac{1}{1-(x+2x^2)} \\ &= \sum_{n=0}^{\infty}(x+2x^2)^2 \\ &= \sum_{n=0}^{\infty} x^n(1+2x)^n \end{aligned}$$

$$
\begin{aligned}
[x^2]A(B(x)) &= [x^2]\sum_{n=0}^{2} x^n(1+2x)^n \\
&= [x^2](1 + x(1+2x) + x^2(1+2x)^2) \\
&= [x^2](1 + x + 2x^2 + x^2 + 4x^3 + 4x^4) \\
&= 3
\end{aligned}
$$

$A(x) = \frac{1}{1-3x+4x^3}$, Find a recurrence relation for the coefficients of $A(x)$
$a_n = a_{n-1} + a_{n-2}$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$, Then $(1x^0 - 3x + 4x^3)(\sum_{n=0}^{\infty} a_n x^n) = 1x^0 + 0x + 0x^2 + \ldots$
$a_0x^0 + (1a_1 - 3_0)x^1 + (1a_2 - 3a_1)x^2 + (1a_3 - 3a_2 + 4a_0)x^3 + (1a_4 - 3a_3 + 4a_1)x^4 + \ldots + (1a_n - 3a_{n-1} + 4a_{n-3})x^n + \ldots$

May 18 2016

Product lamma

Product lamma: Set A, B with lengths α, β. Set $A \times B$ with weight $\omega(a,b) = \alpha(a) + \beta(b)$, Then

$$\Phi_{A\times B}(x) = \Phi_A(x)\Phi_B(x)$$

Example: How many ways can a squence of k non-negative integers sum up to n?

$[k = 4;\ n = 10\ (1,2,3,4)\ (0,0,5,5)]$
$N_0 = \{0,1,2,3,4,\ldots\}$
Consider the set $N_0^k = \{(a_1, a_2, \ldots, a_k) | a_i \in N_0\}$
Define $\omega(a_1, a_2, \ldots, a_k) = a_1 + a_2 + \ldots + a_k$ Use $\alpha(a) = a$ for each N_0
Then product lemma applies: For N_0 with α, $\Phi_{N_0}(x) = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$
By product lemma, $\Phi_{N_0^k}(x) = (\Phi_{N_0}(x))^k = \frac{1}{(1-x)^k}$
Answer is:

$$[x^n]\frac{1}{(1-x)^k}$$

This is

$$\binom{n+k-1}{k-1}$$

Combinatorial proof of $\binom{n+k-1}{k-1}$
$k = 4$, $n = 10$: want $a_1 + a_2 + a_3 + a_4 = 10$
For each k tuple $(a_1, a_2, \ldots, a_k)$ is a non-neg soln to $a_1 + a_2 + \ldots + a_k = n$
Consider binary strings with n 0s and $k-1$ 1s (dividers)
Form a bijection so that $(a_1, a_2, \ldots, a_k)$ is mapped to $0^{a_1}|0^{a_2}|\ldots|0^{a_k}$ where 0^{a_i} represents a_i consecutive 0s
This is reversible, So the # of k-tuples is equal to # of binary str with n 0s and $k-1$ 1s, which is $\binom{n+k-1}{k-1}$

Integer compositions

Definition: A k-tuple $(a_1, \ldots, a_k)$ of positive integers is a composition of n if $a_1 + \ldots + a_k = n$. Such a composition has k parts
Example: composition of 5 includes $(1,3,1)$, (5), $(2,3)$, $(3,2)$, $(1,1,1,1,1)$
Node:

1) parts are at least 1

2) Order does matter

3) There is one composition of 0 which is $()$
Example: How many compositions of n have k parts?
The set N^k represents all compositions with k parts
Let $\omega(a_1, \dots, a_k) = a_1 + \dots + a_k$
Use $\alpha(a) = a$ for each N

$$\Phi_N(x) = x + x^2 + x^3 + \dots = \frac{x}{1-x}$$

So by product lemma:

$$\Phi_{N^k}(x) = (\Phi_N(x))^k = \left(\frac{x}{1-x}\right)^k = \frac{x^k}{(1-x)^k}$$

Our answer is:

$$\begin{aligned} &[x^n]\frac{x^k}{(1-x)^k} \\ =&[x^{n-k}]\frac{1}{(1-x)^k} \\ =&\binom{(n-k)+k-1}{k-1} \\ =&\binom{n-1}{k-1} \end{aligned}$$

