PS 0

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1.

a): Sol:

$$f(x) = \frac{1}{2}x^TAx + b^Tx$$

$$\nabla f(x) = Ax + b.$$

b): $\nabla f(x) = \nabla (h(x)) = g(h(x)) \cdot \nabla h(x)$

$= g(h(x)) [\frac{\partial}{\partial x_1} h(x), \dots \frac{\partial}{\partial x_n} h(x)]^T$

c): $\nabla^2 f(x) = A$ $\quad (a^Tx)'' =$

d): $f(x) = g(a^Tx)$

$\nabla f(x) = g'(a^Tx) \cdot a. \quad (n \times 1)$

$\nabla^2 f(x) = \nabla (\nabla f(x))^T = g''(a^Tx) a a^T$

2.

a): $f(x) = x^TAx = x^TZZ^Tx = (Z^Tx)^T(Z^Tx)$

Z^Tx is a real number : y.

$y^T = y$.

$\therefore f(x) = y^2 \geq 0$. Thus. $A = ZZ^T$ is positive semidefinite.

b): $ZZ^Tx = \vec{0}$

$\text{rank}(A) = \text{rank}(Z \cdot Z^T) \leq \min(\text{rank}(Z), \text{rank}(Z^T)) = 1$

and $ZZ^T \neq 0$ since Z is non-zero vector.

Thus $\text{rank}(A) = 1$.

As for null-space, when $n > 1$, $Ax = 0$ shall have infinite solution. If $n = 1$, the only solution is 0.

3

a): Proof: denote $e(i)$ is the vector that i-th entry is 1 with else 0. Thus, $e(i) = [0, 0, \dots 1, \dots 0, 0]^T$

equation: $A = T\Lambda T^{-1}$

$\Rightarrow AT = T\Lambda$

$ATe(i) = T\Lambda e(i)$

$\Rightarrow At_i = \lambda_i t_i$

Q.E.D.

b): Similarly: $A = U\Lambda U^T$

$AU = U\Lambda$

$AUe(i) = U\Lambda e^{(i)} = U \cdot \lambda_i e^{(i)} = \lambda_i U e^{(i)}$

$Au^{(i)} = \lambda_i u^{(i)}$

c): Proof: Since A is PSD, A is symmetric.

by spectral theorem.

$A = U\Lambda U^T$

and $x^TAx \geq 0$ for any real vector x.

by problem (b):

$Au^{(i)} = \lambda_i u^{(i)}$

let $y = Ux$.

$y^TAy = x^TU^TAUx$

$= x^T\Lambda x = \sum_i \lambda_i x_i^2 \geq 0$.

Q.E.D.