May 25 2016

1. Integer composition
2. Binary Strings

Gen ser for all compositions where every part is odd

$$\frac{1-x^2}{1-x-x^2}$$

Let a_n be the # of comp of n where every part is odd

$$a_n = a_{n-1} + a_{n-2} \quad for\ n \geq 3$$

Combinatiorial proff of this recurrence:
Let S_n be the set of all comp of n where every part is odd
We need to prove that $|S_n| = |S_{n-1}| + |S_{n_2}|$ for $n \geq 3$
We define a bijection $f : S_n \to S_{n-1} \cup S_{n-2}$ as follows:
$n = 5$: $(5), (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 1, 1, 1, 1)$
$n = 4$: $(1, 3), (3, 1), (1, 1, 1, 1)$
$n = 3$: $(3), (1, 1, 1)$
for $n - 1$, add 1 to front
for $n - 2$, add 2 to the first
For each $(a_1 \dots, a_k) \in S_n$ define

$$f(a_1, \dots, a_k) = \begin{cases} (a_2, \dots, a_k) \quad (a_1 = 1) \to inS_{n-1} \\ (a_1 - 2, a_2, \dots, a_k) \quad (a_1 > 1) \to inS_{n-2} \end{cases}$$

Every part is odd after the mapping

The inverse is $f^{-1} : S_{n-1} \cup S_{n-2} \to S_n$ where for each $(b_1, \ldots, a_k) \in S_{n-1} \cup S_{n-2}$, define:

$$f^{-1}(b_1, \ldots, b_k) = \begin{cases} (1, b_1, \ldots, b_k) & b_1 + \ldots + b_k = n-1 \\ (b_1 + 2, b_2, \ldots, b_k) & b_1 + \ldots + b_k = n-2 \end{cases}$$

We can recursively build up S_n based on S_{n-1} and S_{n-2} for each comp

Binary Strings

Definitions: A binary string is a sequence of 0s and 1s.
The length of a string is the total # of 0s and 1s
There is only one string of length 0, the emtpy string or null string, denoted ϵ
The concatenation of two strings a and b is ab
$(a = 101 \quad , b = 0001 \quad , ab = 1010001)$
b is a substring of s if $s = abc$ for some strings a, c (possibly empty)
e.g.: (Substrings of 1001 include ϵ, 001, 10, 1001)
A block is a maximal nonempty substring of all 0s or all 1s.
Example S - 0000 111 0 1 000 1 000 111 (8 blocks)

Main Q - How many binary string of length n satisfy certain properties?
Define a set S of all strings that satisfy the properties
Define the weight of a string to be its length
Answer is

$$[x^n]\Phi_S(x)$$

Example: How many strings of length n has no 0s?
The set of all strings with no 0s is $S = \{\epsilon, 1, 11, 111, 111, \ldots\}$

$$\Phi_S(x) = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}$$

Answer is

$$[x^n]\Phi_S(x)$$

Two operations in regular expression of strings

if A, B are 2 sets of strings, then the concatenation of A, B is:

$$AB = \{ab | a \in A, b \in B\}$$

Example: $A = \{0, 11\},\ B = \{1, 11\}$, $AB = \{01, 011, 111, 1111\}$
They are "like" cartesian products
Powers of strings: $A^2 = AA$, $A^3 = AAA$
Star operator

$$A^* = \{\epsilon\} \cup A \cup A^2 \cup A^3 \cup \ldots = \bigcup_{k \geq 0} A^k$$

Example $\{0, 1\}^*$ is all binary strings: $01101 \in \{0, 1\}^5 \subseteq \{0, 1\}^*$

May 27, 2016

How many binary strings are there of length n?
Generating function for $\{0,1\}$ is $2x^1$
Generating function for $\{0,1\}^n$ is $2^n x^n$
Generating function for binary strings is

$$1x^0 + 2x^1 + 4x^2 + \ldots + (2^n x^n) + \ldots = \frac{1}{1-2x}$$

May 30, 2016

We saw that the binary strings with exactly k 1's decompose as $0^*(10^*)^k$, and that all binary strings decompose as $0^*(10^*)^*$
If A^* gives an unambiguous description of strings obtained by concatenating stings from A, then

$$\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$$

$A = (10^*), A^* = (10^*)^* =$ al binary strings where first bit is 1 (including ϵ)
The non-ambiguity of the decomposition A^* also implies

$$\Phi_{A^k}(x) = (\Phi_A(x))^k$$

A^k is in bijective correspondence with $A \times A \times \ldots \times A$